Computing Kleinian modular forms

Aurel Page

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Warwick, LMFDB workshop

Elliptic curves and automorphic forms

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Where do you find automorphic forms?
In the cohomology of arithmetic groups!

Matsushima's formula: Γ discrete cocompact subgroup of connected Lie group G, E representation of G.

$$H^i(\Gamma, E) \cong \bigoplus_{\pi \in \widehat{G}} \mathrm{Hom}(\pi, L^2(\Gamma \backslash G)) \otimes H^i(\mathfrak{g}, K; \pi \otimes E)$$

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Kleinian case: $\mathcal{O}^\times \subset \mathrm{GL}_2(\mathbb{C})$, $H^1(\mathcal{O}^\times, E)$
Call $H^i(\mathcal{O}^\times, E)$ a space of Kleinian modular forms.

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- Torsion phenomenon.
- Attached Galois representations: open case.

Arithmetic Kleinian groups

Kleinian groups

$\mathbb{H} := \mathbb{C} + \mathbb{C}j$ where $j^2 = -1$ and $jz = \bar{z}j$ for all $z \in \mathbb{C}$.

The upper half-space $\mathcal{H}^3 := \mathbb{C} + \mathbb{R}_{>0}j$.
Metric $\mathrm{d}s^2 = \frac{|\mathrm{d}z|^2 + \mathrm{d}t^2}{t^2}$, volume $\mathrm{d}V = \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t}{t^3}$.

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$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot w := (aw+b)(cw+d)^{-1} \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

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Kleinian group: discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$.
Cofinite if it has finite covolume.

Quaternion algebras

A quaternion algebra B over a field F is a central simple algebra of dimension 4 over F.

Explicitly, $B = \left(\frac{a,b}{F}\right) = F + Fi + Fj + Fij$, $i^2 = a$, $j^2 = b$, $ij = -ji$ (char $F \neq 2$).

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A place v of F is split or ramified according as whether $B \otimes_F F_v$ is $\mathcal{M}_2(F_v)$ or a division algebra.

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Theorem

Γ is a cofinite Kleinian group.
It is cocompact iff B is a division algebra.
If $\mathcal{O}$ is maximal then

$$\mathrm{Covol}(\Gamma) = \frac{|\Delta_F|^{3/2}\zeta_F(2)\prod_{\mathfrak{p}\ \mathit{ram.}}(N(\mathfrak{p})-1)}{(4\pi^2)^{[F:\mathbb{Q}]-1}}.$$

Algorithms

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5. Compute Hecke operator T_δ on $H^1(\mathcal{O}^1)$

Fundamental domains

Γ a Kleinian group. An open subset $\mathcal{F} \subset \mathcal{H}^3$ is a fundamental domain if

- $\Gamma \cdot \overline{\mathcal{F}} = \mathcal{H}^3$
- $\mathcal{F} \cap \gamma\mathcal{F} = \emptyset$ for all $1 \neq \gamma \in \Gamma$.

Dirichlet domains

Dirichlet domains

Let $p \in \mathcal{H}^3$ with trivial stabilizer in Γ. The Dirichlet domain

$$\begin{aligned} D_p(\Gamma) &:= \{x \in X \mid \mathrm{d}(x,p) < \mathrm{d}(\gamma \cdot x, p)\ \forall \gamma \in \Gamma \setminus \{1\}\} \\ &= \{x \in X \mid \mathrm{d}(x,p) < \mathrm{d}(x, \gamma^{-1} \cdot p)\ \forall \gamma \in \Gamma \setminus \{1\}\}. \end{aligned}$$

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is a fundamental domain for Γ that is a hyperbolic polyhedron. If Γ is cofinite, $D_p(\Gamma)$ has finitely many faces.

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- Edges of the domain are grouped into cycles, product of corresponding elements in Γ has finite order.

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Theorem (Poincaré)

- *The elements corresponding to the faces are generators of Γ. The relations corresponding to the edge cycles generate all the relations among the generators.*

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- *The elements corresponding to the faces are generators of Γ. The relations corresponding to the edge cycles generate all the relations among the generators.*
- *If a partial Dirichlet domain $D_p(S)$ has a face-pairing and cycles of edges, then it is a fundamental domain for the group generated by S.*

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$x \in \mathcal{H}^3$, $S \subset \Gamma$ corresponding to the faces of $D_p(\Gamma)$.

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In particular for $\gamma \in \Gamma$, take $x = \gamma^{-1} p$, which will reduce to $x' = p$, to write γ as a product of the generators.

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- an algorithm for computing the volume of a polyhedron.

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- Repeat until D has a face-pairing and $\mathrm{Vol}(D) < 2\,\mathrm{Covol}(\Gamma)$

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- Proved complexity: $\mathrm{Covol}(\Gamma)^{O(1)}$
- Observed complexity: $\mathrm{Covol}(\Gamma)^2$
- Lower bound: $\mathrm{Covol}(\Gamma)$

Group cohomology

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Cohomology :

$$H^1(\Gamma, E) := Z^1(\Gamma, E)/B^1(\Gamma, E)$$

Hecke operators

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Buchmann's algorithm over number fields: subexponential under GRH.
Adapt Buchmann's algorithm over a quaternion algebra: heuristically subexponential.

Examples

A quartic example

Let F the unique quartic field of signature $(2, 1)$ and discriminant -275. Let B be the unique quaternion algebra with discriminant $11\mathbb{Z}_F$, ramified at every real place of F. Let $\mathcal{O}$ be a maximal order in B (it is unique up to conjugation). Then $\mathcal{O}^1$ is a Kleinian group with covolume $93.72\ldots$ The fundamental domain I have computed has 310 faces and 924 edges.

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Have a look at the fundamental domain!

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$N(\mathfrak{p})$	$T_\mathfrak{p}$	characteristic polynomial
9	$\begin{pmatrix} 2 & -4 & -5 \\ -2 & 3 & -2 \\ 0 & 0 & -5 \end{pmatrix}$	$(x+5)(x^2-5x-2)$
9	same	$(x+5)(x^2-5x-2)$
16	$\begin{pmatrix} 1 & 4 & -11 \\ 2 & 0 & -6 \\ 0 & 0 & -8 \end{pmatrix}$	$(x+8)(x^2-x-8)$
19	$\begin{pmatrix} -4 & 0 & 4 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{pmatrix}$	$x(x+4)^2$
19	same	$x(x+4)^2$
25	—	$(x+9)(x^2-x-74)$
29	—	$x(x^2+6x-24)$
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except $(d, k) = (-199, 2)$: dimension 4. Two twin Galois orbits with coefficients in $\mathbb{Q}(\sqrt{13})$, swapped by $\mathrm{Gal}(F/\mathbb{Q})$.

Everywhere good reduction abelian surfaces

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Şengün, Dembélé : the Jacobian J of the hyperelliptic curve

$$\begin{aligned} y^2 &= 33x^6 + 110\sqrt{-223}x^5 + 36187x^4 - 28402\sqrt{-223}x^3 \\ &\quad -2788739x^2 + 652936\sqrt{-223}x + 14157596 \end{aligned}$$

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Looking for more examples in $\mathbb{Q}(\sqrt{-455})$ and $\mathbb{Q}(\sqrt{-571})$.

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Joint with H. Şengün: experimental verification with Hecke operators (still in progress).

Thank you for your attention !