

Rigorous numerical evaluation of D-finite functions in SageMath

Marc Mezzarobba
CNRS, LIP6, Université Paris 6

Séminaire AriC, LIP, ENS Lyon
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[arxiv:1607.01967](https://arxiv.org/abs/1607.01967) [cs.SC]

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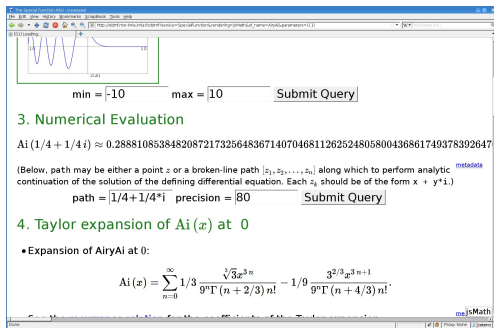
Outline

1. What it IS

2. What it DOES

3. (How it WORKS)

What it Is: A Successor for NumGfun



The screenshot shows a web browser window with the URL <http://ddmf.msr-inria.inria.fr>. The page content includes:

- A plot of the Airy function $Ai(x)$ for x between -10 and 10. The plot shows oscillations for negative x and a smooth decay for positive x .
- Input fields for `min = -10` and `max = 10`, followed by a `Submit Query` button.
- Section **3. Numerical Evaluation**:
 $Ai(1/4 + 1/4i) \approx 0.28881085384820872173256483671407046811262524805800436861749378302647$
A note below states: "(Below, path may be either a point z or a broken-line path $[z_1, z_2, \dots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form $x + y^i$.)" [metadata](#)
Input fields for `path = [1/4+1/4*i]` and `precision = 80`, followed by a `Submit Query` button.
- Section **4. Taylor expansion of $Ai(x)$ at 0**:
• Expansion of AiryAi at 0:
$$Ai(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt[3]{3} x^{3n}}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^{3n+1}}{9^n \Gamma(n + 4/3) n!}$$

The footer of the page includes the logo for `msjMath`.

NumGfun [M. 2010]

- ▶ General D-finite functions
- ▶ Arbitrary precision
- ▶ Rigorous error bounds

- ▶ Maple
- ▶ Oriented towards special functions

<http://ddmf.msr-inria.inria.fr>

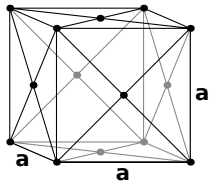
[Benoit, Chyzak, Darrasse, Gerhold, Grégoire, Koutschan, M., Salvy 2010–]

Airy function: $Ai''(z) - z Ai(z) = 0$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$\begin{aligned} \text{dop6} = & 410085196915322880 z^{35} + 112905266474211563520 z^{34} + 1171669263761496 \backslash \\ & 1489920 z^{33} + 690817401287078917363200 z^{32} + 27204862643846611522761600 z^{31} + 7788 \backslash \\ & 11406918247228618497600 z^{30} + 17044384124115240781429792800 z^{29} + 29424523406685 \backslash \\ & 0000428339092800 z^{28} + 4083424587805117060272476125800 z^{27} + 4597302954911979623 \backslash \\ & 5386142827300 z^{26} + 41969559889089825320345876749930 z^{25} + 30642971761740916717 \backslash \\ & 17985958725620 z^{24} + 17169584489259696388755804636033570 z^{23} + 64581771961684810 \backslash \\ & 077279475394020500 z^{22} + 51714221934272099420476126216766700 z^{21} - 1473967391504 \backslash \\ & 437899277380487903179960 z^{20} - 14237554341321335335392023192872385940 z^{19} - 8321 \backslash \\ & 6340134393115016834220980384454340 z^{18} - 36401915432810756256884790682248806 \backslash \\ & 3550 z^{17} - 1261571478513401088177035093275526304300 z^{16} - 3528341032098896995323 \backslash \\ & 439017117956856150 z^{15} - 7964369518593778029521056070442794466900 z^{14} - 14280500 \backslash \\ & 726162786254712841163875001728600 z^{13} - 1953465311568634254358083196094197891 \backslash \\ & 8000 z^{12} - 18398783334222380084238012428704731960000 z^{11} - 755374178599030935723 \backslash \\ & 4054786177488000000 z^{10} + 8887432309419522403983976171775697600000 z^9 + 21137039 \backslash \\ & 158366320685856256980012112000000 z^8 + 2268269355393480 \backslash \\ & 4690446647295508800000000 z^7 + 149381834281462611905463546 \backslash \\ & 71616000000000 z^6 + 469024652858481632994019940044800000 \backslash \\ & 0000 z^5 - 872829008634785738926162452480000000000 z^4 - 11043 \backslash \\ & 27940779745890150773145600000000000 z^3 - 35389870820758085 \backslash \\ & 6772919296000000000000 z^2 - 520793429107744448741376000000 \backslash \\ & 0000000 z - 24287906219318850355200000000000000 + \end{aligned}$$



Face-Centered Cubic Lattices

[Koutschan 2013]

$$\begin{aligned} & (3964156903514787840 z^{36} + 1104718489963413534720 z^{35} + \\ & 117871088739930352834560 z^{34} + 7183287516644479615795200 z^{33} + 29310583 \backslash \\ & 5218942903781855360 z^{32} + 8706572378734984776799502400 z^{31} + 1979497761381158661 \backslash \\ & 33849254880 z^{30} + 3555494624146318548046453851120 z^{29} + 514578986720138650981112 \backslash \\ & 91247320 z^{28} + 606522834979531840684521625237020 z^{27} + 5835366836846027182876920 \backslash \\ & 856348950 z^{26} + 45455202501826358974606219981974015 z^{25} + 2791534044675020629485 \backslash \\ & 31557838260750 z^{24} + 1252275399372837134061507042628908795 z^{23} + 294318280292355 \backslash \\ & 2038161307584706940070 z^{22} - 10483513115206289398510413216920199750 z^{21} - 169948 \backslash \\ & 182933507479161257565568616530700 z^{20} - 115496959477627716064907778598382055 \backslash \\ & 3870 z^{19} - 5548694490781020038019823355124585193590 z^{18} - 2074522951757745127237 \backslash \\ & 7158241970915439245 z^{17} - 62232963928794638659423069651761724690290 z^{16} - 150810 \backslash \\ & 045901978932864163493046405461262105 z^{15} - 292528626523005661629390236883046859 \backslash \\ & 976150 z^{14} - 441395096063183148839008172248580337780300 z^{13} - 484123578764537043 \backslash \\ & 031861206473715269343000 z^{12} - 312976584649334763451810663858004196420000 z^{11} + \\ & 31415133499909950234831915395869293600000 z^{10} + 330738460087674555468491482629 \backslash \\ & 558468000000 z^9 + 391096978918364972225128472061480072000000 z^8 + 23246017042594 \backslash \\ & 8027345434850305279520000000 z^7 + 1806013493488429983484709934556800000000 z^6 - \\ & 100213400891192102370293326036992000000000 z^5 - 8385906451598513649590309945856 \backslash \\ & 000000000 z^4 - 2948705176880404905848732897280000000000 z^3 - 55164827197430422 \backslash \\ & 28472840192000000000000 z^2 - 37955378966350598718455808000000000000 z + 8379327 \backslash \\ & 645665003372544000000000000000) Dz + \end{aligned}$$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$\begin{aligned} & (8133356405487237120 z^{37} + 2294131782043664317440 z^{36} + \\ & 251295328534762193633280 z^{35} + 15795453015240816970091520 z^{34} + 666093618246 \backslash \\ & 765502077439680 z^{33} + 20469375712416843040909376160 z^{32} + 4818173382673386397833 \backslash \\ & 28749120 z^{31} + 8967973126212020032517991216960 z^{30} + 134696914854304536722281866 \backslash \\ & 954300 z^{29} + 1651833654984876079820125885678650 z^{28} + 16607490026343429532811575 \backslash \\ & 311949230 z^{27} + 136263869454304799146859253346813455 z^{26} + 895865319327471447638 \backslash \\ & 111289873238710 z^{25} + 4489711074264384906529925990254793265 z^{24} + 14491852283494 \backslash \\ & 577826654003932547711690 z^{23} - 168509066471194546983174648133542750 z^{22} - 386265 \backslash \\ & 894549826881229123104470731096440 z^{21} - 316325913106056858454611334378198756 \backslash \\ & 1220 z^{20} - 16636182069413821170544684047556220568150 z^{19} - 662467400893936760809 \backslash \\ & 81537130378090658525 z^{18} - 209080245872850631566312137449619561543730 z^{17} - \\ & 529097465740104776391772834675033946593335 z^{16} - 10650386207573139293916393610 \backslash \\ & 32363453750930 z^{15} - 1653651644685620142167009422124022555221700 z^{14} - 182938347 \backslash \\ & 4513975929874027770563298831967800 z^{13} - 10887098968369058066602841492776215683 \backslash \\ & 28000 z^{12} + 437384067337328886944483963336952904080000 z^{11} + 1678380365365432006 \backslash \\ & 625451473236269012000000 z^{10} + 1564385355592027935922683162898655112000000 z^9 + \\ & 303607398715325954207032303663107840000000 z^8 - 929554263384554771136483801 \backslash \\ & 584745600000000 z^7 - 133465853572648237153690804917964800000000 z^6 - 9419775340 \backslash \\ & 0652483718287926356480000000000 z^5 - 3502976538527876775892750166016000000 \backslash \\ & 0000 z^4 - 6428624147389214823464058470400000000000 z^3 - 37458505751548575098585 \backslash \\ & 08800000000000000 z^2 + 6687067779833962474045440000000000000 z + 7286371865795 \backslash \\ & 655106560000000000000000) Dz^2 + \end{aligned}$$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$\begin{aligned} & (6219625486549063680 z^{38} + 1775531336308022522880 z^{37} + \\ & 199409996635132589752320 z^{36} + 12904862497592448920163840 z^{35} + 561222248755 \backslash \\ & 128125708191680 z^{34} + 17798695421072697669468739680 z^{33} + 4325301686047256581892 \backslash \\ & 10596640 z^{32} + 8315189920333341531658617695280 z^{31} + 129103723904595771409928232 \backslash \\ & 487740 z^{30} + 1639190738531986170699647097803790 z^{29} + 17111040709840823035760757 \backslash \\ & 618682440 z^{28} + 146518901443861803658771329866897880 z^{27} + 101553427880666984315 \backslash \\ & 9745327151252620 z^{26} + 5496338276053076075068754467310102760 z^{25} + 2089057420971 \backslash \\ & 4927539267068315744951640 z^{24} + 30164609970591947189827076234922007050 z^{23} - \\ & 289127416281529376142095631015519267120 z^{22} - 30536692798730633467931505919379 \backslash \\ & 74700130 z^{21} - 17566486109105161467894504789161406270600 z^{20} - 73673650638461574 \backslash \\ & 538679743097050051115220 z^{19} - 240407438967557398913317296975336574702980 z^{18} - \\ & 619168293687639511251067273975020197114100 z^{17} - 12412253032904606237959908599 \backslash \\ & 05959226579320 z^{16} - 1835553795134837646262350779261931882894750 z^{15} - 167031460 \backslash \\ & 2837141110845640706031012073555700 z^{14} + 3066874861862296186275087607527571093 \backslash \\ & 2000 z^{13} + 2931594155313390328935716187001614568260000 z^{12} + 4919456458899666498 \backslash \\ & 684069708388548285600000 z^{11} + 3777365646243762653104795884206143332000000 z^{10} + \\ & 68195639154415674514017863641593600000000 z^9 - 31188109375232537262356667826 \backslash \\ & 66096800000000 z^8 - 377183378739961670425890880829428800000000 z^7 - 24884448319 \backslash \\ & 30996824908954989144320000000000 z^6 - 8939591034220938033233052627456000000 \backslash \\ & 0000 z^5 - 10008370571933280667696256102400000000000 z^4 + 2564203084567737476418 \backslash \\ & 01728000000000000000 z^3 + 1028076103183337304001413120000000000000 z^2 + 10337540 \backslash \\ & 08459758568243200000000000000000 z) Dz^3 + \end{aligned}$$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$\begin{aligned} & (2192816677949990400 z^{39} + 633490213477308768000 z^{38} + \\ & 72864986011484455353600 z^{37} + 4847486869795537260532800 z^{36} + 21701401 \backslash \\ & 7048761645614816000 z^{35} + 7088016995800124707996090560 z^{34} + 1774096571316102704 \backslash \\ & 82016190640 z^{33} + 3513089912736584156549238676620 z^{32} + 562015877327404499596706 \backslash \\ & 75451690 z^{31} + 735847759326730529024504240988015 z^{30} + 7934411063073314432988482 \backslash \\ & 485900500 z^{29} + 70405171912630286359945571896774110 z^{28} + 5088828139208506106992 \backslash \\ & 35633677324220 z^{27} + 2913386772290646501282812655546011475 z^{26} + 122379897749644 \backslash \\ & 63385062890926963215950 z^{25} + 27303163874555616052155898475524386210 z^{24} - 84400 \backslash \\ & 724272601405065271773264397209530 z^{23} - 130932954808576856297312907253772416 \backslash \\ & 4955 z^{22} - 8229269199062442444264260234977847805360 z^{21} - 3594874091884447514031 \backslash \\ & 8574840001115213670 z^{20} - 119246681320333134593914488403328142970080 z^{19} - 30427 \backslash \\ & 3308297438630162837099041285050546455 z^{18} - 57714503089590179069731112638689603 \backslash \\ & 6767490 z^{17} - 707778167790136602728038144967670916837350 z^{16} - 15334140655390733 \backslash \\ & 4245470038125935935813900 z^{15} + 1611202920628825942940219406515876419542000 z^{14} + \\ & 4188993616205017046899739124211544543460000 z^{13} + 5699626392082018037453259396 \backslash \\ & 194906388000000 z^{12} + 4145140187203309836183311398252469964000000 z^{11} + 95068892 \backslash \\ & 397133773199630362250506960000000 z^{10} - 322447694713681036240924354097202960000 \backslash \\ & 0000 z^9 - 3636835528138928302767664987399536000000000 z^8 - 210391871123659328537 \backslash \\ & 3196111532800000000000 z^7 - 55159060242541438416411785825280000000000 z^6 + 1162 \backslash \\ & 15925694410902420898178048000000000000 z^5 + 11283852768448237101798156288000000 \backslash \\ & 000000 z^4 + 2664044091169081397355479040000000000000 z^3 + 24272726277931776073 \backslash \\ & 72800000000000000000 z^2) Dz^4 + \end{aligned}$$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$\begin{aligned} & (390720062616543744 z^{40} + 114216661424360307456 z^{39} + \\ & 13440822351615963069696 z^{38} + 917965180366474611870720 z^{37} + 422376739322 \backslash \\ & 63775988570560 z^{36} + 1418218839310481932976078400 z^{35} + 364872068364080011976899 \backslash \\ & 10640 z^{34} + 742504152062765602237759452720 z^{33} + 1220569466691901164246256065 \backslash \\ & 0930 z^{32} + 164251022394443778986763736539405 z^{31} + 18217671156128364340537374047 \backslash \\ & 55054 z^{30} + 16657938302824007267243724365434191 z^{29} + 12453346084962020000971144 \backslash \\ & 5328730256 z^{28} + 743593796442908540070532245488378205 z^{27} + 33360046070881075316 \backslash \\ & 34361889061221370 z^{26} + 9020222161854736084202629824390547047 z^{25} - 919882472294 \backslash \\ & 3404205447421299404277112 z^{24} - 279449868802402514175677041789492570017 z^{23} - \\ & 1907863427661939885576723126598906643790 z^{22} - 8574083646475710050757565542672 \backslash \\ & 979674555 z^{21} - 28405587296847231070183606856583770811720 z^{20} - 6957425817531295 \backslash \\ & 5514440713973653616428745 z^{19} - 114991436892487711669937849824912517430330 z^{18} - \\ & 70378017201579863364495432167182725333675 z^{17} + 261641966501089147843656083216 \backslash \\ & 157842879550 z^{16} + 1049410824795136384837467209810025539400000 z^{15} + 20896637809 \backslash \\ & 64272997600159898811800513390000 z^{14} + 259786002679636380331331345092974504000 \backslash \\ & 0000 z^{13} + 1759136834585156085432113720072647266000000 z^{12} - 1346756149912237131 \backslash \\ & 08740928290811280000000 z^{11} - 157899609879137074628470745343916920000000 z^{10} - \\ & 1556811681322720025894531955998040000000000 z^9 - 67890891761349944176134243479 \backslash \\ & 520000000000 z^8 + 314922159293504699010388187520000000000 z^7 + 192407344459752 \backslash \\ & 425261121833472000000000000 z^6 + 1041268402104447820606289510400000000000 z^5 + \\ & 2098227664304539359901286400000000000000 z^4 + 17874380983279966433280000000000 \backslash \\ & 00000000 z^3) Dz^5 + \end{aligned}$$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$\begin{aligned} & (35882454730090752 z^{41} + 10612604051614486656 z^{40} + 1276532600942212775168 z^{39} + \\ & 89393980129433032096320 z^{38} + 4221606838983473228197008 z^{37} + 145494567985766 \backslash \\ & 484898923048 z^{36} + 3840828004490920060950969480 z^{35} + 8016006238826772717221198 \backslash \\ & 5080 z^{34} + 1350855094398006902682870922050 z^{33} + 1863108289263053682422294940 \backslash \\ & 9585 z^{32} + 211815796834464054711973645322142 z^{31} + 19867083220856675726655250160 \backslash \\ & 37411 z^{30} + 15263082383031406770429022758762048 z^{29} + 94068732852089205756130773 \backslash \\ & 605094705 z^{28} + 441055376229095921513357130918811338 z^{27} + 131963694549876126497 \backslash \\ & 3744224282378779 z^{26} - 137626809673226795399591264079041112 z^{25} - 31072001737970 \backslash \\ & 299221405533198706303141 z^{24} - 226886176666918560987240200768631693150 z^{23} - \\ & 1033954017266382248984767586852072344191 z^{22} - 335673294622437360164908793734 \backslash \\ & 9109785896 z^{21} - 7573126212785007618891225542456994124245 z^{20} - 9076459539413303 \backslash \\ & 184641722134776573895810 z^{19} + 10278671248090335377408918358815408788425 z^{18} + \\ & 85149274357043292385925033653294291853550 z^{17} + 2406893603584982960079390961 \backslash \\ & 87740586134000 z^{16} + 429409878921957648790555775268242743350000 z^{15} + 4957792250 \backslash \\ & 46771906420255540348281344800000 z^{14} + 28712136337931261687156234648446537800 \backslash \\ & 0000 z^{13} - 119682652007548350954457856750250720000000 z^{12} - 39568346559268086740 \backslash \\ & 1293480616198000000000 z^{11} - 32738346275504238594974769124082400000000 z^{10} - \\ & 8664257545050139106678720201952000000000 z^9 + 59704683972170679548931977222400 \backslash \\ & 00000000 z^8 + 7251161027741239099083936307200000000000 z^7 + 338828967558720719 \backslash \\ & 5688626176000000000000 z^6 + 631115677130491732576665600000000000000 z^5 + 51232 \backslash \\ & 3021813756999680000000000000000000 z^4) Dz^6 + \end{aligned}$$

Face-Centered Cubic Lattices

[Koutschan 2013]

$$\begin{aligned} & (1600200173148416 z^{42} + 478782978712278912 z^{41} + 58815380786135567104 z^{40} + \\ & 4218590040421804170816 z^{39} + 204216444469816446653424 z^{38} + 7214118 \backslash \\ & 624119937529541160 z^{37} + 195106070712453547506798168 z^{36} + 416887031952436819753 \backslash \\ & 3959000 z^{35} + 71874412795312940511795668940 z^{34} + 101352803991324920736784237 \backslash \\ & 8270 z^{33} + 11775924181048893848357395670676 z^{32} + 11286205581821339235627976822 \backslash \\ & 5402 z^{31} + 886340494836475569866741139358344 z^{30} + 55926759731865674372797336853 \backslash \\ & 51646 z^{29} + 26983635759333711243427828354079724 z^{28} + 85059388463264142313662526 \backslash \\ & 542420618 z^{27} + 24816956833181644480403313150735864 z^{26} - 1739731529923503295984 \backslash \\ & 796806526752758 z^{25} - 13215685421423157401833903137021991092 z^{24} - 6010151473251 \backslash \\ & 7779329542749898893453858 z^{23} - 187406121933740017212741167478185137320 z^{22} - \\ & 367088786736715063908412462166156515566 z^{21} - 136331238303988349001415414181 \backslash \\ & 532146340 z^{20} + 2052937632229799753666758504303681446150 z^{19} + 89422208647113020 \backslash \\ & 92023950168348534856300 z^{18} + 22112779083456047399791690319673356808000 z^{17} + \\ & 36662299830964853548300895468723502480000 z^{16} + 3866393620905473995570164907678 \backslash \\ & 4708400000 z^{15} + 15575841209632684184725074680551176000000 z^{14} - 233997759271107 \backslash \\ & 78754739301057544560000000 z^{13} - 45957581844555068108338692961807200000000 z^{12} - \\ & 32525005285459811112066289505232000000000 z^{11} - 3661218292337523929046664304 \backslash \\ & 64000000000 z^{10} + 1097039530150661129281453716480000000000 z^9 + 97131124051979 \backslash \\ & 35942595533824000000000000 z^8 + 426097871942078389250377728000000000000 z^7 + \\ & 75646707912253502668800000000000000000 z^6 + 592017714095896977408000000000000 \backslash \\ & 0000 z^5) Dz^7 + \end{aligned}$$

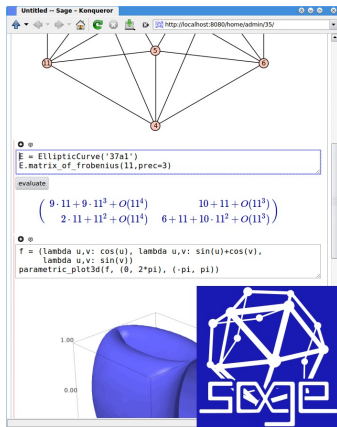
Face-Centered Cubic Lattices

[Koutschan 2013]

$$\begin{aligned} & (27122036833024 z^{43} + 8208413201024064 z^{42} + 1028987679702510976 z^{41} + 7551845 \backslash \\ & 1137118783792 z^{40} + 3743195619381989907184 z^{39} + 135369638077546936261428 z^{38} + \\ & 3745615314367420203992832 z^{37} + 81811619367860049045984675 z^{36} + 1440466637248 \backslash \\ & 203913774334250 z^{35} + 20724331113040275023719172850 z^{34} + 2454466275416520460977 \backslash \\ & 92768214 z^{33} + 2395828801191215780780578117794 z^{32} + 191474074706731112318622494 \backslash \\ & 18166 z^{31} + 122863963621496746370188659696702 z^{30} + 6026212556484859243787006723 \backslash \\ & 31054 z^{29} + 1935192664301617476137337671088360 z^{28} + 694152712036783264243644290 \backslash \\ & 673234 z^{27} - 39030042885818935455901289133872622 z^{26} - 2976459628039331965648737 \backslash \\ & 33670191774 z^{25} - 1329742929728007215704002549281591538 z^{24} - 390398961482564881 \backslash \\ & 9224432657208727646 z^{23} - 6038015534019664017777438417359311914 z^{22} + 7565280951 \backslash \\ & 156009750992823479550694170 z^{21} + 83328126336960183101771239549883786325 z^{20} + \\ & 297859836972471180382017327162905955900 z^{19} + 681226694393685252017130073908 \backslash \\ & 325840500 z^{18} + 1055504316932586226613390044310017920000 z^{17} + 97498262514411065 \backslash \\ & 4834660688990434600000 z^{16} + 80921481727424794623135930623472000000 z^{15} - 124569 \backslash \\ & 2778975371208980497936649580000000 z^{14} - 187761297216604654284152589154800000 \backslash \\ & 0000 z^{13} - 1186201691981014544058180007080000000000 z^{12} - 4424139263701398029187 \backslash \\ & 830400000000000 z^{11} + 526588498855023559134177312000000000000 z^{10} + 410999234738 \backslash \\ & 834010247469568000000000000 z^9 + 17433301221381095805118464000000000000 z^8 + \\ & 29950530354743793153638400000000000000 z^7 + 22769912080611422208000000000000 \backslash \\ & 0000 z^6) Dz^8 \end{aligned}$$

order 8, degree 43, 43-digit coefficients

What it Is: A SageMath Implementation



SageMath

- ▶ Python library
- ▶ “A viable alternative to Magma, Maple, Mathematica and Matlab”

`sage:` $\text{PolS.<z> = PolynomialRing(QQ)}$

`sage:` $(z + 1) * (z - 1)$

$z^2 - 1$



<http://sagemath.org/>

GNU GPL v2+

What it Is: Based on ore_algebra

[Kauers, Jaroschek, Johansson, 2013–]

```
sage: from ore_algebra import OreAlgebra
```

```
sage: DiffOps.<Dz> = OreAlgebra(Pols)
```

```
sage: DiffOps
```

```
Univariate Ore algebra in Dz over Univariate Polynomial Ring  
in z over Rational Field
```

```
sage: Dz*z
```

```
z*Dz + 1
```

Features: Euclidean arithmetic, closure properties, formal solutions, desingularization, first-order factors, guessing...



http://www.risc.jku.at/research/combinat/software/ore_algebra/

What it Is: The -analytic Branch

- ▶ **Symbolic-numeric** extensions for ore_algebra
- ▶ Real & complex arithmetic based on **Arb** [Johansson 2012–]
({Real, Complex}BallField in Sage)
- ▶ Both for “end users” and for prototyping algorithms
- ▶ Development branch, not (yet) integrated into any release of ore_algebra
- ▶ Co-developped with the necessary improvements to Sage



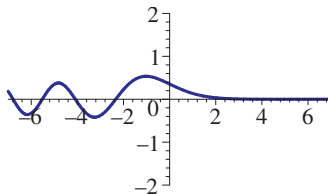
http://marc.mezzarobba.net/code/ore_algebra-analytic
GNU GPL v2+

What it Does: Special Functions

$$\text{Ai}''(z) - z \text{Ai}(z) = 0$$

$$\text{Ai}(0) = \Gamma(2/3)^{-1} 3^{-3/2}$$

$$\text{Ai}'(0) = -\Gamma(1/3)^{-1} 3^{-1/2}$$



```
sage: diffop = Dz^2 - z
```

```
sage: diffop.numerical_solution(  
    [1/(gamma(2/3)*3^(2/3)), -1/(gamma(1/3)*3^(1/3))],  
    [0, i], 1e-40)
```

```
[0.3314933054321411889845293326171343458866 +/- 5.51e-41] +  
[-0.31744985896844377347764292790925852645896 +/- 7.23e-42]*I
```

```
sage: ComplexBallField(138)(i).airy_ai()
```

```
[0.33149330543214118898452933261713434588655 +/- 5.25e-42] +  
[-0.31744985896844377347764292790925852645896 +/- 1.59e-42]*I
```


What it Does: Polynomial Approximations

```
sage: diffop
```

```
Dz^2 - z
```

```
sage: from ore_algebra.analytic import  
      polynomial_approximation as polapprox
```

```
sage: polapprox.on_interval(diffop,  
                             [1/(gamma(2/3)*3^(2/3)), -1/(gamma(1/3)*3^(1/3))],  
                             [[-1,1]], 1e-10)
```

```
[2.0642950800e-7 +/- 6.90e-18]*z^12 + [-1.1239405454e-7 +/-  
3.05e-18]*z^11 + [-5.703884502e-6 +/- 5.09e-16]*z^10 +  
[2.7531935230e-5 +/- 3.10e-16]*z^9 + [-0.0005136124315 +/-  
4.71e-14]*z^7 + [0.0019723755744 +/- 8.14e-14]*z^6 +  
[2.3736844006e-8 +/- 8.48e-19]*z^5 + [-0.021568281826 +/-  
5.47e-13]*z^4 + [0.059171339359 +/- 6.15e-13]*z^3 + [-  
3.780985308e-10 +/- 4.90e-20]*z^2 +  
[-0.25881940369 +/- 8.38e-12]*z + [0.3550280539 +/- 3.86e-11]
```

What it Does: D-Finite Functions

[Stanley, Zeilberger... 1980-]

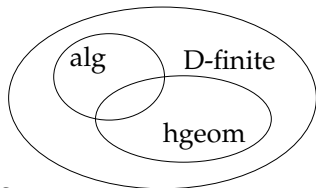
An analytic function $y: \mathbb{C} \rightarrow \mathbb{C}$ is **D-finite** (holonomic) iff it satisfies a linear homogeneous ODE with polynomial coefficients:

$$a_r(z) y^{(r)}(z) + \cdots + a_1(z) y'(z) + a_0(z) y(z) = 0, \quad a_j \in \mathbb{K}[z]$$

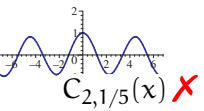
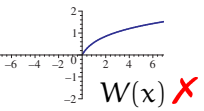
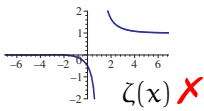
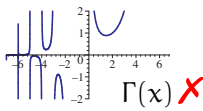
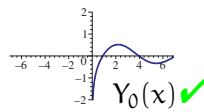
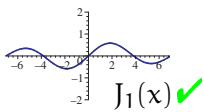
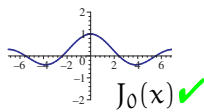
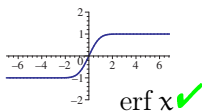
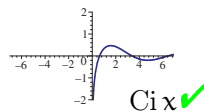
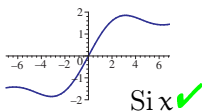
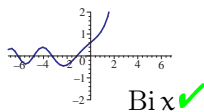
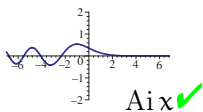
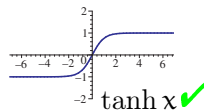
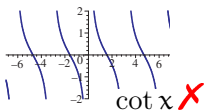
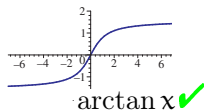
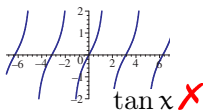
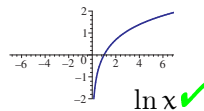
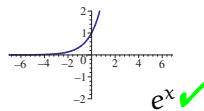
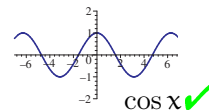
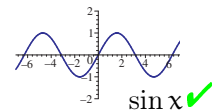
i.e. $(a_r(z) D_z^r + \cdots + a_1(z) D_z + a_0(z))(y) = 0.$

Philosophy:

Provide **general algorithms** for D-finite functions, using { ODE + initial values } as a data structure.



What it Does: D-Finite Functions



What it Does: D-Finite Functions

[Stanley, Zeilberger... 1980-]

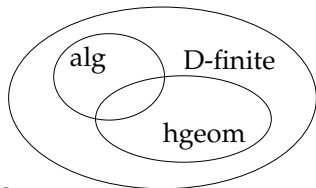
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$$a_r(z) y^{(r)}(z) + \cdots + a_1(z) y'(z) + a_0(z) y(z) = 0, \quad a_j \in \mathbb{K}[z]$$

i.e. $(a_r(z) D_z^r + \cdots + a_1(z) D_z + a_0(z))(y) = 0.$

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i.e.
$$(a_r(z) D_z^r + \cdots + a_1(z) D_z + a_0(z))(y) = 0.$$

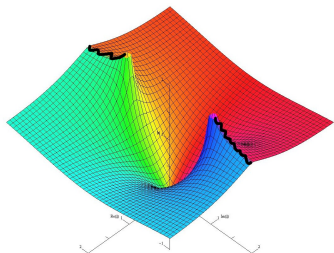
Existence Theorem

[Cauchy]

Let $U \subseteq \mathbb{C}$ be a **simply connected** domain where $a_r(z) \neq 0$.

Then $a_r(z) D_z^r + \cdots + a_0(z)$ admits an r -dimensional vector space of solutions analytic on U . A solution is characterized by the initial values $y(z_0), y'(z_0), \dots, y^{(r-1)}(z_0)$ for any $z_0 \in U$.

What it Does: Analytic Continuation



$$y(z) = \arctan(z)$$

$$(z^2 + 1)y''(z) + 2zy'(z) = 0$$

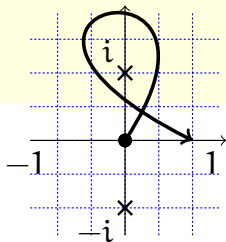
```
sage: dop = (z^2+1)*Dz^2 + 2*z*Dz
```

```
sage: dop.numerical_solution(  
    ini=[0,1], path=[0,1])
```

```
[0.78539816339744831 +/- 1.08e-18]
```

```
sage: dop.numerical_solution(  
    ini=[0,1],  
    path=[0,i+1,2*i,i-1,0,1])
```

```
[3.9269908169872415  
    +/- 4.81e-17]  
+ [+/- 4.63e-21]*I
```



What it Does: Transition Matrices

$$(z^2 + 1)y''(z) + 2zy'(z) = 0 \quad \Rightarrow \quad \begin{bmatrix} y(1) \\ y'(1) \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

↑
"transition matrix"

```
sage: dop.numerical_transition_matrix([0,1])
```

```
[ 1.000000000000000000 [0.7853981633974483 +/- 1.46e-17]]
[                      0 [0.500000000000000000 +/- 1.03e-17]]
```

```
sage: n(pi/4)
```

```
0.785398163397448
```

$$f(z) = 1 \quad = 1 + 0 \cdot z + O(z^2)$$


$$g(z) = \arctan(z) \quad = 0 + 1 \cdot z + O(z^2)$$

$$\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} f(1) & g(1) \\ f'(1) & g'(1) \end{bmatrix}$$

What it Does: Regular Singular Points

The previous examples only involved **ordinary** (= non-singular) points.

$$z^2 y''(z) + z y'(z) + (z^2 - \nu^2) y(z) = 0 \quad (\text{Bessel eq.})$$


singular point at 0
regular (\approx tame) in this case

✓ $z^{-3/2} \log z$

✓ $z^{i\sqrt{2}}$

✗ $e^{\pm 1/z}$

Theorem

[Fuchs, 1866]

Assume that 0 is a regular singular point. Then, for some $D \ni 0$, there exists a basis of solutions defined on $D \setminus \{0\}$ of the form

$$z^\lambda (y_0(z) + y_1(z) \log z + \cdots + y_t(z) \log^t z), \quad \lambda \in \bar{\mathbb{Q}}, \quad y_i \text{ **analytic** on } D.$$

Regular Singular Connection Problems

$$zy''(z) + y'(z) + zy(z) = 0 \quad \Rightarrow \quad \begin{bmatrix} y(1) \\ y'(1) \end{bmatrix} = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

where $y(z) = a \cdot \log z + b \cdot 1 + O(z)$

```
sage: dop = z*Dz^2 + Dz + z
```

```
sage: dop.local_basis_monomials(0)
```

```
[log(z), 1]
```

```
sage: dop.numerical_transition_matrix([0, 1], 1e-10)
```

```
[ [0.22734424279 +/- 4.98e-12] [0.76519768656 +/- 2.04e-12]]  
[ [1.1761104988 +/- 2.83e-11] [-0.44005058574 +/- 4.94e-12]]
```

Applications: special functions, analytic combinatorics, resummation...

Face-Centered Cubic Lattices

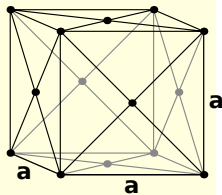
[Koutschan 2013]

```
sage: dop4 = ((-1 + z)*z^3*(2 + z)*(3 + z)*(6 + z)*(8 + z)*(4 +
3*z)^2*Dz^4 + 2*z^2*(4 + 3*z)*(-3456 - 2304*z + 3676*z^2
+ 4920*z^3 + 2079*z^4 + 356*z^5 + 21*z^6)*Dz^3 + 6*z*(-
5376 - 5248*z + 11080*z^2 + 25286*z^3 + 19898*z^4 +
7432*z^5 + 1286*z^6 + 81*z^7)*Dz^2 + 12*(-384 + 224*z
+ 3716*z^2 + 7633*z^3 + 6734*z^4 + 2939*z^5 + 604*z^6
+ 45*z^7)*Dz + 12*z*(256 + 632*z + 702*z^2 + 382*z^3 +
98*z^4 + 9*z^5))
```

```
sage: dop4.local_basis_monomials(0)
[1/6*log(z)^3, 1/2*log(z)^2, log(z), 1]
```

```
sage: dop4.local_basis_monomials(1)
[1, (z - 1)*log(z - 1), z - 1, (z - 1)^2]
```

```
sage: dop4.numerical_transition_matrix([0,1])[0,-1]
[1.1058437979212048 +/- 3.99e-17] + [+/- 6.96e-26]*I
```



What it Does: Asymptotics of Apéry Numbers

$$a_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2 \quad b_n = \sum_{k=1}^n \left(\frac{a_n}{k^3} - \sum_{m=1}^k \frac{(-1)^m \binom{n}{k}^2 \binom{n+k}{k}^2}{2 m^3 \binom{n}{m} \binom{n+m}{m}} \right)$$

(1, 5, 73, 1445, 33001...) (0, 6, 351/4, 62531/36, ...)

- ▶ The OGS $a(z)$ and $b(z)$ are solutions of $L = z^2(z^2 - 34z + 1)D_x^4 + \dots$
- ▶ Singular points: $0, \alpha = (\sqrt{2} + 1)^4 \approx 33.9, \alpha^{-1} = (\sqrt{2} - 1)^4 \approx 0.0294$
- ▶ Prove: $a_n, b_n = \alpha^{n+o(n)} \quad b_n - \zeta(3) a_n = \alpha^{-n+o(n)}$
- ▶ Local expansion at α^{-1} : $a(z) = c_0 f_0(z) + c_1 f_1(z) + c_2 f_2(z) + c_3 f_3(z)$
 where $f_0(\alpha^{-1} + t) = 1 + O(t^3) \quad f_3(\alpha^{-1} + t) = t + O(t^3)$
 $f_1(\alpha^{-1} + t) = \sqrt{t} + O(t^3) \quad f_4(\alpha^{-1} + t) = t^2 + O(t^3)$
- ▶ Singularity analysis: ($c_1 \neq 0$)

$$a(z) \sim c_1 \sqrt{z - \alpha^{-1}} \quad \Rightarrow \quad a_n \sim c_1 [z^n] \sqrt{z - \alpha^{-1}} \sim \frac{c_1 i}{2\sqrt{\alpha\pi}} \alpha^n n^{-3/2}$$

$$\begin{aligned}x^2 Y_1'' + x Y_1' + (x^2 - 1) Y_1 &= 0 \\ Y_1(x) &\sim -\frac{2}{\pi x} + \frac{x \ln x}{\pi} + \dots \\ \text{as } x &\rightarrow 0\end{aligned}$$



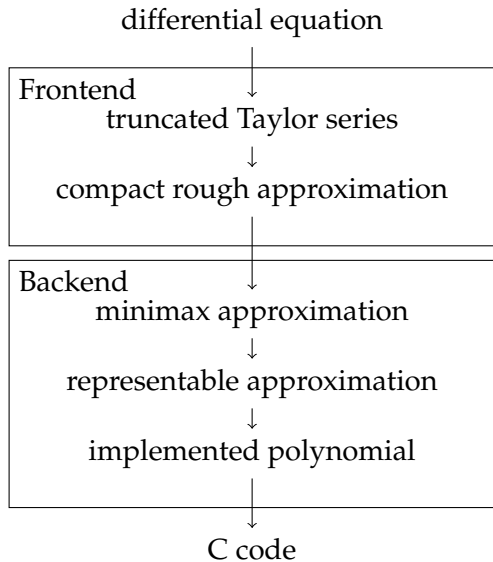
```
double BesselY1 (double x)
{
    // generated code
}
```

Input: differential equation + initial values defining f
domain $[a, b]$
target accuracy
(+ implementation constraints)

Output: C code

$$\left| \frac{\text{implem}(x) - f(x)}{f(x)} \right| \leq \varepsilon \text{ for all } x \in [a, b] \cap \text{double}$$

A Prototype Code Generator



▶ “Rigorous **polynomial approximation** pipeline”

▶ **Interesting points:**

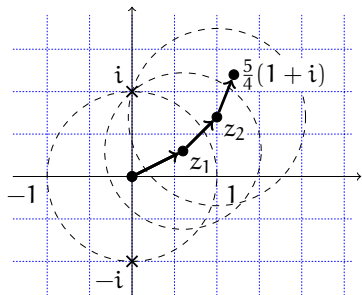
$f(x) = 0$ or $f(x) \approx 0$

$x = 0, x \rightarrow \infty$

later: singularities

▶ Reduce to implementing well-chosen polynomials

How it Works: A Taylor Series Method



$$\arctan\left(\frac{5}{4}(1+i)\right)?$$

$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0.57\dots + 0.22\dots \\ 0 & 0.72\dots - 0.20\dots \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

$$\begin{bmatrix} y(z_2) \\ y'(z_2) \end{bmatrix} = \begin{bmatrix} 1 & 0.36\dots + 0.32\dots \\ 0 & 0.75\dots - 0.07\dots \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

...

At each step, compute the sum of the power series expansion of each entry of the transition matrix.

The idea extends to the regular singular case.

How it Works: Recurrences

The **Taylor coefficients** of a D-finite function $y(z) = \sum_{n=0}^{\infty} y_n z^n$ obey a linear **recurrence relation** with polynomial coefficients:

$$b_s(n) y_{n+s} + \dots + b_1(n) y_{n+1} + b_0(n) y_n = 0.$$

(And conversely, for D-finite formal power series.)

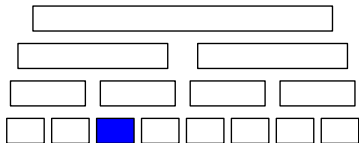
Leads to **fast algorithms** (not fully implemented yet)

[Schroepel 1972; Brent 1976; Chudnovsky & Chudnovsky 1988;
van der Hoeven 1999, 2001; M. 2010, 2012; Johansson 2014]

Best complexity:

time $\mathbf{O}(M(n \log^2 n))$, space $\mathbf{O}(n)$

for fixed z and $\varepsilon = 2^{-n}$



How it Works: Truncation Errors

Problem

$$\sum_{n=0}^{\infty} u_n z^n = \underbrace{\sum_{n=0}^{N-1} u_n z^n}_{\text{known}} + \underbrace{\sum_{n=N}^{\infty} u_n z^n}_{|\cdot| \leq ?}$$

“Adaptive” bounds:

$$A x = b \quad \|A^{-1}\| \leq \mathbf{M}$$

$$A \in GL_n(\mathbb{C})$$

$$A \tilde{x} = \tilde{b} \quad \Rightarrow \quad \|x - \tilde{x}\| \leq \mathbf{M} \cdot \underbrace{\|b - \tilde{b}\|}_{\text{known}}$$

Compute a bound similar to \mathbf{M} when A replaced with a differential operator?

How it Works: Majorant Series

[Cauchy 1842; ...; van der Hoeven 2001; M. & Salvy 2010]

- ▶ Instead of directly bounding $|\sum_{n \geq N} u_n z^n|$, compute a **majorant series**:

$$\sum v_n z^n \in \mathbb{R}_{\geq 0}[[z]] \quad \text{s.t.} \quad \forall n, \quad |u_n| \leq v_n$$

- ▶ Bound the differential equation with a simple “**model equation**”:

$$L(z, D_z) \cdot u = 0 \quad \Leftarrow \quad v'(z) - \frac{1}{(1 - \alpha z)} v(z) = 0$$

for us: always 1st order

- ▶ Solve the model equation and study the solutions:

$$v(z) = \exp \int^z \frac{dt}{1 - \alpha t} \quad \left| \sum_{n=N}^{+\infty} u_n z^n \right| \leq \sum_{n=N}^{+\infty} v_n |z|^n \leq ?$$

How it Works: Adaptive Majorants

$$L(z, D_z) \cdot u = 0$$

Residual: $q(z) := L(z, D_z) \cdot \tilde{u}$

$$u(z) = \sum_{n=0}^{\infty} u_n z^n = \underbrace{\sum_{n=0}^{N-1} u_n z^n}_{\tilde{u}(z)} + \sum_{n=N}^{\infty} u_n z^n$$

► **Model equation**

$$\begin{aligned} q(z) &\ll \hat{q}(z) \\ L(z, D_z) \cdot (\tilde{u} - u) = q &\ll \hat{L}(z, D_z) \cdot v = \hat{q} \end{aligned}$$

► **Majorant property:**

$$(\forall n \leq n_0) \quad |u_n| \leq v_n \quad \Rightarrow \quad (\forall n) \quad |u_n| \leq v_n$$

► **Solving the model equation**

$$v(z) = h(z) \left(\underset{\substack{\nearrow \\ \text{choose } \text{cst} = 0}}{\text{cst}} + \int^z \frac{t^{-1} \hat{q}(t)}{h(t)} dt \right)$$

$= O(z^N)$

$$\text{where } h(z) = \exp \int^z t^{-1} \hat{a}(t) dt$$



Summary

What it is: an extension of ore_algebra written in/for SageMath

What it does: numerical analytic continuation & singular connection, for arbitrary D-finite functions, with rigorous error bounds

How it works: Taylor series, analytic continuation, recurrences, adaptive majorants, ball arithmetic



Code available at

http://marc.mezzarobba.net/code/ore_algebra-analytic



Perspectives

Fast algorithms, lower-level code,
evaluation on wide intervals, D-finite functions as objects,
irregular singular connection problems...

Comments, bug reports, feature requests, examples welcome!

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- ▶ Face centered cubic crystal structure on page 26 from https://commons.wikimedia.org/wiki/File:Lattice_face_centered_cubic.svg

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