Parametrized Surfaces in $\mathbb{P}^3$ of bidegree $(1, 2)$

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Abstract

Parametrized surfaces of low degrees are very useful in applications, specially in Computer Aided Geometric Design and Geometric Modeling. The precise description of their geometry is not easy in general. Here we study surfaces of bidegree $(1, 2)$. We show that, generically up to linear changes of coordinates, they are classified by two continuous parameters (modulus). We present an elegant combinatorial description where these modulus appear as cross ratios. We provide compact implicit equations for these surfaces and for their singular locus together with a geometric interpretation.

1 Introduction

The common representation of surfaces in Solid Modeling and Computer Aided Geometric Design is via parametrized patches, i.e. images of maps

$$\Phi : [0, 1] \times [0, 1] \to \mathbb{R}^3$$
$$(t, u) \mapsto \Phi(t, u) = \left(\frac{\Phi_1(t, u)}{\Phi_0(t, u)}, \frac{\Phi_2(t, u)}{\Phi_0(t, u)}, \frac{\Phi_3(t, u)}{\Phi_0(t, u)} \right)$$

where $\Phi_0, \Phi_1, \Phi_2, \Phi_3$ are polynomials with real coefficients. Such a Φ is classified by the maximal degree (or by the bidegree) of $\Phi_0, \Phi_1, \Phi_2, \Phi_3$; for instance, bicubic (i.e. bidegree $(3, 3)$). Surface patches are encountered in many applications [8, 6]. However self-intersection points of these patches are often

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difficult to detect [1, 7], and more generally the intersection curve of two such surfaces can be very complicated and plays an important role. Therefore it is worthwhile to study systematically parametrized surfaces of low degrees in order to have at our disposal mastered geometric models together with their singular locus.

The surfaces of total degree 1 (i.e. $\max_{0\le i\le 3} \deg \Phi_i = 1$) are planes, while surfaces with parametrization of bidegree $(1, 1)$ are planes or quadrics. The surfaces with a parametrization of total degree 2 are called Steiner surfaces (when they are base-point free) and have been extensively studied [4, 12]. We recall that a base-point of the parametrization Φ is a common root to $\Phi_0, \Phi_1, \Phi_2, \Phi_3$.

The next class to understand is the parametrized surfaces of bidegree $(1, 2)$. This is precisely the object of our study. In this paper, we concentrate on the description and production of a normal form in the generic case of a complex projective surface which is the image $\mathcal{S}$ of a bihomogeneous polynomial map $\Phi : \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^3(\mathbb{C})$ of bidegree $(1, 2)$. This map is given by a matrix $\mathcal{A}_\Phi$ of type 4×6 with complex entries. We will see that for generic $\mathcal{A}_\Phi$, up to projective changes of coordinates in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ and in $\mathbb{P}^3(\mathbb{C})$, the surface $\mathcal{S}$ depends only on two modulus a and b in $\mathbb{P}^1(\mathbb{C})$. This allows us to rewrite the initial parametrization of $\mathcal{S}$ into a much simpler normal form. Then all the geometric properties that we aim to investigate will be deduced from this normal form by formal computations using a computer algebra system (for example MAPLE) guided by an algebro-geometric reasoning.

In forthcoming works we will study the degenerate complex case and especially the real case which is truly important for applications.

The considered surface $\mathcal{S}$ can be seen either as the total space of a family of conics $\mathcal{S} = \cup_t \mathcal{C}_t$ with $t \in \mathbb{P}^1(\mathbb{C})$, or as the total space of a family of lines $\mathcal{S} = \cup_u \mathcal{L}_u$ with $u \in \mathbb{P}^1(\mathbb{C})$. Both interpretations will be useful to describe the geometry of $\mathcal{S}$ and its singular locus. We will provide formulae and also an associated tetrahedral configuration on which one can read most of invariants and constructions. Some examples and graphical illustrations will help the global understanding of this class of surfaces.

The paper is organized as follows. In sections 2 and 3, we define a parametrized surface $\mathcal{S}$ of bidegree $(1, 2)$ and we list some of its properties. In section 4, we give the normal form of this surface by means of two modulus and provide a nice tetrahedral geometric description of it. In section 5, we study the implicit equation of $\mathcal{S}$ and show that it is defined by the determinant of a structured matrix. In sections 6 and 7, we describe the singular locus of $\mathcal{S}$ and we end with an example in order to illustrate our results.

2 Surface of bidegree $(1, 2)$

The complex projective space of dimension $r \in \mathbb{N} \setminus \{0\}$ is denoted by $\mathbb{P}^r = \mathbb{P}^r(\mathbb{C})$.

Consider the map

$$\begin{array}{rcl} \Phi : \mathbb{P}^1 \times \mathbb{P}^1 & \rightarrow & \mathbb{P}^3 \\ \big((t:s), (u:v)\big) & \mapsto & \Phi(t:s; u:v) \end{array}$$

defined by four polynomials $\Phi_0, \Phi_1, \Phi_2, \Phi_3$, which are bihomogeneous of bidegree $(1, 2)$ (i.e. homogeneous in $(t : s)$ of degree 1 and homogeneous in $(u : v)$ of degree 2). In coordinates and with respect to the basis $\{tu^2, 2tuv, tv^2, su^2, 2suv, sv^2\}$ the map Φ writes:

$$\left\{ \begin{array}{l} X = a_1tu^2 + 2b_1tuv + c_1tv^2 + d_1su^2 + 2e_1suv + f_1sv^2 \\ Y = a_2tu^2 + 2b_2tuv + c_2tv^2 + d_2su^2 + 2e_2suv + f_2sv^2 \\ Z = a_3tu^2 + 2b_3tuv + c_3tv^2 + d_3su^2 + 2e_3suv + f_3sv^2 \\ T = a_4tu^2 + 2b_4tuv + c_4tv^2 + d_4su^2 + 2e_4suv + f_4sv^2 \end{array} \right. \tag{1}$$

where $a_i, b_i, c_i, d_i, e_i, f_i$ are complex numbers. To eliminate the trivial cases, we assume that the 4×6 matrix $\mathcal{A}_\Phi$ associated to Φ with entries in $\mathbb{C}$ has rank 4.

Definition 2.1 *The image of Φ in $\mathbb{P}^3$ is a surface $\mathcal{S}$ which is called a parametrized surface of bidegree $(1, 2)$.*

Lemma 2.2 *For a generic choice of entries of $\mathcal{A}_\Phi$ the parametrized surface $\mathcal{S}$ has degree four.*

This is a classical result, but let us give a quick constructive proof: We cut $\mathcal{S}$ by a generic line $\mathcal{L}$ whose equations are

$$\mathcal{L} : \left\{ \begin{array}{l} \alpha_1X + \beta_1Y + \gamma_1Z + \delta_1T = 0 \\ \alpha_2X + \beta_2Y + \gamma_2Z + \delta_2T = 0 \end{array} \right.$$

with $(\alpha_i, \beta_i, \gamma_i, \delta_i) \in \mathbb{C}^4 \setminus \{0\}$. Substituting (1) in $\mathcal{L}$, we get linear forms $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, E_1, E_2, F_1, F_2,$ in $a_i, b_i, c_i, d_i, e_i, f_i,$ with coefficients $\alpha_i, \beta_i, \gamma_i, \delta_i$ such that

$$\left\{ \begin{array}{l} (A_1u^2 + 2B_1uv + C_1v^2)t + (D_1u^2 + 2E_1uv + F_1v^2)s = 0 \\ (A_2u^2 + 2B_2uv + C_2v^2)t + (D_2u^2 + 2E_2uv + F_2v^2)s = 0. \end{array} \right.$$

Eliminating s and t from this system, we obtain the following homogeneous equation of degree four in u and v:

$$(A_1u^2 + 2B_1uv + C_1v^2)(D_2u^2 + 2E_2uv + F_2v^2) - \\ (D_1u^2 + 2E_1uv + F_1v^2)(A_2u^2 + 2B_2uv + C_2v^2) = 0,$$

which has generically four roots in $\mathbb{P}^1$. To each of them corresponds an intersection of the generic line $\mathcal{L}$ and the surface $\mathcal{S}$. Therefore $\mathcal{S}$ has degree four.

Remark 2.3 The lemma 2.2 implies that generically the surface $\mathcal{S}$ admits an implicit equation of degree four in the variables X, Y, Z, T. Theoretically there are several algorithms (resultants, Gröbner basis, moving surfaces, bezoutians, ...) to compute the implicit equation of $\mathcal{S}$. But under the form (1), this equation will be huge and without pertinent structure. So we will first perform linear changes of coordinates in $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^3$ before computing it.

When we restrict our setting to the affine case, we let $x = \frac{X}{T}, y = \frac{Y}{T}, z = \frac{Z}{T}$ and we consider them as functions of $\frac{t}{s}$ and $\frac{u}{v}$, that we rename simply t and u.

First geometric description:

For each fixed value of $(t : s)$ in $\mathbb{P}^1$, we obtain the parametrization in $(u : v)$ of a conic $\mathcal{C}_{(t:s)}$ in $\mathbb{P}^3$; it is included in a plane $\Pi_{(t:s)}$ which varies with $(t : s)$.

For $a_4 = b_4 = c_4 = d_4 = e_4 = 0$ and $f_4 = 1$ we get $T = sv^2$. Considering the affine case with $s = 1, v = 1$ and $T = 1$, we obtain a family of parabolas depending on t.

For each fixed value of $(u : v)$ in $\mathbb{P}^1$, we obtain the parametrization of a line $\mathcal{L}_{(u:v)}$ in $\mathbb{P}^3$.

Therefore the surface $\mathcal{S}$ can be considered either as a family of conics or as a family of lines in $\mathbb{P}^3$.

3 Planes tangent to all conics of the family

We consider a parametrized surface $\mathcal{S}$ of bidegree $(1, 2)$ in $\mathbb{P}^3$ defined as above. We would like to characterize the planes Π in $\mathbb{P}^3$ whose equation writes

$$\alpha X + \beta Y + \gamma Z + \delta T = 0 \tag{2}$$

with $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4 \setminus \{0\}$ such that Π is tangent to any curve $\mathcal{C}_{(t:s)}$ of $\mathcal{S}$ or contains it.

Substituting in (2) the expressions given in (1), we get six linear forms A, B, C, D, E, F in $(\alpha, \beta, \gamma, \delta)$ with coefficients $a_i, b_i, c_i, d_i, e_i, f_i, i = 1, 2, 3, 4$. The intersection of Π and $\mathcal{C}_{(t:s)}$ is given by the equation

$$(At + Bs)u^2 + 2(Ct + Ds)uv + (Et + Fs)v^2 = 0.$$

They are tangent (or $\mathcal{C}_{(t:s)}$ is contained in Π) if and only if the discriminant $(Ct + Ds)^2 - (At + Bs)(Et + Fs) = 0$. This is true for every $(t : s) \in \mathbb{P}^1$ if and only if the following conditions in $(\alpha, \beta, \gamma, \delta)$ are satisfied

$$C^2 = AE \quad , \quad 2CD = AF + BE \quad , \quad D^2 = BF.$$

From this, we obtain four simpler equations

$$C^2 = AE \ , \ D^2 = BF \ , \ CD = AF = BE \ ,$$

which can be expressed by a rank condition

$$\operatorname{rank} \begin{pmatrix} A & C & B & D \\ C & E & D & F \end{pmatrix} \leq 1. \tag{3}$$

In order to interpret this last condition, we introduce the classical rational scroll surface $\mathbb{F}(2, 2)$ in $\mathbb{P}^5$. It is a surface of degree four which has been extensively studied [9]. If we choose a system of coordinates $(w_1 : w_2 : w_3 : w_4 : w_5 : w_6)$ in $\mathbb{P}^5$, the surface $\mathbb{F}(2, 2)$ is given by

$$\operatorname{rank} \begin{pmatrix} w_1 & w_3 & w_2 & w_4 \\ w_3 & w_5 & w_4 & w_6 \end{pmatrix} \leq 1.$$

With this choice of coordinates in $\mathbb{P}^3$ and $\mathbb{P}^5$ the map

$$\begin{array}{rcl} \mathcal{B}_\Phi : \mathbb{P}^3 & \longrightarrow & \mathbb{P}^5 \\ (\alpha : \beta : \gamma : \delta) & \mapsto & (A : C : E : B : D : F) \end{array}$$

is defined by the transpose matrix of $\mathcal{A}_\phi$. For each Φ (with the condition that $\mathcal{A}_\phi$ is of maximal rank), the image of $\mathcal{B}_\Phi$ is a 3-projective plane in $\mathbb{P}^5$.

Definition 3.1 *Let $\mathcal{M}$ be the algebraic set of matrices with four rows and six columns of rank four. For any $\mathcal{A}$ in $\mathcal{M}$, the transpose matrix of $\mathcal{A}$ defines a 3-plane in $\mathbb{P}^5$ denoted by $\Pi_\mathcal{A}$. We define the following Zariski open subset*

$$\mathcal{U} = \{\mathcal{A} \in \mathcal{M} : \Pi_\mathcal{A} \text{ is transverse to } \mathbb{F}(2, 2) \text{ in } \mathbb{P}^5\}$$

of $\mathcal{M}$.

Proposition 3.2 *A matrix $\mathcal{A}_\Phi$ belongs to $\mathcal{U}$ if and only if the surface $\mathcal{S}$ of bidegree $(1,2)$ defined by $\mathcal{A}_\Phi$ admits four distinct planes $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ in $\mathbb{P}^3$ tangent to all conics $\mathcal{C}_{(t:s)}$ drawn on $\mathcal{S}$.*

PROOF Indeed the equations (2) and (3) defining such a plane (Π) in $\mathbb{P}^3$ correspond to the intersection of the image of $\mathcal{B}_\Phi$ with the scroll $\mathbb{F}(2,2)$ in $\mathbb{P}^5$.

The property for the four planes $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ to intersect is a closed condition. So there exists a Zariski open set $\mathcal{V}$ of $\mathcal{U}$ such that these planes do not intersect. In that case, we can choose these four planes as the planes of coordinates $(X=0)$, $(Y=0)$, $(Z=0)$, $(T=0)$ in $\mathbb{P}^3$.

Proposition 3.3 *Assume $\mathcal{A}_\Phi$ is in the Zariski open set $\mathcal{V}$ of $\mathcal{M}$. In the system of coordinates of $\mathbb{P}^3$ given by Π_1, Π_2,
Π_3, Π_4, the equations of $\mathcal{S}$ write*

$$\begin{cases} X = (a_1t + b_1s)(\lambda_1u + \mu_1v)^2 \\ Y = (a_2t + b_2s)(\lambda_2u + \mu_2v)^2 \\ Z = (a_3t + b_3s)(\lambda_3u + \mu_3v)^2 \\ T = (a_4t + b_4s)(\lambda_4u + \mu_4v)^2 \end{cases}$$

where $a_i, b_i, \lambda_i, \mu_i$ are complex numbers.

PROOF Let us prove it, for instance, for the expression of X. Indeed $X = 0$ must be the equation of a plane tangent to all curves $\mathcal{C}_{(t:s)}$ of $\mathcal{S}$. Substituting in (3), the coefficients $a_1, b_1, c_1, d_1, e_1, f_1$ in (1) must satisfy

$$b_1^2 = a_1c_1 \ , \ e_1^2 = d_1f_1 \ , \ c_1d_1 = a_1f_1 = b_1e_1.$$

Solving, we get a couple $(\lambda_1 : \mu_1)$ in $\mathbb{P}^1$ such that
$\lambda_1a_1 = \mu_1b_1, \lambda_1^2a_1 = \mu_1^2c_1, \lambda_1e_1 = \mu_1f_1, \lambda_1^2d_1 = \mu_1^2f_1$. So
$X = (a_1t + d_1s)(\lambda_1^2u^2 + 2\lambda_1\mu_1uv + \mu_1^2v^2)$, and we rename the coefficients.

4 Normal form of the surface $\mathcal{S}$

4.1 Normal form

In order to get a normal form of the equations of $\mathcal{S}$, we perform a projective change of coordinates in the two copies of $\mathbb{P}^1$ which, with the notation of the proposition 3.3, sends $(b_1, -a_1)$ and $(\mu_1, -\lambda_1)$ on $(0:1)$, $(b_2, -a_2)$ and $(\mu_2, -\lambda_2)$ on $(1:1)$, $(b_4, -a_4)$ and $(\mu_4, -\lambda_4)$ on $(1:0)$. Then $(b_3, -a_3)$ must be sent to the cross ration a of the four points

$$(b_1 : -a_1) \ , \ (b_2 : -a_2) \ , \ (b_3 : -a_3) \ , \ (b_4 : -a_4).$$

Similarly $(\lambda_3, -\mu_3)$ must be sent to the cross ratio b of the four points $(\lambda_1, -\mu_1), (\lambda_2, -\mu_2), (\lambda_3, -\mu_3), (\lambda_4, -\mu_4)$. Summarizing, we obtain the following result:

Theorem 4.1 *Assume $\mathcal{A}_\Phi$ is in the Zariski open set $\mathcal{V}$ of $\mathcal{M}$ (i.e. is "generic"), after a suitable change of coordinates the surface $\mathcal{S}$ admits the parametric representation:*

$$\begin{cases} X = tu^2 \\ Y = (t-s)(u-v)^2 \\ Z = (t-as)(u-bv)^2 \\ T = sv^2 \end{cases} \tag{4}$$

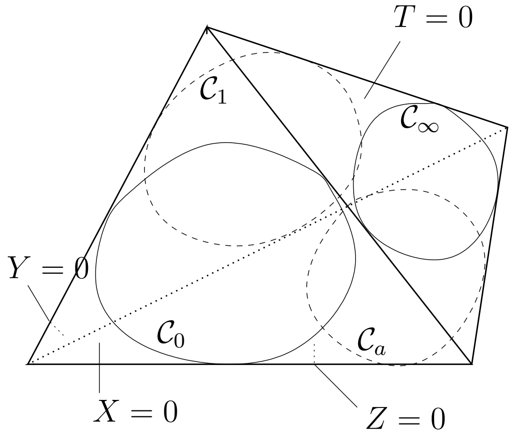
Remark 4.2 In the affine case $T = 1$ with $s = v = 1$, $\mathcal{S}$ is given by

$$x = tu^2 \ , \ y = (t-1)(u-1)^2 \ , \ z = (t-a)(u-b)^2. \tag{5}$$

4.2 Tetrahedron configuration

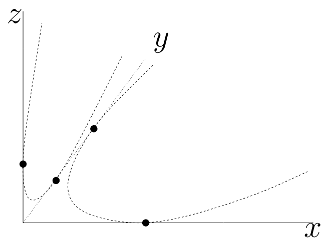
Let us consider the coordinate "tetrahedron" whose vertices are the four points $(0:0:0:1), (0:1:0:0), (0:0:1:0), (0:0:0:1)$, then each of its four faces $(X=0), (Y=0), (Z=0)$ and $(T=0)$ is tangent to all (except one) conic of the family $\mathcal{S}$ and contains a (distinguished) conic tangent to the three boarding edges. For instance for $(X=0)$, the lines D_{XY}, D_{XZ}, D_{XT} defined respectively by $(X=Y=0), (X=Z=0), (X=T=0)$. Therefore each of the six edges is tangent to two (distinguished) conics of the family. These four distinguished conics are $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_a, \mathcal{C}_\infty$. Each of the four faces contains a double line of the family, for example the plane $(X=0)$ contains the line $(u=0)$ whose parametric equation is $X=0, Y=t-s, Z=(t-as)b^2, T=s$. These four distinguished lines are $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_b, \mathcal{L}_\infty$. Moreover the point where one of the four lines cuts an edge is precisely a point where one of the four distinguished conics cuts that edge. On each edge of the tetrahedron we have four points: two vertices and two points tangent to a distinguished conic. So they define up to permutation a cross ratio. We can evaluate it as a function of a and b.

Proposition 4.3 *On the edge $D_{XY} : (X=Y=0)$ the cross ratio defined by the two vertices $(0:0:1:0), (0:0:0:1)$ and the two points of tangency $(0:0:-a(1-b)^2:1), (0:0:(1-a)b^2:1)$ is $(\frac{a}{a-1} : (\frac{b}{b-1})^2)$. Similarly the other five cross ratios are obtained by considering the action of permutations on X, Y, Z, T; we also get $(1-a : (1-b)^2), (\frac{1}{a} : \frac{1}{b^2}), (\frac{a-1}{a} : (\frac{b-1}{b})^2), (\frac{1}{1-a} : (\frac{1}{1-b})^2), (a : b^2)$.*



The configuration is completely defined by the tetrahedron and the $12 = 6 \times 2$ points of tangency on edges linked by the cross ratio relations given by the previous proposition. They also characterize the representation (4). See the last section for a complementary description on an example.

Moreover the family $\mathcal{S}$ is completely defined by two (distinguished) conics tangent to the same edge and their tangencies. In the affine setting this corresponds to the configuration with two parabolas shown on the following picture.



4.3 Lines tangent to conics of the family

With respect to the parametric normal form equations of $\mathcal{S}$, the line $\mathcal{L}$ tangent to $\mathcal{C}_{(t:s)}$ at the point $M(t:s;u:v)$ admits the following parametric equations with $(\lambda:\mu) \in \mathbb{P}^1$:

$X = \lambda tu, Y = (\lambda-\mu)(t-s)(u-v),$
$Z = (\lambda - b\mu)(t-as)(u-bv),\ T = \mu sv.$

In the previous description of the tetrahedral configuration attached to the surface $\mathcal{S}$, one can see that each of the six coordinate edges $D_{XY}, D_{XZ}, D_{XT}, D_{YZ}, D_{YT}, D_{ZT}$ is tangent to two conics of the family. For instance D_{XY} is tangent to $\mathcal{C}_0$ at

$M(0:1;1:1)$ and is tangent to $\mathcal{C}_1$ at $M(1:1;0:1)$. A natural question is: Are there other lines tangent to two conics of the family? For a generic surface $\mathcal{S}$, the answer is no.

Proposition 4.4 *Let $\mathcal{S} = \cup\, \mathcal{C}_{(t:s)}$ be a generic parametrized surface in $\mathbb{P}^3$ of bidegree $(1,2)$. There exist six and only six lines tangent to two conics of the family. Moreover they form a tetrahedron.*

PROOF We check this property by a formal computation. Because of the symmetry of the question, we can restrict ourselves to the affine setting and let $s = v = T = 1$. This will simplify the notations. We have to prove that there are only three such lines at finite distance.

The tangent lines at $M(t_1;u_1)$ to $\mathcal{C}_{t_1}$ and at $M(t_2;u_2)$ to $\mathcal{C}_{t_2}$ coincide if and only if the following 3×3 matrix

$$\begin{pmatrix} t_1u_1 & t_2u_2 & t_1u_1^2 - t_2u_2^2 \\ (t_1-1)(u_1-1) & (t_2-1)(u_2-1) & \alpha_{23} \\ (t_1-a)(u_1-b) & (t_2-a)(u_2-b) & \alpha_{33} \end{pmatrix},$$

with $\alpha_{23} = (t_1-1)(u_1-1)^2 - (t_2-1)(u_2-1)^2$ and $\alpha_{33} = (t_1-a)(u_1-b)^2 - (t_2-a)(u_2-b)^2$, has rank ≤ 1. By standard manipulations in linear algebra, this condition decomposes into three conditions:

$$\begin{cases} t_1u_1 = t_2u_2 = 0 \\ (t_1-1)(u_1-1) = (t_2-1)(u_2-1) = 0 \end{cases}$$

or

$$\begin{cases} t_1u_1 = t_2u_2 = 0 \\ (t_1-a)(u_1-b) = (t_2-a)(u_2-b) = 0 \end{cases}$$

or

$$\begin{cases} (t_1-1)(u_1-1) = (t_2-1)(u_2-1) = 0 \\ (t_1-a)(u_1-b) = (t_2-a)(u_2-b) = 0. \end{cases}$$

They correspond to the lines D_{XY}, D_{XZ} and D_{YZ}. Then we are done.

This last result can be proved directly using general arguments from "modern" algebraic geometry [10].

5 Implicit equation of $\mathcal{S}$

With the result of the subsection 4.1, we can compute the implicit equation of the normal form (5) of the parametric representation of the surface $\mathcal{S}$.

This amounts to eliminate t and u in (5). One could replace t by $\frac{x}{u^2}$ and substitute in the two last equations in (5), we obtain

$$\begin{cases} f := u^4 - 2u^3 + (y - x + 1)u^2 + 2xu - x \\ g := au^4 - 2abu^3 + (z - x + b^2 a)u^2 + 2xbu - b^2 x. \end{cases}$$

Then one can eliminate u from f and g by computing a classical Sylvester resultant. In this case, the Sylvester matrix is of size 8×8 and its determinant is a polynomial of degree six in the variables x, y, z. But we aim to find a polynomial of degree four as we have seen in remark 2.3. We could use a computer algebra system to factorize this polynomial of degree six in order to find the implicit equation of $\mathcal{S}$ we are looking for. However this would give a huge expression made by several monomials without apparent structure.

We prefer to use another matrix formulation to compute the resultant of f and g, and also to have a more compact matrix for the implicit equation. We rely on the Bezout matrix B (which is of size 4×4) to express the resultant of f and g. Its determinant is equal to the determinant of Sylvester matrix. But we notice that we can factorize by x in the first line and in the second line (or in the third and fourth column) of the matrix B. Consequently the implicit equation of the surface $\mathcal{S}$ is given by the determinant of the following matrix $M = (m_{ij})_{1 \leq i,j \leq 4}$:

$$\begin{pmatrix} a - b^2 & 2b(b-a) & m_{13} & 2bx(1-b) \\ 2(b-a) & -b^2 - 4b + a + 4ba & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & 2bx(b-a) \\ 2a(1-b) & m_{42} & 2x(b-a) & x(a-b^2) \end{pmatrix}$$

where $m_{13} = m_{24} = -b^2 y + b^2 x - b^2 + z - x + b^2 a$,
$m_{23} = 2(b^2 + by - bx + b - ba - z + x - b^2 a)$,
$m_{31} = m_{42} = z - x + b^2 a - ay + ax - a$,
$m_{32} = 2(bx - z + x - b^2 a - ax + bay - bax + ba)$,
$m_{33} = x(-b^2 - 4b + a + 4ba) = x m_{22}$.

We can also obtain the matrix M directly by the computation of the Bezoutian of

$$x - tu^2\ ,\ y - (t-1)(u-1)^2\ ,\ z - (t-a)(u-b)^2$$

as polynomials in t and u without simplification by x^2 as in the univariate case. See [5] for more details on this method of elimination.

In other words, the (homogeneous) implicit equation of the surface $\mathcal{S}$ is

$$F(X, Y, Z, T) = \det \begin{pmatrix} T\,\mathtt{A} & \mathtt{B} \\ \mathtt{C} & X\,\mathtt{D} \end{pmatrix},$$

where $\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}$ are the following matrices:

$$\mathtt{A} = \begin{pmatrix} a-b^2 & 2b(b-a) \\ 2(b-a) & -b^2-4b+a+4ba \end{pmatrix},$$

$$\mathtt{D} = \begin{pmatrix} -b^2-4b+a+4ba & 2b(b-a) \\ 2(b-a) & a-b^2 \end{pmatrix},$$

$$\mathtt{B} = \begin{pmatrix} b_{11} & 2b(1-b)X \\ b_{21} & b_{11} \end{pmatrix},$$

with $b_{11} = -b^2Y + (b^2-1)X + Z + b^2(a-1)T$, and $b_{21} = 2[(b^2+b-ba-b^2a)T + bY + (1-b)X - Z]$,

$$\mathtt{C} = \begin{pmatrix} c_{11} & c_{12} \\ 2a(1-b)T & c_{11} \end{pmatrix},$$

with $c_{11} = Z + a(b^2-1)T - aY + (a-1)X$, and $c_{12} = 2[(b-a-ba+1)X - Z + baY + ba(1-b)T]$.

Let us note that in the plane $X = 0$, the implicit equation of $\mathcal{S}$ is $F(0,Y,Z,T) = -\det(B)|_{X=0}\det(C)|_{X=0}$, where $\det(B)|_{X=0} = b_{11}^2|_{X=0} = \big(-b^2Y+Z+b^2(a-1)T\big)^2$ is the equation of the double line $\mathcal{L}_0$, and $\det(C)|_{X=0} = Z^2 + a^2Y^2 + (a^2+6a^2b^2-4a^2b+a^2b^4-4a^2b^3)T^2 + (2a+2ab^2-4ab)TZ + (2a^2+2a^2b^2-4a^2b)TY - 2aZY$ is the equation of the conic $\mathcal{C}_0$. We have a similar remark for $T = 0$.

6 Singularities and double point locus of $\mathcal{S}$

Once we have the implicit equation $F(X,Y,Z,T)$ of $\mathcal{S}$, its singular locus is defined by the set of equations which expresses that all the partial derivatives of F vanish. Since the surface $\mathcal{S}$ is also given by a parametrization we can substitute the variables X, Y, Z, T by the functions given in (4). Then we get four polynomial equations on $\mathbb{P}^1 \times \mathbb{P}^1$.

We will perform this computation below and solve that system of four equations in the case of a generic normal form of the parametrization Φ of bidegree $(1, 2)$.

Another (direct) approach would be to note that a singular point $M(t,u)$ of $\mathcal{S}$ can be detected either locally with respect to the parametrization, i.e. the cross-product in an affine chart, say $s = v = T = 1$, $\frac{\partial M}{\partial t}(t,u) \wedge \frac{\partial M}{\partial u}(t,u)$ must vanish ; or it corresponds to a double point of the parametrization, i.e. $\Phi(s_1 : t_1; u_1 : v_1) = \Phi(s_2 : t_2; u_2 : v_2)$ with $(s_1 : t_1; u_1 : v_1) \neq (s_2 : t_2; u_2 : v_2)$.

In our case, the computation will show that there is a curve of singular points in $\mathcal{S}$ but only four points which correspond to local singularities of the parametrization. Moreover the double-point locus is dense in the singular locus, because these four points belong to the closure of the double-point locus.

6.1 Equation of the singular locus in the parameter space

If we substitute X, Y, Z, T in the partial derivatives of the implicit equation of the surface $\mathcal{S}$ with their expressions given in (4), we get four bihomogeneous polynomials of bidegree $(3, 6)$ in $(t : s)$ and $(u : v)$. Each of these four polynomials decomposes into five factors: Three of bidegree $(0, 1)$, one of bidegree $(1, 1)$ and one of bidegree $(2, 2)$. Their gcd $\mathcal{C}$ is the following factor

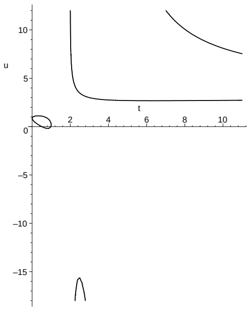
$$\begin{aligned}
&(-4b^2a^2 + 4ba^2 + 8b^3a - 8b^2a - 4b^4 + 4b^3)t^2u^2 + (-3ba^2 + 4b^2a + 10b^2a^2 \\
&+3b^4a - a^3 + b^5 - 10b^3a - 4b^3a^2)tu^2s + (-2b^2a^2 + a^3 - ba^3 + b^4a + 2b^3a^2 \\
&-b^5a)u^2s^2 + (b^6 + 3b^5a - 10b^4a - 4b^4a^2 + 4b^3a + 10b^3a^2 - 3b^2a^2 - \\
&ba^3)stv^2 + (8b^4a^2 + 2ba^3 + 8b^4a - 12b^3a^2 - 8b^2a - 2b^5 + 6ba^2 + 12b^3a - \\
&6b^5a - 8b^2a^2)sutv + (-8b^3a^2 + 4b^2a^3 - 4b^5a + 4b^4a - 4b^3a^3 + 8b^4a^2)s^2v^2 \\
&+(-4b^4a^2 + 4b^2a^2 + 4b^5a + 4b^2a^3 - 4b^4a - 4ba^3)us^2v + (4b^2a^2 + 4b^5 - 4b^4 \\
&+4b^2a - 4b^4a - 4ba^2)vut^2 + (-b^6 - 2b^3a + 2b^4a - b^2a^2 + b^5 + ba^2)v^2t^2.
\end{aligned}$$

It corresponds to a curve of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The other factors give rise to four other singular points which are embedded in $\mathcal{C}$ as we will see.

For a better illustration, we specialize $\mathcal{C}$ for $a = 2$ and $b = 3$ which are "generic" values for this study. We obtain

$$\begin{aligned}
-24t^2u^2 + 145tu^2s - 196u^2s^2 + 435stv^2 - 474sutv - \\
144s^2v^2 + 336us^2v + 168vut^2 - 294v^2t^2.
\end{aligned}$$

Using a plotting device we can display the curve $\mathcal{C}$, for that case, in the affine chart $s = v = 1$.



6.2 Local singular points

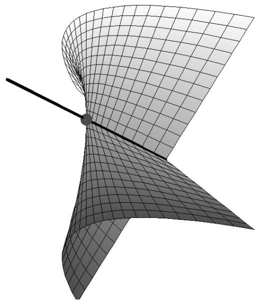
Let us consider the points $M(t,u)$ at "finite distance" (i.e. $s=1, v=1, T=1$) of $\mathcal{S}$. Then $\frac{\partial M}{\partial t} = \big(u^2, (u-1)^2, (u-b)^2\big)$ and $\frac{1}{2}\frac{\partial M}{\partial u} = \big(tu, (t-1)(u-1), (t-a)(u-b)\big)$. So the vector $\frac{1}{2}\frac{\partial M}{\partial t} \wedge \frac{\partial M}{\partial u} = \big((u-1)(u-b)[(1-a)u+(b-1)t+a-b], u(u-b)(au-bt), u(u-1)(t-u)\big)$. It vanishes for the following four values (when a and b are generic):

$u=0$ and $t=\frac{b-a}{b-1}$, $u=1$ and $t=\frac{a}{b}$, $u=b$ and $t=b$.

Computing in another chart we also obtain a point at infinity: $u=\infty$ (i.e. $v=0$) and $t=a\frac{b-1}{b-a}$.

Therefore the singular locus of $\mathcal{S}$ contains four isolated points which correspond to singularities which can be detected locally to the parametrization. The other singular points correspond to double points.

Using the equation of the curve $\mathcal{C}$ given in the subsection 6.1, we check that these four points belong to $\mathcal{C}$.



The singularities near by each of these four points are locally similar to so called Whitney umbrella surface singularity (see the previous picture). More precisely, we will see that the singular locus curve is a twisted cubic $\mathcal{F}$. Moreover, the surface $\mathcal{S}$ is the union of a family of line $\mathcal{L}_u$ joining two points of $\mathcal{F}$. At each point of $\mathcal{F}$ two lines $\mathcal{L}_{u_1}$ and $\mathcal{L}_{u_2}$ intersect with $\mathcal{F}$ except at the four special points where only a (double) line intersect $\mathcal{F}$. Locally this gives rise to a Whitney singularity.

6.3 Double points

The double points of the parametrization of $\mathcal{S}$ at "finite distance" are computed by solving a system of three equations and four unknowns (corresponding to the two couples of parameters). In order to also study the natural involution between these two couples of parameters we denote them by $(\sigma+\mu, k+l)$ and $(\sigma-\mu, k-l)$, where (σ, k) are the coordinates of their middle. Now the system of three equations is the following

$$\begin{cases} (\sigma+\mu)(k+l)^2-(\sigma-\mu)(k-l)^2=0 \\ (\sigma+\mu-1)(k+l-1)^2-(\sigma-\mu-1)(k-l-1)^2=0 \\ (\sigma+\mu-a)(k+l-b)^2-(\sigma-\mu-a)(k-l-b)^2=0. \end{cases}$$

From this, we deduce a bidegree $(1,2)$ relation between σ and k, and two simple relations giving l^2 and μ:

$$\begin{cases} 2(b-a)k^2+(2ab+a-b^2-2b)k+(b-b^2)\sigma+b^2-ab=0 \\ l^2(\sigma+k-1)+k(\sigma+k^2-\sigma k-k)=0 \\ \mu=\frac{2l(\sigma+k-1)}{2k-1}. \end{cases}$$

This provides an interesting representation of the curve $\mathcal{C}$ defined by the implicit equation of bidegree $(2,2)$ given in subsection 6.1. It allows to visualize (for instance for the curve corresponding to the values $a=2, b=3$) the natural involution. The plotting is also more precise because it is guided by a parametrization (even if it is not rational).

6.4 Image of the double-point locus

The image of the curve $\mathcal{C}$ by Φ in $\mathbb{P}^3$ is an algebraic curve that we denote by $\mathcal{F}$. In order to compute the degree of $\mathcal{F}$ we cut it by a generic plane Π. The inverse image of $\mathcal{S}\cap\Pi$ in $\mathbb{P}^1\times\mathbb{P}^1$ by Φ is a generic curve of bidegree $(1,2)$, which intersects $\mathcal{C}$ in $1\times 2+2\times 2=6$ points by a generalized Bezout theorem [13]. As each point in $\mathcal{F}$ corresponds to two couples of parameters on $\mathcal{C}$, we deduce that generically $\Pi\cap\mathcal{F}$ is made of three points. Therefore $\mathcal{F}$ is a cubic curve in $\mathbb{P}^3$ drawn on the surface $\mathcal{S}$. In order to better understand the geometry of the situation, we aim to recognize the intersection of $\mathcal{F}$ with a plane Π_{t_0} containing a conic $\mathcal{C}_{t_0}$ of the family of conics which describes $\mathcal{S}$.

An easy computation provides the following equation for Π_{t_0} for a generic t_0: $\frac{(1-b)x}{t_0}+\frac{by}{t_0-1}-\frac{z}{t_0-a}-b+b^2=0$. Intersecting Π_{t_0} with the surface $\mathcal{S}$ we obtain the curve $\mathcal{C}_{t_0}$ and two lines, that we denote by $\mathcal{L}_{u_1}$ and $\mathcal{L}_{u_2}$, in the family of lines which describes $\mathcal{S}$. The values u_1 and u_2 are the two solutions of the following equation

$$\begin{aligned}[(b-a)t_0+a(1-b)]u^2+2b(a-1)t_0u+(b-b^2)t_0^2 \\ +(b-a)bt_0=0. \end{aligned} \tag{6}$$

obtained by substitution of (5) in the equation of Π_{t_0}. We now show that generically the intersection of $\mathcal{F}$ with Π_{t_0} consists of the following points:

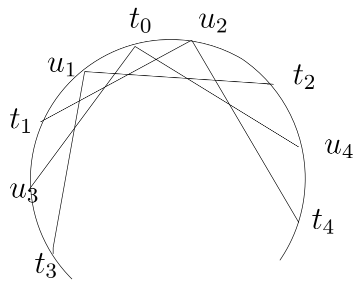
– the intersection point between $\mathcal{L}_{u_1}$ and $\mathcal{L}_{u_2}$

– one of the intersection points between $\mathcal{L}_{u_1}$ and $\mathcal{C}_{t_0}$
– one of the intersection points between $\mathcal{L}_{u_2}$ and $\mathcal{C}_{t_0}$.

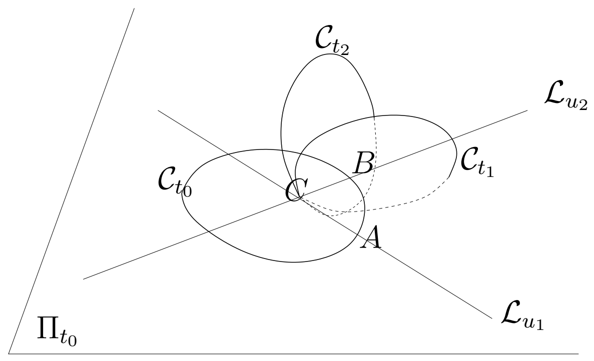
Indeed the intersection point C of $\mathcal{L}_{u_1}$ and $\mathcal{L}_{u_2}$ corresponds to a double point of the parametrization because it is obtained by two couples of parameters $(t_1, u_1) \neq (t_2, u_2)$ for some t_1 and t_2. One of the two intersection points between $\mathcal{L}_{u_1}$ and $\mathcal{C}_{t_0}$ is obtained for the couple (t_0, u_1). The other intersection point A must correspond to another couple, say (t_0, u_3) on $\mathcal{C}_{t_0}$ and say (t_3, u_1) on $\mathcal{L}_{u_1}$; therefore it is a double point of the parametrization. Similarly the "other" intersection point B between $\mathcal{L}_{u_2}$ and $\mathcal{C}_{t_0}$ is obtained for two couples of parameters (t_0, u_4) and (t_4, u_2). As the double-point locus, $\mathcal{F}$ is a curve of degree three, these points are the only three points of $\mathcal{F}$ in Π_{t_0}.

We remark that the lines $\mathcal{L}_{u_1}$ and $\mathcal{L}_{u_3}$ intersect. So they are in a same plane which intersects the surface $\mathcal{S}$ also along a conic ; it is $\mathcal{C}_{t_1}$ and the plane is Π_{t_1}. Similarly $\mathcal{L}_{u_2}$ and $\mathcal{L}_{u_4}$ are included in Π_{t_2}. And similarly $\mathcal{L}_{u_3}$ is included in Π_{t_3} and $\mathcal{L}_{u_4}$ in Π_{t_4}. To summarize these inclusions we draw the following circular diagram where two successive values t and u (or u and t) satisfy the equation (6).

Moreover two values linked by a segment correspond to a couple of double points of the parametrization: These segments jump over two values of the diagram.



To give an idea of the geometry near by the singular locus, let us picture the three conics $\mathcal{C}_{t_0}, \mathcal{C}_{t_1}, \mathcal{C}_{t_2}$, the lines $\mathcal{L}_{u_1}, \mathcal{L}_{u_2}, \mathcal{L}_{u_3}, \mathcal{L}_{u_4}$ and their intersection points. We pointed out the singularities A, B, C.



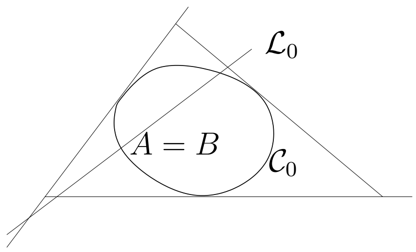
7 Parametrization of the singular locus $\mathcal{F}$

7.1 Implicit equation of $\mathcal{F}$

Once we have the implicit equation $F(X, Y, Z, T)$ of $\mathcal{S}$, the implicit equations of the singular locus $\mathcal{F}$ is given by the vanishing of the derivatives $\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z}, \frac{\partial F}{\partial T}$ which define a homogeneous ideal I in the ring $\mathbb{C}[X, Y, Z, T]$.

Using a computer algebra system with a Groebner package, one can compute a Groebner basis and then we check that, for generic values of the modulus a and b, $\mathcal{F}$ is a curve and has degree three. This is done by computing the Hilbert polynomial of the quotient algebra $\mathbb{C}[X, Y, Z, T]/I$. A primary decomposition program provides the defining ideal of $\mathcal{F}$. We do not display these expressions because they are huge. We will give the results in the subsection 8, for $a = 2$ and $b = 3$.

As $\mathcal{F}$ is a twisted cubic, it admits a parametrization $\theta : \mathbb{P}^1 \to \mathbb{P}^3$ of degree three. We can interpolate this parametrization by choosing special points in $\mathcal{F}$. A natural choice is to take them among the intersection points of $\mathcal{F}$ with the four coordinates planes $(X = 0), (Y = 0), (Z = 0), (T = 0)$. We recall that the intersection of $\mathcal{S}$ with one of these four planes consists of the union of a conic (tangent to the coordinate lines at three distinguished points of the tetrahedron configuration) with a double line (passing by three other distinguished points of the tetrahedron configuration). For example for $(X = 0)$ we have $\mathcal{S} \cap \Pi_0 = \mathcal{C}_0 \cup \mathcal{L}_0$,



$\mathcal{C}_0$ and $\mathcal{L}_0$ intersect at two points, one corresponding to the couple $(t = 0, u = 0)$ is not on $\mathcal{F}$, the other one corresponds to the collapse of the two points A and B described in subsection 6.4 belongs to $\mathcal{F}$. Moreover $\mathcal{F}$ is tangent to $\mathcal{C}_0$ at that point $A = B$. Therefore the coordinates of this point (and of its tangent) are easily computable, they are rational functions of a and b. Together with similar informations attached to the three other coordinate planes this is enough to characterize $\mathcal{F}$ and compute its parametrization. We illustrate below this process in the case where $a = 2$ and $b = 3$.

7.2 Parametrized equations of $\mathcal{F}$

We compute the intersection point $C(x,y,z)$ of the two lines $\mathcal{L}_{u_1} : x = u_1^2 t, y = (u_1-1)^2(t-1), z = (u_1-b)^2(t-a)$ and $\mathcal{L}_{u_2} : x = u_2^2 t, y = (u_2-1)^2(t-1), z = (u_2-b)^2(t-a)$, where u_1 and u_2 are the solutions of the equation (6). So we have $u_1+u_2 = -\frac{2b(a-1)t}{(b-a)t+a(1-b)}$ and $u_1u_2 = \frac{(b-b^2)t^2+(b-a)bt}{(b-a)t+a(1-b)}$. If we substitute $t = \frac{x}{u_1^2}$ (resp. $t = \frac{x}{u_2^2}$) in the other equations of the parameterizations of $\mathcal{L}_{u_1}$ (resp. $\mathcal{L}_{u_2}$), we obtain

$$\begin{array}{rcl} u_1^2 y = (u_1-1)^2(x-u_1^2) & , & u_1^2 z = (u_1-b)^2(x-au_1^2), \\ u_2^2 y = (u_2-1)^2(x-u_2^2) & , & u_2^2 z = (u_2-b)^2(x-au_2^2). \end{array}$$

From this, we deduce that

$$x = \frac{(u_1u_2)^2(u_1+u_2-2)}{2u_1u_2-(u_1+u_2)}, y = \frac{(u_1+u_2)\Big(u_1u_2-(u_1+u_2)+1\Big)^2}{2u_1u_2-(u_1+u_2)}, \text{ and } z = \frac{a(u_1+u_2)\Big(u_1u_2-b(u_1+u_2)+b^2\Big)^2}{b\Big(2u_1u_2-b(u_1+u_2)\Big)}.$$

In this way we get the following affine parametrization of the curve $\mathcal{F}$:

$$\left\{ \begin{array}{c} x = \frac{abt(-t+tb-b+a)^2}{(-tb+ta-a+ba)^2} \\ y = \frac{(a-1)(-tb+a)(t^2b-t^2b^2+tb^2+tba-tb-ta+a-ba)}{(-tb+ta-a+ba)^2} \\ z = -\frac{a(a-1)b(b-t)(-t^2+t^2b+tb+ta-tba-tb^2-ba+b^2a)}{(-tb+ta-a+ba)^2} \end{array} \right.$$

8 Example

We choose the couple of "generic" modulus $a = 2, b = 3$, and we present the details of some constructions and equations in this case. The surface $\mathcal{S}$ is defined by the parametrization

$$X = tu^2, Y = (t-s)(u-v)^2, Z = (t-2s)(u-3v)^2, T = sv^2.$$

It intersects at twelve points denoted by $P_1, \ldots, P_{12}$ the edges of the canonical tetrahedron.
On D_{XY} :
$P_1 = \{t = 0, u = v, X = Y = 0, Z = -8, T = 1\}$,
$P_2 = \{t = s, u = 0, X = Y = 0, Z = -9, T = 1\}$,
on D_{XZ} :
$P_3 = \{t = 0, u = 3v, X = Z = 0, Y = -4, T = 1\}$,
$P_4 := \{t = 2s, u = 0, X = Z = 0, Y = 1, T = 1\}$,
on D_{XT} :
$P_5 = \{t = 0, v = 0, X = T = 0, Y = 1, Z = 2\}$,
$P_6 = \{s = 0, u = 0, X = T = 0, Y = 1, Z = 9\}$,
on D_{YZ} :

$P_7 = \{t = s, u = 3v, Y = Z = 0, X = 1, T = 1\}$,
$P_8 = \{t = 2s, u = v, Y = Z = 0, X = 2, T = 1\}$,
on D_{YT} :
$P_9 = \{t = s, v = 0, Y = T = 0, X = 1, Z = -1\}$,
$P_{10} = \{s = 0, u = v, Y = T = 0, X = 1, Z = 4\}$,
on D_{ZT} :
$P_{11} = \{t = 2s, v = 0, Z = T = 0, X = 2, Y = 1\}$,
$P_{12} = \{s = 0, u = 3v, Z = T = 0, X = 9, Y = 4\}$.

The conic $\mathcal{C}_0$ is in the plane $(X = 0)$, it is tangent at P_1 to D_{XY} at P_3 to D_{XZ}, at P_5 to D_{XT}. The conic $\mathcal{C}_1$ is in the plane $(Y = 0)$, it is tangent at P_2 to D_{XY}, at P_7 to D_{YZ}, at P_9 to D_{YT}. And so on.

The line $\mathcal{L}_0$ is included in the plane $(X = 0)$, it cuts the edges D_{XY} at P_2, D_{XZ} at P_4, D_{XT} at P_6. The line $\mathcal{L}_1$ is included in the plane $(Y = 0)$, it cuts the edges D_{XY} at P_1, D_{YZ} at P_8, D_{YT} at P_{10}. And so on.

Remark 8.1 It is easy to check that with these eight relations of "cocyclicity" and collinearity, one can recover the twelve points from the tetrahedron and five points, say P_1, P_2,
P_3, P_7, P_8. This is true, in general, for a and b generic and shows once again that the configuration is defined by the vertices of the tetrahedron plus three points (e.g. P_8, P_3, P_1) which correspond to a graduation on the coordinates plus two points which correspond to the needed modulus.

Each coordinate plane contains a conic and a line which intersect at two points. One of them is the expected one with parameter t corresponding to the line ; for instance for $(X = 0)$ we get $\{t = 0, u = 0\}$. The other one is a (double) point of the singular locus $\mathcal{F}$. For $(X = 0)$, we get $M_x = (X = 0, Y = -1, Z = -450, T = 49)$. For $(Y = 0)$, we get $M_y = (X = 1, Y = 0, Z = -196, T = 25)$. For $(Z = 0)$, we get $M_z = (X = 450, Y = 196, Z = 0, T = 1)$. For $(T = 0)$, we get $M_t = (X = 49, Y = 25, Z = 1, T = 0)$. By interpolation we obtain a parametric equation of the twisted cubic $\mathcal{F}$:

$$\left\{\begin{array}{c} X = (\frac{3\tau}{250} + \frac{3\sigma}{100})\tau^2 \\ Y = (\frac{3\tau}{490} - \frac{\sigma}{49})(\tau - \sigma)^2 \\ Z = (\frac{3\tau}{12250} - \frac{9\sigma}{2450})(\tau - 50\sigma)^2 \\ T = (\frac{\tau}{20} + \sigma)\sigma^2. \end{array}\right.$$

Moreover we verify easily that the tangent to $\mathcal{F}$ at the point M_x is the tangent to $\mathcal{C}_0$ at M_x, similarly for $\mathcal{C}_1$ at M_y, $\mathcal{C}_a$ at M_z, $\mathcal{C}_\infty$ at M_t.

On the other hand, we compute the implicit equation F of $\mathcal{S}$ given in the section 5. The four derivatives of F span an homogeneous ideal corresponding to some points plus a curve of degree three (its Hilbert polynomial is $5+3n$).

Via Gröbner bases computations, we get rid of the extra points and obtain three quadratic equations:

$$\begin{aligned}
Q_1 &= -20066TZ + 10528XT - 24912YT + 2661YZ \\
&\qquad -1410XZ - 85344T^2 + 6Y^2 - 1185Z^2, \\
Q_2 &= XY - 495XZ + 938YZ - 420Z^2 + 3752XT \\
&\qquad -8910YT - 7111TZ - 30240T^2, \\
Q_3 &= 2X^2 - 2076XZ + 3948YZ - 1778Z^2 + 15966XT \\
&\qquad -38070YT - 30099TZ - 127980T^2.
\end{aligned}$$

An easy computation allows to verify that, as expected, they define the same curve $\mathcal{F}$ as the two parameterizations given above.

8.1 Inversion of the parametrization

On the surface $\mathcal{S}$ and outside the singular locus, the map Φ can be inverted. Moreover the inverse map Ψ of Φ admits a rational representation. In the affine setting, it writes

$$(x, y, z) \in \mathcal{S} \setminus \mathcal{F} \mapsto (t, u) = \Psi(x, y, z).$$

This rational representation is not unique: adding a multiple of the implicit equation of $\mathcal{S}$ does not change its value. We computed "by hand" such a map Ψ by first substituting t by $\frac{x}{u^2}$ in $y(t, u) - y$ and $z(t, u) - z$ and then applying Euclidean division with respect to u. This naive method gave us rational functions of degree three and four in x, y, z. However this inversion procedure for parameterizations has been studied by several authors [11, 3]. We apply the program described in [2] and get the following shorter rational functions of degree two and three. We have $\Psi(x, y, z) = (t, u) = (\frac{t_1}{t_2}, \frac{u_1}{u_2})$, where

$$\begin{aligned}
t_1 &= 768 + z^3 - 12y^3 + 16zy^2 - 7z^2y - 64zy + 52x + \\
&\qquad 192y + 256z + 68xy - 26x^2 + 38xz + 2x^3 - 11x^2y + \\
&\qquad 5x^2z - 18xzy + 4xz^2 + 20xy^2 + 28z^2 - 48y^2 \\
t_2 &= -38x + 444 + 103z - 18y + 6z^2 + 12xz - 24zy + 6x^2 \\
&\qquad -24xy + 24y^2 \\
u_1 &= 40x - 702y + 127z - 77zy + 39xz - 127xy + 32x^2 + \\
&\qquad 126y^2 + 7z^2 + 576 \\
u_2 &= -76x + 888 + 206z - 36y + 12z^2 + 24xz - 48zy + 12x^2 \\
&\qquad -48xy + 48y^2.
\end{aligned}$$

9 Conclusion

In this paper we started the study of parametrized surfaces of bidegree $(1,2)$. We described in detail the generic projective complex case thanks to a simple normal form and a rich tetrahedral configuration naturally attached to such a surface. We proved that the singular locus is a twisted cubic (that we were able to construct from the tetrahedral configuration) with four distinguished points corresponding to Whitney umbrella local singularities. In a future work we will complete this study by describing non generic cases and the real classification. We will also provide a nice gallery of models useful for Computer Aided Geometric Design, together with fast routines to describe the surface, its singular locus and the natural involution on its double point locus.

Acknowledgment: We thank Christian Pauly for fruitful discussions and suggestions.

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