Detailed steps for solving Question 13a in the problem on the Lamb shift

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(i) We wish to apply second–order perturbation theory to the operator H_{I1}. Recall the energy level degeneracies of the hydrogen atom and explain why this calculation must be performed with care.

(ii) We consider an atom whose quantum state is the eigenstate $|n,l,m\rangle$, in the electromagnetic vacuum. Interpret the second–order processes driven by the coupling term H_{I1} in terms of the emission of a photon followed by the absorption of the same photon. Justify that there are a priori four families of such processes, which may be labelled by the evolution of the quantum number l: *(a)* $l \to l+1 \to l+2$, *(b)* $l \to l+1 \to l$, *(c)* $l \to l-1 \to l$, and *(d)* $l \to l-1 \to l-2$.

Hint: Recall why, under the long–wavelength approximation used in the problem, the absorption or the emission of a single photon causes the quantum number l to change by exactly one unit.

(iii) Our goal is to show that the processes *(a)* and *(d)* are actually forbidden: the questions (iv)– (xii) below are a guide towards a mathematical proof of this statement. For the present question, invoke a conservation law to give an intuitive argument why it is not surprising.

(iv) Justify that the role of the energy level degeneracies may be analysed in terms of the hermitian matrix M, defined by the following matrix elements:

$$\langle n,l_1,m_1|M|n,l_2,m_2\rangle = \left(\frac{q}{m}\right)^2 \sum_{n',k'} \frac{1}{E_n - E'_n - \hbar\omega_{k'}} \frac{\hbar}{2\epsilon_0\omega_{k'}L^3} \sum_{l',m'} \sum_{\hat{\mathbf{k}}',j'} \langle n,l_1,m_1|\mathbf{p}\cdot\epsilon_{\mathbf{k}',j'}|n',l',m'\rangle \langle n',l',m'|\mathbf{p}\cdot\epsilon^*_{\mathbf{k}',j'}|n,l_2,m_2\rangle \ . \quad (1)$$

In Eq. (1), the quantum numbers n', l', m' characterise the intermediate atomic state; $\mathbf{k}' = k'\,\hat{\mathbf{k}}'$ is the wavevector of the emitted (and subsequently reabsorbed) photon, k' is its modulus and the unit vector $\hat{\mathbf{k}}'$ gives its direction; the index j' specifies the polarisation of this photon.

We introduce the following three complex, orthogonal, unit vectors:

$$\mathbf{e}_{+1} = -\frac{1}{\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y), \qquad \mathbf{e}_0 = \mathbf{e}_z, \qquad \mathbf{e}_{-1} = +\frac{1}{\sqrt{2}}(\mathbf{e}_x - i\mathbf{e}_y) \ . \quad (2)$$

These vectors have been chosen such that the operators $\mathbf{p}_{+1} = \mathbf{p}\cdot\mathbf{e}_{+1}$, $\mathbf{p}_0 = \mathbf{p}\cdot\mathbf{e}_0$, and $\mathbf{p}_{-1} = \mathbf{p}\cdot\mathbf{e}_{-1}$ are the three components of the vector operator $\mathbf{p}$ used in the Wigner–Eckart theorem (cf. Messiah's Eq. XIII.124).

The present calculation is conveniently carried out in a polarisation basis which *differs* from the one illustrated in Fig. 2 of the problem. We consider a given wavevector $\mathbf{k}'$, and describe its direction $\hat{\mathbf{k}}'$ in terms of spherical coordinates: $\hat{\mathbf{k}}' = \sin\theta\cos\phi\,\mathbf{e}_x + \sin\theta\sin\phi\,\mathbf{e}_y + \cos\theta\,\mathbf{e}_z$.

(v) Justify that the two independent polarisations $\epsilon_{\mathbf{k}',j}$ for the wavevector $\mathbf{k}'$ may be chosen as:

$$\epsilon_{\mathbf{k}',1} = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \qquad \text{and} \qquad \epsilon_{\mathbf{k}',2} = \begin{pmatrix} -\cos\theta\,\cos\phi \\ -\cos\theta\,\sin\phi \\ \sin\theta \end{pmatrix} \ . \quad (3)$$

(vi) Show that the polarisation vectors $\epsilon_{\mathbf{k}',j}$ may be expressed as follows in terms of the three vectors $\mathbf{e}_q$ (where $q = 1$, 0, or -1):

$$\epsilon_{\mathbf{k}',1} = \frac{i}{\sqrt{2}} \left(e^{-i\phi}\,\mathbf{e}_+ + e^{+i\phi}\,\mathbf{e}_- \right), \quad \epsilon_{\mathbf{k}',2} = \frac{\cos\theta}{\sqrt{2}} \left(e^{-i\phi}\,\mathbf{e}_+ - e^{+i\phi}\,\mathbf{e}_- \right) + \sin\theta\,\mathbf{e}_0\ . \tag{4}$$

(vii) Perform the sum over $\hat{\mathbf{k}}'$ and j in Eq. (1):

$$\langle n, l_1, m_1|M|n, l_2, m_2\rangle = \frac{8\pi}{3}\left(\frac{q}{m}\right)^2 \sum_{n',k'} \frac{1}{E_n - E_n' - \hbar\omega_{k'}} \frac{\hbar}{2\epsilon_0\omega_{k'}L^3}$$
$$\sum_{l',m'} \begin{bmatrix} -\ \langle n, l_1, m_1|p_+|n', l', m'\rangle\,\langle n', l', m'|p_-|n, l_2, m_2\rangle \\ -\ \langle n, l_1, m_1|p_-|n', l', m'\rangle\,\langle n', l', m'|p_+|n, l_2, m_2\rangle \\ +\ \langle n, l_1, m_1|p_0|n', l', m'\rangle\,\langle n', l', m'|p_0|n, l_2, m_2\rangle \end{bmatrix} \tag{5}$$

HINT: Mind the following conjugation rules: $p_{+1}^\dagger = -p_{-1}$, $p_0^\dagger = p_0$, $p_{-1}^\dagger = -p_{+1}$.

(viii) Applying the Wigner–Eckart theorem (Messiah's Eq. XIII.125) to the vector operator $\mathbf{p}$, establish the following selection rules on the quantum number m satisfied by the operators p_{+1}, p_0, and p_{-1}: p_{+1} increases m by one unit, p_0 conserves m, p_{-1} decreases m by one unit.
HINT: These rules follow from the properties of the Clebsch–Gordan coefficient appearing in Messiah's equation. Note that he uses the convention $\langle j_1, j_2; m_1, m_2|J, M\rangle$, whereby the quantum numbers j_1, j_2 and m_1, m_2 are collected together.

(ix) Show that the matrix element $\langle n, l_1, m_1|M|n, l_2, m_2\rangle$ may be non–zero only if $m_1 = m_2$.
HINT: Focus on the quantity between the straight brackets in Eq. (5).

We are interested in proving that the processes of types *(a)* and *(d)* are forbidden (see question (ii) above). For these processes, the initial and final values of l differ by ± 2. Exploiting the hermitian character of M, we choose $l_1 = l - 1$ and $l_2 = l + 1$, i.e. we focus on the matrix element $\langle n, l-1, m|M|n, l+1, m\rangle$.

(x) Explain why the relevant intermediate states all carry the quantum number l.
HINT: Exploit the selection rule stated in question (ii).

(xi) Applying the Wigner–Eckart theorem once again, show that $\langle n, l_1, m_1|M|n, l_2, m_2\rangle$ is proportional to the following combination of six Clebsch–Gordan coefficients (written using Messiah's convention $\langle j_1, j_2; m_1, m_2|J, M\rangle$):

$$N_{l,m} = \begin{bmatrix} -\ \langle l, 1; m-1, \quad 1|l-1, m\rangle\,\langle l+1, 1; m, -1|l, m-1\rangle \\ -\ \langle l, 1; m+1, -1|l-1, m\rangle\,\langle l+1, 1; m, +1|l, m+1\rangle \\ +\ \langle l, 1; m \quad , \quad 0|l-1, m\rangle\,\langle l+1, 1; m, \quad 0|l, m\rangle \end{bmatrix} \tag{6}$$

(xii) Prove that $N_{l,m} = 0$ and conclude.

HINTS: $\langle j_1, j_2; m_1, m_2|J, M\rangle = (-)^{j_1-J+m_2}\sqrt{\dfrac{2J+1}{2j_1+1}}\ \langle J, j_2; M, -m_2|j_1, m_1\rangle\ .$

$$\sum_{m_1=-j_1}^{j_1}\ \sum_{m_2=-j_2}^{j_2} \langle j_1, j_2; m_1, m_2|J, M\rangle\ \langle j_1, j_2; m_1, m_2|J', M'\rangle = \delta_{J,J'}\,\delta_{M,M'}\ .$$

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