Algebraic number theory
Revision exercises

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Exercise 1. What is the ring of integers of $\mathbb{Q}(\sqrt{98})$?

Exercise 2. What kinds of number fields have a unit group of rank 1 ?

Exercise 3. Compute the class group of $\mathbb{Q}(\sqrt{-47})$.

Exercise 4. The aim of this exercise is to determine the class group of $K = \mathbb{Q}(\sqrt{82})$, seen as a subfield of $\mathbb{R}$.

1. Prove that the class group of K is either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$.
2. What is the rank of the unit group of K ? Compute a fundamental unit $u > 1$ of K.
3. Suppose that there exist an element $\beta = x + y\sqrt{82} \in \mathbb{Z}_K$ of norm 2. Why may we assume that $\frac{1}{\sqrt{u}} < \beta < \sqrt{u}$? Prove that $x - y\sqrt{82} = \frac{2}{\beta}$, use this to derive bounds on x, and deduce that no such β exists.
4. Prove similarly that no element of $\mathbb{Z}_K$ has norm -2.
5. What is the class group of K ?
6. Was is absolutely necessary that the unit u be fundamental for the above reasoning to be valid ?

Exercise 5. Let $f(x) = x^3 - 4x^2 + 2x - 2$, which is an irreducible polynomial over $\mathbb{Q}$ (why ?), and let $K = \mathbb{Q}(\alpha)$, where α is a root of f.

1. Given that $\operatorname{disc} f = -300$, what can you say about the ring of integers of K and the primes that ramify in K ? What if, on the top of that, you notice that $f(x+3) = x^3 + 5x^2 + 5x - 5$?

2. Prove that $\mathbb{Z}_K$ is a PID.

   Hint: For $n \in \mathbb{Q}$, what relation is there between $f(n)$ and the norm of $n - \alpha$? Use this to find elements of small norm, and thus relations in the class group.

3. Find a generator for each of the primes above 2, 3 and 5.

4. Use the results of the previous question to discover that $u = 2\alpha^2 - \alpha + 1$ is a unit.

5. We use the unique embedding of K into $\mathbb{R}$ to view K as a subfield of $\mathbb{R}$ from now on. Prove that there exists a unit $\varepsilon \in \mathbb{Z}_K^\times$ such that $\mathbb{Z}_K^\times = \{\pm\varepsilon^n, n \in \mathbb{Z}\}$ and $\varepsilon > 1$.

6. By the technique of exercise 2 from exercise sheet number 5, it can be proved that $\varepsilon \geqslant 4.1$. Given that $u \approx 23.3$, prove that u is a fundamental unit.

   Hint : Reduce u modulo the primes above 3 to prove that u is not a square in $\mathbb{Z}_K$.

   What is the regulator of K ?

Exercise 6. Let $f(x) = x^4+3x^3-18x^2-24x+129$, which is an irreducible polynomial over $\mathbb{Q}$ (why ?), and let $K = \mathbb{Q}(\alpha)$, where α is a root of f.

1. If I told you that $\operatorname{disc} f = 930069$, why would not that be very useful to you ? Which information can you get from that nonetheless ?

2. I now tell you that the roots of f are approximately $-4.1 \pm 0.1i$ and $2.6 \pm 1.0i$. What is the signature of K ? Can you compute the trace of α from these approximate values ? Why is the result obvious ?

3. If I now tell you that $\operatorname{disc} f$ factors as $3^3 \cdot 7^2 \cdot 19 \cdot 37$, what can you say about the ring of integers of K and the primes that ramify in K ?

4. In principle (don't actually do it), how could you test whether $\beta = \frac{\alpha^3-2\alpha^2-\alpha+2}{7}$ is an algebraic integer ?

5. If I now tell you that the characteristic polynomial of β is $\chi(\beta) = x^4 + 28x^3 + 207x^2 + 154x + 247$, whose discriminant is $\operatorname{disc} \chi(\beta) = 25364993616$, which conclusions can you draw from that ?

6. Given that $\operatorname{disc} \chi(\beta)$ factors as $2^4 \cdot 3^3 \cdot 17^4 \cdot 19 \cdot 37$, what is the index of the order $\mathbb{Z}[\beta]$? What consequence does this have on the expression of a $\mathbb{Z}$-basis of $\mathbb{Z}_K$ in terms of β ?

7. Let $\gamma = \frac{\beta^2-3\beta-3}{34}$, and let $\delta = \frac{\beta^3-12\beta-9}{34}$, whose respective characteristic polynomials are $\chi(\gamma) = x^4-13x^3+42x^2+8x+1$ and $\chi(\delta) = x^4+139x^3+5163x^2+973$. Prove that $\{1, \beta, \gamma, \delta\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}_K$.

8. Compute explicitly the decomposition of 2, 3, and 7 in K.

9. What is the rank of the unit group of K ? Can you spot a nontrivial (i.e. not ± 1) unit of K ?

10. What can you say about the roots of unity contained in K ? How could you use this to test whether the unit you spotted in the previous question if a root of unity ?