Advanced Quantum Mechanics

Tutorials 2024–2025

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Please ask me MANY questions!

Wednesday, Sept. 18$^{\text{th}}$, 2024

Outline of the tutorials for the first half ot the semester

- **Problem 1:** two–particle interference
- **Problem 2:** coherence and correlations in quantum gases
- **Problem 3:** lattice models, superfluid/Mott insulator transition

All problems describe experiments that have actually been performed

They all contain elements of theory and introduce calculation techniques

They all contain both standard questions and (very?) hard questions

Supplementary exercises

- **Supplementary questions appear in golden throughout my slides.**

  Additionally, I have handed out **a set of 'review' exercises**

  They are not mandatory. I shall not solve them in class.

  I encourage you to ask me for hints and to discuss them with me.

- **Coherent states:** Section 3, pp. 3–4 **standard but useful**

  Quasiclassical properties; fluctuations in the particle number

- **Symmetries of the 2D harmonic oscillator:** Section 2.2, pp. 2–3

  A less standard discussion of the role of conserved quantities

  Link with classical physics; link with the hydrogen atom

- **Identical particles in lower dimensions:** Section 4, p. 4

  Question 2 is difficult: discussions are welcome; **not** exam material.

Mailing list for quantum mechanics tutorials

- If you are *not* enrolled in the Quantum Mechanics track

  [i.e. condensed–matter track, theory track, . . .]

  and wish to receive my problem texts, slides, . . . :

Send me a mail at your earliest convenience

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A bird's eye view of the problem

Problem #1:
Two–particle interference
with bosons and fermions

- Identical quantum particles: bosons, fermions
- Two–particle interference occurs *because the particles are identical*

  The effect depends on whether the particles are bosons or fermions

  Both flavours have been observed!
- Creation and annihilation operators

Review: Young's fringes

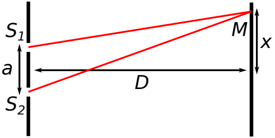
an example of single–particle interference

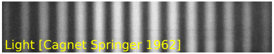
[E. Hecht, *Optics*, 5th edition, Pearson (2017), chapter 9]

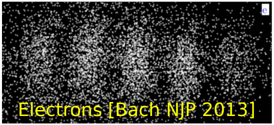
Review: Single–particle interference (questions 1–3)

- The key ingredient is a **wave:** water, laser light, electrons, cold atoms ...

Famous example: Young's fringes







- Assume distance D between mask and screen $\gg$ hole separation a:

$$|\mathbf{S_1M}| = [(x - a/2)^2 + D^2]^{1/2} \approx D + (x - a/2)^2/(2D)$$

$$|\mathbf{S_2M}| = [(x + a/2)^2 + D^2]^{1/2} \approx D + (x + a/2)^2/(2D)$$

- Measured signal: intensity or detection probability

$$I(x) \propto \left| \frac{e^{ik|\mathbf{S_1M}|}}{|\mathbf{S_1M}|} + \frac{e^{ik|\mathbf{S_2M}|}}{|\mathbf{S_2M}|} \right|^2 \approx \frac{1}{D^2} \left| e^{ik|\mathbf{S_1M}|} + e^{ik|\mathbf{S_2M}|} \right|^2 \propto 1 + \cos[k(|\mathbf{S_2M}| - |\mathbf{S_1M}|)]$$

$I(x) \propto 1 + \cos\left[2\pi \frac{a}{\lambda D} x\right]$ so that fringe spacing is $\lambda D/a$

Single–particle interference: discussion

- We are used to thinking of **electrons** as particles

  To prove that they are behaving as waves, make them interfere!

  The wavelength is set by the de Broglie relation $\lambda = h/p$

- We are used to thinking of **light** as a wave

  Reaching and characterising the single–photon regime is challenging

  Photons have no mass and no charge, they go fast

  Lowering the output rate of a laser is not enough

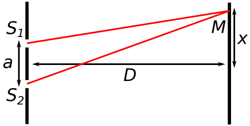
  because laser output state obeys Poisson statistics: $\Delta N = \langle N \rangle^{1/2} \sim 1$ for $\langle N \rangle = 1$

- The interference fringes build up one detection event at a time

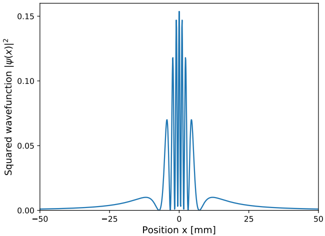
  This is a single–particle effect

  Not affected by the particles being bosons, fermions, or distinguishable!

Young's fringes in single–atom regime: simulation



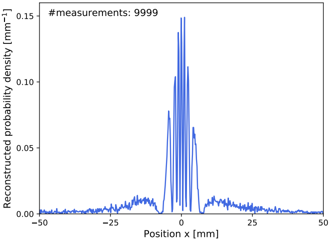
- Relaxing the hypothesis of very large D, calculate squared wavefunction $|\psi(x,D)|^2$ on the screen



- Probability distribution $|\psi(x,D)|^2 \approx 0$ for $|x| \gg D$

$\lambda = 1\,\mu\mathrm{m}$, $a = 4\,\mu\mathrm{m}$, $D = 4\,\mathrm{mm}$

experiment with neon atoms:
[Shimizu, PRA **46**, R17(R) (1992)]



- Draw N random values of x with probability $|\psi(x,D)|^2$

each value of x represents the impact position of 1 electron

After each new draw, update the histogram

What is the correct normalisation
for the histogram to match the probability distribution?

My code for simulation and animation ($<$ 100 *Python lines*) *is available online*

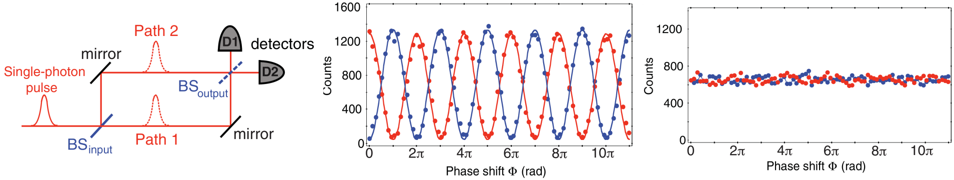
Single–particle quantum interference: 1965 vs 2007

- Strange effects discussed by Feynman as ‘thought experiments’, later observed.
  [Feynman Lectures on Physics vol. III, Addison–Wesley (1965), ch. 1]

- **Compare 2 related single–photon exp. setups** involving 1 or 2 beamsplitters (BS):

$\mathrm{BS}_{\mathrm{input}}$ (solid line) is always present; $\mathrm{BS}_{\mathrm{output}}$ (dashed line) may or may not be present

Are the interference fringes observed with one or two beamsplitters ?



- **Wheeler’s delayed–choice experiment:** (proposed in 1984, realised e.g. 2007)

Decide whether or not to include $\mathrm{BS}_{\mathrm{output}}$ **once the photon has already passed** $\mathrm{BS}_{\mathrm{input}}$

“When the photon exits $\mathrm{BS}_{\mathrm{input}}$, should it behave like a wave or like a particle?”

This experiment probes the non–local character of wave/particle duality

[V. Jacques, A. Aspect et al, Science **315**, 966 (2007)]

Which of these two phenomena are we probing?

1. **Superposition?** $|\psi\rangle = (|\phi_1\rangle + |\phi_2\rangle)/\sqrt{2}$
2. **Entanglement?** $|\psi\rangle = (|\phi_1\rangle\,|\phi_1\rangle + |\phi_2\rangle\,|\phi_2\rangle)/\sqrt{2}$

Review: Bosons and fermions

- **Spin–statistics relation:**

  Particles with half–integer spin are fermions

  (electrons, protons, neutrons all have spin 1/2)

  Particles with integer spin are bosons

  (photons have spin 1)

- **Atoms** $^{A}_{Z}\mathrm{X}$ may be either fermions or bosons

  Z = atomic number = number of electrons; A = mass number = number of nucleons.

  $^{85}_{37}\mathrm{Rb}$ and $^{87}_{37}\mathrm{Rb}$ are both bosons

  $^{7}_{3}\mathrm{Li}$ is a boson, $^{6}_{3}\mathrm{Li}$ is a fermion

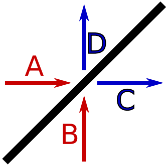
  $^{4}_{2}\mathrm{He}$ is a boson, $^{3}_{2}\mathrm{He}$ is a fermion

- Other bosons in this course: phonons, “Cooper pairs”, . . .

Quantum description of a beamsplitter

[Grynberg, Aspect, Fabre, *Introduction to Quantum Optics*, CUP (2010), chapter 5]

Modelling a beamsplitter (question 4)



- Two input ports A,B, two output ports C,D
- input state $|\psi_{\rm in}\rangle = \alpha\,|A\rangle + \beta\,|B\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$
  output state $|\psi_{\rm out}\rangle = \gamma\,|C\rangle + \delta\,|D\rangle$
- Linear relation between input $|\psi_{\rm in}\rangle$and output $|\psi_{\rm out}\rangle$
  (Maxwell equations or Schrödinger equation)

Write the linear relation in matrix form: $\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = U\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

- The matrix U is unitary: $U\,U^\dagger = \mathbb{1}$
  For Maxwell's equations: conservation of energy
  For the Schrödinger equation: U is an evolution operator $U = \exp(-iHt/\hbar)$
- $1 = \det(U\,U^\dagger) = |\det U|^2$, so $\det U = \exp(i\phi)$
  $U = e^{i\phi/2}U_1$ with $\det U_1 = 1$: the phase ϕ affects both output ports in the same way.

$$U = e^{i\phi/2}\begin{pmatrix} t & -r^* \\ r & t^* \end{pmatrix} \qquad \text{with } |t|^2 + |r|^2 = 1$$

Complex numbers r and t: amplitude transmission and reflection coefficients

A single particle impinges on the beamsplitter (qu. 5a)

- Input state: $|\psi_{\text{in}}\rangle = \alpha\,|A\rangle + \beta\,|B\rangle$

  Output state: $|\psi_{\text{out}}\rangle = \gamma\,|C\rangle + \delta\,|D\rangle$ with $\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = U\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

- **If particle enters through $|A\rangle$:** $|\psi_{\text{in}}\rangle = |A\rangle$, then $|\psi_{\text{out}}\rangle = t\,|C\rangle + r\,|D\rangle$

  It exits through $|C\rangle$ with probability $|t|^2$ and $|D\rangle$ with probability $|r|^2 = 1 - |t|^2$

- Random variable

  $n_C =$ particle exits from port $|C\rangle$ $= \begin{cases} 1 & \text{with probability } |t|^2 \\ 0 & \text{with probability } 1 - |t|^2 \end{cases}$

- Mean value: $\langle n_C\rangle = |t|^2 \times 1 + (1 - |t|^2) \times 0 = |t|^2$

  $$\langle n_C^2\rangle = |t|^2 \times 1^2 + (1 - |t|^2) \times 0^2 = |t|^2$$

  Variance: $\Delta n_C^2 = \langle n_C^2\rangle - \langle n_C\rangle^2 = |t|^2(1 - |t|^2)$

  Mean square deviation: $\Delta n_C = (\Delta n_C^2)^{1/2} = |t|(1 - |t|^2)^{1/2}$

N consecutive measurements (question 5b)

- The N measurements are independent; their outcomes follow the same law

  "Sum of N idependent and identically distributed random variables"

  Total number of particles exiting through $|C\rangle$: $$N_C = \sum_{k=1}^{N} n_C^{(k)}$$

- The mean values add up, and so do the variances

  $$\langle N_C \rangle = N \langle n_C \rangle = N |t|^2 \quad \text{and} \quad \Delta N_C^2 = N \Delta n_C^2 = N |t|^2 (1 - |t|^2)$$

  Beware: Standard deviations do not add up! $\Delta N_C = \sqrt{N}\, |t| (1 - |t|^2)^{1/2} = \sqrt{N}\, \delta n_C$

ADVANCED QUANTUM MECHANICS

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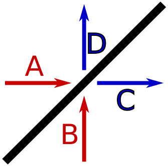
SUMMARY: a single particle impinges on beamsplitter



- Two input ports A,B, two output ports C,D

  input state $|\psi_{\rm in}\rangle = \alpha\,|A\rangle + \beta\,|B\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$

  output state $|\psi_{\rm out}\rangle = \gamma\,|C\rangle + \delta\,|D\rangle$

  Linear, unitary relation between $|\psi_{\rm in}\rangle$ and $|\psi_{\rm out}\rangle$

  If $|\psi_{\rm in}\rangle = |A\rangle$, then $|\psi_{\rm out}\rangle = t\,|C\rangle + r\,|D\rangle$

- Random variable

  $$n_C = \text{ particle exits from port } |C\rangle = \begin{cases} 1 & \text{with probability } |t|^2 \\ 0 & \text{with probability } 1 - |t|^2 \end{cases}$$

  Mean value $\langle n_C\rangle = |t|^2$, variance $\Delta n_C^2 = \langle n_C^2\rangle - \langle n_C\rangle^2 = |t|^2(1-|t|^2)$

- N consecutive measurements:

  the mean values add up, and so do the variances

  $\langle N_C\rangle = N\,\langle n_C\rangle = N\,|t|^2$ and $\Delta N_C^2 = N\,\Delta n_C^2 = N\,|t|^2(1-|t|^2)$

Relevant observables (questions 5c, 5d)

- Fraction of atoms exiting from port C: $\langle N_C \rangle / N = |t|^2$

  Intensive quantity, does not depend on N

- Ratio $\Delta N_C^2 / \langle N_C \rangle = 1 - |t|^2$

  Does not depend on N

  Probes how far N_C is from a Poisson–distributed variable

($P(n) = e^{-\lambda} \lambda^n / n!$: photons coming out of a laser, electronic shot noise in a conductor ...)

How to access them in an experiment ?

- The experimentally accessible quantity is often the current $I(t)$

  The current at output port C satisfies $N_C = \int_0^T dt\, I_C(t)$

  $\langle N_C \rangle = \int_0^T dt\, \langle I_C(t) \rangle$ is related to the average current

  $\Delta N_C^2 = \iint_0^T dt_1 dt_2\, (\langle I(t_1) I(t_2) \rangle - \langle I(t_1) \rangle \langle I(t_2) \rangle)$ probes current fluctuations

Two–particle interference at a beamsplitter

experiments with fermions and bosons

[Liu et al, Nature **391**, 263 (1998)]

[Hong, Ou, & Mandel, Physical Review Letters **59**, 2044 (1987)]

Distinguishable particles

2 distinguishable particles (question 6)

- May be reached both with photons and with electrons

  Send the particles onto the beamsplitter one after the other,

  time delay greater than l/v (coherence length over velocity)

  For electromagnetic waves, l is set by the shape of the wavepacket

  For quantum particles, l is the thermal de Broglie wavelength $\Lambda_T = [2\pi\hbar^2/(mk_\mathrm{B}T)]^{1/2}$

- **Input state** $|1:\phi_1,\ 2:\phi_2\rangle = |1:\phi_1\rangle \otimes |2:\phi_2\rangle$

  specifies the state of particle 1 and particle 2:

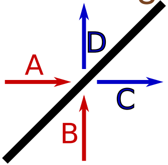
  particle 1 in state $|\phi_1\rangle = \alpha_1 |A\rangle + \beta_1 |B\rangle$, particle 2 in state $|\phi_2\rangle = \alpha_2 |A\rangle + \beta_2 |B\rangle$

- The beamsplitter acts in the same way on each incident particle

  Output state: $|1:U\phi_1,\ 2:U\phi_2\rangle = |1:U\phi_1\rangle \otimes |2:U\phi_2\rangle$

  $U = \begin{pmatrix} t & -r^* \\ r & t^* \end{pmatrix}$, $\quad U|A\rangle = t|C\rangle + r|D\rangle$ and $U|B\rangle = -r^*|C\rangle + t^*|D\rangle$

2 distinguishable particles entering ports $|A\rangle$ & $|B\rangle$ (qu. 7)



- 2–dimensional subspace of input states:

  $|\Psi_{\text{in}}^{\text{dist}}\rangle = u\,|1:A,2:B\rangle + v\,|1:B,2:A\rangle$ with $|u|^2+|v|^2=1$

  Focus on $|\Psi_{\text{in}}^{\text{dist}}\rangle = |1:A,2:B\rangle$

- Output state: **Try reproducing this calculation!**

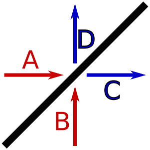
$$|\Psi_{\text{out}}^{\text{dist}}\rangle = -r^*t\,|1:C,2:C\rangle + |t|^2\,|1:C,2:D\rangle - |r|^2\,|1:D,2:C\rangle + rt^*\,|1:D,2:D\rangle$$

- Four possible outcomes:

outcome	probability	$\lvert t\rvert^2 = \lvert r\rvert^2 = 1/2$
$\lvert 1:C,2:C\rangle$	$\lvert r\rvert^2\lvert t\rvert^2$	$1/4$
$\lvert 1:C,2:D\rangle$	$\lvert t\rvert^4$	$1/4$
$\lvert 1:D,2:C\rangle$	$\lvert r\rvert^4$	$1/4$
$\lvert 1:D,2:D\rangle$	$\lvert r\rvert^2\lvert t\rvert^2$	$1/4$

- equiprobable for symmetric beamsplitter $|t|^2 = |r|^2 = 1/2$

2 distinguishable particles: predicted statistics (qu. 8)



outcome	probability	$\lvert t\rvert^2 = \lvert r\rvert^2 = 1/2$
$\lvert 1:C, 2:C\rangle$	$\lvert r\rvert^2\lvert t\rvert^2$	$1/4$
$\lvert 1:C, 2:D\rangle$	$\lvert t\rvert^4$	$1/4$
$\lvert 1:D, 2:C\rangle$	$\lvert r\rvert^4$	$1/4$
$\lvert 1:D, 2:D\rangle$	$\lvert r\rvert^2\lvert t\rvert^2$	$1/4$

- Random variable counting the particles exiting from port C:

$$n_C = \#\text{particles exiting from } |C\rangle \quad = \begin{cases} 2 & \text{with probability } 1/4 \\ 1 & \text{with probability } 1/4 + 1/4 = 1/2 \\ 0 & \text{with probability } 1/4 \end{cases}$$

Average $\langle n_C\rangle = 2 \times 1/4 + 1 \times 1/2 + 0 \times 1/4 \quad = 1$

$$\langle n_C^2\rangle = 2^2 \times 1/4 + 1 \times 1/2 + 0 \times 1/2 \quad = 3/2$$

Variance $\Delta n_C^2 = \langle n_C^2\rangle - \langle n_C\rangle^2 \quad = 1/2$

- Random variable n_{CD} testing for coincidence counts at ports C and D:

$$n_{\text{CD}} = \#\text{coincidence counts} \quad = \begin{cases} 1 & \text{with probability } 1/4 + 1/4 = 1/2 \\ 0 & \text{with probability } 1/4 + 1/4 = 1/2 \end{cases}$$

Average $\langle n_{CD}\rangle = 1 \times 1/2 + 0 \times 1/2 \quad = 1/2$

Relevant observables for two incident particles (qu. 8)

- N independent runs, each with 1 particle entering through $|A\rangle$ and 1 through $|B\rangle$

Number of particles exiting through $|C\rangle$: $\quad N_C = \sum_{k=1}^{N} n_C^{(k)}$

Number of runs yielding coincidence counts at $|C\rangle$ and $|D\rangle$: $\quad N_{CD} = \sum_{k=1}^{N} n_{CD}^{(k)}$

1. Observable related to **average particle number** $\langle N_C \rangle = N \langle n_C \rangle$: $\quad \langle N_C \rangle / (N_A + N_B)$

   For 1 incident particle: $N_A + N_B = N$, we recover the previously introduced observable

   For 2 incident particles: $N_A + N_B = 2N$

2. Observable related to **particle number fluctuations** $\Delta N_C^2 = N \Delta n_C^2$: $\quad \Delta N_C^2 / \langle N_C \rangle$

   Probes how far N_C is from a Poisson distribution

3. Observable related to **average coincidence counts** $N_{CD} = N\, n_{CD}$: $\quad \langle N_{CD} \rangle / N$

   Average fraction of runs leading to coincidence counts

All three selected observables are intensive

No interference with distinguishable particles (qu. 8 & 15)

	1 in A, 0 in B	1 in A, 1 in B disting.	1 in A, 1 in B fermions	1 in A, 1 in B bosons
$\langle N_C \rangle / (N_A + N_B)$	1/2	1/2		
$\Delta N_C^2 / \langle N_C \rangle$	1/2	1/2		
$\langle N_{CD} \rangle / N$	0	1/2		

- $N_{CD} = 0$ in experiments involving one incident particle (because it is indivisible)

  This is the only difference between 1 particle and 2 distinguishable particles

With two **distinguishable particles,** no interference is observed!

Identical particles: FERMIONS

2 fermions entering through A and B: **input** state (qu. 9)

- Distinguishable: input–state space has dimension 2, spanned by $|1:A,2:B\rangle$, $|1:B,2:A\rangle$

- Fermions: the input state must be antisymmetric under particle exchange

  the input–state subspace has dimension 1:

  $$|\Psi_{\mathrm{in}}^{\mathrm{Fermi}}\rangle = \frac{e^{i\theta_{\mathrm{in}}}}{\sqrt{2}}\left(|1:A,2:B\rangle - |1:B,2:A\rangle\right)$$ (the phase θ_{in} plays no role)

- Imposing that the particles are fermions

  reduces the dimension of the relevant Hilbert space from 2 to 1.

- Experimental consequence explained on the next few slides

2 fermions entering through A,B: **output** state (qu. 10)

- Fermions: the output state must be antisymmetric under particle exchange

  the output–state subspace has dimension 1:

$$|\Psi_{\text{out}}^{\text{Fermi}}\rangle = \frac{e^{i\theta_{\text{out}}}}{\sqrt{2}} \left(|1:C,2:D\rangle - |1:D,2:C\rangle \right)$$

- Only the relative phase $\theta_{\text{out}} - \theta_{\text{in}}$ remains to be found

  Check that $\theta_{\text{out}} = \theta_{\text{in}}$ by expanding

$$|\Psi_{\text{out}}^{\text{Fermi}}\rangle = \mathcal{U}\,|\Psi_{\text{in}}^{\text{Fermi}}\rangle \quad = \frac{e^{i\theta_{\text{in}}}}{\sqrt{2}} \left(|1:UA,2:UB\rangle - |1:UB,2:UA\rangle \right)$$

($\mathcal{U}$ generalises U to 2–particle states: see question 17)

- Pauli's exclusion principle fully determines input and output states!

The two fermions always exit from different ports

The values of the coefficients t and r play no role

2 fermions: **predicted statistics** (question 11)

$$|\Psi_{\text{out}}^{\text{Fermi}}\rangle = \frac{1}{\sqrt{2}}\left(|1:C,2:D\rangle - |1:D,2:C\rangle\right)$$

outcome	probability	$\lvert t\rvert^2 = \lvert r\rvert^2 = 1/2$
$\lvert 1:C,2:C\rangle$	0	0
$\lvert 1:C,2:D\rangle$	1/2	1/2
$\lvert 1:D,2:C\rangle$	1/2	1/2
$\lvert 1:D,2:D\rangle$	0	0

- Random variable n_C counting the particles exiting from port C: $n_C = 1$

$$\langle n_C\rangle = 2\times 0 + 1\times 1/2 + 1\times 1/2 + 0\times 0 = 1$$
$$\langle n_C^2\rangle = 4\times 0 + 1\times 1/2 + 1\times 1/2 + 0\times 0 = 1$$
$$\Delta n_C^2 = \langle n_C^2\rangle - \langle n_C\rangle^2 = 0$$

- Random variable n_{CD} testing for coincidence counts at C,D: $n_{CD} = 1$

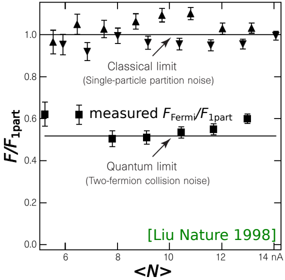
$$\langle n_{CD}\rangle = 0\times 0 + 1\times 1/2 + 1\times 1/2 + 0\times 0 = 1$$

Second–order interference with fermions (qu. 11 & 15)

	1 in A, 0 in B	1 in A, 1 in B disting.	1 in A, 1 in B fermions	1 in A, 1 in B bosons
$\langle N_C \rangle / (N_A + N_B)$	1/2	1/2	1/2	
$\Delta N_C^2 / \langle N_C \rangle$	1/2	1/2	0	
$\langle N_{CD} \rangle / N$	0	1/2	1	

- The first–order observable $\langle N_C \rangle$ does not exhibit interference.

  The second–order observables $\Delta N_c^2 / \langle N_C \rangle$ and $\langle N_{CD} \rangle / N$ exhibit interference!



- Measured observable: current noise $F = \Delta N_c^2 / \langle N_C \rangle$ normalised to expected single–particle value $F_{\rm 1part} = 1/2$
- $F / F_{\rm 1part} \approx 1$ for a single incident particle, $F / F_{\rm 1part}$ **suppressed** for fermions entering through A, B
- The suppression is not complete, probably because the electrons are scattered by many impurities [M. Büttiker Phys. World (1998)]

Identical particles: Bosons

2 bosons entering through A and B: **input** state (qu. 12)

- Distinguishable: input–state space has dimension 2, spanned by $|1:A, 2:B\rangle$, $|1:B, 2:A\rangle$

- Bosons: the input state must be symmetric under particle exchange
  the input–state subspace has dimension 1:

$$|\Psi_{\rm in}^{\rm Bose}\rangle = \frac{e^{i\theta_{\rm in}}}{\sqrt{2}} \left(|1:A, 2:B\rangle + |1:B, 2:A\rangle\right)$$

(the phase $\theta_{\rm in}$ plays no role)

  Unlike for fermions, there are two other allowed states: $|1:A, 2:A\rangle$ and $|1:B, 2:B\rangle$
  They represent both particles entering the beamsplitter through the same port.

- Imposing that the particles are bosons
  reduces the dimension of the relevant Hilbert space from 2 to 1.

2 bosons entering through A,B: **output** state (qu. 13)

- For bosons, the output state subspace has dimension 3, spanned by:
  $|1:C,2:C\rangle$, $|1:D,2:D\rangle$, $\frac{1}{\sqrt{2}}\left(|1:C,2:D\rangle+|1:D,2:C\rangle\right)$

- Calculate the output state corresponding to $|\Psi_{\rm in}^{\rm Bose}\rangle$: $|\Psi_{\rm out}^{\rm Bose}\rangle=\mathcal{U}\,|\Psi_{\rm in}^{\rm Bose}\rangle$

$$|\Psi_{\rm out}^{\rm Bose}\rangle=\frac{1}{\sqrt{2}}[\,2(-r^*t\,|1:C,2:C\rangle+rt^*\,|1:D,2:D\rangle)$$
$$+(|t|^2-|r|^2)(|1:C,2:D\rangle+|1:D,2:C\rangle)\,]$$

- For a symmetric beamsplitter, that is, if $|t|^2=|r|^2=1/2$
  We may write $t=\exp[i(\alpha+\gamma)/2]/\sqrt{2}$, $r=\exp[i(\alpha-\gamma)/2]/\sqrt{2}$

$$|\Psi_{\rm out}^{\rm Bose}\rangle=\frac{1}{\sqrt{2}}\left(-e^{i\gamma}\,|1:C,2:C\rangle+e^{-i\gamma}\,|1:D,2:D\rangle\right)$$

Both bosons always exit from the same output port!

2 bosons: **predicted statistics** (question 14)

$$|\Psi_{\text{out}}^{\text{Bose}}\rangle = \frac{1}{\sqrt{2}}\left(-e^{i\gamma}\,|1:C,2:C\rangle + e^{-i\gamma}\,|1:D,2:D\rangle\right)$$

outcome	probability	$\lvert t\rvert^2 = \lvert r\rvert^2 = 1/2$
$\lvert 1:C,2:C\rangle$	$2\lvert r\rvert^2\lvert t\rvert^2$	$1/2$
$\lvert 1:C,2:D\rangle$	$(\lvert t\rvert^2-\lvert r\rvert^2)^2/2$	0
$\lvert 1:D,2:C\rangle$	$(\lvert t\rvert^2-\lvert r\rvert^2)^2/2$	0
$\lvert 1:D,2:D\rangle$	$2\lvert r\rvert^2\lvert t\rvert^2$	$1/2$

- Random variable n_C counting the particles exiting from port C:

$$\begin{aligned}
\langle n_C\rangle &= 2\times 1/2 + 1\times 0 + 1\times 0 + 0\times 1/2 \quad = 1\\
\langle n_C^2\rangle &= 4\times 1/2 + 1\times 0 + 1\times 0 + 0\times 1/2 \quad = 2\\
\Delta n_C^2 &= \langle n_C^2\rangle - \langle n_C\rangle^2 \quad = 1
\end{aligned}$$

- Random variable n_{CD} testing for coincidence counts at C,D: $n_{CD} = 0$

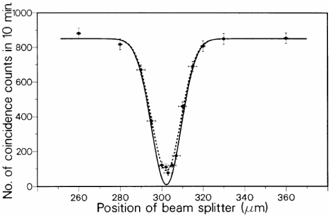
$$\langle n_{CD}\rangle = 0\times 1/2 + 1\times 0 + 1\times 0 + 0\times 1/2 \quad = 0$$

Second–order interference with bosons (qu. 14 & 15)

	1 in A, 0 in B	1 in A, 1 in B disting.	1 in A, 1 in B fermions	1 in A, 1 in B bosons
$\langle N_C \rangle / (N_A + N_B)$	$1/2$	$1/2$	$1/2$	$1/2$
$\Delta N_C^2 / \langle N_C \rangle$	$1/2$	$1/2$	0	1
$\langle N_{CD} \rangle / N$	0	$1/2$	1	0

- The first–order observable $\langle N_C \rangle$ does not exhibit interference.

  The second–order observables $\Delta N_c^2 / \langle N_C \rangle$ and $\langle N_{CD} \rangle / N$ exhibit interference!



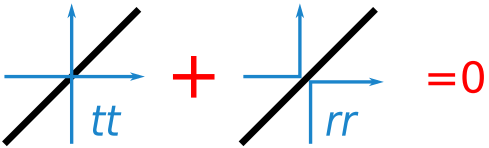
- Measured observable: coincidence counts $\langle N_{CD} \rangle / N$
- Scan **beamsplitter position** to tune the **difference in optical lengths** between $|A\rangle$ and $|B\rangle$
  If they are equal, the coincidence counts vanish
- The **dip width** reflects the **coherence length**:
  wavepacket bandwidth $\delta\omega \quad \longrightarrow \quad \Delta = c/\delta\omega$

Two–particle interference: fermions versus bosons

- Two–particle interference is visible in second–order observables
  current fluctuations, coincidence counts

- **Fermions:** interference leads to **antibunching**, i.e. systematic coincidence counts
  It follows from Pauli's exclusion principle, occurs regardless of the values of t and r

- **Bosons:** interference leads to **bunching**, i.e. no coincidence counts
  Maximal if double–transmission & double–reflection processes are indistinguishable



Beamsplitters and second quantisation

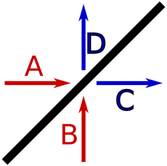
An excellent survey of second quantisation:

[R.P. Feynman, *Statistical Mechanics: a set of lectures*, W.A. Benjamin (1972), chap. 6]

Beware: these are NOT part of the usual *Feynman Lectures on Physics*!

Beamsplitter acting on many–particle states (qu. 17)

- For single–particle states, the beamsplitter is represented a by unitary operator U



$$U = \begin{pmatrix} t & -r^* \\ r & t^* \end{pmatrix}$$

$$U|A\rangle = \ \ t\ |C\rangle + r\ |D\rangle$$
$$U|B\rangle = -r^*|C\rangle + t^*|D\rangle$$

- The operator $\mathcal{U}$ generalises U to input states with arbitrary particle numbers:

$\mathcal{U}|0_A 0_B\rangle = |0_C 0_D\rangle$ (vacuum stays vacuum),

$\mathcal{U}|\phi\rangle = U|\phi\rangle$ for any single–particle state $|\phi\rangle$,

$\mathcal{U}|\phi_1\rangle \dots |\phi_N\rangle = (U|\phi_1\rangle)\dots(U|\phi_N\rangle)$ for N particles in a product state.

From first quantisation to second quantisation

- **First quantisation:** we label each particle and specify its state

  bosons: $|1:A,2:A\rangle$, $|1:B,2:B\rangle$, $(|1:A,2:B\rangle + |1:B,2:A\rangle)/\sqrt{2}$

  fermions: $(|1:A,2:B\rangle - |1:B,2:A\rangle)/\sqrt{2}$

  Tensor product in between particles: $|1:A,2:A\rangle = |1,A\rangle \otimes |2,A\rangle$

- For systems of many identical particles, first quantisation is cumbersome

  1. There are many non–physical states: (anti–)symmetrisation required

  $|1:A,2:B\rangle$ not physical for bosons or fermions

  $|1:A,2:A\rangle$ not physical for fermions, $(|1:A,2:B\rangle - |1:B,2:A\rangle)/\sqrt{2}$ not physical for bosons

  Many unphysical states for $N = 6 \times 10^{23}$ particles

  2. Labelling each particle is redundant

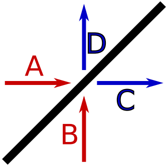
  $|1:A,2:B\rangle$ and $|1:B,2:A\rangle$ lead to the same (anti–)symmetrised states

- **Second quantisation:** specify particle number in each mode (= single–particle state)

  bosons: $|2_A\rangle$, $|2_B\rangle$, $|1_A,1_B\rangle$, fermions: $|1_A,1_B\rangle$

Superposition states involving different particle numbers: $(|\mathrm{vac}\rangle + |1_A\rangle + |1_A,1_B\rangle)/\sqrt{3}$

Annihilation/creation operators for a beamsplitter (qu. 17)



$\mathcal{U}\,|0_A,0_B\rangle = |0_C,0_D\rangle, \quad \mathcal{U}\,|\phi\rangle = U\,|\phi\rangle, \quad \mathcal{U}\,|\phi_1\rangle\cdots|\phi_N\rangle = (U\,|\phi_1\rangle)\cdots(U\,|\phi_N\rangle)$

- Annihilation and creation operators for each of the 4 ports:

  a, b, c, d and $a^\dagger, b^\dagger, c^\dagger, d^\dagger$

- $\tilde{c} = \mathcal{U}^\dagger c\,\mathcal{U}$ and $\tilde{d} = \mathcal{U}^\dagger d\,\mathcal{U}$ are more convenient (see next slide)

  c, d operate on output states but $\tilde{c}$, $\tilde{d}$ operate on input states

$\tilde{c}$, $\tilde{d}$ and $\tilde{c}^\dagger$, $\tilde{d}^\dagger$ are annihilation/creation operators as well

1. **(Anti–)commutation relation:** (use $\tilde{c}\tilde{c}^\dagger = \mathcal{U}^\dagger c\,\mathcal{U}\,\mathcal{U}^\dagger c^\dagger\mathcal{U} \quad = \mathcal{U}^\dagger c c^\dagger \mathcal{U}$)

   $[\tilde{c},\tilde{c}^\dagger]_\pm = \tilde{c}\tilde{c}^\dagger \pm \tilde{c}^\dagger\tilde{c} \quad = \mathcal{U}^\dagger\,[c,c^\dagger]_\pm\,\mathcal{U} \quad = \mathcal{U}^\dagger\mathcal{U} \quad = 1$

2. **Action of $\tilde{c}$ on vacuum:**

   $\tilde{c}\,|0_A,0_B\rangle = \mathcal{U}^\dagger c\,\mathcal{U}\,|0_A,0_B\rangle \quad = \mathcal{U}^\dagger c\,|0_C,0_D\rangle \quad = 0$

3. **Action of $\tilde{c}^\dagger$ on vacuum:**

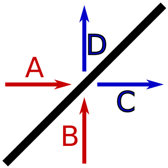
   $\tilde{c}^\dagger\,|0_A,0_B\rangle = \mathcal{U}^\dagger c^\dagger\mathcal{U}\,|0_A,0_B\rangle \quad = \mathcal{U}^\dagger c^\dagger\,|0_C,0_D\rangle \quad = \mathcal{U}^\dagger\,|1_C,0_D\rangle$

   $\tilde{c}^\dagger\,|0_A,0_B\rangle = U^{-1}\,|C\rangle = |C_{\text{in}}\rangle$

   $\tilde{c}^\dagger$ creates a particle in the single–particle input state $|C_{\text{in}}\rangle \quad = \alpha\,|A\rangle + \beta\,|B\rangle$

   such that, after the beamsplitter, there is a single particle in $|C\rangle$

Input states $|C_{\text{in}}\rangle$ and $|D_{\text{in}}\rangle$ created by $\tilde{c}^\dagger$ and $\tilde{d}^\dagger$ (qu. 17)



$$U = \begin{pmatrix} t & -r^* \\ r & t^* \end{pmatrix} \qquad U^{-1} = U^\dagger = \begin{pmatrix} t^* & r^* \\ -r & t \end{pmatrix}$$

- $\tilde{c}^\dagger$ creates a particle in $|C_{\text{in}}\rangle = U^{-1}|C\rangle$, $\tilde{d}^\dagger$ creates a particle in $|D_{\text{in}}\rangle = U^{-1}|D\rangle$

$|C_{\text{in}}\rangle$ is the input state such that, after the beamsplitter, there is a single particle in the output port $|C\rangle$

- $$\begin{aligned} |C_{\text{in}}\rangle &= U^{-1}|C\rangle &= U^\dagger|C\rangle &= t^*|A\rangle - r|B\rangle \\ |D_{\text{in}}\rangle &= U^{-1}|D\rangle &= U^\dagger|D\rangle &= r^*|A\rangle + t|B\rangle \end{aligned}$$

“Creation operators transform like kets”

- Assume we know the creation operators $c^\dagger_{|\phi_1\rangle}$ and $c^\dagger_{|\phi_2\rangle}$ for the states $|\phi_1\rangle$ and $|\phi_2\rangle$

  What is the creation operator $c^\dagger_{|\psi\rangle}$ for the state $|\psi\rangle = \lambda_1 |\phi_1\rangle + \lambda_2 |\phi_2\rangle$?

- **Mnemonic:** act on the vacuum state

  $$c^\dagger_{|\psi\rangle} |\mathrm{vac}\rangle = \lambda_1 |\phi_1\rangle + \lambda_2 |\phi_2\rangle = \lambda_1 c^\dagger_{|\phi_1\rangle} |\mathrm{vac}\rangle + \lambda_2 c^\dagger_{|\phi_2\rangle} |\mathrm{vac}\rangle$$

  $$c^\dagger_{|\psi\rangle} = \lambda_1 c^\dagger_{|\phi_1\rangle} + \lambda_2 c^\dagger_{|\phi_2\rangle}$$

- Beware: transformation law for annihilation operators involves complex conjugates

  $$c_{\langle\psi|} = \lambda_1^* c_{\langle\phi_1|} + \lambda_2^* c_{\langle\phi_2|}$$

Annihilation operators at beamsplitter and matrix U (17)

"Creation operators transform like kets"

- $|C_{\text{in}}\rangle = t^* |A\rangle - r |B\rangle$

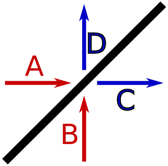
$$\tilde{c}^\dagger = t^* a^\dagger - r b^\dagger$$

$$\tilde{c} = t a - r^* b$$

- $|D_{\text{in}}\rangle = r^* |A\rangle + t |B\rangle$

$$\tilde{d}^\dagger = r^* a^\dagger + t b^\dagger$$

$$\tilde{d} = r a + t^* b$$



- The annihilation operators a, b, $\tilde{c}$, $\tilde{d}$ satisfy:

$$\begin{pmatrix} \tilde{c} \\ \tilde{d} \end{pmatrix} = \begin{pmatrix} t & -r^* \\ r & t^* \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = U \begin{pmatrix} a \\ b \end{pmatrix}$$

same transformation law as wavefunctions, same unitary matrix U

Particle number conservation at the beamsplitter (qu. 18)

- The unitarity of U follows from the conservation of energy
  We now formulate it in terms of particle number conservation

- $$\begin{pmatrix} \tilde{c} \\ \tilde{d} \end{pmatrix} = U \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{is equivalent to} \quad \begin{pmatrix} a \\ b \end{pmatrix} = U^\dagger \begin{pmatrix} \tilde{c} \\ \tilde{d} \end{pmatrix}$$

  Conjugate–transpose the previous relation: $\begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} = \begin{pmatrix} \tilde{c}^\dagger & \tilde{d}^\dagger \end{pmatrix} U$

- $$\begin{aligned} a^\dagger a + b^\dagger b &= \begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \tilde{c}^\dagger & \tilde{d}^\dagger \end{pmatrix} U U^\dagger \begin{pmatrix} \tilde{c} \\ \tilde{d} \end{pmatrix} \\ a^\dagger a + b^\dagger b &= \begin{pmatrix} \tilde{c}^\dagger & \tilde{d}^\dagger \end{pmatrix} \begin{pmatrix} \tilde{c} \\ \tilde{d} \end{pmatrix} = \tilde{c}^\dagger \tilde{c} + \tilde{d}^\dagger \tilde{d} = \mathcal{U}^\dagger \left(c^\dagger c + d^\dagger d \right) \mathcal{U} \end{aligned}$$

- $\mathcal{U}$ and $\mathcal{U}^\dagger$ ensure left– and right–hand sides act on the same Hilbert space (input)

Two–particle states, 2^{nd} quantisation formalism (qu. 19)

- We are interested in the (fermionic or bosonic) input state $|1_A, 1_B\rangle$
  Output state: $\mathcal{U}\,|1_A, 1_B\rangle \;=\; \mathcal{U} a^\dagger b^\dagger\,|\mathrm{O_{AB}}\rangle \;=\; \mathcal{U} a^\dagger b^\dagger \mathcal{U}^\dagger\,|\mathrm{O_{CD}}\rangle$
- Introduce $\tilde{a}^\dagger = \mathcal{U} a^\dagger \mathcal{U}^\dagger$ and $\tilde{b}^\dagger = \mathcal{U} b^\dagger \mathcal{U}^\dagger$ which act on the output Hilbert space
  Starting from $\begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} = \begin{pmatrix} \tilde{c}^\dagger & \tilde{d}^\dagger \end{pmatrix} U,$ show that $\begin{pmatrix} \tilde{a}^\dagger \\ \tilde{b}^\dagger \end{pmatrix} = {}^t U \begin{pmatrix} c^\dagger \\ d^\dagger \end{pmatrix}$
- $\mathcal{U}\,|1_A, 1_B\rangle \;=\; \tilde{a}^\dagger \tilde{b}^\dagger\,|0_{CD}\rangle \;=\; (tc^\dagger + rd^\dagger)(-r^* c^\dagger + t^* d^\dagger)\,|0_{CD}\rangle$
- Fermions: $c^{\dagger 2} = d^{\dagger 2} = 0$ and $[c^\dagger, d^\dagger]_+ = 0$
  $\mathcal{U}\,|1_A, 1_B\rangle \;=\; (|t|^2 c^\dagger d^\dagger - |r|^2 d^\dagger c^\dagger)\,|0_{CD}\rangle \;=\; c^\dagger d^\dagger\,|0_{CD}\rangle \;=\; |1_C, 1_D\rangle$
- Bosons: All terms contribute, $[c^\dagger, d^\dagger]_- = 0$
  $\mathcal{U}\,|1_A, 1_B\rangle \;=\; -tr^*\,|2_C, 0_D\rangle + rt^*\,|0_C, 2_D\rangle + (|t|^2 - |r|^2)\,|1_C, 1_D\rangle$

Both for fermions and bosons, we recover the first–quantisation results

N particles enter through $|A\rangle$ and N through $|B\rangle$ (qu. 20)

- We are interested in the input state $|N_A,N_B\rangle$: only possible with bosons

Beware: when creating N particles in the same state, do not forget prefactor: $|N_C\rangle = c^{\dagger N}\,|0_C\rangle\ /\sqrt{N!}$

$$|N_A,N_B\rangle = \frac{a^{\dagger N}}{\sqrt{N!}}\frac{b^{\dagger N}}{\sqrt{N!}}|0_{AB}\rangle = \frac{1}{N!}a^{\dagger N}b^{\dagger N}|0_{AB}\rangle$$

- Output state: $\mathcal{U}\,|N_A,N_B\rangle = \frac{1}{N!}\mathcal{U}\,a^{\dagger N}b^{\dagger N}\,|0_{AB}\rangle = \frac{1}{N!}\mathcal{U}\,a^{\dagger N}\,b^{\dagger N}\mathcal{U}^{\dagger}\,|0_{CD}\rangle$

$$\mathcal{U}\,|N_A,N_B\rangle = \frac{1}{N!}\,\tilde{a}^{\dagger N}\,\tilde{b}^{\dagger N}\,|0_{CD}\rangle = \frac{1}{N!}\,(tc^{\dagger} + rd^{\dagger})^N(-r^*c^{\dagger} + t^*d^{\dagger})^N\,|0_{CD}\rangle$$

- The *probability amplitude* for all $2N$ atoms to exit from port C is:

$$\langle 2N_C,0_D|\,\mathcal{U}\,|N_A,N_B\rangle = \frac{1}{N!}\,t^N\,(-r^*)^N\,\sqrt{(2N)!} = t^N\,(-r^*)^N\,\sqrt{\frac{(2N)!}{N!\,N!}}$$

The probability for all atoms to exit from port C (= square of probability amplitude)

is **enhanced with respect to distinguishable particles** by a factor $\binom{2N}{N} = \frac{(2N)!}{N!\,N!}$