ICFP M2 Advanced Quantum Mechanics: Problem #1, Solutions to selected questions

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Beamsplitters and unitary matrices

Question 4

We consider either a beamsplitter for the electromagnetic field, or one for matter waves. The electromagnetic waves are solutions of Maxwell's equations or their quantised versions. The matter waves (consisting of atoms or non–relativistic electrons) satisfy Schrödinger's equation. In all cases, the equations governing the fields are linear, so that we may represent the action of the beamsplitter by a matrix U. If a single particle impinges on the beamsplitter with the wavefunction $|\psi_{\text{in}}\rangle = \alpha\,|A\rangle + \beta\,|B\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$, then the wavefunction once the particle exits the beamsplitter is $|\psi_{\text{out}}\rangle = \gamma\,|C\rangle + \delta\,|D\rangle$, where:

$$\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = U\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\ . \tag{1}$$

We assume that the beamsplitter introduces no losses. Then, for electromagnetic waves, the unitarity of U follows from the Stokes relations on the incident, reflected, and refracted field amplitudes near an interface. For matter waves, $U = \exp(-iH\tau/\hbar)$ is an evolution operator corresponding to a Hermitian Hamiltonian H, hence, it is also unitary.

The unitary matrix U satisfies $1 = UU^\dagger$, so that its determinant is constrained by $1 = \det(U)\det(U^\dagger) = |\det(U)|^2$. Hence, $\det(U) = e^{i\phi}$ is a phase, which is not necessarily 1. However, we may write $U = e^{i\phi/2}\,\tilde{U}$. Then, the matrix $\tilde{U}$ is unitary with $\det(\tilde{U}) = 1$. The phase $e^{i\phi/2}$ affects both output states in the same way, so that it has no consequence on any measurement. Therefore, it may be dropped, and we assume from now on that $\det(U) = 1$. Hence, we may write:

$$U = \begin{pmatrix} t & -r^* \\ r & t^* \end{pmatrix} \qquad \text{with } |t|^2 + |r|^2 = 1. \tag{2}$$

In particular, $U\,|A\rangle = t\,|C\rangle + r\,|D\rangle$ and $U\,|B\rangle = -r^*\,|C\rangle + t^*\,|D\rangle$. The complex numbers t and r are the amplitude transmission and reflection coefficients for the (electromagnetic or matter–wave) field. The corresponding intensity transmission and reflection coefficients are the real numbers $T = |t|^2$ and $R = |r|^2$, which satisfy $T + R = 1$. The phases of the complex numbers t and r contain information about optical lengths and reflections (cf. in particular the minus sign affecting the top right coefficient) which are not encoded in the real numbers T and R.

The *2 complex numbers* t and r ($|t|^2 + |r|^2 = 1$) may be parametrised by *3 real numbers*[1] α, β, and γ:

$$\begin{cases} t = \cos(\beta/2)e^{i(\alpha+\gamma)/2} \\ r = \sin(\beta/2)e^{i(\alpha-\gamma)/2}, \end{cases} \qquad \text{where } 0 \le \alpha \le 2\pi, \quad 0 \le \beta \le \pi, \quad 0 \le \gamma \le 2\pi. \tag{3}$$

[1] These three real numbers have a simple physical interpretation. The group SU(2) consisting of the 2×2 unitary matrices with determinant 1 represents all possible 3D rotations. Each 3D rotation may be specified by three Euler's angles, which correspond to the three real numbers α, β, and γ.

Output fields for distinguishable particles, fermions, and bosons

Questions 6–7: distinguishable particles

We first consider the case of distinguishable particles. This regime may be reached using both photons and electrons by sending the two incident particles onto the beamsplitter one after the other, with a time delay in between them which is chosen longer than the ratio l/v of their coherence length l to their velocity v. For photons, $v = c$ and the coherence length l is set by the shape of the wavepacket. For electrons or atoms, $v \ll c$ and $l = \Lambda_T$ is the de Broglie wavelength at the temperature T, given by $\Lambda_T = [2\pi\hbar^2/(mk_\mathrm{B}T)]^{1/2}$.

For distinguishable particles, there is a two–dimensional subspace of input states $|\Psi_\mathrm{in}\rangle$ representing particles entering the beamsplitter through different ports. This subspace is spanned by the orthogonal two–particle input states $|1:A,2:B\rangle$ (particle 1 enters through port A and particle 2 through port B) and $|1:B,2:A\rangle$ (particle 1 enters through port B and particle 2 through port A). Both of these basis states are tensor products involving the states of particles 1 and 2:

$$|1:A,2:B\rangle = |1:A\rangle \otimes |2:B\rangle \quad \text{and} \quad |1:B,2:A\rangle = |1:B\rangle \otimes |2:A\rangle \ . \tag{4}$$

Any normalised linear combination of the two states of Eq. (4) is a suitable input state $|\Psi_\mathrm{in}^\mathrm{dist}\rangle$:

$$|\Psi_\mathrm{in}^\mathrm{dist}\rangle = u\,|1:A,2:B\rangle + v\,|1:B,2:A\rangle \ , \quad \text{with } |u|^2+|v|^2=1. \tag{5}$$

We first consider an input state which is an arbitrary two–particle product state: $|1:\phi_1,2:\phi_2\rangle = |1:\phi_1\rangle \otimes |2:\phi_2\rangle$. Here, $|\phi_1\rangle$ and $|\phi_2\rangle$ are single–particle input states, i.e. arbitrary linear combinations of $|A\rangle$ and $|B\rangle$. The beamsplitter acts in the same way on each incident particle, hence, the corresponding output state is $|1:U\phi_1\rangle \otimes |2:U\phi_2\rangle$.

Then, assuming for instance $u=1$ and $v=0$ in Eq. (5), the obtained output state is:

$$|\Psi_\mathrm{out}^\mathrm{dist}\rangle = -r^*t\,|1:C,2:C\rangle + |t|^2\,|1:C,2:D\rangle - |r|^2\,|1:D,2:C\rangle + rt^*\,|1:D,2:D\rangle \ . \tag{6}$$

If the intensity transmission and reflection coefficients are equal, $|t|^2 = |r|^2 = 1/2$, the beamsplitter is said to be symmetric. Under this assumption, r and t depend only on two of the three Euler angles of Eq. (3): $t = e^{i(\alpha+\gamma)/2}/\sqrt{2}$ and $r = e^{i(\alpha-\gamma)/2}/\sqrt{2}$. Then, Eq. (6) reduces to:

$$|\Psi_\mathrm{out}^\mathrm{dist}\rangle = \frac{1}{2}\left(-e^{i\gamma}\,|1:C,1:C\rangle + |1:C,1:D\rangle - |1:D,1:C\rangle + e^{-i\gamma}\,|1:D,1:D\rangle\right) \ , \tag{7}$$

where γ is one of the Euler angles of Eq. (3). Under this assumption of a symmetric beamsplitter, the four possible detection events (both particles detected at port C; particle 1 detected at port C and particle 2 at port D; particle 1 at port D and particle 2 at port C; both particles detected at port D) will occur with equal probabilities, namely 1/4 for each possible outcome.

Identical particles

We now turn to identical particles. We only consider pure input states. We further assume that the two incident identical particles (2 photons, or 2 electrons, or 2 atoms) enter the beamsplitter through different ports. Both for two fermions (e.g. electrons) and for two bosons (e.g. photons or ^{87}Rb atoms), the subspace of permissible input states $|\Psi_\mathrm{in}\rangle$ is smaller than for distinguishable particles: it now has dimension 1. For fermions, the wavefunction must be antisymmetric with respect to the exchange of the two particles; for bosons, it must be symmetric. In both cases, the permissible input states are proportional to the two–particle wavefunction $|\Psi_\mathrm{in}\rangle$ defined as:

$$|\Psi_\mathrm{in}\rangle = \frac{1}{\sqrt{2}}\left(|1:A,2:B\rangle + \epsilon\,|1:B,2:A\rangle\right) \ , \tag{8}$$

where $\epsilon = -1$ for fermions and $\epsilon = +1$ for bosons.

Exploiting the linearity of the action of the beamsplitter, as well as its action on product states, we find the following output state:

$$|\Psi_{\text{out}}\rangle = \frac{1}{\sqrt{2}}[\ (1+\epsilon)(-r^*t\,|1:C,2:C\rangle + rt^*\,|1:D,2:D\rangle) \\ +(|t|^2-\epsilon|r|^2)(|1:C,2:D\rangle + \epsilon\,|1:D,2:C\rangle)\ ]\ . \tag{9}$$

We now discuss the Fermi and Bose cases separately.

Questions 9–10: fermions

The case of Fermi statistics corresponds to $\epsilon = -1$ in Eqs. (8) and (9). The first line on the right–hand side of Eq. (9) cancels out. The transmission and reflection coefficients drop out as well thanks to the relation $|t|^2 + |r|^2 = 1$. Hence, the fermionic output state reads:

$$|\Psi_{\text{out}}^{\text{Fermi}}\rangle = \frac{1}{\sqrt{2}}\,(|1:C,2:D\rangle - |1:D,2:C\rangle)\ . \tag{10}$$

It is antisymmetric as expected. All antisymmetric output states are proportional to this one (meaning that the subspace of antisymmetric output states has dimension 1), hence, the Pauli exclusion principle fully determines the output state $|\Psi_{\text{out}}^{\text{Fermi}}\rangle$ (up to a phase which does not affect the measurement results). This explains why Eq. (10) holds regardless of the values of t and r.

Questions 12–13: bosons

For bosons, $\epsilon = +1$ in Eqs. (8) and (9). The bosonic output state reads:

$$|\Psi_{\text{out}}^{\text{Bose}}\rangle = \frac{1}{\sqrt{2}}[\ 2(-r^*t\,|1:C,2:C\rangle + rt^*\,|1:D,2:D\rangle) \\ +(|t|^2-|r|^2)(|1:C,2:D\rangle + |1:D,2:C\rangle)\ ]\ . \tag{11}$$

Unlike for fermions, $|\Psi_{\text{out}}^{\text{Bose}}\rangle$ does depend on t and r. This is because the subspace of symmetric output states has dimension 3 (rather than 1 for antisymmetric output states): it is spanned by the three states $|1:C,2:C\rangle$, $|1:D,2:D\rangle$, and $(|1:C,2:D\rangle + |1:D,2:C\rangle)/\sqrt{2}$.

If the beamsplitter is assumed to be symmetric ($|r|^2 = |t|^2$), the second line of Eq. (11) vanishes, and $|\Psi_{\text{out}}^{\text{Bose}}\rangle$ reduces to:

$$|\Psi_{\text{out}}^{\text{Bose}}\rangle = \frac{1}{\sqrt{2}}\left(-e^{i\gamma}\,|1:C,2:C\rangle + e^{-i\gamma}\,|1:D,2:D\rangle\right)\ . \tag{12}$$

For a symmetric beamsplitter, the effect is maximal: No coincidence counts are ever recorded on the output side of the beamsplitter, and both bosons always exit from the same output port (2 bosons exit from C, or 2 bosons exit from D) with equal probabilities, namely 1/2.

This effect may be understood as second–order bosonic interference. A coincidence event may occur in two ways: either both particles are reflected ($A \to C$ and $B \to D$, probability amplitude $tt^* = |t|^2$), or both particles are transmitted ($A \to D$ and $B \to C$, probability amplitude $r(-r^*) = -|r|^2$). These two processes cannot be distinguished by the detectors on the output side of the beamsplitter, therefore they interfere. The key point is that the two probability amplitudes have opposite signs, so that the interference is destructive, leading to a total suppression of the coincidence events for a symmetric beamsplitter.

Observing second–order interference with fermions and bosons

First–order and second–order observables: Questions 5, 8, 11, 14

The observables appearing in Table 1 are chosen as follows.

	1 in A, 0 in B	1 in A, 1 in B disting.	1 in A, 1 in B fermions	1 in A, 1 in B bosons
$\langle N_C \rangle/(N_A + N_B)$	1/2	1/2	1/2	1/2
$\Delta N_C^2/\langle N_C \rangle$	1/2	1/2	0	1
$\langle N_{CD} \rangle/N$	0	1/2	1	0

Table 1 First– and second–order observables characterising the output states after the beamsplitter for *(i)* a single incident particle, *(ii)* two distinguishable particles, *(iii)* two fermions, *(iv)* two bosons. In the three cases where two particles collide on the beamsplitter, they enter through different ports. In all cases except the fermionic one, the beamsplitter is symmetric ($|t|^2 = |r|^2$).

For each of the four considered cases (one particle; two distinguishable particles entering through different ports; two fermionic particles entering through different ports; two bosonic particles entering through different ports), we perform N successive runs of the experiment. These realisations are independent and identically distributed, hence, their contributions to the average particle count $\langle N_C \rangle$ in the output port C, the variance ΔN_C^2 on the particle number in the same port, and the average coincidence count $\langle N_{CD} \rangle$ add up. Therefore, these three quantities are proportional to N. In order to eliminate their dependence on N, we wish to divide them by other quantities which also scale with N.

- We choose to divide the average value $\langle N_C \rangle$ by $(N_A + N_B)$ so as to obtain the fraction of incident particles which exit the beamsplitter through port C. The denominator $(N_A + N_B)$ is equal to N for single–particle input states (second column of Table 1), and to $2N$ in all cases where two particles collide at the beamsplitter (columns 3–5 of Table 1).
- We now turn to the variance ΔN_C^2. We wish to consider a local observable, which depends on measurements at port C but not at port D: this leaves a choice between dividing it by N or by the average value $\langle N_C \rangle$. We choose to divide it by $\langle N_C \rangle$ so as to probe how far the random variable N_C is from following a Poisson distribution (for which $\Delta N_C^2 = N_C$).
- The ratio $\langle N_{CD} \rangle /N$ represents the fraction of the runs leading to coincidence detections.

We now assume the beamsplitter to be symmetric ($|t|^2 = |r|^2$), except for the fermionic case where the values of t and r play no role (see Eq. (10)). The first–order observable $\langle N_C \rangle /(N_A+N_B)$ does not discriminate between any of the four columns of Table 1 (neither would its counterpart $\langle N_D \rangle /(N_A+N_B)$). The second–order local observable $\Delta N_C^2/ \langle N_C \rangle$, which characterises the fluctuations (or "noise") on the particle number, does discriminate between two distinguishable particles, two fermions, and two bosons. For two distinguishable particles, it takes the same value 1/2 as when a single particle arrives on the beamsplitter. For two identical fermions, it vanishes (see the experimental Fig. 3 in the problem set). Finally, for two identical bosons, it is equal to 1.

Coincidence counts: Questions 15 and 16

One way to distinguish between experiments involving one and two distinguishable particles is to examine the coincidence counts $\langle N_{\text{CD}} \rangle /N$. This is also a second–order observable, but it is non–local, in the sense that it requires simultaneous measurements at the two output ports C and D. The quantity $\langle N_{\text{CD}} \rangle /N = 0$ for any experiment involving a single particle: it is "indivisible" and must therefore exit either through port C or through port D. By contrast, non–zero coincidence counts are obtained with two distinguishable particles ($\langle N_{\text{CD}} \rangle /N = 1/2$) and with two fermions ($\langle N_{\text{CD}} \rangle /N = 1$, meaning that two fermions always exit through different output ports). However, for two bosons, the coincidence counts vanish as a consequence of destructive interference: this is the Hong–Ou–Mandel effect, illustrated by Fig. 4 in the problem set which results from an experiment with photons.

The experimental signal shown on that figure is the photon coincidence number $\langle N_{\text{CD}} \rangle /N$ plotted as a function of the position of the beamsplitter. This position determines the difference in optical paths between the photons entering the beamsplitter through ports A and B. If this difference is chosen to be zero, the two incident photons are indistinguishable and the bosonic interference effect occurs, leading to $\langle N_{\rm CD}\rangle/N = 0$. On the contrary, if the difference in optical paths is greater than the coherence length of the photons, they behave as distinguishable particles, so that $\langle N_{\rm CD}\rangle/N = 1/2$. Hence, the width of the Hong–Ou–Mandel dip is a measurement of the coherence length of the incident photons, which is directly related to the spread in momenta, or in frequencies, of the photon wavepackets.

Beamsplitters and second quantisation

Question 17

We start by introducing the operator $\mathcal{U}$, which generalises the role of U to input (and output) states containing an arbitrary number of particles:

$$\mathcal{U}\,|0_A 0_B\rangle = |0_C 0_D\rangle \quad \text{(vacuum stays vacuum)}, \tag{13}$$

$$\mathcal{U}\,|\phi\rangle = U\,|\phi\rangle \quad \text{for any single–particle state } |\phi\rangle, \tag{14}$$

$$\mathcal{U}\,|\phi_1\rangle \ldots |\phi_N\rangle = (U\,|\phi_1\rangle)\ldots(U\,|\phi_N\rangle) \quad \text{for } N \text{ particles in a product state.} \tag{15}$$

The operator $\mathcal{U}$ represents the effect of the beamsplitter. It transforms an input state (single–particle states are linear combinations of the input ports $|A\rangle$ and $|B\rangle$) into an output state (single–particle states are linear combinations of the output ports $|C\rangle$ and $|D\rangle$).

We use $\mathcal{U}$ to define the operators $\tilde{c} = \mathcal{U}^\dagger c\,\mathcal{U}$ and $\tilde{d} = \mathcal{U}^\dagger d\,\mathcal{U}$. Note that the operators c and d act on the output states whereas the operators $\tilde{c}$ and $\tilde{d}$ act on the input states.

We now wish to express the annihilation operators $\tilde{c}$ in terms of the input–state operators a and b. For this purpose, we first construct the single–particle input state $|C_{\rm in}\rangle$ such that, after the beamsplitter, the output state is $|C\rangle$. This definition is summarised by the following relation: $U\,|C_{\rm in}\rangle = |C\rangle$. Just like any input state, $|C_{\rm in}\rangle$ is a linear combination of $|A\rangle$ and $|B\rangle$: $|C_{\rm in}\rangle = \alpha\,|A\rangle + \beta\,|B\rangle$. According to Eq. (1), α and β satisfy:

$$U\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix} \quad , \quad \text{so that} \quad \begin{pmatrix}\alpha\\ \beta\end{pmatrix} = U^\dagger\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}t^*\\ -r\end{pmatrix} \ . \tag{16}$$

Therefore, $|C_{\rm in}\rangle = t^*\,|A\rangle - r\,|B\rangle$.

We now use the key property that creation operators depend linearly on single–particle states ("creation operators transform like kets"): $a^\dagger_{\lambda_1|u_1\rangle+\lambda_2|u_2\rangle} = \lambda_1 a^\dagger_{|u_1\rangle} + \lambda_2 a^\dagger_{|u_2\rangle}$. When it is applied to the state $|C_{\rm in}\rangle$, this linearity property yields:

$$\tilde{c}^\dagger = t^* a^\dagger - r b^\dagger \ . \tag{17}$$

Similarly, we introduce the input state $|D_{\rm in}\rangle = \gamma\,|A\rangle + \delta\,|B\rangle$ such that, after the beamsplitter, the output state is $U\,|D_{\rm in}\rangle = |D\rangle$. Thanks to Eq. (1), the coefficients γ and δ satisfy:

$$U\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix} \quad , \quad \text{so that} \quad \begin{pmatrix}\gamma\\ \delta\end{pmatrix} = U^\dagger\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}r^*\\ t\end{pmatrix} \ . \tag{18}$$

Therefore, $|D_{\rm in}\rangle = r^*\,|A\rangle + t\,|B\rangle$, and the creation operator $\tilde{d}^\dagger$ reads:

$$\tilde{d}^\dagger = r^* a^\dagger + t b^\dagger \ . \tag{19}$$

Finally, we transpose Eqs. (17) and (19), and collect the results in matrix form:

$$\begin{pmatrix}\tilde{c}\\ \tilde{d}\end{pmatrix} = \begin{pmatrix}t & -r^*\\ r & t^*\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = U\begin{pmatrix}a\\ b\end{pmatrix} \ . \tag{20}$$

As a sanity check, we note that in the quasi–classical limit (all states are coherent states, and the annihilation operators a, b, c, d, may be replaced by the field amplitudes $\mathcal{E}_A$, $\mathcal{E}_B$, $\mathcal{E}_C$, $\mathcal{E}_D$), Eq. (20) reduces to the expected classical law:

$$\begin{pmatrix} \mathcal{E}_C \\ \mathcal{E}_D \end{pmatrix} = \begin{pmatrix} t & -r^* \\ r & t^* \end{pmatrix} \begin{pmatrix} \mathcal{E}_A \\ \mathcal{E}_B \end{pmatrix} = U \begin{pmatrix} \mathcal{E}_A \\ \mathcal{E}_B \end{pmatrix} \ . \tag{21}$$

Question 20

The input state $|\Psi_{\rm in}\rangle$ satisfies:

$$|\Psi_{\rm in}\rangle = \frac{a^{\dagger N}}{\sqrt{N!}} \frac{b^{\dagger N}}{\sqrt{N!}} |0_A 0_B\rangle \ . \tag{22}$$

The output state $|\Psi_{{\rm out},C}\rangle$ is given by:

$$|\Psi_{{\rm out},C}\rangle = \frac{c^{\dagger 2N}}{\sqrt{(2N)!}} |0_C 0_D\rangle = \frac{c^{\dagger 2N}}{\sqrt{(2N)!}} \mathcal{U} |0_A 0_B\rangle \ , \tag{23}$$

where the last step follows from Eq. (13).

Therefore, the probability amplitude for going from $|\Psi_{\rm in}\rangle$ to $|\Psi_{{\rm out},C}\rangle$ is:

$$\langle \Psi_{{\rm out},C} | \mathcal{U} | \Psi_{\rm in} \rangle = \langle 0_A 0_B | \mathcal{U}^\dagger \frac{c^{2N}}{\sqrt{(2N)!}} \mathcal{U} \frac{a^{\dagger N}}{\sqrt{N!}} \frac{b^{\dagger N}}{\sqrt{N!}} |0_A 0_B\rangle \ . \tag{24}$$

Now, we use $\mathcal{U}\mathcal{U}^\dagger = \mathbb{1}$ to replace the operator c in Eq. (24) by $\tilde{c} = \mathcal{U}^\dagger c \mathcal{U}$:

$$\langle \Psi_{{\rm out},C} | \mathcal{U} | \Psi_{\rm in} \rangle = \langle 0_A 0_B | \frac{\tilde{c}^{2N}}{\sqrt{(2N)!}} \frac{a^{\dagger N}}{\sqrt{N!}} \frac{b^{\dagger N}}{\sqrt{N!}} |0_A 0_B\rangle \ . \tag{25}$$

Finally, we use Eq. (20) to replace $\tilde{c} = ta - r^* b$:

$$\langle \Psi_{{\rm out},C} | \mathcal{U} | \Psi_{\rm in} \rangle = \langle 0_A 0_B | \frac{(ta - r^* b)^{2N}}{\sqrt{(2N)!}} \frac{a^{\dagger N}}{\sqrt{N!}} \frac{b^{\dagger N}}{\sqrt{N!}} |0_A 0_B\rangle \ . \tag{26}$$

In order to finish the calculation, we expand the binomial factor $(ta - r^* b)^{2N}$ and only retain the terms which contain $a^N b^N$: there are $\binom{2N}{N}$ such terms, each having the same contribution which is proportional to $t^N r^{*N}$. The vacuum averages $\langle 0_A | a^N a^{\dagger N} |0_A\rangle = \langle 0_B | b^N b^{\dagger N} |0_B\rangle = N!$. Therefore, the probability amplitude reduces to:

$$\langle \Psi_{{\rm out},C} | \mathcal{U} | \Psi_{\rm in} \rangle = t^N r^{*N} \sqrt{\frac{(2N)!}{N! N!}} \ , \tag{27}$$

leading to the probability $|\langle \Psi_{{\rm out},C} | \mathcal{U} | \Psi_{\rm in} \rangle|^2 = |rt|^{2N} \frac{(2N)!}{N!N!}$ for all particles exiting through port C. Note the enhancement by a factor $\binom{2N}{N}$ compared to the case of distinguishable particles.