Isospectrality, regulators and special value formulas

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November 16, 2016

ICERM peer-to-peer seminar

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Answer:

Milnor 1964: No! (dimension 16)

Sunada 1985: No! (dimension d)

Gordon, Webb, Wolpert 1992: No! (domains of the plane)

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Dimension 3 orientable $\Rightarrow$ torsion-free H_0, H_2 and H_3

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Theorem (P., Bartel)

For all primes $p \leq 37$, there exist pairs of compact hyperbolic 3-manifolds M, M' that are strongly isospectral and cover a common manifold, but such that

$$|H_1(M,\mathbb{Z})[p^\infty]| \neq |H_1(M',\mathbb{Z})[p^\infty]|$$

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Same class number?
Dyer 1999: No!

Existing examples: $p = 2, 3, 5$.

Special value formulas

Analytic class number formula:

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} h_K R_K}{w_K |D_K|^{1/2}}$$

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Spectrum of Δ on i-forms: $\zeta_{M,i}(s) = \sum \lambda^{-s}$.

Cheeger–Müller theorem (conjectured by Ray–Singer):

$$\prod_i \left(R_i(M)\cdot|H_i(M,\mathbb{Z})_{tors}|\right)^{(-1)^i} = \prod_i \exp\left(\tfrac{1}{2}\zeta'_{M,i}(0)\right)^{(-1)^i}$$

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Birch and Swinnerton-Dyer !

Construction of isospectral objects

Gassman triple (1925):
G finite group and H, H' subgroups such that

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Sunada: if $X \to Y$ is a Galois covering with Galois group G
$\Rightarrow X/H$ and X/H' are strongly isospectral.

Cohomological Mackey functors

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Map: $\mathcal{F} : \{\text{subgroups of } G\} \longrightarrow R$-modules, and R-linear maps

- $c_H^g : \mathcal{F}(H) \to \mathcal{F}(H^g)$ conjugation
- $r_K^H : \mathcal{F}(H) \to \mathcal{F}(K)$ restriction
- $t_K^H : \mathcal{F}(K) \to \mathcal{F}(H)$ transfer

with natural axioms, among which

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Proposition (P., Bartel)

$H \mapsto H_i(X/H, \mathbb{Z})$ is a cohomological Mackey functor. In particular, if $\mathbb{Z}_p[G/H] \cong \mathbb{Z}_p[G/H']$ then

$$H_i(X/H, \mathbb{Z}) \otimes \mathbb{Z}_p \cong H_i(X/H', \mathbb{Z}) \otimes \mathbb{Z}_p.$$

Smallest Gassman triple

Theorem (de Smit)

Let p be an odd prime. If G, H, H' is a Gassman triple such that

$$\mathbb{Z}_p[G/H] \not\cong \mathbb{Z}_p[G/H']$$

and $[G:H] \leq 2p+2$, then there is an isomorphism

$$G \cong \mathrm{GL}_2(\mathbb{F}_p)/(\mathbb{F}_p^\times)^2$$

sending H, H' to

$$\begin{pmatrix} \square & * \\ 0 & * \end{pmatrix} \text{ and } \begin{pmatrix} * & * \\ 0 & \square \end{pmatrix}.$$

Regulator constants

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G, H, H' Gassman triple, ρ representation of G over $R = \mathbb{Z}$ or $\mathbb{Q}$. $\langle\cdot,\cdot\rangle$ G-invariant pairing on $\rho \otimes \mathbb{C}$.

$$\mathcal{C}(\rho) = \frac{\det(\langle\cdot,\cdot\rangle | (\rho^H)_{free})}{\det(\langle\cdot,\cdot\rangle | (\rho^{H'})_{free})} \in R/(R^\times)^2.$$

Theorem (Dokchitser, Dokchitser)

$\mathcal{C}(\rho)$ is independent of the pairing.

Example of regulator constants

$G = \mathrm{GL}_2(\mathbb{F}_p)/\square$, $H_+ = \begin{pmatrix} \square & * \\ 0 & * \end{pmatrix}$, $H_- = \begin{pmatrix} * & * \\ 0 & \square \end{pmatrix}$.

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset \mathrm{GL}_2(\mathbb{F}_p)$, $r : \begin{pmatrix} a & * \\ 0 & c \end{pmatrix} \mapsto \left(\frac{a}{p}\right)$.

$I = \mathrm{Ind}_B^G r$ irreducible.

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Proposition (P., Bartel)

For all irreducible representation ρ of G over $\mathbb{Q}$, we have $\mathcal{C}(\rho) = 1$, except $\mathcal{C}(I) = p$.

Comparison of regulators

Theorem (P., Bartel)

$X \to Y$ Galois covering of hyperbolic 3-manifolds with Galois group G. Gassman triple G, H, H' and p prime number. Assume $|H^{ab}|$ and $|H'^{ab}|$ coprime to p.
$M :=$ G-module $H_2(X, \mathbb{Z})$. Then

$$\frac{R(X/H')}{R(X/H)} = \mathcal{C}(M) \cdot u.$$

for some $u \in \mathbb{Z}_{(p)}^{\times}$.

Computations: fundamental domains

Good supply of 3-manifold: arithmetic Kleinian groups!

F number field with $r_2 = 1$.

B quaternion algebra over F ramified at the real places.

$\mathcal{O}$ order in B.

$\Gamma = \mathcal{O}^1/\{\pm 1\} \subset \mathrm{PSL}_2(\mathbb{C})$ torsion-free.

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Computation: fundamental domain and finite presentation for Γ.

Computations: covering, homology

$h : \Gamma \to G$ is surjective, $Y = \mathcal{H}^3/\Gamma$ and $X = \mathcal{H}^3/\ker h$,
$\Rightarrow X \to Y$ is a Galois covering with Galois group G.

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$h : \Gamma \to G$ is surjective, $Y = \mathcal{H}^3/\Gamma$ and $X = \mathcal{H}^3/\ker h$,
$\Rightarrow X \to Y$ is a Galois covering with Galois group G.

$H_1(X/H, R) \cong H_1(h^{-1}(H), R) \cong H_1(\Gamma, R[G/H])$,
where $H_1(\Gamma, M) = M/\langle \gamma m - m \rangle$.

Can be computed by linear algebra.

Example

$F = \mathbb{Q}(t)$ with $t^4 - t^3 + 2t^2 - 1$.

$B = \left(\frac{-1,-1}{F}\right)$.

$\mathcal{O}$ an Eichler order of level norm 71.

Γ has volume $27.75939054\ldots$, and a presentation with 5 generators and 7 relations.

We found a surjective $\Gamma \to \mathrm{GL}_2(\mathbb{F}_7)$, yielding two isospectral manifolds with homology

$$\mathbb{Z}^3 + \mathbb{Z}/4 + \mathbb{Z}/4 + \mathbb{Z}/12 + \mathbb{Z}/12 + \mathbb{Z}/(2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 23), \text{ and}$$

$$\mathbb{Z}^3 + \mathbb{Z}/4 + \mathbb{Z}/4 + \mathbb{Z}/12 + \mathbb{Z}/(12 \cdot 7) + \mathbb{Z}/(2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 23).$$

Questions?

Thank you!