Computing good covers of compact arithmetic manifolds

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joint work with Michael Lipnowski

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Automorphic forms: theory and computation
King’s College London
$G = \text{semisimple algebraic group over } \mathbb{Q}$
$\Gamma = \text{arithmetic group } G(\mathbb{Z})$
$G = G(\mathbb{R}), \ K \subset G \text{ maximal compact, } X = G/K$

Hecke operators on $H^i(\Gamma, M)$ for certain $M$ realise spaces of automorphic forms or torsion analogues.

Cohomological automorphic forms should be the most accessible ones.

Problem

Compute $H^i(\Gamma, M)$ together with the action of Hecke operators.
Existing algorithms

Direct homology computations:

- \( \mathbb{G}(\mathbb{R}) \) compact: lattice methods (Dembélé, Donnelly, Loeffler, Voight, Greenberg);
- \( \mathbb{G} = \text{SL}_2 / \mathbb{F} \) where \( \mathbb{F} = \mathbb{Q} \) or imaginary quadratic: modular symbols (Stein, Cremona+), group cohomology (Şengün);
- \( \mathbb{G} = \text{SL}_n / \mathbb{F} \) and \( H^i(\Gamma) \) where \( i = \text{vcd} \) or \( i = \text{vcd} - 1 \): Sharblies + Voronoï algorithm (Ash+, Gunnells+, Yasaki);
- \( \mathbb{G} = B^1 \) where \( B = \) quaternion algebra ramified at every infinite place except one: Dirichlet domains and group cohomology (Voight, P.).

Other methods:

- Jacobi and Siegel modular forms (Ryan, Sirolli, Skoruppa, Tornaría, Stroemberg, Poor, Yuen, ...);
- Trace formulas (Zagier, Cohen, Booker, Lee, Chenevier, Renard, Mégabarné, ...).
I will describe an algorithm that can compute the cohomology of an *arbitrary cocompact* arithmetic group with the action of Hecke operators.

- solves the problem in principle, explicit complexity bound
- speed in practice not clear.
$\mathbf{G}(\mathbb{R})$ compact: $\dim X = 0$, work with a finite set of points.

Each point corresponds to a (totally definite) lattice $\sim$; elementary operation is **testing isometry of lattices** (Plesken–Souvignier algorithm).

Hecke operators send a lattice $L$ to its **Kneser neighbours**: sublattices $L' \subset L$ such that $L/L'$ is a prescribed finite abelian group.
If $G = G(\mathbb{R})$ is not compact, need to take into account the symmetric space $X \rightarrow \text{study } \Gamma \backslash X$.

Immitate the lattice methods: approximate $\Gamma \backslash X$ by a finite set.

Assume that $M = \Gamma \backslash X$ is compact, and choose $r > 0$. An $r$-cover of $M$ is a finite set of points $x_1, \ldots, x_n$ such that $M = \sum_{i=1}^{n} B_r(x_i)$.

Goal: compute an $r$-cover of $M$ and extract information about $M$ from the cover.
Fact: the orbit of a point under the action of large degree Hecke operators equidistributes in $M$:

$$Tf(x) = \frac{1}{\deg T} \sum_{g \in C_T} f(gx).$$

We have $\|T\| \leq \frac{c}{(\deg T)^\alpha}$ on $L^2_0(M)$ for explicit $c, \alpha$ depending only on $G$. 

Equidistribution of Hecke orbits
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Let $B$ be a ball of radius $r$ in $M$, $f = 1_B$.

We have $Tf(x) > 0$ for all $x \in M$ if and only if the orbit of $B$ under the action of $T$ cover $M$. 

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Write

\[ f = f^0 + \frac{\text{Vol}(B)}{\text{Vol}(M)} 1. \]

Let \( x \in M \).

\[ Tf(x) \geq \left\| \frac{\text{Vol}(B)}{\text{Vol}(M)} 1 \right\| - \| Tf^0 \| \geq \frac{\text{Vol}(B)}{\text{Vol}(M)} - \frac{c}{(\text{deg } T)^{\alpha}}. \]

If \( \text{deg } T > \frac{1}{c} \left( \frac{\text{Vol}(M)}{\text{Vol}(B)} \right)^{1/\alpha} \), the orbit under \( T \) of the center of \( B \) is an \( r \)-cover of \( M \).
Good covers and nerve

How do we obtain information about $\Gamma$ from an $r$-cover?

An $r$-cover $(x_i)_i$ is **good** if every finite intersection $\bigcap_{i=1}^k B_r(x_i)$ is contractible.

The **nerve** $\mathcal{N}$ of the cover is the simplicial complex with one vertex for each $x_i$, and such that $\{x_1, \ldots, x_k\}$ is a simplex in $\mathcal{N}$ if and only if $\bigcap_{i=1}^k B_r(x_i) \neq \emptyset$.
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Proposition

*The nerve of a good cover of $M$ is homotopy equivalent to $M$.*

If the restriction of the projection $\pi : X \to M$ to every ball of radius $r$ is injective, then every $r$-cover is good.
Computational problem

Problem

Given a finite set of points in $X$ whose projections in $M$ form an $r$-cover, compute the corresponding nerve $\mathcal{N}$.

Example: 1-skeleton of $\mathcal{N}$.

Given $x, y \in X$, do we have $d(x, y) \leq 2r$ in $M$?

Equivalently, does there exist $\gamma \in \Gamma$ such that $d(\gamma x, y) \leq 2r$?
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$X = G/K$ is the set of majorants of $Q$, i.e. positive definite quadratic forms $Q_+: V_\mathbb{R} \to \mathbb{R}$ such that

$$|Q| \leq Q_+$$

and minimal for this property.

Example: if $Q(x) = \sum_{i=1}^{n} a_i x_i^2$, then $Q_+(x) = \sum_{i=1}^{n} |a_i| x_i^2$ is a majorant.
Let $Q_+, Q'_+ \in X$ be two majorants of $Q$.

Observation: for all $x \in V$,

$$|\log Q_+(x) - \log Q'_+(x)| \leq d(Q_+, Q'_+).$$
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If $\gamma \in \text{SO}(L)$ is such that $d(\gamma Q_+, Q'_+) \leq 2r$, then for all $x \in L$ we have

$$Q'_+(x)e^{-2r} \leq Q_+(\gamma^{-1} x) \leq Q'_+(x)e^{2r}.$$  

i.e. $\gamma$ is an $e^{2r}$-quasi-isometry.

$\leadsto$ finitely many possibilities for the image of a basis, do a Plesken–Souvignier enumeration.
Assume the injectivity radius of $M$ is $> 2r$. Then, given $k$ balls that pairwise intersect in $M$, the problem of whether they have empty intersection can be lifted to $X$. Let $x_1, \ldots, x_k \in X$. Define

$$f(x) = \max_i d(x, x_i)^2.$$ 

The balls intersect if and only if $\exists x^* \in X$ such that $f(x^*) \leq r^2$. 

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$X$ has negative curvature $\Rightarrow$ $f$ is strictly convex on $X$.

- there exists a unique minimum $x^*$ of $f$;
- we can approximate $x^*$ by convex optimisation methods.
There currently exists two ways of computing Hecke operators:

- explicit local-global principle (indefinite methods);
- Kneser neighbours (lattice methods).

We can use Kneser neighbours: instead of fixing the lattice \( L \) and varying the majorant \( Q^+ \in X \), consider pairs \((L, Q^+)\).
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Recurrent problem when computing Hecke action on homology: The Hecke operators do not stabilise the finite complex computing the homology. We need a second, infinite-dimensional complex having the same homology, on which Hecke operators act, and computable maps between them.
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Our case: there is a subcomplex of “Riemannian simplices” of the singular simplices:

\[ C_{\text{Riem}} \subset C_{\text{sing}} \]

having the same homology, and computable maps

\[ \mathcal{N} \leftrightarrow C_{\text{Riem}} \]

inducing isomorphisms on homology.
We have a partial, toy implementation in magma. We were able to recover known examples (but slower) in small dimension.

Many possible improvements:
- quasi-isometry testing should be in C and optimised;
- easy to parallelise;
- other metrics might be better than the Riemannian one;
- tune the radius $r$ better;
- simplicial complex simplification.
Thank you!