ICFP M2 Advanced Quantum Mechanics: Exercises on identical particles

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1 Symmetrised wavefunctions for identical particles

1.1 Two particles

1. Consider a system of two identical particles with spin s. Find the number of different spin states which are *(a)* symmetric and *(b)* antisymmetric with respect to the exchange of the spin variables.

2. Specialise the results of the previous question to the case of $s = 1/2$.
   In this case, determine the total spin of each of the two–particle states you have found.

1.2 Three particles

3. We consider three identical particles.
   Let $(\psi_n(\boldsymbol{r}))$ be a basis of normalised wavefunctions for single—particle states.
   a) Using the first quantisation formalism, write down a normalised and properly symmetrised wavefunction for the state composed of three particles with the quantum numbers n_1, n_2, and n_3, *(i)* in the fermionic case and *(ii)* in the bosonic case.
   b) Express these wavefunctions using creation operators acting on the vacuum state.

   Discuss what happens if two or more of the indices n_j are equal.

1.3 Dimension of the symmetrised subspace for identical bosons

4. We consider N identical bosons, each having Ω accessible internal states (e.g. $\Omega = 3$ for a spin–1 boson). How many symmetrised N–particle internal states are there?

2 Symmetries, selection rules, and conserved quantities

2.1 A selection rule for ${}^{2S+1}L$ states

We consider two identical particles in free space, each with mass m and spin s, which interact via the potential $V(r) = V(|\boldsymbol{r_2} - \boldsymbol{r_1}|)$ depending only on their relative distance.

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1. Recall why the motion of the two particles may be separated into *(i)* the motion of the centre of mass and *(ii)* the relative motion.

We focus on the relative motion and assume that the corresponding angular momentum has the well–defined value L (i.e. $\boldsymbol{L}^2 = \hbar^2 L(L+1)$).

2. We consider the total spin operator $\boldsymbol{S} = \boldsymbol{s_1} + \boldsymbol{s}_2$. Depending on whether the atoms are fermions or bosons and on the value of L, what are the allowed values for the quantum number S such that $\boldsymbol{S}^2 = \hbar^2 S(S+1)$?
   HINT: Show that the two–particle spin state $|S, M_S\rangle$, which has well–defined values of $\boldsymbol{S}^2$ and S_z, is either symmetric or antisymmetric upon exchange of the two particles depending on the parity of S.

3. **Application to quantum collisions.** We consider two ^{87}Rb atoms in their hyperfine ground state $f = 1$. These (neutral, non–dipolar) atoms are identical bosons. We consider the total hyperfine spin $\boldsymbol{F} = \boldsymbol{f}_1 + \boldsymbol{f}_2$. Depending on the value of F, which partial waves may intervene in a quantum collision involving these two atoms (i.e. which values of L are allowed)? Comment on the limit of zero incident energy.

2.2 Symmetries of the two–dimensional harmonic oscillator

We consider a two–dimensional harmonic oscillator described by the following Hamiltonian:

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega_1^2 x^2 + \frac{p_y^2}{2m} + \frac{1}{2}m\omega_2^2 y^2 \ . \tag{1}$$

1. What is the energy spectrum of this system?

2. In the case where ω_1 and ω_2 are incommensurate (i.e. ω_2/ω_1 is not a rational number), justify that this spectrum is non–degenerate.

3. We now assume $p\omega_1 = q\omega_2$, where p and q are (relatively prime) integers. Show that the energy spectrum is degenerate. Construct two Hermitian operators Q_- and Q_+ which commute with H and which describe the symmetry causing the degeneracy.

For now on, we focus on **the isotropic case $\omega_1 = \omega_2 = \omega$.**

4. Show that the three following operators $(T_i)_{1\leq i\leq 3}$ are constants of motion in the quantum sense, i.e. $[H, T_i] = 0$:

$$T_1 = \frac{1}{2}\left(m\omega\, xy + \frac{p_x p_y}{m\omega}\right), \quad T_2 = \frac{1}{2}\left(xp_y - yp_x\right), \quad T_3 = \frac{1}{4}\left(m\omega(x^2 - y^2) + \frac{p_x^2 - p_y^2}{m\omega}\right) . \tag{2}$$

   Show that they obey the commutation relations characterising angular momenta:

$$[T_i, T_j] = i\hbar\epsilon_{ijk}T_k \ . \tag{3}$$

The potential term $m\omega^2(x^2 + y^2) = m\omega^2 r^2$ is invariant under the 2D spatial rotations. However, Eq. (3) shows that the full Hamiltonian H, which includes the (2D) kinetic term $(p_x^2 + p_y^2)/(2m)$, actually exhibits as much symmetry as a free particle in 3D space[1].

[1] In the language of group theory, the 'dynamical' symmetry group is SO(3), which is larger than the spatial symmetry group SO(2). This phenomenon is known as 'dynamical symmetry enlargement'.

5. For the 2D isotropic harmonic oscillator, identify the physical manifestation of this enlarged symmetry.
   HINT: Discuss the degeneracies in the energy spectrum.

6. Name another system presenting an analogously enlarged dynamical symmetry. From the point of view of classical mechanics, what do these two systems have in common?
   HINT: What do astronomy and the Hydrogen atom have in common?

3 Coherent states

3.1 Definition and basic properties

We consider a physical system which can be modelled as a 1D harmonic oscillator, such as one phononic mode of the vibrations of a solid, or one photonic mode of the electromagnetic field. This system may contain an arbitrary number of "particles" (phonons or photons). Its Hamiltonian is $H = \hbar\omega a^\dagger a$, where a and $a^\dagger$ are the annihilation and creation operators for the mode. We define the coherent states as the eigenstates $|\alpha\rangle$ of the annihilation operator:

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle \ . \tag{4}$$

1. Determine all allowed eigenvalues α for which the coherent state $|\alpha\rangle$ exists, and express this state in the Fock state basis. Ensure that it is correctly normalised.

2. Is the annihilation operator a hermitian? Point out an important property of the eigenvalue spectrum obtained in the previous question which illustrates your answer.

3. If, at the time $t = 0$, the system is prepared in the coherent state $|\psi(t = 0)\rangle = |\alpha\rangle$, what is the state $|\psi(t)\rangle$ of the system at the time t?

4. Which is the only coherent state that carries a well–defined phonon (or photon) number?

5. For a given state $|\alpha\rangle$, show that the probability distribution of the phonon number n is:

$$P_\alpha(n) = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!} \ . \tag{5}$$

   What is the name of this probability distribution?
   Give the mean photon value $\overline{n}$ and its standard deviation $\Delta n = \left(\overline{n^2} - \overline{n}^2\right)^{1/2}$.

6. Name two other physical systems characterised by the probability distribution of Eq. (5).

3.2 Closure relation

7. Calculate the scalar product $\langle\alpha|\beta\rangle$ of two coherent states $|\alpha\rangle$ and $|\beta\rangle$.
   Are the coherent states an orthonormal set?

8. Show the following closure relation for the coherent states:

$$\frac{1}{\pi}\int d^2\alpha \; |\alpha\rangle \langle\alpha| = \mathbb{1} \;, \tag{6}$$

where the integral is taken over the whole complex plane. Use Eq. (6) to write the Fock state $|n\rangle$ as a superposition of coherent states. Is such a decomposition unique?
HINT: Concerning whether or not the expansion is unique, consider using your answer to Question 4.

3.3 Quasi–classical properties

9. In terms of the operators $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$, the electric field operator reads: $\boldsymbol{E} = i\mathcal{E}_0[\boldsymbol{\epsilon}\boldsymbol{a} - \boldsymbol{\epsilon}^*\boldsymbol{a}^\dagger]$, where $\mathcal{E}_0$ is a normalisation factor and the vector $\boldsymbol{\epsilon}$ represents the polarisation of the field. What is the average value $\langle\alpha|\,\boldsymbol{E}\,|\alpha\rangle$ of the electric field in a coherent state?

10. The photon number is not an exact quantum number for coherent states. However, using the results of question 5 above, show that in the limit of large mean photon numbers $\overline{n}$, the photon number is well defined. In this limit, summarise the results of questions 5 and 3 on a diagram representing the complex plane $(\mathrm{Re}(\alpha), \mathrm{Im}(\alpha))$.

4 Identical particles and dimensionality

1. **In three–dimensional space,** use a simple symmetry argument to justify that there are two families of identical particles.

2. **In two–dimensional space,** why is the above argument not applicable?
HINT: Think in terms of topology.