I

Cohomology of unit groups of quaternionic orders I

Aurel Page.

I. Quaternion algebras and arithmetic groups

Def K field, $\mathrm{char}\, K \neq 2$.

A quaternion algebra over K is a K-algebra of the form

$$A = K + Ki + Kj + Kij$$

with $i^2 = a$, $j^2 = b$, $ij = -ji$ $\quad a, b \in K^\times$.

Notation: $A = \left(\frac{a,b}{K}\right)$.

Examples: $\left(\frac{1,1}{K}\right) \cong M_2(K)$.

$i \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$j \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\left(\frac{-1,-1}{\mathbb{R}}\right) =: \mathbb{H}$ is a division algebra.

Hamilton quaternions.

$\left(\frac{u^2 a, v^2 b}{K}\right) \xrightarrow{\sim} \left(\frac{a,b}{K}\right)$ $\quad i \longmapsto ui$, $j \longmapsto vj$.

Def Conjugation $A \longrightarrow A$ $\quad w \longmapsto \bar{w}$ $\quad K$-linear, $i \mapsto -i$, $j \mapsto -j$, $\overline{ww'} = \bar{w}'\bar{w}$.

Reduced trace $\mathrm{trd} : A \longrightarrow K$, $w \longmapsto w + \bar{w}$ $\quad K$-linear

Reduced norm $\mathrm{nrd} : A \longrightarrow K$, $w \longmapsto w\bar{w}$ $\quad$ multiplicative.

Examples $\left(\frac{1,1}{K}\right) \cong M_2(K)$

$\mathrm{trd} \longleftrightarrow \mathrm{tr}$

$\mathrm{nrd} \longleftrightarrow \det$

$w \in A^\times \Leftrightarrow \mathrm{nrd}(w) \in K^\times$

$w^{-1} = \dfrac{\bar{w}}{\mathrm{nrd}(w)}$.

Prop A quaternion algebra $\Rightarrow \begin{cases} A \cong M_2(K), \text{ or} \\ A \text{ is a division algebra.} \end{cases}$

Def K number field, A quaternion algebra $/K$.
v place of K : $\sigma : K \hookrightarrow \mathbb{C}$ or $\mathfrak{p}$ prime ideal of $\mathbb{Z}_K$.

v is : • <u>ramified in A</u> if $A \otimes_K K_v$ is a division algebra
• <u>unramified in A</u> if $A \otimes_K K_v \cong M_2(K_v)$

Thm

$$\{\text{quaternion algebras}/K\}/\text{isomorphism} \xleftrightarrow{\sim} \left\{\begin{matrix}\text{finite sets } R \text{ of places of } K \\ \# R \text{ even}\end{matrix}\right\}$$

$$A \longmapsto \mathrm{Ram}(A)$$

Example $\mathrm{Ram}\left(\left(\frac{-1,-1}{\mathbb{Q}}\right)\right) = \{2, \infty\}$.

Def K nf, A quat alg $/K$.
An <u>order</u> in A is a subring $\mathcal{O} \subseteq A$ such that $K\mathcal{O} = A$ and $\mathcal{O}$ is finitely generated as a $\mathbb{Z}_K$-module.
<u>Maximal order</u> = maximal for inclusion.

<u>Ex</u>: $a, b \in \mathbb{Z}_K$: $\mathcal{O} = \mathbb{Z}_K + \mathbb{Z}_K i + \mathbb{Z}_K j + \mathbb{Z}_K ij \subseteq \left(\frac{a,b}{K}\right)$ is an order.
never maximal. $\mathcal{O} = M_2(\mathbb{Z}_K) \subseteq M_2(K)$.

(r,s) signature of K, u = number of unramified real places of K.

$X = \mathcal{H}_2^u \times \mathcal{H}_3^s$. $\mathcal{O}$ order in A.

$$\Gamma(\mathcal{O}) = \mathcal{O}_+^\times / \mathbb{Z}_K^\times \subseteq \underbrace{PGL_2^+(\mathbb{R})^u \times PGL_2(\mathbb{C})^s}_{G} \curvearrowright X$$

nrd > 0 at all real places

Thm $\Gamma(\mathcal{O})$ is a lattice in G, [redacted] and it is cocompact if and only if A is a division algebra.

$Y(\mathcal{O}) = \Gamma(\mathcal{O}) \backslash X$ arithmetic manifold
orbifold

Maximal order $\mathcal{O}$ is not unique: $\mathcal{O} \rightsquigarrow x\mathcal{O}x^{-1}$, but finitely many up to conjugation.

Thm

$$\{\text{maximal orders}\}/\text{conjugation} \xrightarrow{\sim} Cl_A(K) / \langle \mathfrak{a}^2, \mathfrak{p} \text{ ramified in } A \rangle = C(A)$$

$$\mathcal{Y}(A) := \bigsqcup_{\mathfrak{a} \in C(A)} Y(\mathcal{O}_{\mathfrak{a}}).$$ (better: adélic interpretation)

$\mathcal{O}_{\mathfrak{a}}$: maximal order

Hecke operators: $\Gamma, \Gamma' \leq G$ lattices, $\delta \in G$

$$\begin{tikzcd}
\Gamma \cap \delta^{-1}\Gamma'\delta \backslash X \arrow[r, "x \mapsto \delta x"] \arrow[d, "\text{finite}"'] & \delta\Gamma\delta^{-1} \cap \Gamma' \backslash X \arrow[d, "\text{finite}"] \\
\Gamma \backslash X \arrow[r, dashed, "T_\delta"'] & \Gamma' \backslash X
\end{tikzcd}$$

$\Gamma = \Gamma(\mathcal{O}_{\mathfrak{a}})$ $\quad \Gamma' = \Gamma(\mathcal{O}_{\mathfrak{b}})$ $\quad \mathfrak{a}, \mathfrak{b}$ reps of $C(A)$

$\delta \in A^\times$ s.t. $\mathrm{nrd}(\delta)\mathfrak{a} = \mathfrak{p}\mathfrak{b}$

$$T_{\mathfrak{p}} : Y(\mathcal{O}_{\mathfrak{a}}) \dashrightarrow Y(\mathcal{O}_{\mathfrak{b}})$$

$$T_{\mathfrak{p}} \circlearrowright \mathcal{Y}(A)$$

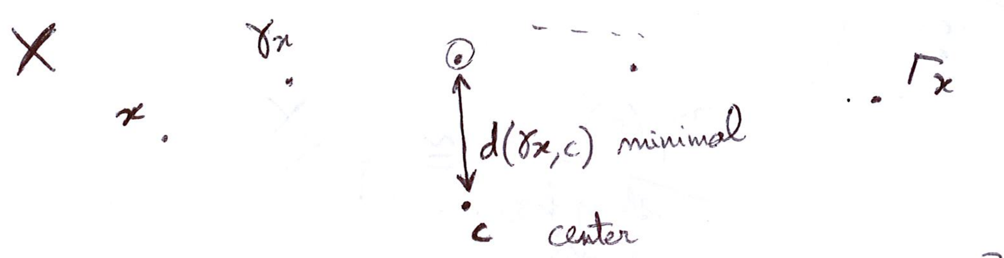
$\mathbb{T}$ = Hecke algebra generated by $T_{\mathfrak{p}}$, $\mathfrak{p}$ unramified prime.

II - Dirichlet domains and algorithms

Here assume $X = \mathcal{H}_3$. $(r=0, s=1)$

For computations, need a good model for $Y(\mathcal{O}) \to$ fundamental domain.

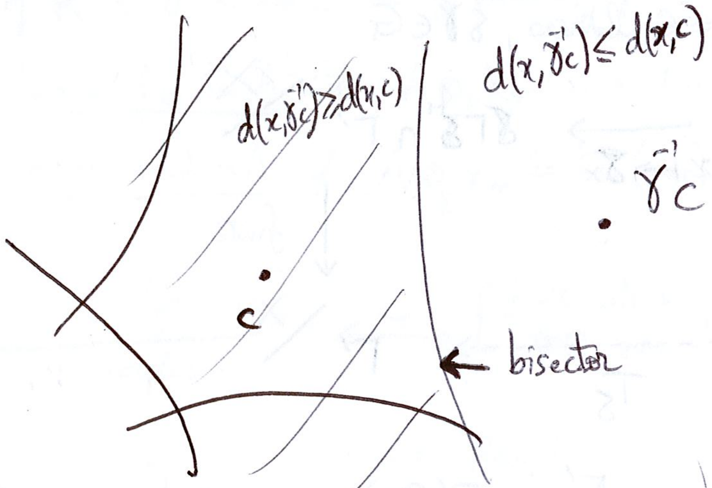
Fundamental domain $\approx$ set of representatives of $\Gamma \backslash X$ (up to boundary) = pick a point in each orbit



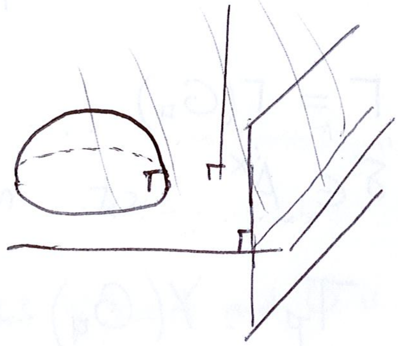
$$D_c(\Gamma) = \{ x \in X \mid d(\gamma x, c) \geq d(x, c) \ \forall \gamma \in \Gamma \}$$

Dirichlet domain: fundamental domain if $\mathrm{Stab}_\Gamma(c) = \{1\}$.

$$d(\gamma x, c) \gtrless d(x,c) \iff d(x, \gamma^{-1} c) \gtrless d(x,c)$$

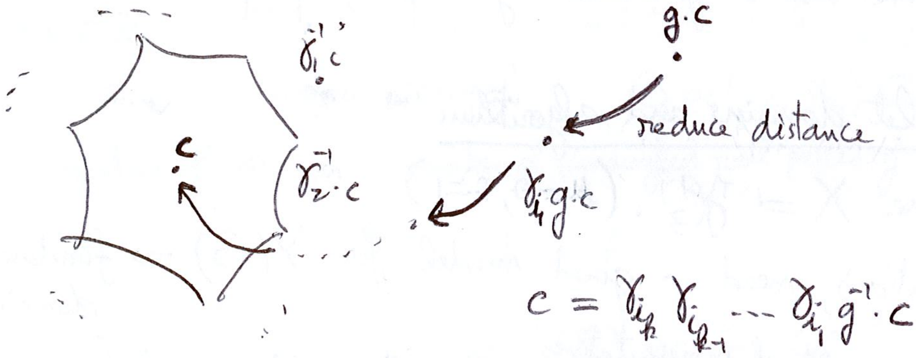


In $\mathcal{H}_3$, bisectors are totally geodesic.



If Γ is a lattice in $PGL_2(\mathbb{C})$,
$D_p(\Gamma)$ is a polyhedron with finitely many sides.

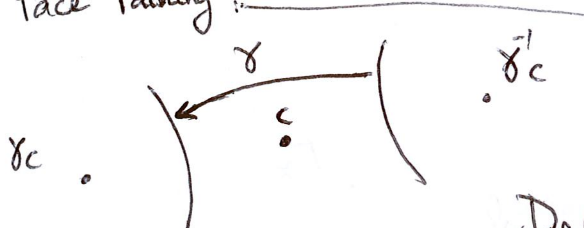
Faces of $D_p(\Gamma)$ $\longrightarrow$ generators of Γ : $g \in \Gamma$



$$c = \gamma_{i_k} \gamma_{i_{k-1}} \cdots \gamma_{i_1} g^{-1} \cdot c$$

$$\Longrightarrow g = \gamma_{i_k} \gamma_{i_{k-1}} \cdots \gamma_{i_1}.$$

Face Pairing $\longrightarrow$ Model of $\Gamma\backslash X$



$$\underset{\text{gluing faces under this pairing}}{\underline{D_p(\Gamma)}} \cong \Gamma \backslash X.$$

$\rightarrow$ $D_p(\Gamma)$ allows us to compute generators, $H(\omega)$ homology, Hecke action.

Algorithms: measure of complexity

$$V = \mathrm{vol}(Y(\mathcal{O}))$$

$$n = [K:\mathbb{Q}].$$

Best possible?

N sides for $D_p(\Gamma) \Rightarrow$ triangulation of $\Gamma\backslash X$ with $O(N)$ tetrahedra.

$\mathrm{vol}(\text{tetrahedron}) \leq \text{constant} \Rightarrow N \geq \text{constant}\cdot V$.

Hope: $O(V)$ complexity.

Need to find elements in $\mathcal{O}^\times$.

Fact: there exists a family of positive-definite quadratic form $Q_{x_1,x_2} : A\otimes\mathbb{R} \to \mathbb{R}$, $x_1, x_2 \in X$

making $\mathcal{O}$ a lattice, s.t. $\forall g \in \mathcal{O}\setminus\{0\}$

$$Q_{x_1,x_2}(g) = 2\cosh(d(gx_1, x_2)) + \mathrm{tr}_{K/\mathbb{Q}}(\mathrm{nrd}(g)) - 2.$$

Algorithm 1

$R \leftarrow 1 \quad S \leftarrow \emptyset$

while $D_c(S)$ is not compact:

- $S \leftarrow \{\gamma \in \mathcal{O}^\times \mid d(\gamma c, c) \leq R\}$
- $R \leftarrow 2R+1$

$R \leftarrow \sup\{d(c,x) : x \in D_c(S)\}$

$S \leftarrow \{\gamma \in \mathcal{O}^\times \mid d(\gamma c, c) \leq 2R\}$

Output $D_c(S)$

Correctness: if $\mathrm{bisector}(c, \gamma' c)$ intersects $D_c(S)$, then $d(c, \gamma' c) \leq 2R$.

Running time Last $R \approx V^{O(1)} \Rightarrow$ time $V^{O(n)}$.

Uses too little of the group structure: does not create new γ's from products of previous ones.

Poincaré's theorem: Combinatorial conditions on the polyhedron $D_p(S)$ that imply $D_p(S) = D_c(\langle S\rangle)$.

Refinement: condition that fails $\Rightarrow$ new element γ s.t. $D_c(S\cup\{\gamma\}) \subsetneq D_c(S)$.

VI

Algorithm 2 (Voight)

$R \leftarrow 1 \quad S \leftarrow \emptyset$

while $D_c(S) \neq D_c(\langle S \rangle)$ or $\mathrm{vol}(D_c(S)) \geq \underbrace{2\,\mathrm{vol}(Y(\mathcal{O}))}_{\text{from arithmetic formula}}$

  increase S from failed conditions

  $S \leftarrow S \cup \{\gamma \in \mathcal{O}^\times \mid d(\gamma c, c) \leq R\}$

  $R \leftarrow R+1$

Output $D_c(S)$

Correctness If $D_c(S) = D_c(\langle S \rangle)$ then $\mathrm{vol}(D_c(S)) = \underbrace{[\Gamma(\mathcal{O}) : \langle S \rangle]}_{\text{integer}} \mathrm{vol}(Y(\mathcal{O}))$.

Running time $V^{O(n)}$ ← smaller constant.

~~Butt~~ Problem: enumeration for large R is too costly.

More aggressive probabilistic algorithm:

Algorithm 3 (P.)

$S \leftarrow \emptyset$

while $D_c(S) \neq D_c(\langle S \rangle)$ or $\mathrm{vol}(D_c(S)) \geq 2V$

  increase S from failed conditions

  repeat V^2 times:

    ~~$S \leftarrow S \cup \{\gamma \in \mathcal{O}^\times \mid$~~

    $x \leftarrow$ random element s.t. $d(c, x) \leq V^C$

    $S \leftarrow S \cup \{\gamma \in \mathcal{O}^\times \mid d(x, \gamma^{-1} c) \leq 1\}$

Output $D_c(S)$

Running time:
- success with proba $\approx \frac{1}{V}$
- $\approx V$ random ~~elements~~ small elements of Γ should generate it

Heuristically $\tilde{O}_n(V^2)$.

Provable variant $O_n(V^{O(1)})$.

Cohomology of unit groups of quaternionic orders II

Aurel Page

III _ Galois representations and functoriality (torsion/mod ℓ)

G group. 2-dimensional representations $\rho : G \to GL_2(E)$ can be of 3 types:

- $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ decomposable ⟵ semisimple
  semisimplification.
- $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ reducible but indecomposable
- $\begin{pmatrix} * & * \\ * & * \end{pmatrix}$ irreducible ($\Rightarrow$ indecomposable)

Example: $G = \mathbb{Z}$, $E = \mathbb{Q}$.

- $n \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix}$
- $n \longmapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ with semisimplification $n \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $n \longmapsto \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}^n$ (becomes decomposable over $\mathbb{Q}(\sqrt{2})$)

Semisimple representations in characteristic 0 are determined by the $\mathrm{tr}\,\rho(g)$ for $g \in G$.

Langlands conjecture for GL_2:

$$\left\{ \begin{array}{c} \text{semisimple} \\ \text{Galois representations} \\ \rho : \mathrm{Gal}(\overline{K}/K) \to GL_2(\overline{\mathbb{Q}}_\ell) \\ + \text{conditions} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathbb{T}\text{-eigenvalue} \\ \text{systems in} \\ H^*(\mathcal{Y}(M_2(K)), \mathbb{C}) \end{array} \right\}$$

$$\mathrm{tr}\,\rho(\mathrm{Frob}_\mathfrak{p}) = a_\mathfrak{p} \text{ eigenvalue of } T_\mathfrak{p}.$$

decomposable ⟷ Eisenstein

irreducible. ⟷ cuspidal

II

A/K division quaternion algebra.

$$\left\{\begin{array}{c}\text{irreducible} \\ \text{Galois representations} \\ \rho : Gal(\overline{K}/K) \longrightarrow GL_2(\overline{\mathbb{Q}}_\ell) \\ + \text{ conditions}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\cancel{\text{Eisenstein}} \\ \Pi\text{-eigenvalues systems} \\ \text{in } H^*(Y(A), \mathbb{C})\end{array}\right\}$$

no condition on $\rho \mid Gal(\overline{K}_p/K_p)$ $\longleftarrow$	p unramified in A	$\longrightarrow A \otimes K_p \cong M_2(K_p)$
(swap)		
$\rho \mid Gal(\overline{K}_p/K_p)$ indecomposable $\longleftarrow$	p ramified in A	$\longrightarrow A \otimes K_p$ division algebra

Functoriality: maps on the Galois representations side should induce a corresponding map on the automorphic/arithmetic group side. Sometimes, we can prove it without having the correspondence with Galois representations.

Thm (Jacquet–Langlands, special case)

There exists

$$H^{u+s}(Y(A), \mathbb{C}) \xrightarrow{\sim} H^{r+s}(Y_0(M_2(K), Ram(A)), \mathbb{C})^{new}$$

Π-equivariant.

new Π-eigenvalue systems = don't appear in lower levels.

Computation: dim $2u+3s$ $\leqslant$ dim $2r+3s$.

More general automorphic forms: differential p-forms and their Laplace eigenvalues, etc.

III

Mod ℓ / torsion version?

$$\left\{ \begin{array}{c} \text{semisimple Galois representations} \\ \rho : \mathrm{Gal}(\bar{K}/K) \to GL_2(\bar{\mathbb{F}}_\ell) \\ + \text{conditions} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathbb{T}\text{-eigenvalue systems} \\ \text{in } H^*(\mathcal{Y}(M_2(K)), \bar{\mathbb{F}}_\ell) \end{array} \right\}$$

What about $\mathcal{Y}(A)$?

Complications:

1) ρ can be reducible since ~~mod ℓ~~ (irreducible mod ℓ) can be reducible

2) even for ρ irreducible, $\rho|\mathrm{Gal}(\bar{K}_\mathfrak{p}/K_\mathfrak{p})$ can be decomposable since (indecomposable mod ℓ) can be decomposable.

So new $\neq$ does not appear in lower level.

Assume $X = \mathcal{H}_3$. $(r=0, s=1)$ [Calegari-Venkatesh]

Def $H_1(\mathcal{Y}_0(M_2(K), \mathrm{Ram}(A)), \mathbb{Z})^{new}$

$= H_1(\mathcal{Y}_0(M_2(K), \mathrm{Ram}(A)), \mathbb{Z}) /$ image from lower levels.

$$H_1(\mathcal{Y}(A), \mathbb{Z}) \underset{\mathbb{T}}{\cong} H_1(\mathcal{Y}_0(M_2(K), \mathrm{Ram}(A)), \mathbb{Z})^{new} \ ?$$

No: parasite classes corresponding to "congruence homology" and "K-theoretic classes".

decomposable representations:

Maximal ideal $\mathfrak{m}$ of $\mathbb{T}$:

~~generated by some (prime ideal) $T_\mathfrak{p} \in \mathbb{Z}_\mathfrak{p}$~~

$$\mathbb{T}/\mathfrak{m} \cong \mathbb{F}_{\ell^k}$$

$$\ell \longmapsto 0$$

$$T_\mathfrak{p} \longmapsto a_\mathfrak{p} \in \mathbb{F}_{\ell^k}$$

$$\left\{ \begin{array}{c} \text{maximal ideal} \\ \text{of } \mathbb{T} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Galois orbit of} \\ \mathbb{T}\text{-eigenvalue} \\ \text{system in } \bar{\mathbb{F}}_\ell \end{array} \right\}$$

If $\mathbb{T} \circlearrowright M$ then $M \otimes \mathbb{Z}_\ell \cong \bigoplus_{\mathfrak{m} \ni \ell} M_{\mathfrak{m}}$.

Consider only $\mathfrak{m}$ corresponding to irreducible $\bar{\rho}$.

$$H_1(\mathcal{Y}(A), \mathbb{Z})_{\mathfrak{m}} \cong H_1(\mathcal{Y}_0(\Pi_2(K), \mathrm{Ram}(A)), \mathbb{Z})_{\mathfrak{m}}^{\mathrm{new}} \ ?$$

No! There are counter-examples.

<u>Conjecture (Calegari-Venkatesh):</u>

$$\# H_1(\mathcal{Y}(A), \mathbb{Z})_{\mathfrak{m}} = \# H_1(\mathcal{Y}_0(M_2(K), \mathrm{Ram}(A)), \mathbb{Z})_{\mathfrak{m}}^{\mathrm{new}}$$

⚠ If they are infinite, don't expect the torsion subgroups to have the same size.

IV _ Partial results / Evidence

* Numerical computations.

<u>Regulators</u> Y Riemannian manifold (closed, orientable)

$$H_i(M, \mathbb{R}) \cong \mathcal{H}^i(M)$$ ← canonical inner product coming from metric.

↑ de Rham + Hodge; ↑ harmonic i-forms

<u>Def</u> $$\mathrm{Reg}_i(M) = \mathrm{vol}\left(\frac{H_i(M, \mathbb{R})}{H_i(M, \mathbb{Z})}\right)$$ analogy: regulators of number fields.

<u>Examples</u> • $\mathrm{Reg}_0(M) = \mathrm{vol}(M)^{-1/2}$

• $\mathrm{Reg}_{\dim M - i}(M) = \mathrm{Reg}_i(M)^{-1}$.

<u>Analytic torsion</u> $\Delta \circlearrowright \Omega^i(M)$ discrete spectrum.

$$\zeta_{M,i}(s) = \sum_{\substack{\lambda > 0 \\ \text{ev of } \Delta \circlearrowright \Omega^i(M)}} \lambda^{-s} \quad \text{for } \mathrm{Re}(s) \gg 0.$$

$s \in \mathbb{C}$

analogy: Dedekind zeta function.

Thm (Cheeger–Müller)

$$\prod_{i=0}^{\dim M} \exp(\mathcal{T}'_{Y,i}(0))^{i(-1)^i} = \prod_{i=0}^{\dim M}\left(\frac{\# H_i(Y,\mathbb{Z})_{tor}}{Reg_i(Y)}\right)^{(-1)^i}$$

analogy: analytic class number formula

analytic torsion — Reidemeister torsion

Since classical Jacquet–Langlands gives something about analytic torsion, get partial results, mostly without information about Hecke action.

Sample thm (Calegari–Venkatesh)

Assume $H_1(Y_0(M_2(K), Ram(A)), \mathbb{C}) = 0$.

Then

$\# H_1^{E^*}(Y(A),\mathbb{Z})$ divides $\# H_1^{E^*}(Y_0(M_2(K), Ram(A)),\mathbb{Z})^{new}$.

Simpler but somewhat analogous situation: Vignéras pairs.

Recall components of $Y(A) \longleftrightarrow$ class group $C(A)$.

$\mathbb{T} = \langle T_{\mathfrak{p}} : \text{all unramified } \mathfrak{p} \rangle \circlearrowright Y(A)$

$\cup$

$\mathbb{T}_0 = \langle T_{\mathfrak{p}} : [\mathfrak{p}]=0 \text{ in } C(A) \rangle \circlearrowright Y(G)$

Philosophy:

$\rho : Gal(\bar{K}/K) \to GL_2 \longleftrightarrow \{\mathbb{T}\text{-eigenvalue systems}\}$ GL$_2$ automorphic forms

$(P\rho : Gal(\bar{K}/K) \to PGL_2) \longleftrightarrow \{\mathbb{T}_0\text{-eigenvalue systems}\}$ SL$_2$ automorphic forms.

Twist-classes of ρ by $\chi : C(A) \to \mathbb{C}^\times$

$P\rho \cong P\rho' \Longleftrightarrow \rho' \cong \rho \otimes \chi$

<u>Generically</u> all $\rho \otimes \chi$ are <u>distinct</u>.

$\mathbb{P}_\rho$ contributes to all $c_{\mathfrak{a}} \in H^i(Y(\mathcal{O}_{\mathfrak{a}}), \mathbb{C})$ $\quad [\mathfrak{a}] \in C(A)$

$\mathbb{T}_0$ - eigenclass.

$$\Downarrow$$

$$\operatorname{span}\langle c_{\mathfrak{a}} \rangle \subseteq H^i(Y(A), \mathbb{C})$$

$$\|$$

$$\rho, \rho \otimes \chi, \dots \longleftrightarrow \bigoplus f \quad \mathbb{T}\text{- eigenclasses}$$

same for other automorphic forms.

Non-generic case: $\rho \cong \rho \otimes \chi \Rightarrow \rho \cong \operatorname{ind}_{L/K} \psi$. (Labesse–Langlands) "endoscopic"

What about torsion?

<u>Thm</u> (Bartel-P.)

Let $\mathfrak{m}$ maximal ideal of $\mathbb{T}_0$ not corresponding to $\rho \cong \rho \otimes \chi$.

Then $H_i(Y(\mathcal{O}_{\mathfrak{a}}), \mathbb{Z})_{\mathfrak{m}} \cong H_i(Y(\mathcal{O}_{\mathfrak{b}}), \mathbb{Z})_{\mathfrak{m}}$.