Computing fundamental domains for arithmetic Kleinian groups

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Definition

The **upper half-space** is the set $\mathcal{H}^3 = \mathbb{C} \times \mathbb{R}_{>0}$ with the metric

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}$$

where $(z, t) \in \mathcal{H}^3$, $z = x + iy$ and $t > 0$. The set $\hat{\mathbb{C}} = \mathbb{P}^1(\mathbb{C})$ is the **sphere at infinity**, and the **completed upper half-space** is $\hat{\mathcal{H}}^3 = \mathcal{H}^3 \cup \hat{\mathbb{C}}$. 
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Definition (Action of $\text{PSL}_2(\mathbb{C})$ on $\mathcal{H}^3$)

Let $\gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{C})$. Using $\mathcal{H}^3 \cong \mathbb{C} + \mathbb{R}_{>0}j \subset \mathbb{H}$, we define

$$\gamma \cdot z = (az + b)(cz + d)^{-1}.$$
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Classification of elements $\gamma \neq 1 \in \text{PSL}_2(\mathbb{C})$:

- $\text{tr}(\gamma) \in \mathbb{C} \setminus [-2, 2]$: $\gamma$ is loxodromic.
- $\text{tr}(\gamma) \in (-2, 2)$: $\gamma$ is elliptic.
- $\text{tr}(\gamma) = \pm 2$: $\gamma$ is parabolic.
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A Kleinian group is a discrete subgroup of $\text{PSL}_2(\mathbb{C})$.

Example

If $F \subset \mathbb{C}$ is a quadratic imaginary field and $\mathbb{Z}_F$ is the ring of integers of $F$, then the Bianchi group $\text{PSL}_2(\mathbb{Z}_F)$ is a Kleinian group.
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If $F \subset \mathbb{C}$ is a quadratic imaginary field and $\mathcal{O}_F$ is the ring of integers of $F$, then the Bianchi group $\text{PSL}_2(\mathcal{O}_F)$ is a Kleinian group.
Definition

Let $\Gamma$ be a Kleinian group. A **fundamental domain** for $\Gamma$ is an open connected subset $\mathcal{F}$ of $\mathcal{H}^3$ such that

(i) $\bigcup_{\gamma \in \Gamma} \gamma \cdot \mathcal{F} = \mathcal{H}^3$;

(ii) For all $\gamma \in \Gamma \setminus \{1\}$, $\mathcal{F} \cap \gamma \cdot \mathcal{F} = \emptyset$;

(iii) $\text{Vol}(\partial \mathcal{F}) = 0$.

A fundamental domain that is a polyhedron is a **fundamental polyhedron**, it is **finite** if it has finitely many faces. If $\Gamma$ admits a fundamental domain with compact closure, then we say $\Gamma$ is **cocompact**. If $\Gamma$ admits a fundamental domain $\mathcal{F}$ with finite volume, then we say $\Gamma$ has **finite covolume** and we let

$$\text{Covol}(\Gamma) = \text{Vol}(\mathcal{F}).$$
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Goal

Given a Kleinian group $\Gamma$, compute a fundamental domain for $\Gamma$. 
Proposition

Let $\Gamma$ be a Kleinian group. Let $p \in \mathcal{H}^3$ be a point with trivial stabilizer in $\Gamma$. Then the set

$$D_p(\Gamma) = \{ x \in \mathcal{H}^3 \mid \text{for all } \gamma \in \Gamma \setminus \{1\}, \ d(x, p) < d(\gamma \cdot x, p) \}$$

is a convex fundamental polyhedron for $\Gamma$. If it has finite covolume, then it is finite.

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The domain $D_p(\Gamma)$ is a Dirichlet domain for $\Gamma$. 
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Definition

Let $\mathcal{F}$ be a Dirichlet domain for a Kleinian group $\Gamma$, and $F$ the set of faces of $\mathcal{F}$. A **face pairing** is a map $\cdot^* \times g : F \to F \times \Gamma$ which assigns to every face $f$ a face $f^*$ and an element $g(f) \in \Gamma$ s.t.

(a) $g(f) \cdot f = f^*$;
(b) $\cdot^* : F \to F$ is an involution;

The elements $g(f)$ where $f$ is a face of $\mathcal{F}$ are the **pairing transformations**.

Proposition

Any Dirichlet domain admits a face pairing. The pairing transformations are generators for the group $\Gamma$. 
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Any Dirichlet domain admits a face pairing. The pairing transformations are generators for the group $\Gamma$. 
Definition (Cycles)

Let $\mathcal{F}$ be a finite Dirichlet domain for a Kleinian group $\Gamma$. Define by induction:

- Pick $e_1$ an edge of $\mathcal{F}$;
- $e_1 = f \cap f_1$;
- $g_1 = g(f_1)$;
- $e_{i+1} = g_i \cdot e_i$;
- $e_{i+1} = f_{i+1} \cap f_i^*$;
- $g_{i+1} = g(f_{i+1})$.

The sequence $(e_i)$ is periodic; let $m$ be its period. $C = (e_1, \ldots, e_m)$ is a cycle of edges. The cycle transformation at $e_1$ is $h = g_m g_{m-1} \ldots g_1$. 

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Proposition

Let $\mathcal{F}$ be a finite Dirichlet domain for a Kleinian group $\Gamma$.

- If a face $f$ has $f^* = f$, then we have the reflection relation $g(f)^2 = 1$;
- For all edges $e$ with cycle transformation $h$ and for all $x \in e$ we have $h \cdot x = x$; $h$ satisfies the cycle relation $h^\nu = 1$.

The reflection relations and the cycle relations form a complete set of relations for $\Gamma$. 
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Goal
Given a Kleinian group \( \Gamma \), compute a Dirichlet domain for \( \Gamma \) with a face pairing, and a presentation for \( \Gamma \).

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Compute the inverse isomorphism with the abstract finitely presented group.
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Compute the inverse isomorphism with the abstract finitely presented group.
Definition

Let $F$ be a field with $\text{char } F \neq 2$ and $a, b \in F^\times$. An $F$-algebra admitting a presentation of the form

$$\langle i, j \mid i^2 = a, j^2 = b, ij = -ji \rangle$$

is the quaternion algebra $(\frac{a,b}{F})$ over $F$.

Example

The matrix algebra is $\mathcal{M}_2(F) \cong \left( \frac{1,1}{F} \right)$ via the homomorphism

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mapsto j$. 
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Definition

Let $B = \left(\frac{a,b}{F}\right)$ be a quaternion algebra and $\beta = x + yi + zj + tij \in B$. The conjugate, reduced trace and reduced norm of $\beta$ are $\overline{\beta} = x + yi + zj + tij$, $\text{trd}(\beta) = \beta + \overline{\beta} = 2x$ and $\text{nrd}(\beta) = \beta\overline{\beta} = x^2 - ay^2 - bz^2 + abt^2$.

Example

In the matrix ring, the reduced trace is the usual trace and the reduced norm is the determinant.
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Definition

Let $B = \left( \frac{a, b}{F} \right)$ be a quaternion algebra over a number field $F$. A place $\nu$ of $F$ is **split** or **ramified** according as $B \otimes_{F} F_{\nu} = \mathcal{M}_2(F_{\nu})$ or not, where $F_{\nu}$ denotes the completion of $F$ at $\nu$. The product of all ramified primes $p \subset \mathbb{Z}_F$ is the **discriminant** $\Delta_B$ of $B$. 
**Definition**

Let $F$ be a number field and $\mathbb{Z}_F$ its ring of integers. Let $B$ be a quaternion algebra over $F$. An order in $B$ is a finitely generated $\mathbb{Z}_F$-submodule $\mathcal{O} \subset B$ with $F\mathcal{O} = B$ which is also a subring. We write $\mathcal{O}_1^\times$ the group of units in $\mathcal{O}$ with reduced norm 1.

**Example**

Let $F$ and $\mathbb{Z}_F$ be as above, $a, b \in \mathbb{Z}_F \setminus \{0\}$ and $B = \left( \frac{a, b}{F} \right)$. Then the $\mathbb{Z}_F$-module $\mathcal{O} = \mathbb{Z}_F + \mathbb{Z}_Fi + \mathbb{Z}_Fj + \mathbb{Z}_Fij$ is an order in $B$ and the $\mathbb{Z}_F$-module $\mathcal{M}_2(\mathbb{Z}_F)$ is an order in $\mathcal{M}_2(F)$. 
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Definition

Let $F$ be a number field. We say $F$ is **quasi totally real** or **QTR** if $F$ has exactly one complex place. A **Kleinian quaternion algebra** is a quaternion algebra over a QTR number field, ramified at every real place.

Example

A quadratic imaginary field is a QTR number field. For any positive cubefree integer $d \neq 1$, $\mathbb{Q}(\sqrt[3]{d})$ is a QTR number field.
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Theorem

Let $B$ be a Kleinian quaternion algebra over a QTR number field $F$. Then there is a discrete embedding

$$
\rho : \mathcal{O}_1^\times \hookrightarrow \text{SL}_2(\mathbb{C}).
$$

The Kleinian group $\Gamma = \rho(\mathcal{O}_1^\times) / \pm 1$ has finite covolume and it is cocompact if and only if $B$ is a division algebra. If $B$ is not a division algebra then the base field $F$ of $B$ is a quadratic imaginary field and $B \cong M_2(F)$. If $\mathcal{O}$ is maximal, then we have

$$
\text{Covol}(\Gamma) = \frac{|\Delta_F|^{3/2} \zeta_F(2) \prod_{p|\Delta_B} (N(p) - 1)}{(4\pi^2)^{n-1}}
$$

where $\Delta_F$ is the discriminant of $F$, $\zeta_F$ is the Dedekind zeta function of $F$ and $\Delta_B$ is the discriminant of $B$. 
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The Kleinian group $\Gamma = \rho(\mathcal{O}_1^\times)/\pm 1$ has finite covolume. and it is cocompact if and only if $B$ is a division algebra. If $B$ is not a division algebra then the base field $F$ of $B$ is a quadratic imaginary field and $B \cong M_2(F)$. If $\mathcal{O}$ is maximal, then we have

$$\text{Covol}(\Gamma) = \frac{|\Delta_F|^{3/2} \zeta_F(2) \prod_{p|\Delta_B} (N(p) - 1)}{(4\pi^2)^{n-1}}$$

where $\Delta_F$ is the discriminant of $F$, $\zeta_F$ is the Dedekind zeta function of $F$ and $\Delta_B$ is the discriminant of $B$. 
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The Kleinian group $\Gamma = \rho(O_1^\times) / \pm 1$ has finite covolume, and it is cocompact if and only if $B$ is a division algebra. If $B$ is not a division algebra then the base field $F$ of $B$ is a quadratic imaginary field and $B \cong M_2(F)$.

If $O$ is maximal, then we have

$$\text{Covol}(\Gamma) = \frac{|\Delta_F|^{3/2} \zeta_F(2) \prod_{p|\Delta_B} (N(p) - 1)}{(4\pi^2)^{n-1}}$$

where $\Delta_F$ is the discriminant of $F$, $\zeta_F$ is the Dedekind zeta function of $F$ and $\Delta_B$ is the discriminant of $B$. 
An Kleinian group is **arithmetic** if it is commensurable with some $\rho(O_1^\times)/\pm 1$ as in the previous theorem. For example, we recover the Bianchi groups by taking $F$ a quadratic imaginary field, $B = M_2(F)$ and $O = M_2(\mathbb{Z}_F)$ so that $O_1^\times = \text{SL}_2(\mathbb{Z}_F)$.

**Goal**

Given an arithmetic Kleinian group $\Gamma$, compute a Dirichlet domain for $\Gamma$ with a face pairing, and a presentation for $\Gamma$. 
An Kleinian group is **arithmetic** if it is commensurable with some $\rho(\mathcal{O}_1^\times)/\pm 1$ as in the previous theorem. For example, we recover the Bianchi groups by taking $F$ a quadratic imaginary field, $B = M_2(F)$ and $\mathcal{O} = M_2(\mathbb{Z}_F)$ so that $\mathcal{O}_1^\times = \text{SL}_2(\mathbb{Z}_F)$.

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Given an arithmetic Kleinian group $\Gamma$, compute a Dirichlet domain for $\Gamma$ with a face pairing, and a presentation for $\Gamma$. 
Applications

- Compute the structure of unit groups $\mathcal{O}^\times$ where $\mathcal{O}$ is an order in a Kleinian quaternion algebra;
- Compute the cohomology of arithmetic Kleinian groups with the action of Hecke operators;
- Study a large class of compact hyperbolic 3-manifolds.
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Applications

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Definition

The unit ball $\mathcal{B}$ is the open ball of center 0 and radius 1 in $\mathbb{R}^3 \cong \mathbb{C} + \mathbb{R}j \subset \mathbb{H}$ with the metric

$$ds^2 = \frac{4(dx^2 + dy^2 + dt^2)}{(1 - |w|^2)^2}$$

where $w = (z, t) \in \mathcal{B}$, $z = x + iy$ and $|w|^2 = x^2 + y^2 + t^2 \leq 1$. The sphere at infinity $\partial \mathcal{B}$ is the sphere of center 0 and radius 1. We let $\hat{\mathcal{B}} = \mathcal{B} \cup \partial \mathcal{B}$ be the closed ball of radius 1.
Proposition

The map

\[ \eta : \mathcal{H}^3 \longrightarrow \mathcal{B} \]

\[ z \longmapsto (z - j)(1 - jz)^{-1} \]

is a bijective isometry;

For all

\[ w, z \in \mathcal{B}, \ d(w, z) = \cosh^{-1} \left( 1 + 2 \frac{|w - z|^2}{(1 - |w|^2)(1 - |z|^2)} \right). \]
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For all \( w \in B \), \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{C}) \), let

\[ g \cdot w = \eta(g \cdot \eta^{-1}(w)). \]

**Proposition**

\[ g \cdot w = (Aw + B)(Cw + D)^{-1} \]

where

\[ A = a + d + (b - c)j, \quad B = b + \bar{c} + (a - \bar{d})j, \]
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Suppose $g \in \text{SL}_2(\mathbb{C})$ does not fix 0 in $\mathcal{B}$. Then let

- $I(g) = \{ w \in \mathcal{B} \mid d(w, 0) = d(g \cdot w, 0) \}$;
- $\text{Ext}(g) = \{ w \in \mathcal{B} \mid d(w, 0) < d(g \cdot w, 0) \}$;
- $\text{Int}(g) = \{ w \in \mathcal{B} \mid d(w, 0) > d(g \cdot w, 0) \}$;

$I(g)$ is the isometric sphere of $g$. For $S \subset \text{SL}_2(\mathbb{C})$ with no element fixing 0, the exterior domain of $S$ is

$$\text{Ext}(S) = \bigcap_{g \in S} \text{Ext}(g).$$

For $S$ a Euclidean sphere, define $\text{Ext}(S)$ (resp. $\text{Int}(S)$) to be the intersection of $\mathcal{B}$ and the exterior (resp. the interior) of the sphere.
Proposition

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{C})$ and $A, B, C, D$ as in the previous proposition. Then $g \cdot 0 = 0$ if and only if $C = 0$. If $g$ does not fix 0, then $l(g)$ is the intersection of $B$ and the Euclidean sphere of center $-C^{-1}D$ and radius $2/|C|$, and we have $\text{Int}(g) = \text{Int}(l(g))$ and $\text{Ext}(g) = \text{Ext}(l(g))$. 
Remark

If $0 \in B$ has a trivial stabilizer in the Kleinian group $\Gamma$, then we have $D_0(\Gamma) = \text{Ext}(\Gamma \setminus \{1\})$.

Lemma

Let $\gamma \in \Gamma$ and $\mathcal{F} = \text{Ext}(\Gamma \setminus \{1\})$. Then $\gamma \cdot \mathcal{I}(\gamma) = \mathcal{I}(\gamma^{-1})$, and $\mathcal{I}(\gamma)$ contributes to the boundary of $\mathcal{F}$ if and only if $\mathcal{I}(\gamma^{-1})$ does.

Remark

The face pairing of an exterior domain is given by the isometric spheres.
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Lemma
Let $\gamma \in \Gamma$ and $\mathcal{F} = \text{Ext}(\Gamma \setminus \{1\})$. Then $\gamma \cdot I(\gamma) = I(\gamma^{-1})$, and $I(\gamma)$ contributes to the boundary of $\mathcal{F}$ if and only if $I(\gamma^{-1})$ does.

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**Goal**

Given an arithmetic Kleinian group $\Gamma$, compute the exterior domain of $\Gamma$ with the face pairing, and a presentation for $\Gamma$.

**Algorithm**

Enumerate the element of $\Gamma$ in a finite set $S$ until we have

$$\text{Vol}(\text{Ext}(S)) = \text{Covol}(\Gamma).$$
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The integral

$$-\int_0^\theta \ln |2 \sin u| \, du$$

converges for $\theta \in \mathbb{R} \setminus \pi \mathbb{Z}$ and admits a continuous extension to $\mathbb{R}$, which is odd and periodic with period $\pi$.

Definition

This extension is called the Lobachevsky function $\mathcal{L}(\theta)$. 
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Proposition

The Lobachevsky function admits a power series expansion:

\[ \mathcal{L}(\theta) = \theta \left( 1 - \ln(2\theta) + \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{2n(2n+1)!} \theta^{2n} \right) \]

where the \( B_n \) are the Bernoulli numbers defined by

\[ \frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \]
Proposition

Let $T_{\alpha,\gamma}$ be the tetrahedron in $\mathcal{H}^3$ with one vertex at $\infty$ and the other vertices $A, B, C$ on the unit hemisphere such that they project vertically onto $A', B', C'$ in $\mathbb{C}$ with $A' = 0$ to form a Euclidean triangle, with angles $\frac{\pi}{2}$ at $B'$ with and $\alpha$ at $A'$, and such that the angle along $BC$ is $\gamma$. Then $T_{\alpha,\gamma}$ is unique up to isometry and

$$\text{Vol}(T_{\alpha,\gamma}) = \frac{1}{4} \left[ \mathcal{L}(\alpha + \gamma) + \mathcal{L}(\alpha - \gamma) + 2\mathcal{L}\left(\frac{\pi}{2} - \alpha\right) \right].$$

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Every polyhedron $P$ can be decomposed in a way such that the volume of $P$ is a sum of volumes of tetrahedra $T_{\alpha,\gamma}$. 
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Definition

Let $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$ and define $\text{invrad}(m) = |C|^2$.

For $\gamma \in \text{SL}_2(\mathbb{C})$ with $\gamma \cdot 0 \neq 0$, let $\text{rad}(\gamma)$ be the radius of $I(\gamma)$.

Proposition

Suppose $\rho : \mathcal{O}_1^\times \hookrightarrow \text{SL}_2(\mathbb{C})$ is a discrete embedding. The absolute reduced norm $Q : B \to \mathbb{R}$ defined by

$$Q(x) = \text{invrad}(\rho(x)) + \text{tr}_{F/Q}(\text{nrd}(x))$$

for all $x \in B$ gives $\mathcal{O}$ the structure of a lattice, and we have

$$\text{for all } x \in \mathcal{O}_1^\times, \quad Q(x) = \frac{4}{\text{rad}(\rho(x))^2} + n.$$
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Definition

Let $\Gamma$ be a Kleinian group and $S \subset \Gamma$. A point $z \in \mathcal{B}$ is $S$-reduced if for all $g \in S$, we have $d(z, 0) \leq d(g \cdot z, 0)$, i.e. if $z \in \overline{\text{Ext}(S)}$.

Algorithm

Input: $z \in \mathcal{B}$.

Let $z' = z$. If $d(g \cdot z', 0) < d(z', 0)$ for some $g \in S$, then set $z' = g \cdot z'$ and repeat.

Output: $z' \in \mathcal{B}$, $S$-reduced and $\delta \in \langle S \rangle$ s.t. $z' = \delta \cdot z$. 
**Definition**

Let $\Gamma$ be a Kleinian group and $S \subset \Gamma$. A point $z \in B$ is *S-reduced* if for all $g \in S$, we have $d(z, 0) \leq d(g \cdot z, 0)$, i.e. if $z \in \text{Ext}(S)$.

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Computing fundamental domains for Kleinian groups
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**Definition**

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**Definition**

Let $\Gamma$ be a Kleinian group, $S \subset \Gamma$ and $z \in \mathcal{B}$. An element $\gamma \in \Gamma$ is $(S, z)$-reduced if $\gamma \cdot z$ is $S$-reduced, i.e. if $\gamma \cdot z \in \text{Ext}(S)$. The reduced element computed by the previous algorithm is written $\text{Red}_S(\gamma; z)$ and simply $\text{Red}_S(\gamma) = \text{Red}_S(\gamma; 0)$.

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Algorithm

do

- Enumerate some elements of $\Gamma$ in $S$
- $(S, 0)$-reduce the elements of $S$
- For every $z \in I(\gamma)$ s.t. $\gamma \cdot z \notin \text{Ext}(S)$, add $\text{Red}_S(\gamma; z)$ to $S$

until $\text{Vol}(\text{Ext}(S)) = \text{Covol}(\Gamma)$
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Proposition

Let $\Gamma = \text{PSL}_2(\mathbb{Z}[\frac{1+\sqrt{-163}}{2}])$. Then the Bianchi group $\Gamma$ has covolume $\text{Covol}(\Gamma) \approx 57.435648$, and $\Gamma$ admits a presentation with 10 generators, 7 elliptic relations and 10 other relations.

The fundamental polyhedron that was computed has 111 faces and 306 edges, and the maximum absolute reduced norm of the elements that bound this exterior domain is 4536. In the lattice, 70 millions of vectors were enumerated, and 8500 of them had norm 1.
Proposition

Let $F = \mathbb{Q}(\sqrt[3]{11})$ with discriminant $-3267$ and class number 2, $\alpha = \sqrt[3]{11}$, $B = \left(\frac{-2, -4\alpha^2 - \alpha - 2}{F}\right)$, $\mathcal{O}$ a maximal order in $B$ and $\Gamma = \mathcal{O}_1^\times / \pm 1$. The quaternion algebra $B$ has discriminant $\mathfrak{p}_2$ where $N(\mathfrak{p}_2) = 2$. Then the group $\Gamma$ has covolume $\text{Covol}(\Gamma) \approx 206.391784$, and $\Gamma$ admits a presentation with 17 generators, 11 elliptic relations and 21 other relations.

The fundamental polyhedron that was computed has 647 faces and 1877 edges, and the maximum absolute reduced norm of the elements that bound this exterior domain is 5802. In the lattice, 80 millions of vectors were enumerated, and 300 of them had norm 1.
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