Algebraic number theory
Exercise sheet for chapter 1

Nicolas Mascot (n.a.v.mascot@warwick.ac.uk)
Aurel Page (a.r.page@warwick.ac.uk)
TA: Pedro Lemos (lemos.pj@gmail.com)

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Answers must be submitted by Friday January 29, 14:00

Exercise 1 (10 points)

Let $K = \mathbb{Q}[\sqrt[3]{2}]$, and let $\beta = 1 + \sqrt[3]{2} \in K$. Use a Bézout identity[1] to compute $1/\beta$ as a polynomial in $\sqrt[3]{2}$ with coefficients in $\mathbb{Q}$.

Exercise 2 (20 points)

Let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}^*$ be algebraic numbers. Use resultants to prove that α/β is also an algebraic number.

Exercise 3 (25 points)

Let L/K be a finite extension such that $[L : K]$ is a prime number.

1. (15 points) Prove that if E is a field such that $K \subset E \subset L$, then $E = K$ or $E = L$.
2. (10 points) Deduce that every $\alpha \in L \setminus K$ is a primitive element for the extension L/K.

Exercise 4 (45 points)

1. (10 points) Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{-5})$. Compute $[L : \mathbb{Q}]$.
2. (5 points) What is the signature of L ?

[1]That is to say, use successive Euclidian divisions to find $U, V \in \mathbb{Q}[x]$ such that

$$(x^3 - 2)U(x) + (1 + x)V(x) = 1.$$

3. (15 points) Let $\beta = \sqrt{2} + \sqrt{-5}$. Compute the characteristic polynomial $\chi_{\mathbb{Q}}^{L}(\beta)$ of β with respect to the extension $L/\mathbb{Q}$.

4. (15 points) Is this polynomial squarefree ? What does this tell us about β ?

UNASSESSED QUESTIONS

The next questions are not worth any points. I still recommend you to try to solve them, for practice. Correction will be available online, just like for the marked questions.

Exercise 5

1. Let $K = \mathbb{Q}(\sqrt{-5})$, and let $\alpha = a + b\sqrt{-5}$ $(a, b \in \mathbb{Q})$ be an element of K. Compute the trace, norm, and characteristic polynomial of α in terms of a and b.

2. Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{-5})$, and let $\beta = \sqrt{2} + \sqrt{-5}$. Compute the characteristic polynomial $\chi_{K}^{L}(\beta)$ of β with respect to the extension L/K.

Exercise 6

Let $K = \mathbb{Q}(\alpha)$ be a number field, let $A(x) \in \mathbb{Q}[x]$ be the minimal polynomial of α, and let $\beta = B(\alpha) \in K$, where $B(x) \in \mathbb{Q}[x]$ is some polynomial. Express the characteristic polynomial $\chi_{\mathbb{Q}}^{K}$ of β in terms of a resultant involving A and B.