Torsion homology of arithmetic Kleinian groups

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Five College Number Theory Seminar

Plan

- Arithmetic Kleinian groups
- Torsion Jacquet–Langlands conjecture
- Isospectrality and torsion homology

Arithmetic Kleinian Groups

Arithmetic groups

Arithmetic group $\approx \mathbb{G}(\mathbb{Z})$ for $\mathbb{G}$ linear algebraic group over $\mathbb{Q}$.

Examples: $\mathrm{SL}_n(\mathbb{Z}_F), O(q_{\mathbb{Z}})$.

Motivation:

- Classical reduction theories: Gauss, Minkowski, Siegel.
- Interesting class of lattices in Lie groups.
- Automorphisms of natural objects: quadratic forms, abelian varieties.
- Modular forms / Automorphic forms.
- Parametrize structures: Shimura varieties, Bhargava's constructions.

Arithmetic Kleinian groups

Arithmetic Kleinian group $=$ arithmetic subgroup of $\mathrm{PSL}_2(\mathbb{C})$.

Why this case?

- small dimension: easier geometry but still rich arithmetic.
- 3-dimensional hyperbolic manifolds.
- related to units in quaternion algebras.

Arithmetic Kleinian groups

F number field with $r_2 = 1$. Example: $F = \mathbb{Q}(\sqrt{-d})$.

B **quaternion algebra** over F:
$B = F + Fi + Fj + Fij$ with $i^2 = a$, $j^2 = b$, $ij = -ij$.
Ramified at the real places: $a, b \ll 0$
Example: $B = \mathcal{M}_2(F)$ ($a = b = 1$).

Reduced norm:
$\mathrm{nrd} : B \to F$ multiplicative
$\mathrm{nrd}(x + yi + zj + tij) = x^2 - ay^2 - bz^2 + abt^2$.
Example: $\mathrm{nrd} = \det$

$\mathcal{O}$ **order** in B: subring, f.g. $\mathbb{Z}$-module, $\mathcal{O}F = B$.
Example: $\mathcal{O} = \mathcal{M}_2(\mathbb{Z}_F)$.

$\Gamma = \mathcal{O}^1/\{\pm 1\} \subset \mathrm{PSL}_2(\mathbb{C})$

Dirichlet domains

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$$D_p(\Gamma) = \{x \in \mathcal{H}^3 \mid d(x,p) \leq d(\gamma x, p) \text{ for all } \gamma \in \Gamma\}$$

is a fundamental domain, finite volume, finite-sided, provides a presentation of Γ.

Example:

$$D_{2i}(\mathrm{PSL}_2(\mathbb{Z})) = \text{ usual fundamental domain.}$$

Algorithms

Basic algorithm:

- Enumerate elements of Γ and compute partial Dirichlet domain.
- Stop when the domain cannot get smaller.

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Efficient algorithm:

- Efficient enumeration of Γ.
- Enough to find any generators.
- Stopping criterion using volume formula and combinatorial structure of Dirichlet domain.

Torsion Jacquet–Langlands

joint work with Haluk Şengün

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- Cohomology and Galois representations
- The torsion Jacquet–Langlands conjecture
- Examples

Cohomology and automorphic forms

Matsushima's formula: Γ discrete cocompact subgroup of connected Lie group G, E representation of G.

$$H^i(\Gamma, E) \cong \bigoplus_{\pi \in \widehat{G}} \mathrm{Hom}(\pi, L^2(\Gamma \backslash G)) \otimes H^i(\mathfrak{g}, K; \pi \otimes E)$$

The cohomology has an action of Hecke operators, corresponding to the one on the automorphic forms.

$\rightsquigarrow$ Hecke eigenclasses should have attached Galois representations.

Torsion and Galois representations

Theorem (Scholze, conjecture of Ash)

Let Γ be a congruence subgroup of $\mathrm{GL}_n(\mathbb{Z}_F)$ with F a CM field. Then for any system of Hecke eigenvalues in $H^i(\Gamma, \mathbb{F}_p)$, there exists a continuous semisimple representation $\mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ such that Frobenius and Hecke eigenvalues match.

Classical Jacquet–Langlands

$F = \mathbb{Q}(\sqrt{-d})$.
B quaternion algebra over F with discriminant $\mathfrak{D}$ (ideal: set of bad primes). $\mathfrak{N}$ ideal coprime to $\mathfrak{D}$.

Get two arithmetic Kleinian groups:

- $\Gamma_0(\mathfrak{N}\mathfrak{D}) \subset \mathrm{PSL}_2(\mathbb{Z}_F)$
- $\Gamma_0^{\mathfrak{D}}(\mathfrak{N}) \subset B^1/\{\pm 1\}$

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Theorem (Jacquet–Langlands)

There exists a Hecke-equivariant isomorphism

$$H_1(\Gamma_0^{\mathfrak{D}}(\mathfrak{N}), \mathbb{C}) \to H_{1,cusp}(\Gamma_0(\mathfrak{N}\mathfrak{D}), \mathbb{C})^{\mathfrak{D}-new}$$

Torsion Jacquet–Langlands

$\mathfrak{m}$ maximal ideal of the Hecke algebra $=$ system of Hecke eigenvalues modulo some prime p.

Conjecture (Calegari–Venkatesh)

If $\mathfrak{m}$ is not Eisenstein, then

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Theorem (Calegari–Venkatesh): numerical version (without Hecke operators) in some cases.

Torsion Jacquet–Langlands, subtleties

- Eisenstein: eigenvalue of $T_{\mathfrak{p}}$ is $\chi_1(\mathfrak{p}) + \chi_2(\mathfrak{p})N(\mathfrak{p})$ for characters χ_1, χ_2 of ray class groups.
- Congruence classes, such as $\Gamma_0(\mathfrak{N})/\Gamma_1(\mathfrak{N}) \to (\mathbb{Z}_F/\mathfrak{N})^\times$
- "new" is the **quotient** by the oldforms $\rightsquigarrow$ level-raising.
- Cannot expect an isomorphism of Hecke-modules, multiplicity one can fail.

Example

(on the blackboard)

Isospectral manifolds and torsion homology

joint work with Alex Bartel

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- Isospectral manifolds
- Tools to study their torsion homology
- Computations and examples

Can you hear the shape of a drum?

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Mathematical question (Kac 1966):
M, M' same spectrum for Laplace operator (**isospectral**)
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Answer:

Milnor 1964: No! (dimension 16)

Sunada 1985: No! (dimension d)

Gordon, Webb, Wolpert 1992: No! (domains of the plane)

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Sunada: No! (dimension 4)

Tighter question: small dimension, special classes of manifolds

Dimension 2 orientable $\Rightarrow$ torsion-free homology

Dimension 3 orientable $\Rightarrow$ torsion-free H_0, H_2 and H_3

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Theorem (P., Bartel)

For all primes $p \leq 37$, there exist pairs of compact hyperbolic 3-manifolds M, M' that are strongly isospectral and cover a common manifold, but such that

$$|H_1(M,\mathbb{Z})[p^\infty]| \neq |H_1(M',\mathbb{Z})[p^\infty]|$$

Arithmetically equivalent number fields

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Same class number?
Dyer 1999: No!

Existing examples where $v_p(h_{K_1}) \neq v_p(h_{K_2})$: $p = 2, 3, 5$.

Special value formulas

Analytic class number formula:

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{w_K|D_K|^{1/2}}$$

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Spectrum of Δ on i-forms: $\zeta_{M,i}(s) = \sum \lambda^{-s}$.
Cheeger–Müller theorem (conjectured by Ray–Singer):

$$\prod_i \left(R_i(M)\cdot|H_i(M,\mathbb{Z})_{tors}|\right)^{(-1)^i} = \prod_i \exp(\tfrac{1}{2}\zeta'_{M,i}(0))^{(-1)^i}$$

$R_i(M)$ regulator of $H_i(M,\mathbb{Z})/H_i(M,\mathbb{Z})_{tors}$.

Examples of regulators

$$R_0(M) = \mathrm{Vol}(M)^{-1/2}$$

$$R_d(M) = \mathrm{Vol}(M)^{1/2}$$

Construction of isospectral objects

Gassmann triple (1925):
G finite group and H, H' subgroups such that

$$\mathbb{C}[G/H] \cong \mathbb{C}[G/H'].$$

Equivalently, for every conjugacy class C, $|C \cap H| = |C \cap H'|$.

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$$\Rightarrow \zeta_{K^H}(s) = L(\mathbb{C}[G/H], s).$$

Sunada: if $X \to Y$ is a Galois covering with Galois group G
$\Rightarrow X/H$ and X/H' are strongly isospectral.

Example of a Gassmann triple

$G = \mathrm{SL}_3(\mathbb{F}_2)$ acting on $\mathbb{P}^2(\mathbb{F}_2)$.

$H =$ stabilizer of a point

$H' =$ stabilizer of a line

Representation theory

$$\mathbb{C}[G/H] \cong \mathbb{C}[G/H']$$

$$\iff \mathbb{Q}[G/H] \cong \mathbb{Q}[G/H']$$

$$\iff \mathbb{Q}_p[G/H] \cong \mathbb{Q}_p[G/H']$$

$$\Longleftarrow \mathbb{Z}_p[G/H] \cong \mathbb{Z}_p[G/H']$$

and $\iff$ if $p \nmid |G|$.

Cohomological Mackey functors

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Map: $\mathcal{F} : \{\text{subgroups of } G\} \longrightarrow R$-modules, and R-linear maps

- $c_H^g : \mathcal{F}(H) \to \mathcal{F}(H^g)$ conjugation
- $r_K^H : \mathcal{F}(H) \to \mathcal{F}(K)$ restriction
- $t_K^H : \mathcal{F}(K) \to \mathcal{F}(H)$ transfer

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Proposition (P., Bartel)

$H \mapsto H_i(X/H, \mathbb{Z})$ is a cohomological Mackey functor. In particular, if $\mathbb{Z}_p[G/H] \cong \mathbb{Z}_p[G/H']$ then

$$H_i(X/H, \mathbb{Z}) \otimes \mathbb{Z}_p \cong H_i(X/H', \mathbb{Z}) \otimes \mathbb{Z}_p.$$

Smallest Gassmann triple

Theorem (de Smit)

Let p be an odd prime. If G, H, H' is a Gassmann triple such that

$$\mathbb{Z}_p[G/H] \not\cong \mathbb{Z}_p[G/H']$$

and $[G : H] \leq 2p + 2$, then there is an isomorphism

$$G \cong \mathrm{GL}_2(\mathbb{F}_p)/(\mathbb{F}_p^\times)^2$$

sending H, H' to

$$\begin{pmatrix} \square & * \\ 0 & * \end{pmatrix} \text{ and } \begin{pmatrix} * & * \\ 0 & \square \end{pmatrix}.$$

Regulator constants

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G, H, H' Gassmann triple, ρ representation of G over $R = \mathbb{Z}$ or $\mathbb{Q}$. $\langle\cdot,\cdot\rangle$ G-invariant nondegenerate pairing on $\rho \otimes \mathbb{C}$.

$$\mathcal{C}(\rho) = \frac{\det(\langle\cdot,\cdot\rangle|\rho^H/(\rho^H)_{tors})}{\det(\langle\cdot,\cdot\rangle|\rho^{H'}/(\rho^{H'})_{tors})} \in \mathbb{C}/(R^\times)^2.$$

Theorem (Dokchitser, Dokchitser)

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Example of units

$K/\mathbb{Q}$ Galois with group G. Let G, H_1, H_2 Gassmann triple. Let $\rho = \mathbb{Z}_K^\times$ as a G-module. $K_i = K^{H_i}$. Then

$$\mathcal{C}(\rho) = \frac{R_{K_1}}{R_{K_2}} = \frac{h_{K_2}}{h_{K_1}}.$$

Example of regulator constants

$G = \mathrm{GL}_2(\mathbb{F}_p)/\square$, $H_+ = \begin{pmatrix} \square & * \\ 0 & * \end{pmatrix}$, $H_- = \begin{pmatrix} * & * \\ 0 & \square \end{pmatrix}$.

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset \mathrm{GL}_2(\mathbb{F}_p)$, $r : \begin{pmatrix} a & * \\ 0 & * \end{pmatrix} \mapsto \left(\frac{a}{p}\right)$.

$I = \mathrm{Ind}_B^G r$ irreducible, of dimension $p + 1$.

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Proposition (P., Bartel)

For all irreducible representation ρ of G over $\mathbb{Q}$, we have $\mathcal{C}(\rho) = 1$, except $\mathcal{C}(I) = p$.

Comparison of regulators

Theorem (P., Bartel)

$X \to Y$ Galois covering of hyperbolic 3-manifolds with Galois group G. Gassmann triple G, H, H' and p prime number. Assume $|H^{ab}|$ and $|H'^{ab}|$ coprime to p.
$M :=$ G-module $H_2(X, \mathbb{Z})$. Then

$$\frac{R(X/H')}{R(X/H)} = \mathcal{C}(M) \cdot u.$$

for some $u \in \mathbb{Z}_{(p)}^{\times}$.

Computations

Good supply of 3-manifold: arithmetic Kleinian groups!

$h : \Gamma \to G$ is surjective, $Y = \mathcal{H}^3/\Gamma$ and $X = \mathcal{H}^3/\ker h$,
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$\Rightarrow X \to Y$ is a Galois covering with Galois group G.

$H_1(X/H, R) \cong H_1(h^{-1}(H), R) \cong H_1(\Gamma, R[G/H])$.

Example

$F = \mathbb{Q}(t)$ with $t^4 - t^3 + 2t^2 - 1$.

$B = \left(\frac{-1,-1}{F}\right)$.

$\mathcal{O}$ an Eichler order of level norm 71.

Γ has volume $27.75939054\ldots$, and a presentation with 5 generators and 7 relations.

We found a surjective $\Gamma \to \mathrm{GL}_2(\mathbb{F}_7)$, yielding two isospectral manifolds with homology

$$\mathbb{Z}^3 + \mathbb{Z}/4 + \mathbb{Z}/4 + \mathbb{Z}/12 + \mathbb{Z}/12 + \mathbb{Z}/(2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 23), \text{ and}$$

$$\mathbb{Z}^3 + \mathbb{Z}/4 + \mathbb{Z}/4 + \mathbb{Z}/12 + \mathbb{Z}/(12 \cdot 7) + \mathbb{Z}/(2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 23).$$

Questions?

Thank you!