Algorithms for arithmetic Kleinian groups

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Definition

A quaternion algebra over a field F is a central simple algebra of dimension 4. Explicitely, if char $F \neq 2$ it admits a presentation of the form

$$\langle\, i, j \mid i^2 = a,\ j^2 = b,\ ij = -ji \,\rangle$$

with $a, b \in F^\times$.

Example

The matrix algebra $\mathcal{M}_2(F)$ is a quaternion algebra over F. The ring of Hamiltonians $\mathbb{H}$ is a division quaternion algebra over $\mathbb{R}$.

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Let B be a quaternion algebra and $\beta = x + yi + zj + tij \in B$. The conjugate, reduced trace and reduced norm of β
are $\overline{\beta} = x - yi - zj - tij$, $\mathrm{trd}(\beta) = \beta + \overline{\beta}$ and $\mathrm{nrd}(\beta) = \beta\overline{\beta}$.
Group of norm 1 elements : $B_1^\times \subset B^\times$.

Example

In the matrix ring $B = \mathcal{M}_2(F)$, the reduced trace is the usual trace, the reduced norm is the determinant and $B_1^\times = \mathrm{SL}_2(F)$.

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If F is a number field, $B_1^\times$ is an algebraic group over F, we can consider $\mathbf{G} = \mathrm{Res}_{F/\mathbb{Q}}(B_1^\times)$.

Goal : compute arithmetic subgroups of $\mathbf{G}$. Ultimately, compute automorphic forms for $\mathbf{G}$.

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Let F be a number field, $\mathbb{Z}_F$ its ring of integers, and B a quaternion algebra over F. An order in B is a finitely generated $\mathbb{Z}_F$-submodule $\mathcal{O} \subset B$ with $F\mathcal{O} = B$ which is also a subring.

Example

The subring $\mathcal{M}_2(\mathbb{Z}_F)$ is an order in $\mathcal{M}_2(F)$.

The arithmetic subgroups of $\mathbf{G}$ are the groups commensurable with $\mathcal{O}_1^\times$ where $\mathcal{O}$ is any order in B.

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Let B be a quaternion algebra over a number field F. A place v of F is split or ramified according as $B \otimes_F F_v = \mathcal{M}_2(F_v)$ or not, where F_v is the completion of F at v. The product of all ramified primes $\mathfrak{p} \subset \mathbb{Z}_F$ is the discriminant Δ_B of B.

Symmetric space

Let F be a number field, B a quaternion algebra over F, $\mathbf{G}$ the associated algebraic group. We have

$$F \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}^c \times \mathbb{R}^r.$$

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$$\mathbf{G}(\mathbb{R})/K \cong (\mathrm{SL}_2(\mathbb{C})/\,\mathrm{SU}_2(\mathbb{C}))^c \times (\mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}_2(\mathbb{R}))^s.$$

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Let F be a number field, B a quaternion algebra over F, $\mathbf{G}$ the associated algebraic group. We have

$$\mathbf{G}(\mathbb{R})/K \cong \mathcal{H}_3^c \times \mathcal{H}_2^s.$$

Symmetric space

Let F be a number field with exactly one complex place, B a quaternion algebra over F ramified at every real place, $\mathbf{G}$ the associated algebraic group. We have

$$\mathbf{G}(\mathbb{R})/K \cong \mathcal{H}_3.$$

Definition

The unit ball $\mathcal{B}$ is the open ball of center 0 and radius 1 in $\mathbb{R}^3$ with the metric

$$ds^2 = \frac{4(dx^2 + dy^2 + dz^2)}{(1 - (x^2 + y^2 + z^2))^2}$$

where $(x, y, z) \in \mathcal{B}$.

Proposition

The unit ball is a model for the hyperbolic 3-space.

$\mathrm{SL}_2(\mathbb{C})$ acts by isometries on $\mathcal{B}$ and the stabilizer of the point $0 \in \mathcal{B}$ is $\mathrm{SU}_2(\mathbb{C})$.

We have explicit formulas for the action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathcal{B}$, the distance between two points, geodesics, etc.

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Definition

A Kleinian group is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$.

Theorem

Let F be a number field with exactly one complex place, B a quaternion algebra over F ramified at every real place, $\mathcal{O}$ an order in B, and $\rho : B \hookrightarrow \mathcal{M}_2(\mathbb{C})$ an embedding. Then $\Gamma = \rho(\mathcal{O}_1^\times)/\pm 1$ is a Kleinian group. It has finite covolume, and it is cocompact if and only if B is a division algebra. If $\mathcal{O}$ is maximal, then we have

$$\mathrm{Covol}(\Gamma) = \frac{|\Delta_F|^{3/2}\zeta_F(2)\prod_{\mathfrak{p}|\Delta_B}(N(\mathfrak{p})-1)}{(4\pi^2)^{n-1}}$$

where Δ_F is the discriminant of F, ζ_F is the Dedekind zeta function of F and Δ_B is the discriminant of B.

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Definitions

Let Γ be a Kleinian group. A fundamental domain for Γ is an open connected subset $\mathcal{F}$ of $\mathcal{B}$ such that

(i) $\bigcup_{\gamma\in\Gamma} \gamma \cdot \overline{\mathcal{F}} = \mathcal{B}$;

(ii) For all $\gamma \in \Gamma \setminus \{1\}$, $\mathcal{F} \cap \gamma \cdot \mathcal{F} = \emptyset$;

(iii) $\mathrm{Vol}(\partial\mathcal{F}) = 0$.

A fundamental domain that is a polyhedron is a fundamental polyhedron, it is finite if it has finitely many faces.

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Proposition

Let Γ be a Kleinian group. Let $p \in \mathcal{B}$ be a point with trivial stabilizer in Γ. Then the set

$$D_p(\Gamma) = \{x \in \mathcal{B} \mid \text{for all } \gamma \in \Gamma \setminus \{1\},\ \mathrm{d}(x,p) < \mathrm{d}(\gamma \cdot x, p)\}$$

is a convex fundamental polyhedron for Γ. If Γ has finite covolume, then $D_p(\Gamma)$ is finite.

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The domain $D_p(\Gamma)$ is a Dirichlet domain for Γ.

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Definitions

Suppose $g \in \mathrm{SL}_2(\mathbb{C})$ does not fix 0 in $\mathcal{B}$. Then let

- $\mathrm{I}(g) = \{w \in \mathcal{B} \mid \mathrm{d}(w, 0) = \mathrm{d}(g \cdot w, 0)\}$;
- $\mathrm{Ext}(g) = \{w \in \mathcal{B} \mid \mathrm{d}(w, 0) < \mathrm{d}(g \cdot w, 0)\}$;

$\mathrm{I}(g)$ is the isometric sphere of g. For $S \subset \mathrm{SL}_2(\mathbb{C})$ with no element fixing 0, the exterior domain of S is

$$\mathrm{Ext}(S) = \bigcap_{g \in S} \mathrm{Ext}(g).$$

For $g \in \mathrm{SL}_2(\mathbb{C})$, $I(g)$ is the intersection of an explicit Euclidean sphere with $\mathcal{B}$.

Given a finite set $S \subset \mathrm{SL}_2(\mathbb{C})$, we can compute the combinatorial structure of the polyhedron $\mathrm{Ext}(S)$.

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Given a finite set $S \subset \mathrm{SL}_2(\mathbb{C})$, we can compute the combinatorial structure of the polyhedron $\mathrm{Ext}(S)$.

Remark : If $0 \in \mathcal{B}$ has a trivial stabilizer in the Kleinian group Γ, then we have $D_0(\Gamma) = \mathrm{Ext}(\Gamma \setminus \{1\})$.

If Γ has finite covolume, there exists a finite subset $S \subset \Gamma$ such that

$$\mathrm{Ext}(\Gamma \setminus \{1\}) = \mathrm{Ext}(S).$$

Such a set S generates Γ, and we can describe the relations in terms of the combinatorial structure of $\mathrm{Ext}(S)$.

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Algorithm

Enumerate the elements of Γ in a finite set S until we have

$$\mathrm{Ext}(\Gamma \setminus \{1\}) = \mathrm{Ext}(S).$$

When do you stop ?

We can compute the volume of a finite polyhedron.

If Γ is given by a maximal order, we know a priori the volume of $\mathrm{Ext}(\Gamma \setminus \{1\})$.

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How do you enumerate elements ?

Proposition

There exists an explicit positive definite quadratic form Q on $B \otimes_{\mathbb{Q}} \mathbb{R}$ that gives $\mathcal{O}$ the structure of a lattice, and such that

$$\textit{for all } x \in \mathcal{O}_1^\times,\ Q(x) = \frac{4}{\mathrm{rad}(\rho(x))^2} + n$$

where $\mathrm{rad}(g)$ *denotes the radius of* $I(g)$ *for* $g \in \mathrm{SL}_2(\mathbb{C})$.

Use lattice enumeration techniques.

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Use lattice enumeration techniques.

How do you write an element as a word in the generators ?

Algorithm

1. Let $z = \gamma \cdot 0$ and $w = 1$.
2. If possible, pick $g \in S$ and let $z = g \cdot z$ and $w = wg^{-1}$ s.t. $\mathrm{d}(z, 0)$ decreases.
3. repeat.

Can also be used to speed up the algorithm.

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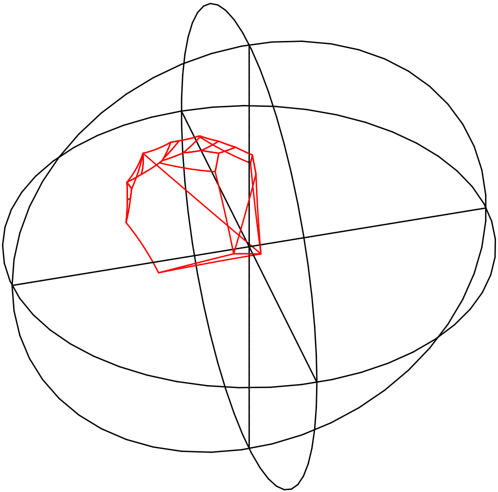
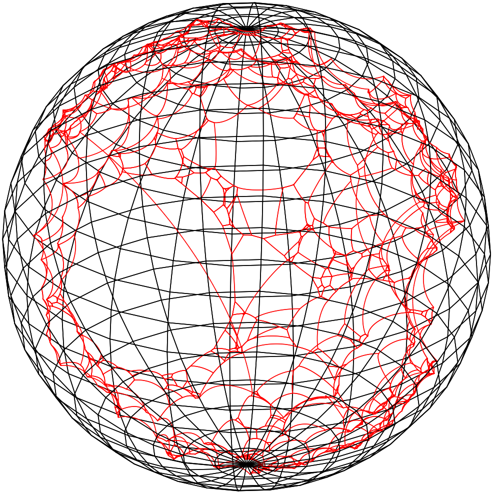


Proposition

Let $F = \mathbb{Q}(\sqrt[3]{11})$ *with discriminant* -3267, $\alpha = \sqrt[3]{11}$, $B = \left(\frac{-2,-4\alpha^2-\alpha-2}{F}\right)$, $\mathcal{O}$ *a maximal order in* B *and* $\Gamma = \mathcal{O}_1^\times / \pm 1$. *The quaternion algebra* B *has discriminant* $\mathfrak{p}_2$ *where* $\mathrm{N}(\mathfrak{p}_2) = 2$. *Then the group* Γ *has covolume* $\mathrm{Covol}(\Gamma) \approx 206.391784$, *and* Γ *admits a presentation with* 17 *generators and* 32 *relations.*

The fundamental polyhedron that was computed has 647 faces and 1877 edges. In the lattice, 80 millions of vectors were enumerated, and 300 of them had norm 1.



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Hecke operators

(work in progress)

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