Algebraic number theory
Exercise sheet for chapter 1

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Answers must be submitted by Tuesday January 31, 14:00

Exercise 1 (10 points)

Let $K = \mathbb{Q}(\sqrt{3})$, and let $\alpha = a + b\sqrt{3}$ $(a, b \in \mathbb{Q})$ be an element of K. Compute the trace, norm, and characteristic polynomial of α in terms of a and b

1. (5 points) by writing down the matrix of the multiplication-by-α map with respect to the $\mathbb{Q}$-basis of K of your choice,
2. (5 points) by considering complex embeddings.

Exercise 2 (40 points)

In this exercise, you may assume[1] *that the polynomial $x^3 - 2$ is irreducible over $\mathbb{Q}$.*

1. (8 points) Let $K = \mathbb{Q}(\sqrt[3]{2})$, and let $\alpha = \frac{\sqrt[3]{2}+1}{\sqrt[3]{2}-1} \in K$. Find $a, b, c \in \mathbb{Q}$ such that $\alpha = a + b\sqrt[3]{2} + c(\sqrt[3]{2})^2$.
2. (4 points) Are these rational numbers a, b, c unique ?
3. (2 points) What is the degree of K ?
4. (7 points) Prove that $\sqrt{2} \notin K$.
5. (7 points) Prove that $K = \mathbb{Q}(\alpha)$.
6. (12 points) Compute the trace, norm, and characteristic polynomial of α, and use the previous question to prove that this polynomial is irreducible over $\mathbb{Q}$.

[1] We will see an efficient way (Eisenstein's criterion) to prove this in chapter 3.

Exercise 3 (30 points)

1. (3 points) Let $K = \mathbb{Q}(\sqrt{2})$. Prove that $i \notin K$.

2. (5 points) Let $L = \mathbb{Q}(\sqrt{2}, i)$. Compute $[L : \mathbb{Q}]$.

3. (5 points) What is the signature of L ?

4. (6 points) Let $\alpha = \sqrt{2} + i \in L$. Compute the characteristic polynomial $\chi_{\mathbb{Q}}^{L}(\alpha)$ of α with respect to the extension $L/\mathbb{Q}$.

5. (5 points) Is the polynomial $\chi_{\mathbb{Q}}^{L}(\alpha)$ squarefree? What does this tell us about α?

6. (6 points) Compute the characteristic polynomial $\chi_{K}^{L}(\alpha)$ of α with respect to the extension L/K.

Exercise 4 (20 points)

In this exercise, you may freely assume that π and e are both transcendental over $\mathbb{Q}$.

1. (5 points) Prove that e and π are both algebraic over the field $\mathbb{Q}(e + \pi, e\pi)$.

2. (15 points) Deduce that at least one of the numbers $e + \pi$ and $e\pi$ is transcendental over $\mathbb{Q}$.

UNASSESSED QUESTIONS

The next questions are not worth any points. I still recommend you to try to solve them, for practice. Correction will be available online, just as for the marked questions.

Exercise 5

Let K be a field, L a finite extension of K of degree n, and $f(x) \in K[x]$ a polynomial of degree m which is irreducible over K.

1. Prove that if m and n are coprime, then $f(x)$ remains irreducible over L. *Hint: Consider a root α of $f(x)$ in some large enough field containing L, what is the degree of $L(\alpha)$ over K ?*

2. Is the conclusion the same if m and n are not coprime ?

Exercise 6

Let $K = \mathbb{Q}(\alpha)$ be a number field, let $A(x) \in \mathbb{Q}[x]$ be the minimal polynomial of α, and let $\beta = B(\alpha) \in K$, where $B(x) \in \mathbb{Q}[x]$ is some polynomial. Express the characteristic polynomial $\chi_{\mathbb{Q}}^{K}$ of β in terms of a resultant involving A and B.