Computing good covers of compact arithmetic manifolds

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Automorphic forms: theory and computation
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Arithmetic groups and cohomology

$\mathbf{G}$ = semisimple algebraic group over $\mathbb{Q}$
Γ = arithmetic group $\mathbf{G}(\mathbb{Z})$
$G = \mathbf{G}(\mathbb{R})$, $K \subset G$ maximal compact, $X = G/K$

Hecke operators on $H^i(\Gamma, M)$ for certain M realise spaces of automorphic forms or torsion analogues.

Cohomological automorphic forms should be the most accessible ones.

Problem

Compute $H^i(\Gamma, M)$ together with the action of Hecke operators.

Existing algorithms

Direct homology computations:

- $\mathbf{G}(\mathbb{R})$ compact: lattice methods (Dembélé, Donnelly, Loeffler, Voight, Greenberg);
- $\mathbf{G} = \mathrm{SL}_2 / F$ where $F = \mathbb{Q}$ or imaginary quadratic: modular symbols (Stein, Cremona+), group cohomology (Şengün);
- $\mathbf{G} = \mathrm{SL}_n / F$ and $H^i(\Gamma)$ where $i = vcd$ or $i = vcd - 1$: Sharblies + Voronoï algorithm (Ash+, Gunnells+, Yasaki);
- $\mathbf{G} = B^1$ where $B =$ quaternion algebra ramified at every infinite place except one: Dirichlet domains and group cohomology (Voight, P.).

Other methods:

- Jacobi and Siegel modular forms (Ryan, Sirolli, Skoruppa, Tornaría, Stroemberg, Poor, Yuen, ...);
- Trace formulas (Zagier, Cohen, Booker, Lee, Chenevier, Renard, Mégabarné, ...).

Today

I will describe an algorithm that can compute the cohomology of an **arbitrary cocompact** arithmetic group with the action of Hecke operators.

- solves the problem in principle, explicit complexity bound
- speed in practice not clear.

Lattice methods

$\mathbf{G}(\mathbb{R})$ compact: $\dim X = 0$, work with a finite set of points.

Each point corresponds to a (totally definite) lattice $\rightsquigarrow$ elementary operation is **testing isometry of lattices** (Plesken–Souvignier algorithm).

Hecke operators send a lattice L to its **Kneser neighbours**: sublattices $L' \subset L$ such that L/L' is a prescribed finite abelian group.

Covers of arithmetic manifolds

If $G = \mathbf{G}(\mathbb{R})$ is not compact, need to take into account the symmetric space $X \rightsquigarrow$ study $\Gamma\backslash X$.

Immitate the lattice methods: approximate $\Gamma\backslash X$ by a finite set.

Assume that $M = \Gamma\backslash X$ is **compact**, and choose $r > 0$.

An r**-cover** of M is a finite set of points $x_1, \ldots, x_n$ such that $M = \sum_{i=1}^{n} B_r(x_i)$.

Goal: compute an r-cover of M and extract information about M from the cover.

Equidistribution of Hecke orbits

Fact: the orbit of a point under the action of large degree Hecke operators **equidistributes** in M:

$$Tf(x) = \frac{1}{\deg T} \sum_{g \in C_T} f(gx).$$

We have $\|T\| \leq \frac{c}{(\deg T)^{\alpha}}$ on $L_0^2(M)$ for explicit c, α depending only on G.

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Let B be a ball of radius r in M, $f = \mathbf{1}_B$.

We have $Tf(x) > 0$ for all $x \in M$ if and only if the orbit of B under the action of T cover M.

Equidistribution of Hecke orbits (II)

Write

$$f = f^0 + \frac{\mathrm{Vol}(B)}{\mathrm{Vol}(M)}\mathbf{1}.$$

Let $x \in M$.

$$Tf(x) \geq \left\| \frac{\mathrm{Vol}(B)}{\mathrm{Vol}(M)}\mathbf{1} \right\| - \| Tf^0 \| \geq \frac{\mathrm{Vol}(B)}{\mathrm{Vol}(M)} - \frac{c}{(\deg T)^\alpha}.$$

If $\deg T > \frac{1}{c}\left(\frac{\mathrm{Vol}(M)}{\mathrm{Vol}(B)}\right)^{1/\alpha}$, the orbit under T of the center of B is an r-cover of M.

Good covers and nerve

How do we obtain information about Γ from an r-cover?

An r-cover $(x_i)_i$ is **good** if every finite intersection $\cap_{i=1}^{k} B_r(x_i)$ is **contractible**.

The **nerve** $\mathcal{N}$ of the cover is the simplicial complex with one vertex for each x_i, and such that $\{x_1, \ldots, x_k\}$ is a simplex in $\mathcal{N}$ if and only if $\cap_{i=1}^{k} B_r(x_i) \neq \emptyset$.

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Proposition

The nerve of a good cover of M is homotopy equivalent to M.

If the restriction of the projection $\pi : X \to M$ to every ball of radius r is injective, then every r-cover is good.

Computational problem

Problem

Given a finite set of points in X whose projections in M form an r-cover, compute the corresponding nerve $\mathcal{N}$.

Example: 1-skeleton of $\mathcal{N}$.

Given $x, y \in X$, do we have $d(x, y) \leq 2r$ in M?

Equivalently, does there exist $\gamma \in \Gamma$ such that $d(\gamma x, y) \leq 2r$?

Majorants

Assume $\mathbf{G} = \mathrm{SO}(Q)$ where (V, Q) is a quadratic space over $\mathbb{Q}$, and $\Gamma = \mathrm{SO}(L)$ where $L \subset V$ is a lattice.

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$X = G/K$ is the set of **majorants** of Q, i.e. positive definite quadratic forms $Q_+ : V_{\mathbb{R}} \to \mathbb{R}$ such that

$$|Q| \leq Q_+$$

and minimal for this property.

Example: if $Q(x) = \sum_{i=1}^{n} a_i x_i^2$, then $Q_+(x) = \sum_{i=1}^{n} |a_i| x_i^2$ is a majorant.

Indefinite Plesken–Souvignier algorithm

Let $Q_+, Q'_+ \in X$ be two majorants of Q.

Observation: for all $x \in V$,

$$\left|\log Q_+(x) - \log Q'_+(x)\right| \leq d(Q_+, Q'_+).$$

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If $\gamma \in \mathrm{SO}(L)$ is such that $d(\gamma Q_+, Q'_+) \leq 2r$, then for all $x \in L$ we have

$$Q'_+(x)e^{-2r} \leq Q_+(\gamma^{-1}x) \leq Q'_+(x)e^{2r}.$$

i.e. γ is an e^{2r}-quasi-isometry.

$\rightsquigarrow$ finitely many possibilities for the image of a basis, do a Plesken–Souvignier enumeration.

Higher degree skeletons of the nerve

Assume the injectivity radius of M is $> 2r$.

Then, given k balls that pairwise intersect in M, the problem of whether they have empty intersection can be lifted to X.

Let $x_1, \ldots, x_k \in X$. Define

$$f(x) = \max_i d(x, x_i)^2.$$

The balls intersect if and only if $\exists x^* \in X$ such that $f(x^*) \leq r^2$.

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X has negative curvature $\Rightarrow$ f is strictly convex on X.

- there exists a unique minimum x^* of f;
- we can approximate x^* by convex optimisation methods.

Hecke operators: Kneser neighbours

There currently exists two ways of computing Hecke operators:

- explicit local-global principle (indefinite methods);
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We can use Kneser neighbours: instead of fixing the lattice L and varying the majorant $Q_+ \in X$, consider pairs (L, Q_+)!

Hecke operators: simplicial complexes

Recurrent problem when computing Hecke action on homology:

The Hecke operators **do not stabilise** the finite complex computing the homology. We need a second, **infinite-dimensional** complex having the same homology, on which Hecke operators act, and computable maps between them.

Hecke operators: simplicial complexes

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Our case: there is a subcomplex of ”Riemannian simplices” of the singular simplices:

$$C_\bullet^{Riem} \subset C_\bullet^{sing}$$

having the same homology, and computable maps

$$\mathcal{N} \rightleftarrows C_\bullet^{Riem}$$

inducing isomorphisms on homology.

Implementation???

We have a partial, toy implementation in magma. We were able to recover known examples (but slower) in small dimension.

Many possible improvements:

- quasi-isometry testing should be in C and optimised;
- easy to parallelise;
- other metrics might be better than the Riemannian one;
- tune the radius r better;
- simplicial complex simplification.

Questions?

Thank you!