ICFP M2 Advanced Quantum Mechanics Homework Problem

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Radiative corrections and the Lamb shift in atomic hydrogen

The Lamb shift is an effect whereby the energy levels of an atom are shifted with respect to the fine–structure prediction. It is due to the coupling of the atom to the electromagnetic vacuum. For instance, in the case of hydrogen, the electron orbiting around the nucleus may emit and reabsorb photons, causing some energy lines to be shifted.

The fine structure of an atom is described in terms of the principal quantum number n, the angular quantum number l, and the quantum number j giving the squared modulus $\hbar^2 j(j+1)$ of the total angular momentum $\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s}$, with $\boldsymbol{s}$ being the electron spin (so that $s = 1/2$). The historical example for the Lamb shift is that of the hydrogen levels $|2s_{1/2}\rangle = |n = 2, l = 0, j = 1/2\rangle$ and $|2p_{1/2}\rangle = |n = 2, l = 1, j = 1/2\rangle$: the fine structure theory predicts that they should have the same energy, but spectroscopy experiments have revealed a shift in between the two levels which is of the order of 1058 MHz*, as shown in Fig. 1 where the relevant transition appears in red. The precision measurement and calculation of this shift have provided first an incentive for the creation of quantum electrodynamics, and then a stringent test of its predictions (see Ref. [1] for a historical introduction to the spectroscopy of atomic hydrogen).*

The goal of this problem is to explain the Lamb shift in terms of a simplified approach introduced by Hans Bethe in 1947. Though non–relativistic, it provides a good first approximation to the value of the energy shift between $|2s_{1/2}\rangle$ and $|2p_{1/2}\rangle$. It may be discussed conveniently using the formalism of the resolvent and projection operators.

The problem consists of four parts. The first part introduces the Hamiltonians describing the atom, the electromagnetic field, and their coupling. The second part provides detailed steps for the non–relativistic calculation of the Lamb shift, on the example of the atomic ground state coupled to the electromagnetic vacuum. The third part generalises the previous results to an arbitrary discrete atomic level. Finally, in the fourth part, a numerical value is obtained for the Lamb shift on the $1s$–$2s$ transition. ***The answers to all questions and all required equations are given in the text, so that you may explore the whole problem even without providing a full answer to each question.***

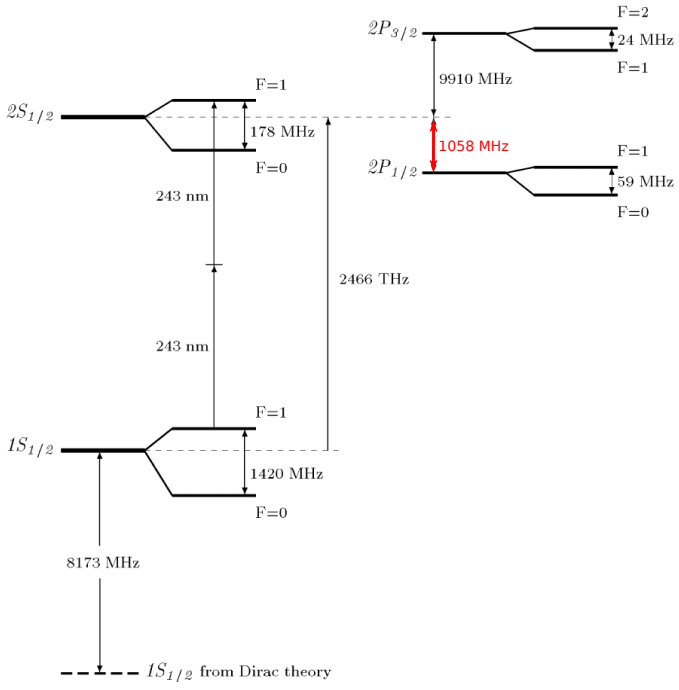

Figure 1 The level structure of atomic hydrogen with recent experimental values for the energy splittings. This problem deals with the Lamb shift, which is a shift of some energy levels (the s–states, i.e. those with the quantum number $l = 0$) due to the coupling between the atom and the electromagnetic vacuum. The shift between the levels $|2s_{1/2}\rangle$ and $|2p_{1/2}\rangle$ is highlighted in red: an order of magnitude for this shift will be calculated in Sec. 4. (Adapted from Ref. [2].)

1 Hamiltonians for the atom, the electromagnetic field, and their coupling

The system we wish to describe consists of a single hydrogen atom in the presence of the electromagnetic field vacuum. Its Hamiltonian H reads:

$$H = \frac{1}{2m}(\boldsymbol{p} - q\boldsymbol{A})^2 - \frac{q^2}{4\pi\epsilon_0 r} + H_{\text{field}}, \quad \text{with} \quad H_{\text{field}} = \sum_{\boldsymbol{k},j} \hbar\omega_k \, a^\dagger_{\boldsymbol{k},j} a_{\boldsymbol{k},j} \ . \tag{1}$$

In Eq. (1), $\boldsymbol{r}$ is the position of the single electron, $\boldsymbol{p}$ its momentum, q its charge, and m its mass. The operator $\boldsymbol{A}(\boldsymbol{r})$ is the vector potential describing the electromagnetic field. The term H_{field} describes the free electromagnetic field, which is understood as a collection of harmonic oscillators. Each oscillator corresponds to a given mode, labelled by the wavevector $\boldsymbol{k}$ and the polarisation index[1] $j = 1, 2$. The mode frequency satisfies the relation $\omega_k = ck$. The operator $a_{\boldsymbol{k},j}$ destroys a photon with the wavevector $\boldsymbol{k}$ and the polarisation j. The annihilation operators $a_{\boldsymbol{k},j}$ and the corresponding creation operators $a^\dagger_{\boldsymbol{k},j}$ satisfy the bosonic commutation rules, e.g. $[a_{\boldsymbol{k},j}, a^\dagger_{\boldsymbol{k}',j'}] = \delta_{\boldsymbol{k},\boldsymbol{k}'}\,\delta_{j,j'}$.

We expand the term $(\boldsymbol{p} - q\boldsymbol{A})^2/(2m)$ in Eq. (1), so as to split the complete Hamiltonian $H = H_0 + H_I$ into the sum of a *(i)* a 'bare' Hamiltonian H_0 representing the uncoupled atom and field, and *(ii)* an interaction term H_I. The bare Hamiltonian H_0 reads:

$$H_0 = H_A + H_{\text{field}} \ , \quad \text{with} \quad H_A = \frac{\boldsymbol{p}^2}{2m} - \frac{q^2}{4\pi\epsilon_0 r} \ . \tag{2}$$

We focus on the bound levels of the atomic Hamiltonian H_A, which are characterised by the usual wavefunctions $|\phi_{n,l,m}\rangle$ for the hydrogen atom, neglecting all aspects due to spin or radiation. The 'bare' states $|\phi_{n,l,m}\rangle$ are labelled by the principal quantum number n, the angular quantum number l setting the squared modulus $\hbar^2 l(l+1)$ of the orbital angular momentum $\boldsymbol{l}$, and the magnetic quantum number m giving the projection $\hbar m$ of $\boldsymbol{l}$ along the quantisation axis.

We now turn to the interaction Hamiltonian H_I. We use the Coulomb gauge, that is, we impose that the vector potential $\boldsymbol{A}(r)$ satisfies $\nabla \cdot \boldsymbol{A}(\boldsymbol{r}) = 0$.

1. Show that, thanks to the choice of the Coulomb gauge: $\boldsymbol{p} \cdot \boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{A}(\boldsymbol{r}) \cdot \boldsymbol{p}$.
   HINT: For any function f, the commutator $[p_x, f(\boldsymbol{r})] = -i\hbar\,\partial f/\partial x$.

We write the interaction Hamiltonian H_{I1} as a sum of two terms. The first one, H_{I1}, is proportional to q, whereas the second one, H_{I2}, is proportional to q^2:

$$H_I = H_{I1} + H_{I2} \quad \text{with} \quad H_{I1} = -\frac{q}{m}\boldsymbol{p} \cdot \boldsymbol{A}(\boldsymbol{0}) \quad \text{and} \quad H_{I2} = \frac{q^2}{2m}\boldsymbol{A}^2(\boldsymbol{0}) \ . \tag{3}$$

In Eq. (3), we have performed the long–wavelength approximation, whereby the spatial dependence of the electromagnetic field (and, hence, of $\boldsymbol{A}(\boldsymbol{r}) \approx \boldsymbol{A}(\boldsymbol{0})$) is neglected over the size of the atom. In other words, we focus on wavevectors k such that $ka_0 \ll 1$, where $a_0 = 4\pi\epsilon_0\hbar^2/(mq^2) = 5.29 \times 10^{-11}$ m is the Bohr radius. Then, the vector potential operator reads:

$$\boldsymbol{A}(\boldsymbol{0}) = \sum_{\boldsymbol{k},j} \left(\frac{\hbar}{2\epsilon_0\omega_k L^3}\right)^{1/2} \boldsymbol{\epsilon}_{\boldsymbol{k},j} \, (a_{\boldsymbol{k},j} + a^\dagger_{\boldsymbol{k},j}) \ , \tag{4}$$

where L^3 is a fictitious quantisation volume which will not appear in the final results, and the vector $\boldsymbol{\epsilon}_{\boldsymbol{k},j}$ characterises the polarisation j. Each wavevector $\boldsymbol{k}$ is orthogonal to both $\boldsymbol{\epsilon}_{\boldsymbol{k},1}$ and $\boldsymbol{\epsilon}_{\boldsymbol{k},2}$.

2. Three questions about the orders of magnitude involved:
   a) Explain why the motion of the proton is neglected in the Hamiltonian of Eq. (1).
   b) Recover the expression for the Bohr radius a_0 using dimensional analysis.
   c) Justify that the long–wavelength approximation is compatible with the non–relativistic approximation.
   HINT: The fine structure constant $\alpha = q^2/(4\pi\epsilon_0\hbar c) \approx 1/137$.

2 Shift of the atomic ground state coupled to the electromagnetic vacuum

We shall investigate the role of the coupling between the atom and the electromagnetic field through the formalism based on the resolvent and projection operators. We now recall the key result of this formalism (Eq. (5) below), which is the only one used in this problem. We split the complete Hamiltonian $H = H_0 + V$ into the sum of the 'bare' Hamiltonian H_0 and a small coupling term V. If $|\phi_0\rangle$ is an eigenstate of H_0 with the energy E_0, then, the coupling V

[1] Do not confuse the polarisation index j, which is either 1 or 2, with a quantum number related to the fine–structure angular momentum $\boldsymbol{j} = \boldsymbol{l} + \boldsymbol{s}$: the latter is mentioned in the introduction but does not play any role in the problem.

has two consequences: *(i)* a dissipative effect: it endows the level with a linewidth $\Gamma > 0$ (that is, a lifetime $\propto 1/\Gamma$), and *(ii)* a reactive effect: it shifts the energy level by an amount Δ. The quantities Γ and Δ are both numbers (i.e. not operators) carrying the unit of energy; they are both real and positive. We introduce an (orthogonal) basis $\{|\phi_a\rangle\}$ of eigenstates of H_0 which contains $|\phi_0\rangle$, and the projector $Q = 1 - |\phi_0\rangle\langle\phi_0|$. Then, to second order in the coupling V, and neglecting the slow energy dependence of the linewidth Γ and the shift Δ, they are given by:

$$\Gamma = 2\pi \sum_{a\neq 0} |\langle\phi_a|V|\phi_0\rangle|^2 \, \delta(E_0 - E_a) \quad \text{and} \quad \Delta = \langle\phi_0|V|\phi_0\rangle + \mathcal{P} \sum_{a\neq 0} \frac{|\langle\phi_a|V|\phi_0\rangle|^2}{E_0 - E_a} \, , \tag{5}$$

where the sums $\sum_{a\neq 0}$ are carried out over all basis states $|\phi_a\rangle \neq |\phi_0\rangle$, and $\mathcal{P}$ stands for the Cauchy principal value.

We now apply Eq. (5) to the calculation of the Lamb shift in atomic hydrogen. Its full calculation requires a relativistic framework which relies on the framework of quantum electrodynamics. In this problem, we shall describe an approximate approach due to Hans Bethe (and concisely sketched in Ref. [3, §19β]), which relies on the non–relativistic approximation. This approximation already provides results that are in reasonable agreement with experiments.

We first consider the ground state $|g\rangle = |\phi_{1,0,0}\rangle$ of the Hamiltonian H_A. It is characterised by the quantum numbers $n = 1$, $l = 0$, $m = 0$. The value $l = 0$ means that the state $|g\rangle$ carries no angular momentum and, hence, is non–degenerate. Furthermore, we choose the electromagnetic field to be in the vacuum state $|\text{vac}\rangle$. Hence, Eq. (5) is applicable with $H_0 = H_A + H_{\text{field}}$, $V = H_I$, and $|\phi_0\rangle = |g\rangle \otimes |\text{vac}\rangle$, corresponding to the bare energy $E_0 = E_{1,0,0}$ which is the ground–state energy of an isolated hydrogen atom.

3. Explain why the lifetime of the configuration considered here is expected to remain infinite ($\Gamma = 0$) even in the presence of the coupling to the electromagnetic field.

We now turn to the radiative shift Δ, which we shall evaluate up to the order q^2. We write $\Delta = \Delta^{(i)} + \Delta^{(ii)}$ as a sum of two contributions, corresponding to the first and second term in the expression for Δ given by Eq. (5):

$$\Delta = \Delta^{(i)} + \Delta^{(ii)} \, , \quad \text{where} \quad \Delta^{(i)} = \langle\phi_0| H_I |\phi_0\rangle \quad \text{and} \quad \Delta^{(ii)} = \mathcal{P} \sum_{a\neq 0} \frac{|\langle\phi_a| H_I |\phi_0\rangle|^2}{E_0 - E_a} \, . \tag{6}$$

We start by considering the contribution $\Delta^{(i)}$.

4. Explain why H_{I1} does not contribute to $\Delta^{(i)}$, and show that:

$$\Delta^{(i)} = \frac{q^2}{m} \sum_{\boldsymbol{k}} \frac{\hbar}{2\epsilon_0 \omega_k L^3} \, . \tag{7}$$

5. Justify that replacing the sum $\sum_{\boldsymbol{k}}$ in Eq. (7) by the integral $\int d^3k/(2\pi/L)^3$ leads to a divergent result.

In order to eliminate the divergence, we truncate the integral over k: we discard all large values of k and keep only those wavevectors k smaller than some cut–off value k_M.

6. Under this prescription, calculate $\Delta^{(i)}$ explicitly:

$$\frac{\Delta^{(i)}}{\text{Ry}} = \frac{1}{\pi\,\alpha} \left(\frac{\hbar k_M}{mc}\right)^2 \qquad \text{where} \quad \text{Ry} = \frac{\alpha^2}{2} mc^2 = h \times 3.3 \times 10^{15}\,\text{Hz} \quad \text{is the Rydberg energy.} \tag{8}$$

We now turn to the second contribution, $\Delta^{(ii)}$, appearing in Eq. (6). It involves the squared matrix elements $|\langle\phi_a|H_I|g,\text{vac}\rangle|^2$, where $|\phi_a\rangle$ is an eigenstate of $H_A + H_{\text{field}}$ other than $|\phi_0\rangle = |g,\text{vac}\rangle$.

7. In view of the calculation of $\Delta^{(ii)}$, show the following properties:
   a) The denominator in the expression for $\Delta^{(ii)}$ never changes signs, so that the principal part $\mathcal{P}$ plays no role.
   b) The contribution to $\Delta^{(ii)}$ up to order q^2 comes from H_{I1} only.
   c) In order for $\langle\phi_a|H_{I1}|g,\text{vac}\rangle$ to be nonzero, $|\phi_a\rangle$ must involve an excited atomic state $|e\rangle$ and exactly one photon: $|\phi_a\rangle = |e\rangle \otimes |\boldsymbol{k}, j\rangle$, where $|k\rangle$ and j are the wavevector and the polarisation of the photon.
   HINT: Use an argument involving parity to prove that, for any atomic state $|a\rangle$, $\langle a|\boldsymbol{p}|a\rangle = \boldsymbol{0}$.
   d) The matrix element $\langle e;\boldsymbol{k},j|H_{I1}|g;\text{vac}\rangle$ is given by:

$$\langle e;\boldsymbol{k},j|H_{I1}|g,\text{vac}\rangle = -\frac{q}{m} \left(\boldsymbol{p}_{ge} \cdot \boldsymbol{\epsilon}_{\boldsymbol{k},j}\right) \left(\frac{\hbar}{2\epsilon_0 \omega_k L^3}\right)^{1/2} \qquad \text{where} \quad \boldsymbol{p}_{ge} = \langle e|\boldsymbol{p}|g\rangle \, . \tag{9}$$

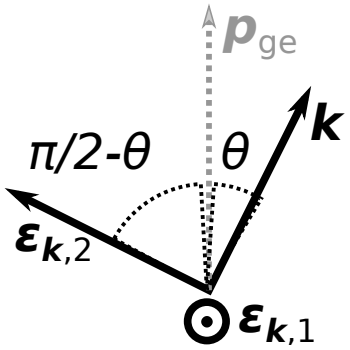

Figure 2 For a given excited atomic state and photon wavevector $\boldsymbol{k}$, the polarisation vectors $\epsilon_{\boldsymbol{k},1}$ and $\epsilon_{\boldsymbol{k},2}$ sketched on the figure may be used to prove Eq. (11).

e) The contribution $\Delta^{(ii)}$ is a sum over excited atomic states $|e\rangle$, photon wavevectors $\boldsymbol{k}$ and polarisations j:

$$\Delta^{(ii)} = -\frac{1}{2L^3}\,\frac{q^2}{m^2\epsilon_0 c^2}\,\sum_e \sum_{\boldsymbol{k}} \frac{1}{k(k_{ge}+k)} \sum_{j=1,2} |\boldsymbol{p}_{ge}\cdot\boldsymbol{\epsilon}_{\boldsymbol{k},j}|^2 \quad \text{where} \quad k_{ge} = \frac{E_e - E_g}{\hbar c}\ . \tag{10}$$

f) For a given wavevector $\boldsymbol{k}$ and excited atomic state $|e\rangle$, we call θ the angle between $\boldsymbol{k}$ and $\boldsymbol{p}_{ge}$. Show that:

$$\sum_{j=1,2} |\boldsymbol{p}_{ge}\cdot\epsilon_{\boldsymbol{k},j}|^2 = |\boldsymbol{p}_{ge}|^2 \sin^2\theta\ . \tag{11}$$

HINT: First, justify that the polarisations $\boldsymbol{\epsilon}_{\boldsymbol{k},1}$ and $\boldsymbol{\epsilon}_{\boldsymbol{k},2}$ may be chosen as on Fig. 2.

8. Use the results of question 7 to calculate $\Delta^{(ii)}$, using the same cut–off value k_M as in question 6. Express $\Delta^{(ii)} = \Delta^{(ii,a)} + \Delta^{(ii,b)}$ as a sum of two terms, with $\Delta^{(ii,a)}$ being proportional to k_M and $\Delta^{(ii,b)}$ to its logarithm:

$$\Delta^{(ii,a)} = -\frac{2\alpha}{3\pi}\,\hbar c k_M\,\frac{\sum_e |\boldsymbol{p}_{ge}|^2}{(mc)^2} \quad \text{and} \quad \Delta^{(ii,b)} = \frac{2\alpha}{3\pi}\,\frac{\hbar}{m^2 c}\sum_e |\boldsymbol{p}_{ge}|^2\, k_{ge}\,\ln\left(\frac{k_M}{k_{ge}}\right)\ . \tag{12}$$

HINT: The final step in the derivation of the second half of Eq. (12) is to assume $k_M \gg k_{ge}$.

2.1 Analysis of $\Delta^{(ii,a)}$: radiative correction to the electron mass

9. Cast $\Delta^{(ii,a)}$ into the following form:

$$\Delta^{(ii,a)} = -\frac{\langle g|\boldsymbol{p}^2|g\rangle}{2m}\,\frac{\delta m}{m} \quad \text{with} \quad \frac{\delta m}{m} = \frac{4\alpha}{3\pi}\,\frac{\hbar k_M}{mc}\ . \tag{13}$$

HINT: Use the closure relation on the atomic states, $1 = |g\rangle\langle g| + \sum_e |e\rangle\langle e|$, and the hint for question 7c above.

In order to understand the meaning of Eq. (13), we consider — **in question 10 only** — the case of a free electron. The atomic Hamiltonian H_A is replaced by $H_A^{\rm free} = \boldsymbol{p}^2/(2m)$, which contains no Coulomb trapping term and, hence, reduces to the kinetic energy.

10. We consider the bare state $|\boldsymbol{p}_0, \text{vac}\rangle$, which represents the free electron in the plane–wave state with the momentum $\boldsymbol{p}_0$, in the absence of any photon.
    a) Using Eq. (5), show that the energy shift $\Delta^{\rm free}$ of the state $|\boldsymbol{p}_0, \text{vac}\rangle$, due to the coupling between the electron and the electromagnetic field, is given by:

$$\Delta^{\rm free} = -\frac{\boldsymbol{p}_0^2}{2m}\,\frac{\delta m}{m}\ , \qquad \text{where } \delta m \text{ is defined in Eq. (13).} \tag{14}$$

    What is the corresponding linewidth $\Gamma^{\rm free}$?

    b) Working up to first order in $\alpha \approx 1/137$, explain why this shift may be understood as a radiative correction to the mass of the free electron. In other words, justify that the Hamiltonian for the free electron may be corrected to include the radiative shift by changing the electron mass slightly:

$$\widetilde{H}^{\rm free} = \frac{\boldsymbol{p}^2}{2\widetilde{m}}\ , \qquad \text{with} \qquad \widetilde{m} = m + \delta m\ . \tag{15}$$

11. Using question 10, interpret the contribution $\Delta^{(ii,a)}$ to the Lamb shift for the full Hamiltonian H_A (cf. Eq. (13)).

2.2 Analysis of $\Delta^{(ii,b)}$: main contribution to the Lamb shift

We now turn to $\Delta^{(ii,b)}$. First, we introduce the 'Bethe logarithm' $\ln\left(K_g^{\rm Bethe} a_0\right)$, defined as:

$$\ln\left(K_g^{\rm Bethe} a_0\right) = \frac{\sum_e |\boldsymbol{p}_{ge}|^2 k_{ge} \ln(k_{ge} a_0)}{\sum_e |\boldsymbol{p}_{ge}|^2 k_{ge}} \tag{16}$$

The Bethe logarithm measures the average excitation energy. It is defined for all atomic levels $|n,l,m\rangle$ and depends only on the quantum numbers n and l (see Eq. (22) and question 14 below). Its numerical values are tabulated in the literature (see e.g. [4, Table I]).

12. In order to write $\Delta^{(ii,b)}$ in a more explicit form:
    a) First, rewrite $\Delta^{(ii,b)}$ in terms of the Bethe logarithm:

$$\Delta^{(ii,b)} = \frac{2\alpha}{3\pi} \frac{\hbar}{m^2 c} \left(\sum_e |\boldsymbol{p}_{ge}|^2 k_{ge} \right) \ln\left(\frac{k_M}{K_g^{\rm Bethe}} \right) \tag{17}$$

    b) Calculate the average, in the atomic state $|g\rangle$, of the following double commutator involving p_x and H_A:

$$\langle g| \left[p_x, [p_x, H_A]\right] |g\rangle = -2\hbar c \sum_e k_{ge} \, |\, \langle e|p_x|g\rangle \,|^2 \ . \tag{18}$$

    c) Justify the following identity:

$$[p_x, [p_x, H_A]] = -\hbar^2 \frac{\partial^2}{\partial x^2} V_{\rm Coulomb}(\boldsymbol{r}) \quad \text{where} \quad V_{\rm Coulomb}(\boldsymbol{r}) = -\frac{q^2}{4\pi\epsilon_0 r} \ . \tag{19}$$

    HINT: Use the hint for question 1 above.

    d) Combining Eqs. (18) and (19), show that:

$$\sum_e |\boldsymbol{p}_{ge}|^2 k_{ge} = \frac{\hbar}{2c} \ \langle g|\nabla^2 V_{\rm Coulomb}(\boldsymbol{r})|g\rangle \quad = 2\pi\alpha^4 \ \frac{(mc)^3}{\hbar} \ a_0^3 \ \langle g|\delta(\boldsymbol{r})|g\rangle \ . \tag{20}$$

    HINT: $\nabla^2(1/r) = -4\pi\,\delta(\boldsymbol{r})$, an equation closely linked to Poisson's equation on the electrostatic field.

    e) Conclude that $\Delta^{(ii,b)}$ may be cast into the following form:

$$\frac{\Delta^{(ii,b)}}{\rm Ry} = \frac{8}{3} \, \alpha^3 a_0^3 \, \langle g|\delta(\boldsymbol{r})|g\rangle \ \ln\left(\frac{k_M}{K_g^{\rm Bethe}} \right) \ . \tag{21}$$

3 Extension to excited atomic levels

In this Section, we extend the results of Sec. 2 to an arbitrary discrete level $|\phi_{n,l,m}\rangle$ of the (spin–less) hydrogen atom described by the Hamitonian H_A.

13. In order to show that the previous results are applicable with minimal changes, justify the following two points:
    a) In the considered approximation, the electromagnetic field does not couple atomic levels with different angular momenta. Hence, one may work in a multiplicity with given l and m, which eliminates the degeneracies.
    b) Excited atomic levels have finite lifetimes. Correspondingly, the principal part in Eq. (5) now plays a role.

We generalise the Bethe logarithm of Eq. (16) to an excited atomic state $|e\rangle$ through the formula:

$$\ln\left(K_e^{\rm Bethe} a_0\right) = \frac{\sum_a |\boldsymbol{p}_{ea}|^2 |k_{ea}| \ln(|k_{ea}| a_0)}{\sum_a |\boldsymbol{p}_{ea}|^2 |k_{ea}|} \ , \quad \text{where} \quad \boldsymbol{p}_{ea} = \langle a|\boldsymbol{p}|e\rangle \ \text{and} \ k_{ea} = (E_a - E_e)/(\hbar c) \ . \tag{22}$$

In Eq. (22), the sum over a covers all atomic states $|\phi_{n,l,m}\rangle$, including the ground state.

14. Justify that, if $|e\rangle = |\phi_{n,l,m}\rangle$, $K_e^{\rm Bethe}$ does not depend on the quantum number m giving the projection of the angular momentum. Hence, we may write $K_e^{\rm Bethe} = K_{nl}^{\rm Bethe}$.

15. Write the total shift Δ_e for the atomic level $|e\rangle = |\phi_{n,l,m}\rangle$ in the following form:

$$\frac{\Delta_e}{\mathrm{Ry}} \;=\; \frac{1}{\pi\,\alpha}\left(\frac{\hbar k_M}{mc}\right)^2 \;-\; \frac{\langle e|\,\boldsymbol{p}^2/(2m)\,|e\rangle}{\mathrm{Ry}}\,\frac{\delta m}{m} \;+\; \frac{8}{3}\,\alpha^3 a_0^3\,\langle e|\delta(\boldsymbol{r})|e\rangle\,\ln\left(\frac{k_M}{K_a^{\mathrm{Bethe}}}\right)\,. \qquad (23)$$

Let us now investigate the dependence of the three terms in Eq. (23) on the quantum numbers n, l, m. The first term, proportional to k_M^2, is the same for all atomic levels.

16. Show that the second term, proportional to k_M (through δm), depends only on n.
HINT: You may use a well–known theorem which applies both to the hydrogen atom and to the harmonic oscillator.

17. Question 14 shows that the third term in Eq. (23), containing the logarithm of k_M, depends on only n and l.
    a) Recall why the eigenstates $|\phi_{n,l,m}\rangle$ of H_{A} may be sought as $\phi_{n,l,m}(\boldsymbol{r}) = \chi_{n,l,m}(r)\,Y_l^m(\boldsymbol{r})/r$, where $Y_l^m(\boldsymbol{r})$ is a spherical harmonic, and the function $\chi_{n,l,m}(r)$ is regular for $r \to 0$.
    b) Write down the complete equation satisfied by $\chi_{n,l,m}(r)$.
    Show that, for small r, this equation reduces to: $d^2\chi_{n,l,m}/dr^2 = l(l+1)\,\chi_{n,l,m}(r)/r^2$.
    c) Using questions 17a & 17b, show that, for small r, $\phi_{n,l,m}(\boldsymbol{r})$ behaves like r^l. Recall the name of this effect.
    d) Conclude that the third term in Eq. (23) is non–zero only for atomic states $|\phi_{n,0,0}\rangle$ with the angular quantum number $l = 0$ (atomic 's–wave' states).

4 Lamb shift of the $2s$–$2p$ transition

In this final section, we calculate the numerical value of the Lamb shift for the $2s$–$2p$ transition, that is, the relative shift between the two levels $|\phi_{n=2,l=0,m=0}\rangle$ and $|\phi_{n=2,l=1,m=0}\rangle$.

18. Thanks to question 16, justify that the relative shift between these two levels depends only logarithmically on the value of the cut–off k_M.

19. In keeping with the non–relativistic approximation, explain why we may choose $k_M = mc/\hbar$ (thus $k_M a_0 = 1/\alpha$).

20. For the state $|e\rangle = |\phi_{n=2,l=0,m=0}\rangle$, the Bethe logarithm $\ln(K_e^{\mathrm{Bethe}} a_0) = -4.64$, and the atomic wavefunction satisfies $|\phi_a(\boldsymbol{0})|^2 = 1/(8\pi a_0^3)$. Calculate the Lamb shift between the states $|n=2, l=0, m=0\rangle$ and $|n=2, l=1, m=0\rangle$ using Eq. (23). Compare the obtained numerical value with the measured one, $h\times 1058\,\mathrm{MHz}$.

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A more accurate evaluation of the Lamb shift (see e.g. Ref. [3, §18 & 19]) includes the (relativistic) contribution of the high–frequency photons, which we have neglected in our non–relativistic approach. This more accurate approach yields an expression for the shift which does not depend on the cut–off wavevector k_M. The improved theoretical value, of the order of 1052 MHz*, is in very good agreement with the experimental result (*1058 MHz*, see Fig. 1).*

Further reading

- A concise and rigorous presentation of the techniques associated with the resolvent may be found in Messiah's book [5]: §15–17 in chapter XVI, and §13 and §15 in chapter XXI, are especially relevant.
- The formulas pertaining to the quantised electromagnetic field and its coupling to atoms are collected in the Appendix to Ref. [6].
- Reference [1] is an excellent historical introduction to the spectroscopy of atomic hydrogen.

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