Isospectrality, regulators and special value formulas

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joint work with Alex Bartel
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ICERM peer-to-peer seminar
Can you hear the shape of a drum?

Mathematical question (Kac 1966):

\[ M, M' \text{ same spectrum for Laplace operator (isospectral)} \Rightarrow M, M' \text{ isometric?} \]

Answer:

Milnor 1964: No! (dimension 16)

Sunada 1985: No! (dimension \(d\))

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What properties of drums can you hear?

Weyl's law

Betti numbers (if strongly isospectral)

Torsion in the homology?

Sunada: No! (dimension 4)

Tighter question: small dimension, special classes of manifolds

Dimension 2 orientable

⇒ torsion-free homology

Dimension 3 orientable

⇒ torsion-free $H_0$, $H_2$ and $H_3$

Theorem (P. Bartel)

For all primes $p \leq 37$, there exist pairs of compact hyperbolic 3-manifolds $M, M'$ that are strongly isospectral and cover a common manifold, but such that $|H_1(M, \mathbb{Z})[\overline{p}]| \neq |H_1(M', \mathbb{Z})[\overline{p}]|$. 

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$$|H_1(M, \mathbb{Z})[p^\infty]| \neq |H_1(M', \mathbb{Z})[p^\infty]|$$
Number fields $K, K'$ are **arithmetically equivalent** if $\zeta_K = \zeta_{K'}$ but $K \not\cong K'$. 

- Same degree, same signature.
- Same discriminant.
- Same largest subfield that is Galois over $\mathbb{Q}$.
- Same roots of unity.
- Same product class number $\times$ regulator.

Dyer 1999: No!

Existing examples: $p = 2, 3, 5$. 

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Analytic class number formula:

$$\lim_{{s \to 1}} (s - 1)\zeta_K(s) = \frac{2^{r_1} (2\pi)^{r_2} h_K R_K}{w_K |D_K|^{1/2}}$$
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Spectrum of $\Delta$ on $i$-forms: $\zeta_{M,i}(s) = \sum \lambda^{-s}$.

Cheeger–Müller theorem (conjectured by Ray–Singer):

$$\prod_i \left( R_i(M) \cdot |H_i(M, \mathbb{Z})_{tors}| \right)^{(-1)^i} = \prod_i \exp(\frac{1}{2} \zeta'_{M,i}(0))^{(-1)^i}$$

$R_i(M)$ regulator of $H_i(M, \mathbb{Z})/H_i(M, \mathbb{Z})_{tors}$. 
Special value formulas

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Birch and Swinnerton-Dyer !
Gassman triple (1925): 

$G$ finite group and $H, H'$ subgroups such that 

$$\mathbb{C}[G/H] \cong \mathbb{C}[G/H'].$$
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If $K$ Galois number field with Galois group $G$

$$\Rightarrow \zeta_{K^H}(s) = L(\mathbb{C}[G/H], s).$$
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If $K$ Galois number field with Galois group $G$

$$\Rightarrow \zeta_{KH}(s) = L(\mathbb{C}[G/H], s).$$

Sunada: if $X \to Y$ is a Galois covering with Galois group $G$

$$\Rightarrow X/H \text{ and } X/H' \text{ are strongly isospectral.}$$
Cohomological Mackey functors

Map:
\[ F: \{ \text{subgroups of } G \} \to \text{R-modules}, \] and \[ \text{R-linear maps } c: F(H) \to F(H^g) \] conjugation
\[ r: F(H) \to F(K) \] restriction
\[ t: F(K) \to F(H) \] transfer

with natural axioms, among which
\[ r_{H,L} \circ t_{H,K} = \sum_{g \in L \setminus H/K} \text{"usual formula"} \]

Proposition (P., Bartel)
\[ H \mapsto H_i(X/H, Z) \] is a cohomological Mackey functor. In particular, if \[ Z_p[G/H] \cong Z_p[G/H'] \] then
\[ H_i(X/H, Z) \otimes Z_p \cong H_i(X/H', Z) \otimes Z_p. \]
Cohomological Mackey functors

Map: $\mathcal{F} : \{\text{subgroups of } G\} \longrightarrow R$-modules, and $R$-linear maps
- $c^g_H : \mathcal{F}(H) \rightarrow \mathcal{F}(H^g)$ conjugation
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Proposition (P., Bartel)

$H \mapsto H_i(X/H, \mathbb{Z})$ is a cohomological Mackey functor. In particular, if $\mathbb{Z}_p[G/H] \cong \mathbb{Z}_p[G/H']$ then

\[ H_i(X/H, \mathbb{Z}) \otimes \mathbb{Z}_p \cong H_i(X/H', \mathbb{Z}) \otimes \mathbb{Z}_p. \]
Smallest Gassman triple

Theorem (de Smit)

Let $p$ be an odd prime. If $G, H, H'$ is a Gassman triple such that

$$\mathbb{Z}_p[G/H] \ncong \mathbb{Z}_p[G/H']$$

and $[G : H] \leq 2p + 2$, then there is an isomorphism

$$G \cong \text{GL}_2(\mathbb{F}_p)/(\mathbb{F}_p^\times)^2$$

sending $H, H'$ to

$$\begin{pmatrix} \Box & * \\ 0 & * \end{pmatrix} \text{ and } \begin{pmatrix} * & * \\ 0 & \Box \end{pmatrix}.$$
Regulator constants

Regulators: transcendental, arithmetic, hard.
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$G, H, H'$ Gassman triple, $\rho$ representation of $G$ over $R = \mathbb{Z}$ or $\mathbb{Q}$. $\langle \cdot, \cdot \rangle$ $G$-invariant pairing on $\rho \otimes \mathbb{C}$.

\[
C(\rho) = \frac{\det(\langle \cdot, \cdot \rangle | (\rho^H)_{\text{free}})}{\det(\langle \cdot, \cdot \rangle | (\rho^{H'})_{\text{free}})} \in R/(R^\times)^2.
\]

Theorem (Dokchitser, Dokchitser)

$C(\rho)$ is independent of the pairing.
Example of regulator constants

\[ G = \text{GL}_2(\mathbb{F}_p)/\Box, \quad H_+ = \begin{pmatrix} \Box & \ast \\ 0 & \ast \end{pmatrix}, \quad H_- = \begin{pmatrix} \ast & \ast \\ 0 & \Box \end{pmatrix}. \]

\[ B = \begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix} \subset \text{GL}_2(\mathbb{F}_p), \quad r : \begin{pmatrix} a & \ast \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} a \\ p \end{pmatrix}. \]

\[ I = \text{Ind}_B^G r \text{ irreducible}. \]
Example of regulator constants

\[ G = \text{GL}_2(\mathbb{F}_p)/\square, \quad H_+ = \begin{pmatrix} \square & \ast \\ 0 & \ast \end{pmatrix}, \quad H_- = \begin{pmatrix} \ast & \ast \\ 0 & \square \end{pmatrix}. \]

\[ B = \begin{pmatrix} \ast & \ast \\ 0 & \ast \end{pmatrix} \subset \text{GL}_2(\mathbb{F}_p), \quad r : \begin{pmatrix} a & \ast \\ 0 & c \end{pmatrix} \mapsto \left( \frac{a}{p} \right). \]

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**Proposition (P., Bartel)**

For all irreducible representation \( \rho \) of \( G \) over \( \mathbb{Q} \), we have \( C(\rho) = 1 \), except \( C(I) = p \).
Comparison of regulators

Theorem (P., Bartel)

\( X \rightarrow Y \) Galois covering of hyperbolic 3-manifolds with Galois group \( G \). Gassman triple \( G, H, H' \) and \( p \) prime number. Assume \( |H^{ab}| \) and \( |H'^{ab}| \) coprime to \( p \). 

\( M := G \)-module \( H_2(X, \mathbb{Z}) \). Then

\[
\frac{R(X/H')}{R(X/H)} = C(M) \cdot u.
\]

for some \( u \in \mathbb{Z}_x^{(p)} \).
Good supply of 3-manifold: arithmetic Kleinian groups!

$F$ number field with $r_2 = 1$.

$B$ quaternion algebra over $F$ ramified at the real places.

$\mathcal{O}$ order in $B$.

$\Gamma = \mathcal{O}^1 / \{\pm 1\} \subset \text{PSL}_2(\mathbb{C})$ torsion-free.

$\Rightarrow Y = \mathcal{H}^3 / \Gamma$ hyperbolic 3-manifold with $\pi_1(Y) \cong \Gamma$. 
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Computation: fundamental domain and finite presentation for $\Gamma$. 
$h : \Gamma \to G$ is surjective, $Y = \mathcal{H}^3/\Gamma$ and $X = \mathcal{H}^3/\ker h$,
$\Rightarrow X \to Y$ is a Galois covering with Galois group $G$. 

$H_1(\frac{X}{\mathcal{H}}, R) \cong H_1(\frac{h^{-1}(\mathcal{H})}{R}) \cong H_1(\Gamma, R[\frac{G}{\mathcal{H}}])$,
where $H_1(\Gamma, M) = M/\langle \gamma m - m \rangle$. Can be computed by linear algebra.
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Can be computed by linear algebra.
Example

\[ F = \mathbb{Q}(t) \text{ with } t^4 - t^3 + 2t^2 - 1. \]

\[ B = \left( \frac{-1}{F}, -1 \right). \]

\( \mathcal{O} \) an Eichler order of level norm 71.

\( \Gamma \) has volume 27.75939054 \ldots, and a presentation with 5 generators and 7 relations.

We found a surjective \( \Gamma \to \text{GL}_2(\mathbb{F}_7) \), yielding two isospectral manifolds with homology

\[ \mathbb{Z}^3 + \mathbb{Z}/4 + \mathbb{Z}/4 + \mathbb{Z}/12 + \mathbb{Z}/12 + \mathbb{Z}/(2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 23), \text{ and} \]

\[ \mathbb{Z}^3 + \mathbb{Z}/4 + \mathbb{Z}/4 + \mathbb{Z}/12 + \mathbb{Z}/(12 \cdot 7) + \mathbb{Z}/(2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 23). \]
Thank you!