ICFP M2 Advanced Quantum Mechanics: Problem #1: two–particle interference with bosons and fermions

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1 Single–particle interference: Young's fringes

We start this problem by revisiting Young's double–slit interference experiment (see Fig. 1).

1. The most familiar version of this experiment is performed with intense laser light. Sketch the interferogram and justify that the fringe spacing is $\lambda D/a$, where λ is the incident light wavelength, and the parameters D and a are defined in Fig. 1.

2. We now perform the experiment using a beam of slow electrons.
   a) In this matter–wave experiment, which parameter determines the wavelength?
   b) What is the key difference with respect to the previous optical experiment?

3. Experiments performed in the single–particle regime reveal that the interference fringes build up detection event by detection event. Does this interference effect probe single–particle or many–particle physics? Are quantum statistics involved?

2 Quantum description of a beamsplitter

We consider a single quantum particle (for example, a photon or an electron), which arrives on the beamsplitter represented in Fig. 2. The particle may enter the beamsplitter through the input ports A or B, corresponding to the input states $|A\rangle$ or $|B\rangle$. It may exit the beamsplitter through the output ports C or D, corresponding to the output states $|C\rangle$ and $|D\rangle$. If the wavefunction of

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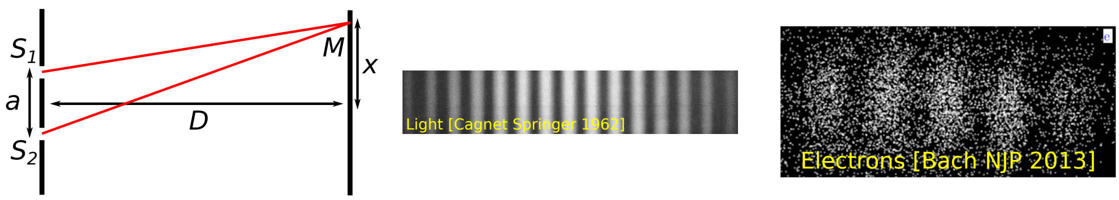

Figure 1 Left: Schematics of Young's double–slit interference experiment. Centre: Interference fringes obtained with intense laser light [1]. Right: Analogous fringes obtained with electrons [2].

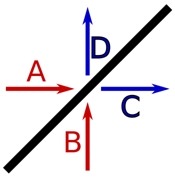

Figure 2 The beamsplitter (black), its input ports A and B (red), and its output ports C and D (blue).

the incident particle is $|\psi_{\text{in}}\rangle = \alpha\,|A\rangle + \beta\,|B\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$, then the wavefunction once the particle exits the beamsplitter is $|\psi_{\text{out}}\rangle = \gamma\,|C\rangle + \delta\,|D\rangle$, where:

$$\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = U\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\ . \tag{1}$$

4. Which property must the 2×2 matrix U satisfy? Assuming that the elements of U are all real, show that, up to an unimportant choice of signs, U may be written as:

$$U = \begin{pmatrix} t & -r\\ r & t\end{pmatrix}\ , \tag{2}$$

   with $t^2 + r^2 = 1$. Give the physical meaning of the parameters t and r.

5. We assume that the beamsplitter is symmetric: $t^2 = r^2 = 1/2$.
   a) A single particle arrives at the input port A. We call n_C the random variable representing the measurement at the outport port C: The variable $n_C = 1$ if the particle exits through the port C and $n_C = 0$ if it exits through the port D. Calculate the mean value $\langle n_C\rangle$ and the variance $\Delta n_c^2 = \langle n_C^2\rangle - \langle n_C\rangle^2$.
   b) We perform N measurements $k = 1,\dots,N$, and define the random variable $N_C = \sum_{k=1}^{N} n_C^{(k)}$. Calculate the mean value $\langle N_C\rangle$ and the variance ΔN_C^2.
   c) Explain why $\langle N_C\rangle/N$ and $F_{\text{1part}} = \Delta N_C^2/\langle N_C\rangle$ are relevant observables.
   Check that $\langle N_C\rangle/N = 1/2$ and $\Delta N_C^2/\langle N_C\rangle = 1/2$.
   d) In an experiment, justify that the particle current at the output port C is related to N_C and that the ratio F represents the current fluctuations.

3 Two–particle interference at a beamsplitter

We now consider two incident particles which *enter the beamsplitter through different ports*. Both particles are in the same spin state which remains constant during the process.

3.1 Distinguishable particles

6. For the input state $|1:\phi_1, 2:\phi_2\rangle$, express the output state using the matrix U of Eq. 1.

7. What are the four possible results for the measurement of the output states?
   Give their probabilities assuming that the beamsplitter is symmetric.

8. We consider a single experimental run where two particles enter through A and B, and call n_C the random variable giving the result of the particle detection at the port C. Then, we perform N such experimental runs and introduce the random variable $N_C = \sum_{k=1}^{N} n_C^{(k)}$. Following Question 5, calculate the ratio $F_{\text{classical}} = \Delta N_C^2/\langle N_C\rangle$.
   Does the particle current exhibit interference? What about the current noise?
   Hint: The number of incident particles is twice as large compared to Question 5.

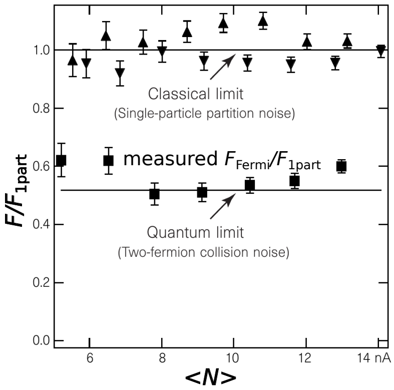

Figure 3 Measured reduction of the current noise for two electrons colliding on a beamsplitter. (Reproduced from Ref. [3].)

3.2 Identical fermions: reduction of the noise on the current

9. What is the only allowed two–particle input state?

10. May the particles ever exit from the same output port? Conclude that, even if the beamsplitter is not symmetric, the output state is $(|1:C,2:D\rangle - |1:D,2:C\rangle)/\sqrt{2}$.

11. We define n_C and N_C as in Question 8.
Show that $\langle N_C\rangle/(2N) = 1/2$ and $F_{\text{Fermi}} = \Delta N_C^2/\langle N_C\rangle = 0$.
Conclude that the current does not exhibit interference, but that the current noise does.
Compare with the experimental results for two colliding electrons shown on Fig. 3.

3.3 Identical bosons: Hong–Ou–Mandel effect

12. What is the only allowed two–particle input state?

13. Calculate the two–particle output state. For a symmetric beamsplitter, check that both bosons always exit from the same output port.

14. We define n_C and N_C as in Question 8. For a symmetric beamsplitter, show that $\langle N_C\rangle/(2N) = 1/2$ and $F_{\text{Bose}} = \Delta N_C^2/\langle N_C\rangle = 1$. Conclude that the current exhibits no interference, but that the fluctuations do. Justify the difference with respect to identical fermions.

In the optical experiment of Ref. [4], the bosonic interference effect was observed by measuring the coincidence count N_{CD}, i.e. the number of experimental runs which led to one photon being detected at the output port C and the other at the output port D.

15. Calculate the value of N_{CD}/N *(i)* for N experimental runs involving two distinguishable particles; *(ii)* for N experimental runs involving two identical bosons.

16. The position of the beamsplitter, labelling the horizontal axis of Fig. 4, sets the difference in the optical paths of the incident photons. Interpret the experimental results thanks to Question 15. Which parameter controls the width of the dip in the curve?

4 Beamsplitters and second quantisation

Finally, we turn to the case of an arbitrary number of incident identical particles (fermions or bosons) using second quantisation. Hence, we introduce the annihilation operators a, b and c, d corresponding to the input states $|A\rangle$, $|B\rangle$ and output states $|C\rangle$, $|D\rangle$ introduced in Sec. 2. We also introduce the operator $\mathcal{U}$, which generalises U so that it relates input and output wavefunctions involving an arbitrary number of particles.

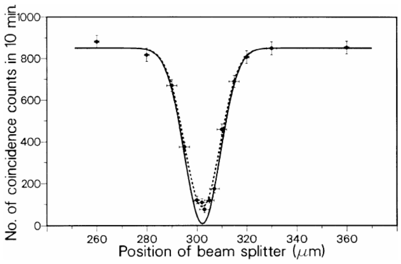

Figure 4 Measured coincidence count at the output ports C and D of a nearly symmetric eamsplitter for two photons entering through different input ports. (Reproduced from Ref. [4].)

17. Show that the operators a, b, $\tilde{c} = \mathcal{U}^\dagger c\mathcal{U}$, $\tilde{d} = \mathcal{U}^\dagger d\mathcal{U}$ satisfy a relation similar to Eq. (1):

$$\begin{pmatrix} \tilde{c} \\ \tilde{d} \end{pmatrix} = U \begin{pmatrix} a \\ b \end{pmatrix} . \tag{3}$$

Hints: First, define $\mathcal{U}$ using a suitable basis of many–particle states.
Creation operators depend linearly on single–particle states: $a^\dagger_{\lambda_1|u_1\rangle+\lambda_2|u_2\rangle} = \lambda_1 a^\dagger_{|u_1\rangle} + \lambda_2 a^\dagger_{|u_2\rangle}$.

18. Reinterpret the symmetry property of the matrix U in terms of a relation between the number operators $n_a = a^\dagger a$, $n_b = b^\dagger b$, $n_c = c^\dagger c$, and $n_d = d^\dagger d$.

19. Rederive the results of Questions 10 and 13 using the second quantisation formalism.

20. We choose the $2N$–particle input state $|\Psi_{\text{in}}\rangle$ which describes N particles entering through port $|A\rangle$ and N particles entering through port $|B\rangle$. Let $|\Psi_{\text{out},C}\rangle$ and $|\Psi_{\text{out},D}\rangle$ be the output states which represent $2N$ particles exiting through ports $|C\rangle$ and $|D\rangle$, respectively.
    a) For $N \geq 2$, which quantum statistics is/are compatible with $|\Psi_{\text{in}}\rangle$ and $|\Psi_{\text{out},C}\rangle$, $|\Psi_{\text{out},D}\rangle$? Write these three states in terms of $a^\dagger$, $b^\dagger$, and the vacuum state $|0_a, 0_b\rangle$.
    b) Calculate the probability of measuring all $2N$ particles exiting from the same port. Compare this probability with the result for distinguishable particles.
    Hint: For N large, Stirling's approximation holds for the factorial function: $N! \sim (N/e)^n \sqrt{2\pi N}$.

Further reading

Introductory

- Young's double–slit experiment with intense light is presented in Ref. [5, §9.3]. Its quantum matter–wave version [6, chap. 1] may be used to illustrate both the superposition principle and complementarity. It has been performed with (fermionic) electrons; the closely related Mach–Zender interference effect has been implemented with (bosonic) helium atoms.
- The interference experiments presented here are used as building blocks in many recent experiments. For instance, the double–slit experiment is at the heart of the "quantum eraser" investigations [7]. Second–order interference with twin photons is used in high–precision metrology [8]; it raises interesting questions related to quantum information [9].

More advanced

- One–photon wavepackets, two–photon interference, and their differences with respect to quasiclassical effects are discussed in [10, Chap. 5B].
- The suppression of the current noise due to fermionic two–particle interference has been observed with electrons in quantum point contacts [11]. The bosonic interference between two photons has been observed and used to measure photon arrival time differences with subpicosecond accuracies [10, §7.4].

References

[1] Cagnet M., Françon M., Thrierr J. *Atlas of optical phenomena.* Springer (1962).

[2] Bach R., et al. New J. Phys. 15, 033018 (2013).

[3] Liu R.C., et al. Nature 391, 263 (1998).

[4] Hong C.K., Ou Z.Y., Mandel L. Phys. Rev. Lett. 59, 2044 (1987).

[5] Hecht E. *Optics.* Addison Wesley, 4 ed. (2002).

[6] Feynman R.P., Leighton R.B., Sands M. *The Feynman Lectures on Physics, volume III.* Addison-Wesley (1965).

[7] Hillmer R., Kwiat P. Scientific American 296:5, 90 (2007).

[8] Migdall A. Phys. Today 52:1, 41 (1999).

[9] Chiao R.Y., Kwiat P.G., Steinberg A.M. Scientific American 269, 52 (1993).

[10] Grynberg G., Aspect A., Fabre C. *Introduction to quantum optics.* Cambridge University Press (2010).

[11] Büttiker M. Phys. World 11, 30 (1998).