ICFP M2 Advanced Quantum Mechanics: Homework problem

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THE JOSEPHSON EFFECT IN A SUPERFLUID BOSE GAS

The Josephson effect is a macroscopic quantum phenomenon affecting systems of many interacting particles, whereby the amplitude of the current tunnelling through a potential barrier is enhanced thanks to the interactions. This effect has first been discovered in the context of superconductors, where the supercurrent crossing the barrier consists of interacting electrons which may be described using the Bardeen–Cooper–Schrieffer theory for superconductivity [1, chap. 21]. In this problem, we analyse its experimental observation with an ultracold bosonic gas trapped in a double–well potential: here, the superfluid whose constituents tunnel through the barrier is an interacting Bose–Einstein condensate described by the Gross–Pitaevskii equation.

In the first part of the problem, we establish two 'hydrodynamic' equations describing the macroscopic behaviour of many interacting particles. In the second part, we focus on the case of a Bose–Einstein condensate at $T = 0$ trapped in a double–well potential, which we describe using the Gross–Pitaevskii equation. We derive the two Josephson equations and point out their analogy with the mechanical equations for the classical pendulum. In the third and final part, we identify two regimes for the Josephson effect, and compare our theoretical predictions to the experimental results of Ref. [2].

The problem is entirely self–contained. Solving it does not at all require reading the references given at the end. All of the results required for proceeding with the solution are explicitly given. Four questions require background knowledge about superfluids; they are identified with 'dangerous bend' symbols ☡ and their solution has no incidence on the rest of the problem.

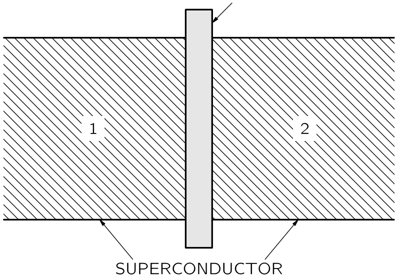


Figure 1 In solid–state systems, the Josephson effect is investigated in Josephson junctions, i.e. two superconductors separated by a thin insulating barrier. The supercurrent crossing the barrier consists of interacting electrons. Reproduced from [1, chap. 21].

1 Hydrodynamic description of a superfluid

We consider a three–dimensional system of N identical bosonic particles. These particles each have the mass m and interact via the two–body potential $V(|\boldsymbol{r}_i - \boldsymbol{r}_j|)$, where $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ are the spatial coordinates of the particles i and j. At this point, we make no assumption on the range of the interaction. We also assume for now that the particles are trapped in a box–like potential, the size of the box being much larger than all relevant lengthscales. In terms of first quantisation operators, the many–body Hamiltonian reads:

$$H = \sum_{i=1}^{N} \frac{\boldsymbol{p}_i^2}{2m} \quad + \quad \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i} V(|\boldsymbol{r}_i - \boldsymbol{r}_j|) \ . \tag{1}$$

1.1 Equation for the time dependence of the phase

1. Express the Hamiltonian of Eq. (1) in terms of the field operators $\hat{\Psi}(\boldsymbol{r})$, $\hat{\Psi}^\dagger(\boldsymbol{r})$, and their spatial derivatives. Briefly recall why, in the Heisenberg representation, the time–dependent field operator $\hat{\Psi}(\boldsymbol{r},t)$ satisfies the following equation:

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}(\boldsymbol{r},t) = \left[-\frac{\hbar^2\nabla^2}{2m} + \int d^3r'\ \hat{\Psi}^\dagger(\boldsymbol{r'},t)V(\boldsymbol{r'}-\boldsymbol{r})\hat{\Psi}(\boldsymbol{r'},t)\right]\hat{\Psi}(\boldsymbol{r},t) \ . \tag{2}$$

For a given velocity $\boldsymbol{v}$, we seek how $\hat{\Psi}(\boldsymbol{r},t)$ is affected by the Galilean transformation representing a change of referentials from $(\boldsymbol{r},t)$ to $(\boldsymbol{R}=\boldsymbol{r}-\boldsymbol{v}t, T=t)$. Let $\hat{\Psi}(\boldsymbol{R},T)$ be a solution of Eq. (2):

$$i\hbar\frac{\partial}{\partial T}\hat{\Psi}(\boldsymbol{R},T) = -\frac{\hbar^2\nabla^2}{2m}\hat{\Psi}(\boldsymbol{R},T) + \int d^3R'\ \hat{\Psi}^\dagger(\boldsymbol{R'},t)V(\boldsymbol{R'}-\boldsymbol{R})\hat{\Psi}(\boldsymbol{R'},T)\hat{\Psi}(\boldsymbol{R},T) \ . \tag{3}$$

We construct the function $\hat{\Psi}'(\boldsymbol{r},t)$ defined by:

$$\hat{\Psi}'(\boldsymbol{r},t) = \hat{\Psi}(\boldsymbol{R},T)\exp\left[\frac{i}{\hbar}\left(m\boldsymbol{v}\cdot\boldsymbol{r} - \frac{1}{2}mv^2t\right)\right] \quad \text{with} \quad \boldsymbol{R}=\boldsymbol{r}-\boldsymbol{v}t \text{ and } T=t \ . \tag{4}$$

2. Show that $\hat{\Psi}'(\boldsymbol{r},t)$ is also a solution of Eq. (2).
   HINT: First, express $i\hbar\frac{\partial}{\partial t}\hat{\Psi}'(\boldsymbol{r},t)$ in terms of the derivatives $\frac{\partial}{\partial T}\hat{\Psi}(\boldsymbol{R},T)$ and $\nabla_{\boldsymbol{R}}\hat{\Psi}(\boldsymbol{R},T)$. Then, use Eq. (3). Finally, express $\hat{\Psi}(\boldsymbol{R},T)$ and its spatial derivatives in terms of $\hat{\Psi}'(\boldsymbol{r},t)$ and its spatial derivatives.

3. We introduce the order parameter Ψ, defined as the expectation value $\Psi(\boldsymbol{r},t) = \langle\hat{\Psi}(\boldsymbol{r},t)\rangle$. We assume that $\Psi(\boldsymbol{r},t)\neq 0$. Recall how this assumption is linked to Bose–Einstein condensation.

4. Using Eq. (4), justify that Ψ obeys the same transformation law as $\hat{\Psi}$:

$$\Psi'(\boldsymbol{r},t) = \Psi(\boldsymbol{R},T)\exp\left[\frac{i}{\hbar}\left(m\boldsymbol{v}\cdot\boldsymbol{r} - \frac{1}{2}mv^2t\right)\right] \ . \tag{5}$$

5. We recall that, in the reference frame where the fluid is at rest, the order parameter is $\Psi_0 = \sqrt{n}e^{-i\mu t/\hbar}$, where n is the uniform fluid density and μ is its chemical potential. Using Eq. (5), show that, in the frame where the fluid moves with the velocity $\boldsymbol{v}$, $\Psi(\boldsymbol{r},t)$ reads:

$$\Psi(\boldsymbol{r},t) = \sqrt{n}e^{iS(\boldsymbol{r},t)} \quad \text{with} \quad \hbar S = m\boldsymbol{v}\cdot\boldsymbol{r} - \left(\frac{1}{2}mv^2 + \mu\right)t \ . \tag{6}$$

6. Starting from Eq. (6), show that the velocity $\boldsymbol{v}$ is given by the gradient of the phase S:

$$\boldsymbol{v} = \frac{\hbar}{m}\nabla S \ . \tag{7}$$

7. Show that the phase of the order parameter obeys the law:

$$\hbar \frac{\partial S}{\partial t} = -\left(\frac{1}{2} m v^2 + \mu\right) \ . \tag{8}$$

Finally, we relax the hypothesis of a uniform fluid. The atoms are now trapped in a potential $V_{\text{trap}}(\boldsymbol{r})$. Therefore, the density $n(\boldsymbol{r})$ and the velocity $\boldsymbol{v}(\boldsymbol{r})$ now depend on position.

8. Justify that, if n and $\boldsymbol{v}$ vary sufficiently slowly in space and time, Eq. (8) becomes:

$$m \frac{\partial \boldsymbol{v}}{\partial t} + \nabla \left[\frac{1}{2} m v^2 + \mu(n(\boldsymbol{r})) + V_{\text{trap}}(\boldsymbol{r})\right] = 0 \ . \tag{9}$$

9. Using Eq. (7), explain why the motion of the fluid is said to be irrotational.
   May we conclude that the fluid cannot support angular momentum?
   HINT: What are the excitations associated with rotating superfluids?

10. Which lengthscale should the variations of n and $\boldsymbol{v}$ be compared to when applying Eq. (9)? Is the hydrodynamic approach more robust for interaction energies which are larger or smaller compared to the kinetic energy?
    HINT: For example, in a weakly–interacting Bose gas, the chemical potential is $\mu = gn$.

1.2 Continuity equation

11. We focus on the zero–temperature case $T = 0$.
    Invoke a general argument which justifies the following equation satisfied by the density:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \boldsymbol{v}) = 0 \ . \tag{10}$$

12. Explain why Eq. (10) is valid only at zero temperature.

A short comment: *For a Bose condensate at $T = 0$, the relation between the velocity and the gradient of the phase of Eq. (7), as well as the hydrodynamic Eqs. (9) and (10), all follow from the Gross–Pitaevskii equation [3, §22.7.2], and may be derived by writing the condensate wavefunction Ψ in terms of its modulus and phase: $\Psi(\boldsymbol{r}, t) = \sqrt{n(\boldsymbol{r}, t)} e^{iS(\boldsymbol{r}, t)}$. However, these equations are actually more general than the Gross–Pitaevskii theory. They hold both at $T = 0$ and at $T > 0$ (in which case the continuity Eq. (10) should be suitably generalised to account for the non–condensed atoms). They describe not only bosonic gases, but also helium and fermionic superfluids (where the dependence of the chemical potential on the density is very different).*

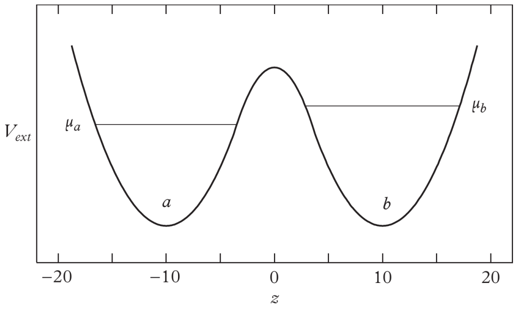


Figure 2 The considered geometry: a 3D Bose–Einstein condensate described by the Gross–Pitaevskii equation is trapped inside a double–well potential. The left and right wells respectively contain $N_a(t)$ and $N_b(t)$ atoms, corresponding to the chemical potentials $\mu_a(t)$ and $\mu_b(t)$, both of which always remain smaller than the barrier height.

2 The two Josephson equations

2.1 Derivation of the equations starting from the Gross–Pitaevskii equation

We focus on the case of a Bose–Einstein condensate at $T = 0$. We consider the 3D double–well geometry represented on Fig. 2, where the trapping potential $V_{\text{ext}}(\boldsymbol{r})$ is symmetric with respect to the centre of the barrier: $V_{\text{ext}}(x, y, z) = V_{\text{ext}}(x, y, -z)$. We describe the condensate using the macroscopic wavefunction $\Psi(\boldsymbol{r}, t)$, which is normalised to the total number N of particles: $\int d^3r\, |\Psi(\boldsymbol{r}, t)|^2 = N$. The function $\Psi(\boldsymbol{r}, t)$ satisfies the Gross–Pitaevskii equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V_{\text{ext}}(\boldsymbol{r})\Psi + g|\Psi(\boldsymbol{r}, t)|^2\Psi\ . \tag{11}$$

We first consider stationary solutions. We call $\Psi_a(\boldsymbol{r}, N_a)$ the (real) ground state solution for N_a atoms localised in the left well (a), and $\Psi_b(\boldsymbol{r}, N_b)$ the (real) ground state solution for N_b atoms localised in the right well (b):

$$\mu_\eta\Psi_\eta(\boldsymbol{r}) = -\frac{\hbar^2}{2m}\nabla^2\Psi_\eta + V_{\text{ext}}(\boldsymbol{r})\Psi_\eta + g|\Psi_\eta(\boldsymbol{r})|^2\Psi_\eta \quad \text{with} \int d^3r|\Psi_\eta|^2 = N_\eta \quad (\eta = a \text{ or } b). \tag{12}$$

We assume that the chemical potentials $\mu_a(N/2) = \mu_b(N/2)$, calculated for $N/2$ atoms localised in a given well a or b, are smaller than the height of the barrier separating the two wells.

13. We neglect the overlap between the functions $\Psi_a(\boldsymbol{r}, N/2)$ and $\Psi_b(\boldsymbol{r}, N/2)$. Justify that any linear combination of the form $\Psi_{\alpha,\beta}(\boldsymbol{r}, t) = [\alpha\Psi_a(\boldsymbol{r}, N/2) + \beta\Psi_b(\boldsymbol{r}, N/2)]e^{-i\mu t/\hbar}$, where α and β are (constant) complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 2$, is a stationary solution of Eq. (11) corresponding to the total atom number N.

14. We focus on the case where $\alpha = \beta = 1$, and consider the wavefunction $\Psi(\boldsymbol{r}, t)$ defined as:

$$\Psi(\boldsymbol{r}, t) = \left[\Psi_a\left(\boldsymbol{r}, \frac{N}{2}\right) + \Psi_b\left(\boldsymbol{r}, \frac{N}{2}\right)\right] e^{-i\mu t/\hbar}\ . \tag{13}$$

    Justify that $\Psi(\boldsymbol{r}, t)$ is the ground–state solution of Eq. (11).

    Hint: Thinking in terms of Eq. (7), what extra energy would a phase gradient near $z = 0$ yield?

15. For a stationary solution $\Psi(\boldsymbol{r})e^{-i\mu t/\hbar}$ of Eq. (11) representing N atoms, show that:

$$N\mu = -\frac{\hbar^2}{2m}\int d^3r\ \Psi^*\Delta\Psi + \int d^3r\ V(\boldsymbol{r})|\Psi|^2 + g\int d^3r\ |\Psi|^4\ . \tag{14}$$

    Conclude that, in Eq. (13), $\mu = \mu_a(N/2) = \mu_b(N/2)$.

We now seek time–dependent solutions of Eq. (11) in the form:

$$\Psi(\boldsymbol{r}) = \Psi_a(\boldsymbol{r}, N_a)\, e^{iS_a} + \Psi_b(\boldsymbol{r}, N_b)\, e^{iS_b}\ , \tag{15}$$

where the real functions Ψ_a and Ψ_b have been introduced in Eq. (12). The atom numbers $N_a(t)$ and $N_b(t)$ depend on time; the phases $S_a(t)$ and $S_b(t)$ depend on time but not on $\boldsymbol{r}$.

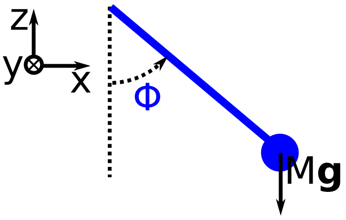


Figure 3 A classical, mechanical pendulum oscillating in the (xz) plane.

16. In Eq. (15), what is the sum $N_a + N_b$ constantly equal to?
Conclude that the atomic current I satisfies $I = \partial N_b/\partial t = -\partial N_a/\partial t = \partial[(N_b - N_a)/2]/\partial t$.

17. We write $\Psi(\boldsymbol{r}, t) = n(\boldsymbol{r}, t)e^{iS(\boldsymbol{r},t)}$. According to Eq. (7), the velocity $\boldsymbol{v}(\boldsymbol{r}, t) = \hbar\nabla S(\boldsymbol{r}, t)/m$. Justify that the current density $\boldsymbol{j}(\boldsymbol{r}, t) = n\boldsymbol{v}$ is along the z axis, and show that $I(t)$ is related to the relative phase $\Phi(t) = S_b - S_a$ through:

$$I = -I_J \sin(\Phi), \quad \text{with} \quad I_J = \frac{\hbar}{m}\int dxdy \left[\Psi_a \frac{\partial \Psi_b}{\partial z} - \Psi_b \frac{\partial \Psi_a}{\partial z}\right]_{z=0} . \tag{16}$$

Hint: First, establish $\boldsymbol{j}(\boldsymbol{r}, t) = \hbar(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)/(2im)$.

18. Using the expression for I_J in Eq. (16), justify that it is a positive quantity.
Hint: Do $\Psi_a(N/2, z)$ and $\Psi_b(N/2, z)$ increase or decrease when z increases?

19. We assume that the kinetic energy $mv^2/2$ is small compared to the chemical potentials $\mu_a(t)$ and $\mu_b(t)$. Starting from Eq. (8), show that the relative phase $\Phi(t)$ satisfies:

$$\hbar\frac{\partial \Phi}{\partial t} = -(\mu_b - \mu_a) . \tag{17}$$

20. We assume that $N_a(t)$ and $N_b(t)$ always remain close to their equilibrium values $N_a^0 = N_b^0 = N/2$ (but $\Phi(t)$ may become large). We introduce $\delta N(t) = (N_b - N_a)/2 \ll N$, $E_C = 2d\mu_a/dN_a|_{N_a=N/2}$, and $E_J = \hbar I_J$. Show that Eqs. (16) and (17) reduce to:

$$\hbar\frac{\partial}{\partial t}\delta N = -E_J \sin\Phi \quad \text{and} \quad \hbar\frac{\partial \Phi}{\partial t} = E_C\,\delta N . \tag{18}$$

Hint: For the second relation, write $\mu_b - \mu_a = [\mu_b - \mu_b(N/2)] - [\mu_a - \mu_a(N/2)]$.

2.2 Analogy with the classical pendulum

We consider a classical, mechanical pendulum of length L oscillating in the (xz) plane. No assumption is made on the amplitude of the oscillations of the angle ϕ. We call σ_y the (classical) angular momentum of the mass M with respect to the axis (Oy).

21. Show that the equations of motion may be written in terms of ϕ and σ_y as:

$$\frac{\partial}{\partial t}\sigma_y = -MgL\sin\phi \quad \text{and} \quad \frac{\partial}{\partial t}\phi = \frac{1}{ML^2}\sigma_y . \tag{19}$$

22. Comparing Eqs. (18) and (19), show that the oscillations of the Bose–Einstein condensate described by $(\delta N, \Phi)$ are analogous to the dynamics of the pendulum described by (σ_y, Φ).

23. We introduce the energy $H_J(\Phi, \delta N)$ defined as follows:

$$H_J = \frac{1}{2}E_C\,(\delta N)^2 - E_J\,\cos\Phi . \tag{20}$$

Show that H_J is a conserved quantity for the time evolution described by Eq. (18).
What is the corresponding quantity for the classical mechanical oscillator of Fig. 3?

3 Two regimes for the Josephson effect

3.1 Plasma oscillations

In this regime, the phase Φ appearing Eq. (18) always remains small compared to 2π.

24. What does this regime correspond to in the case of the mechanical oscillator of Fig. 3?
25. Show that, in this regime, both the atom number $\delta N(t)$ and the phase $\Phi(t)$ undergo sinusoidal oscillations at the same frequency ω_{pl} given by:

$$\hbar\omega_{\mathrm{pl}} = \sqrt{E_C E_J} \ . \tag{21}$$

3.2 Self–trapping

In this second regime, the phase $\Phi(t)$ undergoes at least one full rotation, i.e. it explores at least all possible values from 0 to 2π.

26. What does this regime correspond to in the case of the mechanical oscillator of Fig. 3?
27. We assume that the initial phase difference is $\Phi_0 = \Phi(t = 0) = 0$. Using the conserved energy H_J of Eq. (20), show that for the phase Φ to undergo at least one full rotation, the initial atom number difference $\delta N_0 = \delta N(t = 0)$ must satisfy:

$$|\delta N_0| \geq (2E_J/E_C)^{1/2} \ . \tag{22}$$

28. We wish the atom number to remain almost constant: $\delta N(t) \approx \delta N_0$. Under this condition, show from Eq. (18) that the phase $\Phi(t)$ and the atom number difference $\delta N(t)$ are given by:

$$\Phi \simeq \frac{E_C}{\hbar}\,\delta N_0\, t \quad \text{and} \quad \delta N \simeq \delta N_0 \left[1 + \frac{1}{(\delta N_0)^2}\frac{E_J}{E_c}\cos\left(\frac{E_C}{\hbar}\delta N_0\, t\right)\right] \ . \tag{23}$$

29. Justify that the validity of Eq. (23) requires a more stringent condition than Eq. (22), namely:

$$(\delta N_0)^2 \gg \frac{E_J}{E_C} \ . \tag{24}$$

30. Finally, exploit the condition for small atom number oscillations $|\delta N| \ll N$ to show that the self–trapping regime is possible only if the total atom number is large enough: $N^2 \gg E_J/E_C$.

3.3 Comparison with the experiment

31. Explain why the left column of Fig. 4 right illustrates plasma oscillations, and give an estimate for the observed plasma oscillation period $T_{\mathrm{pl}} = 2\pi/\omega_{\mathrm{pl}}$.
32. Explain why the right column of Fig. 4 right illustrates self–trapping.

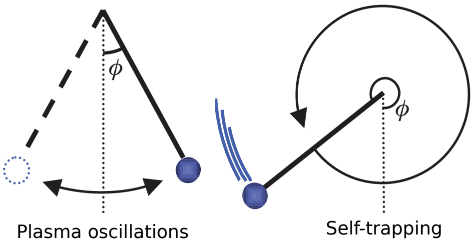
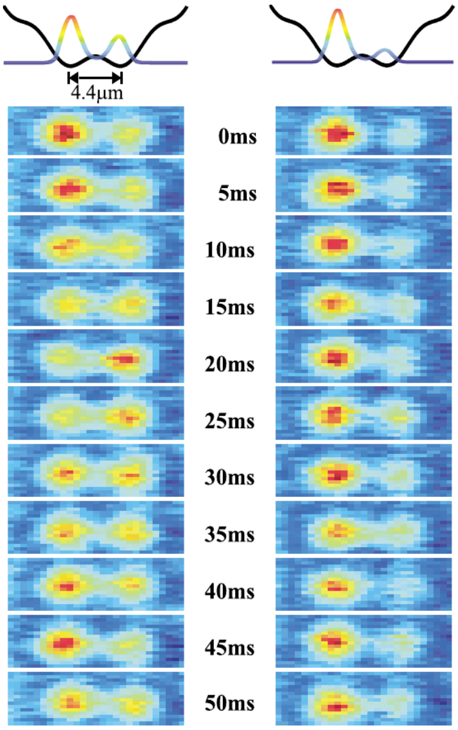


Figure 4 Above: interpretation of the two regimes (plasma oscillations, self–trapping) in terms of the analogy with the classical pendulum (adapted from Ref. [4]). Right: experimental absorption images showing the tunnelling dynamics of a Bose–Einstein condensate in a double–well potential. The left column shows plasma oscillations and the right column shows self–trapping. The phases in the two wells are initially equal; the dynamics are caused by an initial atom number imbalance δN_0 in between the two wells which is small for plasma oscillations and large for self–trapping. (Reproduced from Ref. [2]).



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