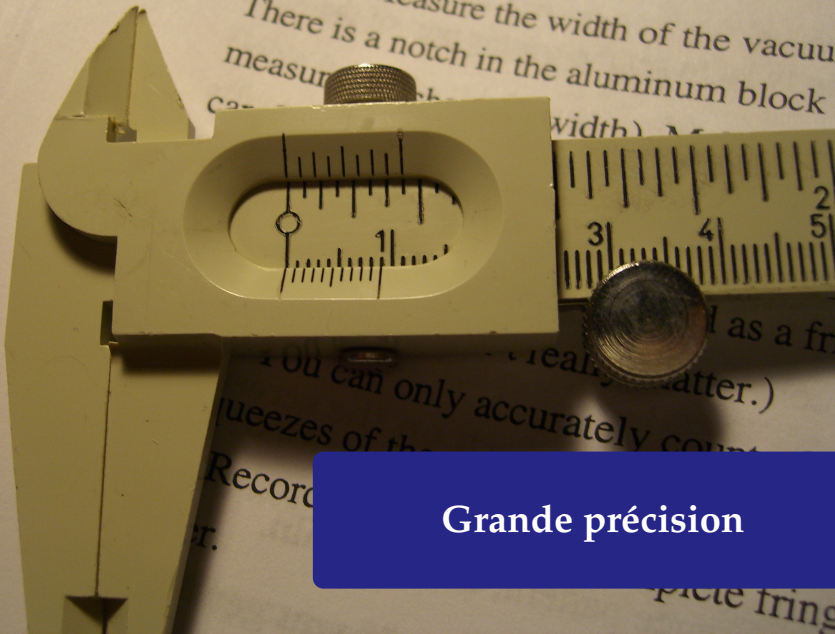


Autour de l'évaluation numérique des fonctions D-finies

Marc MEZZAROBBA



Journées nationales de calcul formel
16 novembre 2011



Grande précision

Un dictionnaire des fonctions D-finies

Dynamic Dictionary of Mathematical Functions - Iceweasel

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http://ddmf.msr-inria.fr/ddmf/service=MainIndex&rendering=jsMath

Wikipedia.com

Home Glossary

Dynamic Dictionary of Mathematical Functions

rendering [link](#)

Welcome to this interactive site on [Mathematical Functions](#), with properties, truncated expansions, numerical evaluations, plots, and more. The functions currently presented are elementary functions and special functions of a single variable. More functions — special functions with parameters, orthogonal polynomials, sequences — will be added with the project advances.

Contents

Select a special function from the list

- [Help](#) on selecting and configuring the mathematical rendering
- DDMF [developers](#) list
- [Motivation](#) of the project

- The [inverse cosecant](#) $\operatorname{arccsc}(x)$
- The [inverse cosine](#) $\operatorname{arccos}(x)$
- The [inverse cotangent](#) $\operatorname{arccot}(x)$
- The [inverse hyperbolic cosecant](#) $\operatorname{arcsch}(x)$
- The [Airy function of the first kind](#) $\operatorname{Ai}(x)$
- The [inverse secant](#) $\operatorname{arcsec}(x)$
- The [inverse sine](#) $\operatorname{arcsin}(x)$
- The [inverse tangent](#) $\operatorname{arctan}(x)$
- The [Airy function \(of the second kind\)](#) $\operatorname{Bi}(x)$
- The [hyperbolic cosine integral](#) $\operatorname{Chi}(x)$
- The [cosine integral](#) $\operatorname{Ci}(x)$
- The [cosine](#) $\cos(x)$
- The [exponential integral](#) $\operatorname{Ei}(x)$
- The [error function](#) $\operatorname{erf}(x)$
- The [complementary error function](#) $\operatorname{erfc}(x)$
- The [imaginary error function](#) $\operatorname{erfi}(x)$

jsMath

Done Proxy: None zotero

Un dictionnaire des fonctions D-finies

<http://ddmf.msr-inria.inria.fr>

Dynamic Dictionary of Mathematical Functions

Home Glossary

Dynamic Dictionary of Mathematical Functions

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Select a special function from the list

Benoit, Chyzak, Darrasse, Gerhold, M. & Salvy (2010)

- [DDMF developers list](#)
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rendering [link](#)

Contents

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jsMath

Done Proxy: None zotero

Un dictionnaire des fonctions D-finies

The Special Function Ai(x) - Iceweasel

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http://ddmf.msr-inria.fr/ddmf/service=SpecialFunction&rendering=jsMath&st_name=AiyAi¶meters={ }

[01] Loading...

Home Glossary

The Special Function $Ai(x)$

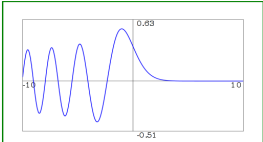
1. Differential equation rendering [link](#)

The function $Ai(x)$ satisfies

$$\frac{d^2}{dx^2}y(x) - xy(x) = 0$$

with initial values $y(0) = 1/3 \frac{\sqrt[3]{3}}{\Gamma(2/3)}$, $(y')(0) = -1/2 \frac{\sqrt[6]{3}\Gamma(2/3)}{\pi}$. [metadata](#)

2. Plot of $Ai(x)$



Done Proxy: None | zotero

jsMath

Un dictionnaire des fonctions D-finies

The Special Function Ai(x) - Iceweasel

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http://ddmf.msr-inria.fr/ddmf/service=SpecialFunction&rendering=jsMath&st_name=AiyAi¶meters={}

[01] Loading...

Home Glossary

The Special

Données :
EDL à coeff. polynomiaux
+ conditions initiales
(fonction D-finie)

1. Differential equation

The function $Ai(x)$ satisfies

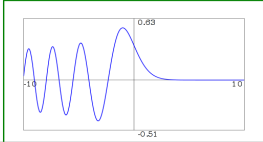
$$\frac{d^2}{dx^2}y(x) - xy(x) = 0$$

with initial values $y(0) = 1/3 \frac{\sqrt[3]{3}}{\Gamma(2/3)}$, $(y')(0) = -1/2 \frac{\sqrt[6]{3}\Gamma(2/3)}{\pi}$.

[rendering link](#)

[metadata](#)

2. Plot of $Ai(x)$

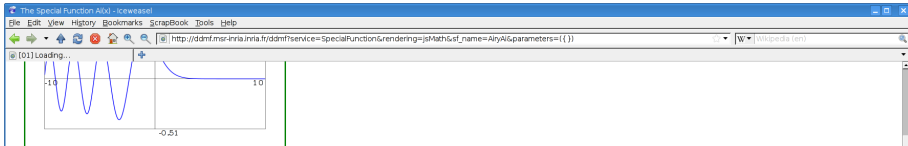


The plot shows the function $Ai(x)$ on a coordinate system. The x-axis ranges from -10 to 10, and the y-axis ranges from -0.51 to 0.83. For $x < 0$, the function oscillates with increasing amplitude as x becomes more negative. For $x > 0$, the function decays monotonically towards zero.

jsMath

Done Proxy: None | zotero

Un dictionnaire des fonctions D-finies



min =

max =

3. Numerical Evaluation

$$\text{Ai}(1/4 + 1/4i) \approx 0.28881085 - 0.06285935i.$$

(Below, path may be either a point z or a broken-line path $[z_1, z_2, \dots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form $x + y*i$.) [metadata](#)

path =

precision =

4. Taylor expansion of $\text{Ai}(x)$ at 0

- Expansion of $\text{Ai}(x)$ at 0:

$$\text{Ai}(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt[3]{3} x^{3n}}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^{3n+1}}{9^n \Gamma(n + 4/3) n!}.$$

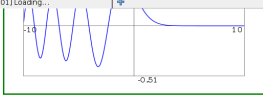
Un dictionnaire des fonctions D-finies

The Special Function Ai(x) - Iceweasel

File Edit View History Bookmarks ScrapBook Tools Help

http://ddmf.msr-inria.fr/ddmf/service=SpecialFunction&rendering=jsMath&st_name=AiryAi¶meters={}

[01] Loading...



min = max =

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path = precision =

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jsMath

Done Proxy: None zotero

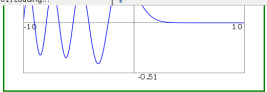
Un dictionnaire des fonctions D-finies

The Special Function Axi - Iceweasel

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http://ddmf.msr-inria.fr/ddmf/service=SpecialFunction&rendering=jsMath&st_name=AiryAi¶meters={}

[01] Loading...



min = max =

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$Ai(1/4 + 1/4i) \approx 0.28881085 - 0.06285935i$.

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path = precision =

4. Taylor expansion of $Ai(x)$ at 0

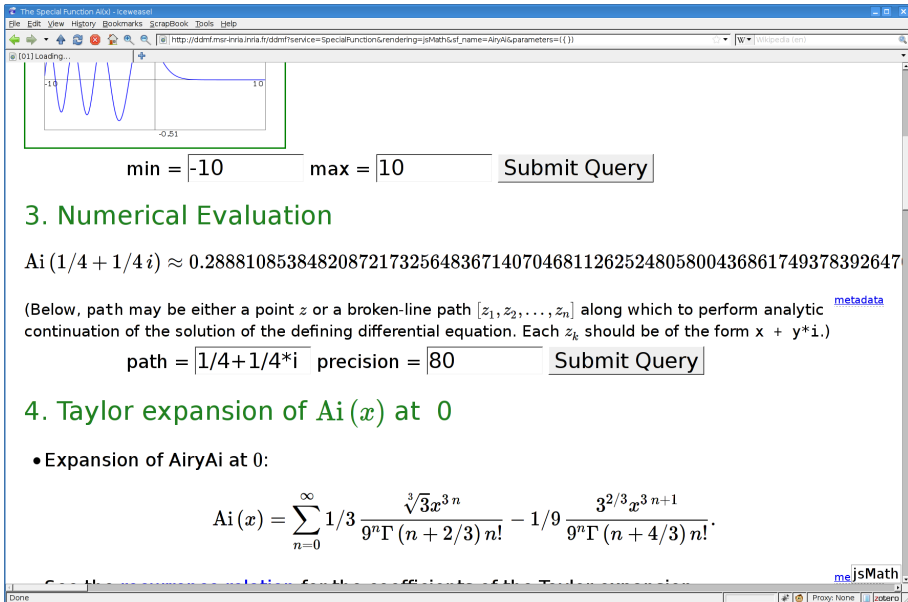
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me jsMath

Done Proxy: None zotero

Un dictionnaire des fonctions D-finies



The screenshot shows a web browser window titled "The Special Function Ai(x) - Ickweasel". The address bar contains a URL from ddmf.msr-inria.fr. The main content area displays a plot of the Airy function $Ai(x)$ with a green box highlighting the region where x ranges from -10 to 10. Below the plot, there are input fields for "min = -10" and "max = 10", followed by a "Submit Query" button.

3. Numerical Evaluation

$Ai(1/4 + 1/4i) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647$

(Below, path may be either a point z or a broken-line path $[z_1, z_2, \dots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form $x + y*i$.)

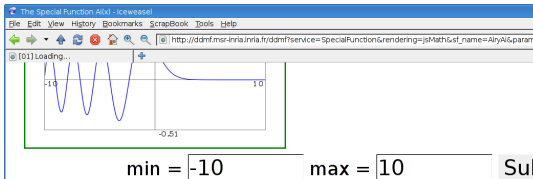
path = $1/4+1/4*i$ precision = 80

4. Taylor expansion of $Ai(x)$ at 0

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$$Ai(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt[3]{3} x^{3n}}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^{3n+1}}{9^n \Gamma(n + 4/3) n!}.$$

Un dictionnaire des fonctions D-finies



- ▶ à partir de l'équa. diff.
- ▶ résultats garantis
- ▶ précision arbitraire
- ▶ efficace

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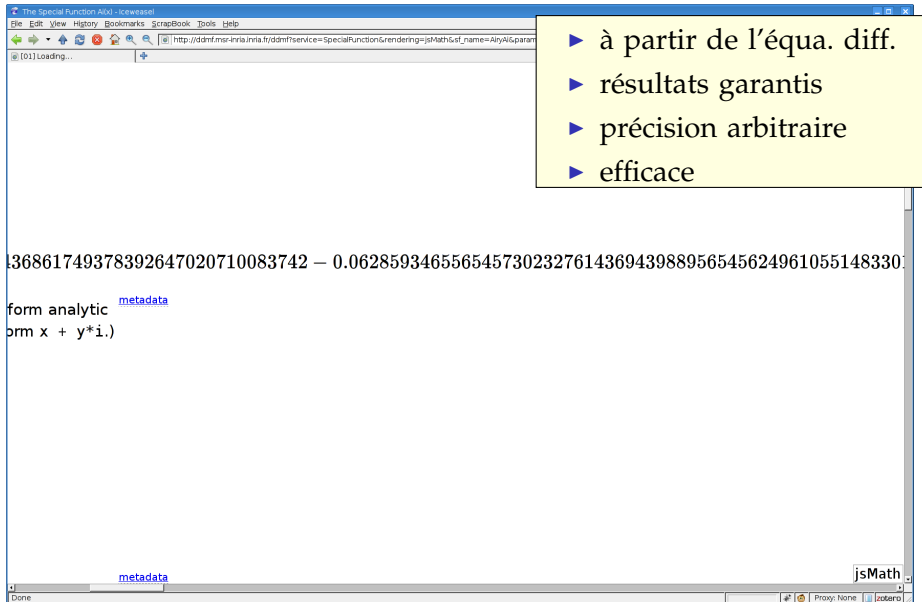
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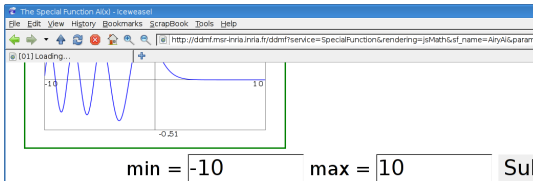
Un dictionnaire des fonctions D-finies



The screenshot shows a web browser window titled "The Special Function API - Iceweasel". The address bar contains the URL: `http://ddmf.msr-inria.fr/ddmf/service=SpecialFunction&rendering=jsMath&st_name=AlyA¶m`. The main content area displays a long numerical value: `36861749378392647020710083742 - 0.062859346556545730232761436943988956545624961055148330`. Below this, the text "form analytic" is followed by a blue link "metadata". The bottom of the page shows another blue link "metadata" and the "jsMath" logo. The browser's status bar at the bottom indicates "Done" and "Proxy: None".

- ▶ à partir de l'équa. diff.
- ▶ résultats garantis
- ▶ précision arbitraire
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Un dictionnaire des fonctions D-finies



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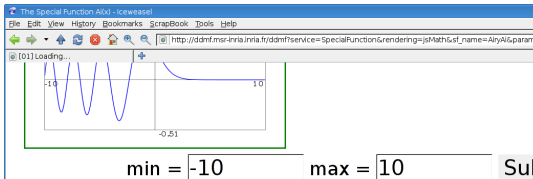
path = `1/4+1/4*i` precision = `80` `Submit Query`

4. Taylor expansion of $Ai(x)$ at 0

- Expansion of $AiryAi$ at 0:

$$Ai(x) = \sum_{n=0}^{\infty} \frac{1}{3} \frac{\sqrt[3]{3} x^{3n}}{9^n \Gamma(n + 2/3) n!} - \frac{1}{9} \frac{3^{2/3} x^{3n+1}}{9^n \Gamma(n + 4/3) n!}.$$

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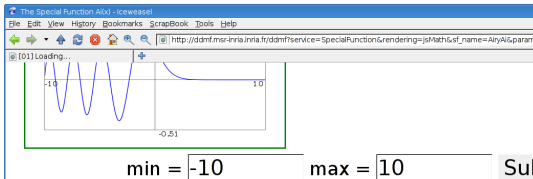
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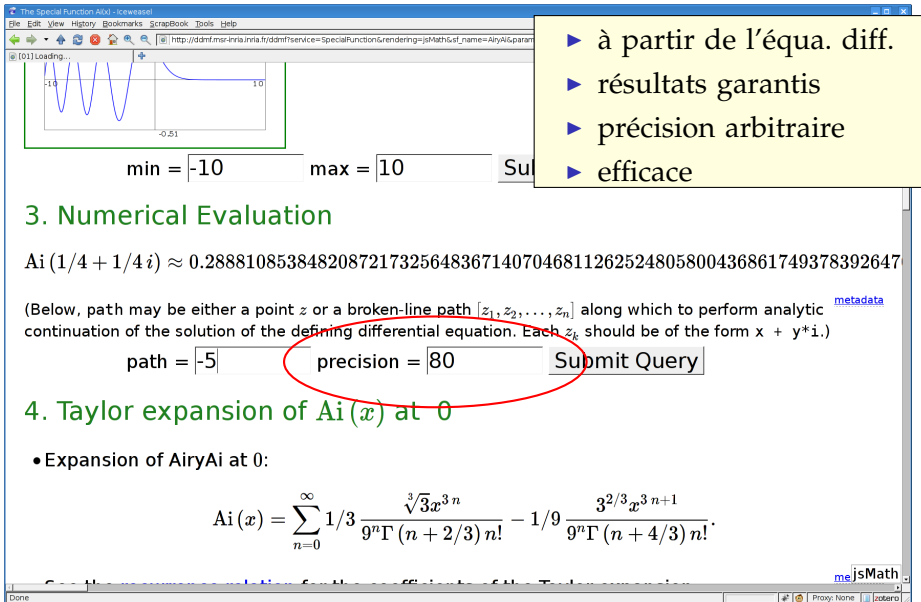
path = -5 precision = 80 Submit Query

4. Taylor expansion of $Ai(x)$ at 0

- Expansion of $AiryAi$ at 0:

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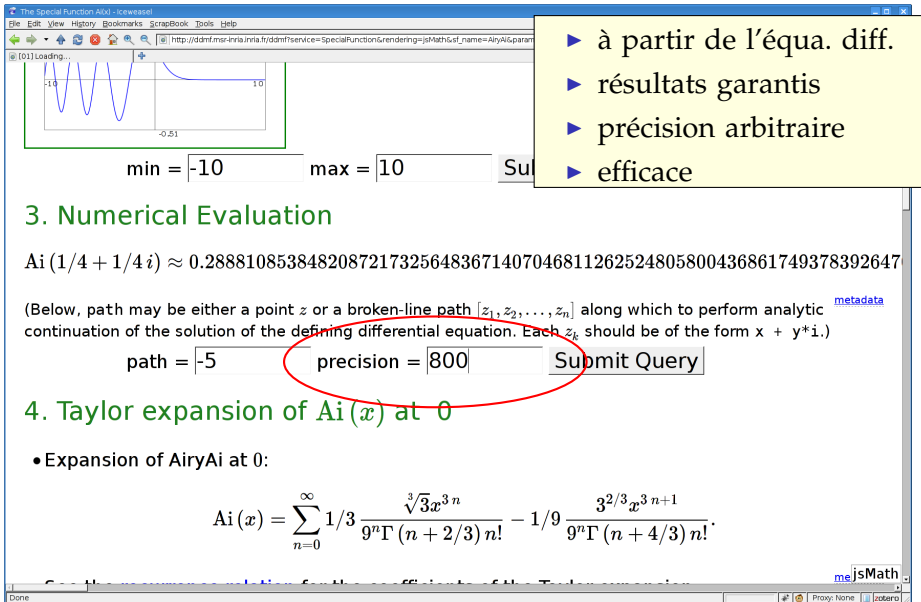
path = precision =

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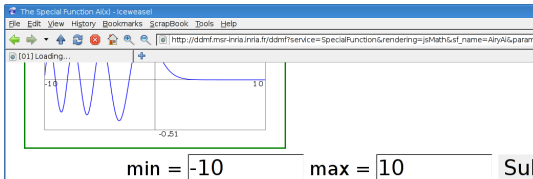
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Un dictionnaire des fonctions D-finies



- ▶ à partir de l'équa. diff.
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- ▶ précision arbitraire
- ▶ efficace

3. Numerical Evaluation

$Ai(-5) \approx 0.350761009024114319788016327696742221484443250893087208211128178049911192682$

(Below, path may be either a point z or a broken-line path $[z_1, z_2, \dots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form $x + y*i$.) [metadata](#)

path = precision =

4. Taylor expansion of $Ai(x)$ at 0

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NumGfun



<http://algo.inria.fr/libraries/> (LGPL)



B. Salvy and P. Zimmermann. Gfun : a Maple package for the manipulation of generating and holonomic functions in one variable. ACM TOMS, 1994.



M. Mezzarobba. NumGfun : a Package for Numerical and Analytic Computation with D-finite functions. ISSAC 2010.

Un exemple aléatoire

```
[> diffeq := random_diffeq(3, 2);
```

Un exemple aléatoire

```
> diffeq := random_diffeq(3, 2);
```

$$\begin{aligned} \text{diffeq} := & \left\{ \left(\frac{13}{30} + \frac{8}{15} z + \frac{7}{30} z^2 \right) y(z) + \left(-\frac{9}{20} + \frac{29}{30} z \right. \right. \\ & - \frac{1}{12} z^2 \left. \right) \left(\frac{d}{dz} y(z) \right) + \left(-\frac{43}{60} + \frac{49}{60} z \right. \\ & + \frac{11}{30} z^2 \left. \right) \left(\frac{d^2}{dz^2} y(z) \right) + \left(-\frac{7}{12} + \frac{17}{30} z \right. \\ & - \frac{3}{5} z^2 \left. \right) \left(\frac{d^3}{dz^3} y(z) \right), y(0) = 0, D(y)(0) = \frac{7}{30}, D^{(2)}(y)(0) = \\ & \left. -\frac{43}{60} \right\} \end{aligned}$$

Un exemple aléatoire

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$$\begin{aligned} \text{diffeq} := & \left\{ \left(\frac{13}{30} + \frac{8}{15} z + \frac{7}{30} z^2 \right) y(z) + \left(-\frac{9}{20} + \frac{29}{30} z \right. \right. \\ & \left. \left. - \frac{1}{12} z^2 \right) \left(\frac{d}{dz} y(z) \right) + \left(-\frac{43}{60} + \frac{49}{60} z \right. \right. \\ & \left. \left. + \frac{11}{30} z^2 \right) \left(\frac{d^2}{dz^2} y(z) \right) + \left(-\frac{7}{12} + \frac{17}{30} z \right. \right. \\ & \left. \left. - \frac{3}{5} z^2 \right) \left(\frac{d^3}{dz^3} y(z) \right), y(0) = 0, D(y)(0) = \frac{7}{30}, D^{(2)}(y)(0) = \right. \\ & \left. -\frac{43}{60} \right\} \end{aligned}$$

```
> evaldiffeq(diffeq, y(z), (1+I)/5, 40);
```

Un exemple aléatoire

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$$\begin{aligned} \text{diffeq} := & \left\{ \left(\frac{13}{30} + \frac{8}{15} z + \frac{7}{30} z^2 \right) y(z) + \left(-\frac{9}{20} + \frac{29}{30} z \right. \right. \\ & - \left. \frac{1}{12} z^2 \right) \left(\frac{d}{dz} y(z) \right) + \left(-\frac{43}{60} + \frac{49}{60} z \right. \\ & + \left. \frac{11}{30} z^2 \right) \left(\frac{d^2}{dz^2} y(z) \right) + \left(-\frac{7}{12} + \frac{17}{30} z \right. \\ & - \left. \frac{3}{5} z^2 \right) \left(\frac{d^3}{dz^3} y(z) \right), y(0) = 0, D(y)(0) = \frac{7}{30}, D^{(2)}(y)(0) = \\ & \left. -\frac{43}{60} \right\} \end{aligned}$$

```
> evaldiffeq(diffeq, y(z), (1+I)/5, 40);
```

```
0.0448555748776784313189330814759311548663
```

```
+ 0.0199048983021280530504789772581099788282 I
```

Grande précision

```
[> evaldiffeq(diffeq, y(z), 1/5, 1000000);
```


Grande précision

```
[> evaldiffeq(diffeq, y(z), 1/5, 1000000);
```

(29 min plus tard...)

Grande précision

```
> evaldiffeq(diffeq, y(z), 1/5, 1000000);  
0.033253281257567506772459381920024394391065961347292863\  
13611785593075654371610784719859620906805710762776061\  
65993844793918297941976188620650536691082179149605904\  
31080482988558239935175505111768194891591740446771304\  
74730251896359727561534310095807343639273056518962333\  
97217595138842309884016425632431029577130431472108646\  
95485154767624024297343851584414126056237771911489680\  
.....  
97933258259972366466573219602501650218139747781157348\  
78322628655747195818205282428148240800376913561455564\  
29598794491231828039584256430669932365880956101719727\  
33806130243940574539991121877851105270752378138422728\  
76176859592508040781771637205060431902227437673286901\  
71292574098466950906705927590030494460150099288210121\  
868701569
```

Algorithme d'évaluation [Chudnovsky & Chudnovsky 1988]

Idées principales

0 multiplication rapide

2 prolongement analytique

1 scindage binaire

3 *bit burst*

On peut multiplier deux entiers de n bits en $O(n \log n 2^{O(\log^* n)})$ opérations binaires [Fürer 2007].

Algorithme d'évaluation [Chudnovsky & Chudnovsky 1988]

Idées principales

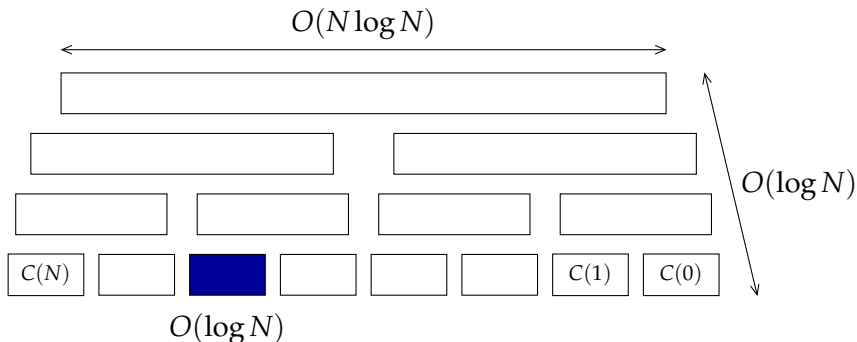
0 multiplication rapide

2 prolongement analytique

1 scindage binaire

3 *bit burst*

1. Dans le disque de convergence d'un développement de Taylor : sommer efficacement la série (récurrence)



Algorithme d'évaluation [Chudnovsky & Chudnovsky 1988]

Idées principales

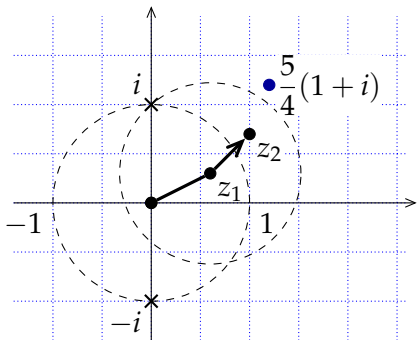
0 multiplication rapide

2 prolongement analytique

1 scindage binaire

3 *bit burst*

2. Évaluation hors du disque de convergence :
transporter les conditions initiales



$$\arctan\left(\frac{5}{4}(1+i)\right) = ?$$

$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0,570\dots+0,220\dots i \\ 0 & 0,728\dots-0,206\dots i \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

$$\begin{bmatrix} y(z_2) \\ y'(z_2) \end{bmatrix} = \begin{bmatrix} 1 & 0,365\dots+0,329\dots i \\ 0 & 0,751\dots-0,079\dots i \end{bmatrix} \begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix}$$

Algorithme d'évaluation [Chudnovsky & Chudnovsky 1988]

Idées principales

0 multiplication rapide

2 prolongement analytique

1 scindage binaire

3 *bit burst*

3. Points donnés à grande précision :
prolongement analytique même si la série converge

$$z_0 = 10_2 \rightarrow z_1 = 10,1_2$$

$$\rightarrow z_2 = 10,101_2 \quad \sin(e) = \sin(2,718\dots) = ?$$

$$\rightarrow z_3 = 10,1011011_2$$

$$\rightarrow z_4 = 10,101101110010100_2$$

$$\rightarrow \dots$$

$$\rightarrow z = 10.101101110010100110000\dots_2 \simeq e$$

Algorithme d'évaluation [Chudnovsky & Chudnovsky 1988]

Idées principales

0 multiplication rapide

2 prolongement analytique

1 scindage binaire

3 *bit burst*

Théorème (Chudnovsky²)

À z fixé, on peut calculer $y(z)$ à 2^{-n} près en

$$O\left(M\left(n \cdot (\log n)^3\right)\right)$$

opérations binaires.

Algorithme d'évaluation [Chudnovsky & Chudnovsky 1988]

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opérations binaires.

Améliorations

Suivi des erreurs

- ▶ Précision des calculs intermédiaires
- ▶ Bornes fines pour les troncatures de séries

« Facteur constant »

- ▶ Structure des matrices de récurrence
- ▶ Calcul simultané efficace de plusieurs dérivées

Points singuliers réguliers

- ▶ Méthode de Heffter-Poole simplifiée « à la Ore »
- ▶ Algorithme de scindage binaire explicite
- ▶ Traitement plus efficace des « logarithmes »

II
MEILEN
BIS
BERLIN

PROTECTOR
MILITÄRISCHER
LIGA 1944
1945

Bornes

Motivation (I) : évaluation numérique

$$\sum_{n=0}^{\infty} y_n z^n = \sum_{n=0}^{N-1} y_n z^n + \underbrace{\sum_{n=N}^{\infty} y_n z^n}_{?}$$

Calculer les ordres de troncature et autres bornes ?



Chudnovsky & Chudnovsky — Ordres de grandeur uniquement



van der Hoeven (1999, 2001, 2003) — Bornes données par la formule de Cauchy

Bornes asymptotiquement optimales ?

Motivation (II) : suites récurrentes

Permutations de Baxter

(OEIS A001181)

- ▶ $(n+2)(n+3)B_n = (7n^2 + 7n - 2)B_{n-1} + 8(n-1)(n-2)B_{n-2}$,
 $B_0 = B_1 = 1$
- ▶ $B_n \leq 2,9 \cdot 8^n$

Formule de Chudnovsky et Chudnovsky pour π

- ▶ $\frac{1}{\pi} = \frac{12}{640320^{3/2}} \sum_{k=0}^{\infty} t_k$
où $t_k = \frac{(-1)^k (6k)! (13591409 + 545140134k)}{(3k)! (k!)^3 640320^{3k}}$
- ▶ $\left| \frac{640320^{3/2}}{12\pi} - \sum_{k=0}^{n-1} t_k \right| \leq 10^6 (2,3n^3 + 13,6n^2 + 25n + 13,6) \alpha^n$
où $\alpha = \frac{1}{151931373056000} \simeq 0,66 \cdot 10^{-14}$

Bornes « fines »

Entrée Réurrence + conditions initiales

$$\{p_s(n) y_{n+s} + \dots + p_0(n) y_n = 0, \quad y_0 = \dots\}$$

Sortie $|y_n| \leq n!^{p/q} \alpha^n \varphi(n)$

avec φ sous-exponentielle, i.e. $\varphi(n) = e^{o(n)}$

- ▶ borne correcte
- ▶ pour des conditions initiales génériques :
 p/q et α optimaux (voire $\varphi(n) = n^{O(1)}$)

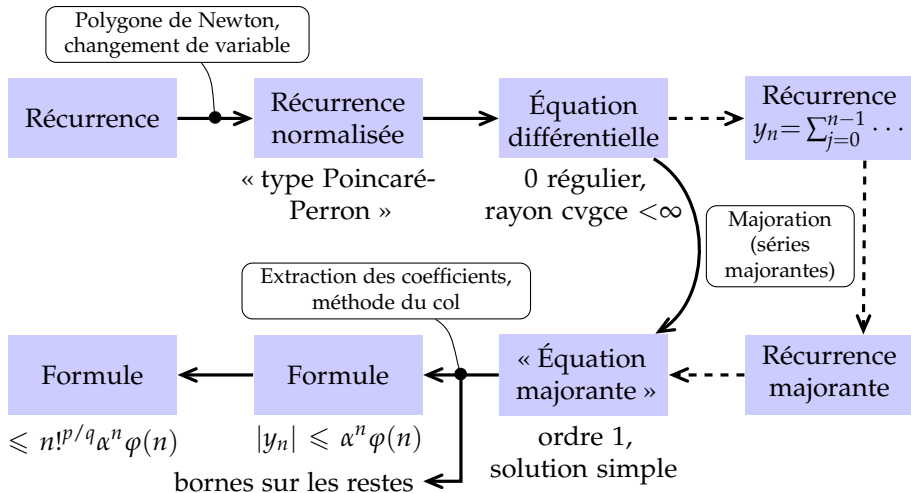
Théorème

On peut calculer p/q , α , φ remplissant ces conditions.



M. Mezzarobba and B. Salvy. Effective bounds for P-recursive sequences.
JSC, 2010.

Démarche



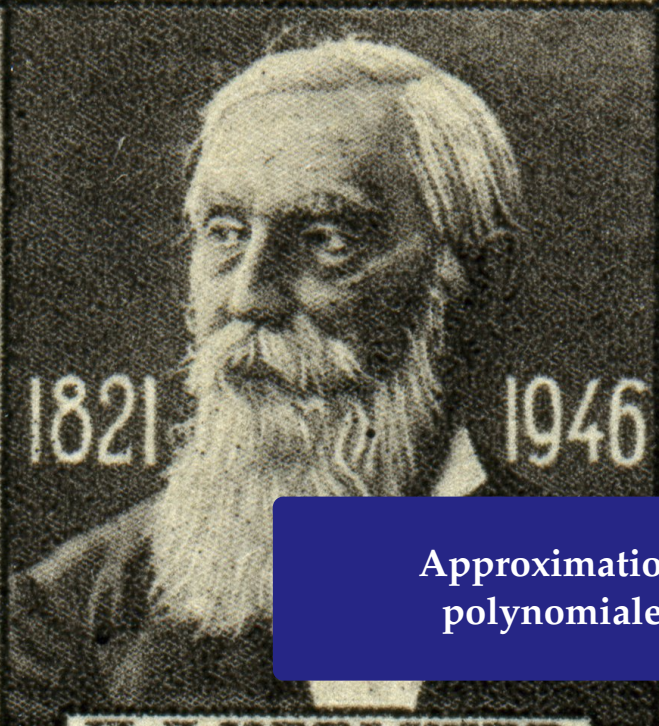
Influence sur l'évaluation numérique

	$\frac{\operatorname{arccot}(z)}{(z^2-1)(z^2+5)} @ \frac{1}{2}$	$\psi(1/2)$	$\arctan \frac{9}{10}$	$\arctan \frac{99}{100}$
10^{-10}	64/27	40/23	336/164	4238/1496
10^{-100}	380/321	342/313	2338/2108	25210/21848
10^{-1000}	3392/3307	3336/3293	22050/21754	231844/227810

	$\frac{\exp(1/(1-z))}{(1-z)} @ \frac{1}{2}$	$\operatorname{Bi}\left(\frac{1}{1-z}\right) @ \frac{1}{2}$	$\operatorname{Ai}\left(\frac{1}{1-z}\right) @ \frac{3}{4}$	$\operatorname{Ai}\left(\frac{1}{1-z}\right) @ \frac{7}{8}$
10^{-10}	70/54	148/56	1558/77	23818/215
10^{-100}	418/387	664/416	3430/879	29258/2025
10^{-1000}	3568/3490	4700/3645	16284/8372	69594/18529

	e^{-100}	$\operatorname{erf}^2(1)$	$\operatorname{erf}(10)$	$\operatorname{erf}(100)$
10^{-10}	298/291	60/33	628/574	54492/54388
10^{-100}	456/450	190/163	936/894	54904/54800
10^{-1000}	1406/1402	1036/1011	2828/2800	58870/58772

nombre de termes calculé / nombre de termes minimal nécessaire



Approximation
polynomiale

Évaluations répétées

```
[> deq := holerptodiffeq(AiryAi(z), y(z)):
```

```
[>
```

Évaluations répétées

```
[> deq := holexprtodiffeq(AiryAi(z), y(z)):  
[> myAi := diffeqtoproc(deq, y(z),  
                        prec=12, disks=[[0,6]]):
```

Évaluations répétées

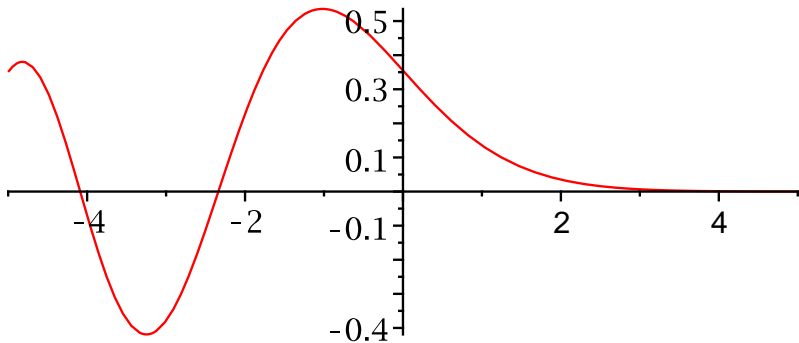
```
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[>
```

Évaluations répétées

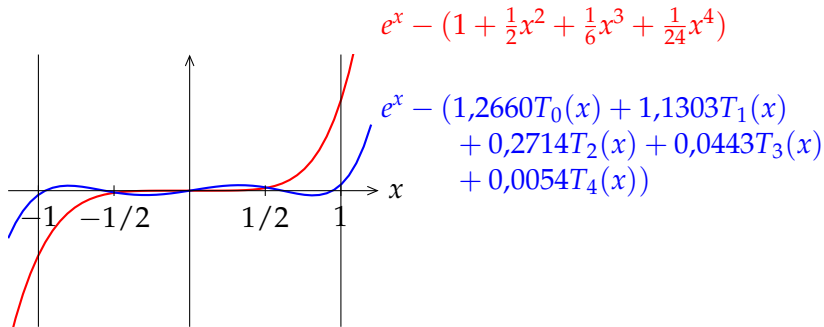
```
[> deq := holexprtodiffeq(AiryAi(z), y(z)):  
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[> plot(myAi, -5..5);
```

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[> plot(myAi, -5..5);  
[>
```



Séries de Taylor et séries de Tchebycheff










Approximation quasi-minimax

Pour toute fonction f continue sur $[-1; 1]$,

$$\|f - p_d\|_\infty \leq \left(\frac{4}{\pi^2} \log(d+1) + 4 \right) \|f - p_d^*\|_\infty$$


Historique & enjeux

- ▶ Calcul des coefficients de Tchebycheff
 -  Lánzos (1938) – méthode τ
 -  Clenshaw (1957) – calcul itératif à la Miller
- ▶ Récurrence
 -  Fox & Parker (1968) – petits ordres, lien avec Clenshaw
 -  Paszkowski (1975) – cas général
 -  Geddes (1977), Rebillard (1998), Benoit & Salvy (2009) – calcul formel
- ▶ Calcul par intervalles sur les séries de Tchebycheff
 -  Kaucher & Miranker (1984) – ultra-arithmétique
 -  Brisebarre & Joldeş (2010) – ChebModels

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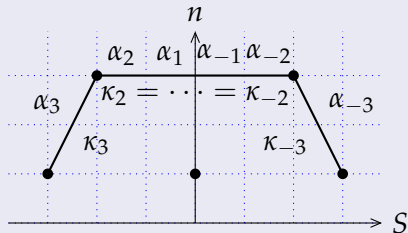
exposé de Mioara Joldeş vendredi

 -  Brisebarre & Joldeş (2010) – ChebModels

Séries de Tchebycheff D-finies

Difficultés

- ▶ Solutions (toujours) divergentes
- ▶ Conditions initiales ?
- ▶ Certification du résultat



Démarche

1. Calcul des coefficients
2. Validation séparée



A. Benoit, M. Joldes and M. Mezzarobba. Rigorous uniform approximation of D-finite functions using Chebyshev expansions. In preparation.

Calcul des coefficients à la Miller

Exemple

$$y(x) = e^x = \sum_{n=-\infty}^{\infty} c_n T_n(x)$$

$$c_{n+1} + 2n c_n - c_{n-1} = 0$$

$$u_0 \approx -4,40 \cdot 10^{81}$$

$$u_1 \approx 1,96 \cdot 10^{81}$$


$$u_2 \approx -4,72 \cdot 10^{80}$$

⋮

$$u_{50} \approx 1,02 \cdot 10^2$$

$$u_{51} = 1$$

$$u_{52} = 0$$

$$c_n := u_n / S$$


$$c_0 \approx 1,27$$

$$c_1 \approx -5,65 \cdot 10^{-1}$$

$$c_2 \approx 1,36 \cdot 10^{-1}$$

⋮

$$c_{50} \approx 2,93 \cdot 10^{-80}$$

$$c_{51} \approx 2,88 \cdot 10^{-82}$$

$$c_{52} \approx 2,77 \cdot 10^{-84}$$

$$S = \sum_{n=-50}^{50} u_n T_n(0) \approx -3,48 \cdot 10^{81}$$

Calcul des coefficients

Complexité **linéaire** en l'indice de départ N .

Proposition (sous hypothèses simplificatrices)

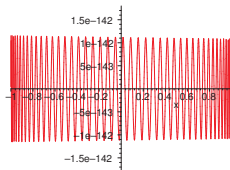
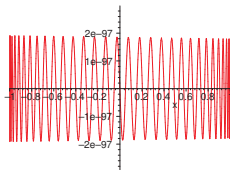
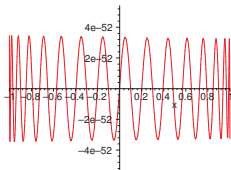
L'erreur sur les coefficients calculés

$$\max_{n=0}^N |c_n^{[N]} - c_n|$$

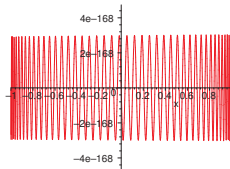
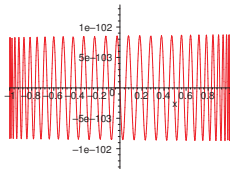
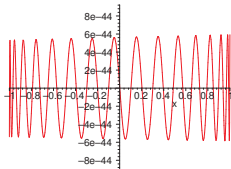
décroît exponentiellement avec N .

Qualité des polynômes calculés

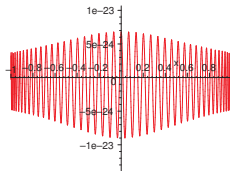
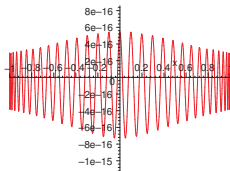
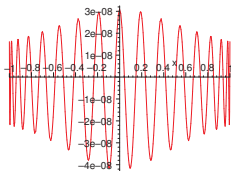
$$\frac{e^{x/2}}{\sqrt{x+16}}$$



$$\frac{3 \cos x - \sin x}{2}$$



$$e^{1/(1+2x^2)}$$



degré = 30

degré = 60

degré = 90

Validation

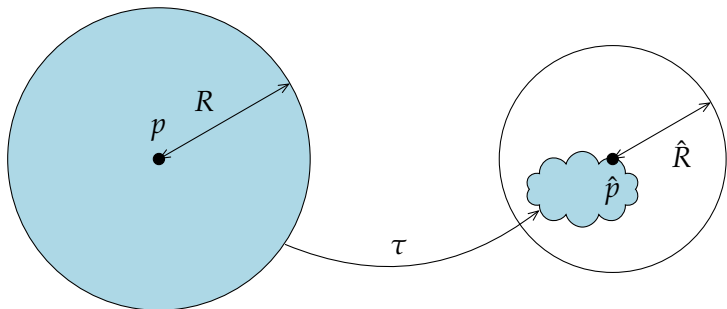
Entrée Opérateur différentiel, conditions initiales,
polynôme p de degré d , précision ε

Sortie R tel que $\|y - p\|_\infty \leq R = O(\sqrt{d} (\|y^{(r-1)} - p^{(r-1)}\|_\infty + \varepsilon))$

$$\tau(y) := \left(x \mapsto y_0 + \int_0^x \frac{a(t)}{b(t)} y(t) dt \right)$$

$$\|\tau(f) - \tau(g)\|_\infty \leq \gamma \|f - g\|_\infty$$

$\gamma < 1$



Validation

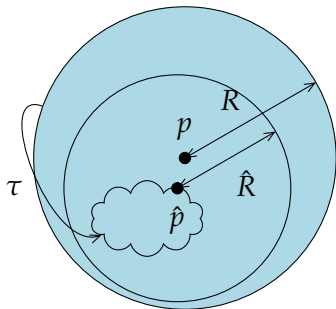
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$$\|p - \hat{p}\|_\infty + \hat{R} \leq R$$

Validation

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Algorithme

- ▶ Prendre i assez grand
- ▶ Calculer $p_i \approx \tau^i(p)$
- ▶ Renvoyer $R \geq \frac{\|p - p_i\|_\infty + (\text{erreurs})}{1 - \gamma_i}$

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Algorithme

- ▶ Prendre i assez grand
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- ▶ Renvoyer $R \geq \frac{\|p - p_i\|_\infty + (\text{erreurs})}{1 - \gamma_i}$ O(d) ops

Qualité des bornes validées

$$\log_{10} \frac{\text{(borne calculée)}}{\|y - p\|_{\infty}}$$

$\frac{e^{x/2}}{\sqrt{x+16}}$	4,8	0,58	0,57
$\frac{3 \cos x - \sin x}{2}$	3,1	3,7	4,1
$e^{1/(1+2x^2)}$	0,57	0,56	0,56
	degré = 30	degré = 60	degré = 90



En bref

- ▶ Prolongement analytique numérique multiprécision général – garanti – automatique – rapide
- ▶ Bornes fines
suites – séries majorantes – restes de séries
- ▶ Approximants polynomiaux sur la base de Tchebycheff
complexité linéaire en le degré – bornes d'erreur fines



Code disponible

<http://algo.inria.fr/libraries/> (GNU LGPL)



Perspectives

- ▶ NumGfun 1.0 et au-delà
- ▶ Travaux en cours sur l'approximation polynomiale
- ▶ Calcul formel pour l'arithmétique des ordinateurs



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Merci !



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