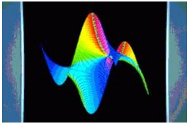
Évaluation numérique de fonctions spéciales et combinatoire analytique avec

NumGfun

Marc Mezzarobba

Projet Algorithms



INRIA Paris Rocquencourt

Séminaire Calin, 15 mars 2011

Dynamic Dictionary of Mathematical Functions - Iceweasel

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[01] Dynamic Dictionary of Ma...

Home   Glossary

Dynamic Dictionary of Mathematical Functions

Welcome to this interactive site on Mathematical Functions, with properties, truncated expansions, numerical evaluations, plots, and more. The functions currently presented are elementary functions and special functions of a single variable. More functions — special functions with parameters, orthogonal polynomials, sequences — will be added with the project advances.

Select a special function from the list

- Help on selecting and configuring the mathematical rendering
- DDMF developers list
- Motivation of the project

rendering link

Contents

- The inverse cosecant $\operatorname{arccsc}(x)$
- The inverse cosine $\arccos(x)$
- The inverse cotangent $\operatorname{arccot}(x)$
- The inverse hyperbolic cosecant $\operatorname{arccsch}(x)$
- The Airy function of the first kind $\operatorname{Ai}(x)$
- The inverse secant $\operatorname{arcsec}(x)$
- The inverse sine $\arcsin(x)$
- The inverse tangent $\arctan(x)$
- The Airy function (of the second kind) $\operatorname{Bi}(x)$
- The hyperbolic cosine integral $\operatorname{Chi}(x)$
- The cosine integral $\operatorname{Ci}(x)$
- The cosine $\cos(x)$
- The exponential integral $\operatorname{Ei}(x)$
- The error function $\operatorname{erf}(x)$
- The complementary error function $\operatorname{erfc}(x)$
- The imaginary error function $\operatorname{erfi}(x)$

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Home Glossary

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Select a special function from

Benoit, Chyzak, Darrasse, Gerhold, M. & Salvy (2010)

- DDMF developers list
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Contents

rendering link

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The Special Function Ai(x) - Iceweasel

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[01] Loading...

Home Glossary

The Special Function $\mathrm{Ai}\,(x)$

1. Differential equation

rendering link

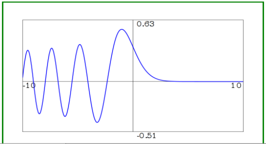
The function $\mathrm{Ai}\,(x)$ satisfies

$$\frac{d^2}{dx^2} y\,(x) - xy\,(x) = 0$$

with initial values $y\,(0) = 1/3\,\frac{\sqrt[3]{3}}{\Gamma(2/3)}$, $(y')\,(0) = -1/2\,\frac{\sqrt[6]{3}\Gamma(2/3)}{\pi}$.

metadata

2. Plot of $\mathrm{Ai}\,(x)$



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The Special Function Ai(x) - Iceweasel

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Home

The Special Function Ai(x)

Structure de données:
EDL à coeff. polynomiaux
+ conditions initiales
(fonctions D-finies)

1. Differential equation

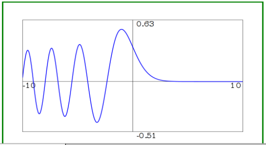
The function $\mathrm{Ai}(x)$ satisfies

$$\frac{d^2}{dx^2}y(x) - xy(x) = 0$$

with initial values $y(0) = 1/3\,\frac{\sqrt[3]{3}}{\Gamma(2/3)}$, $(y')(0) = -1/2\,\frac{\sqrt[6]{3}\Gamma(2/3)}{\pi}$.

metadata

2. Plot of $\mathrm{Ai}(x)$



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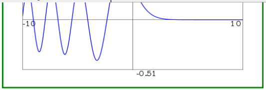

min = -10 max = 10 Submit Query

3. Numerical Evaluation

$$\mathrm{Ai}\,(1/4+1/4\,i)\approx 0.28881085-0.06285935\,i.$$

(Below, path may be either a point z or a broken-line path $[z_1, z_2, \ldots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form x + y*i.) metadata

path = 1/4+1/4*i precision = 8 Submit Query

4. Taylor expansion of $\mathrm{Ai}\,(x)$ at 0

- Expansion of AiryAi at 0:

$$\mathrm{Ai}\,(x)=\sum_{n=0}^{\infty} 1/3\,\frac{\sqrt[3]{3}x^{3\,n}}{9^n\Gamma\,(n+2/3)\,n!}-1/9\,\frac{3^{2/3}x^{3\,n+1}}{9^n\Gamma\,(n+4/3)\,n!}.$$

me jsMath

Done Proxy: None zotero

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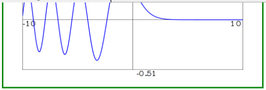

min = -10 max = 10 Submit Query

3. Numerical Evaluation

$$\mathrm{Ai}\,(1/4 + 1/4\,i) \approx 0.28881085 - 0.06285935\,i.$$

metadata

(Below, path may be either a point z or a broken-line path $[z_1, z_2, \ldots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form x + y*i.)

path = 1/4+1/4*i precision = 8 Submit Query

4. Taylor expansion of $\mathrm{Ai}\,(x)$ at 0

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me jsMath

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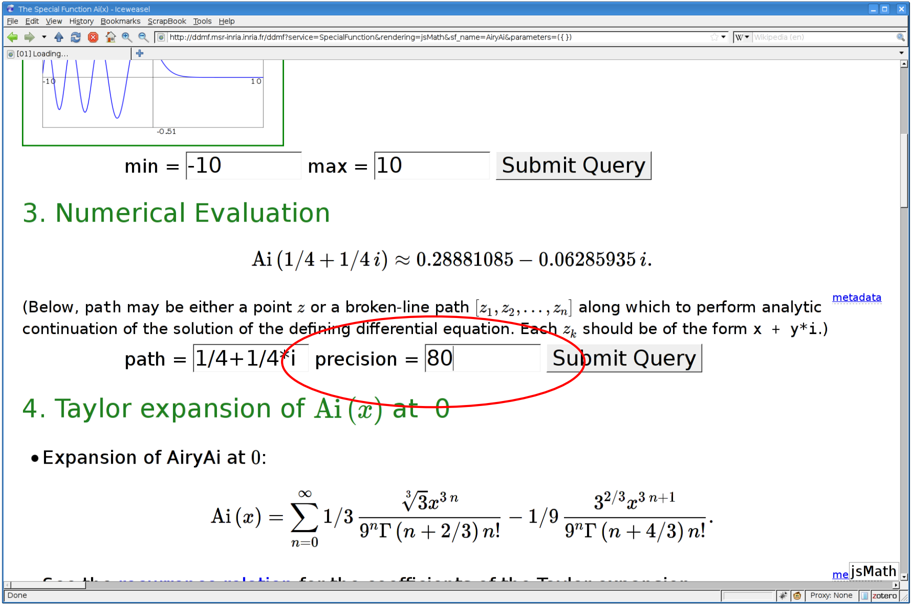
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[01] Loading...
-10
10
-0.51
min = -10 max = 10 Submit Query
3. Numerical Evaluation
$\mathrm{Ai}\,(1/4+1/4\,i) \approx 0.28881085 - 0.06285935\,i.$
metadata
(Below, path may be either a point z or a broken-line path $[z_1, z_2, \ldots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form x + y*i.)
path = 1/4+1/4*i precision = 80 Submit Query
4. Taylor expansion of $\mathrm{Ai}\,(x)$ at 0
• Expansion of AiryAi at 0:
$\mathrm{Ai}\,(x) = \sum_{n=0}^{\infty} 1/3\,\frac{\sqrt[3]{3}x^{3\,n}}{9^n\Gamma\,(n+2/3)\,n!} - 1/9\,\frac{3^{2/3}x^{3\,n+1}}{9^n\Gamma\,(n+4/3)\,n!}.$
me jsMath
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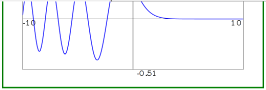

min = -10 max = 10 Submit Query

3. Numerical Evaluation

$$\mathrm{Ai}\,(1/4+1/4\,i) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647$$

(Below, path may be either a point z or a broken-line path $[z_1, z_2, \ldots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form `x + y*i`.) metadata

path = 1/4+1/4*i precision = 80 Submit Query

4. Taylor expansion of $\mathrm{Ai}\,(x)$ at 0

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$$\mathrm{Ai}\,(x) = \sum_{n=0}^{\infty} 1/3\,\frac{\sqrt[3]{3}x^{3\,n}}{9^n\Gamma\,(n+2/3)\,n!} - 1/9\,\frac{3^{2/3}x^{3\,n+1}}{9^n\Gamma\,(n+4/3)\,n!}.$$

me jsMath

Done Proxy: None zotero

- précision arbitraire
- résultats garantis
- à partir de l'équa. diff.

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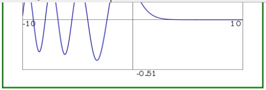

min = -10 max = 10 Su

3. Numerical Evaluation

$$\mathrm{Ai}\,(1/4+1/4\,i) \approx 0.2888108538482087217325648367140704681126252480580043686174937839264\ldots$$

(Below, path may be either a point z or a broken-line path $[z_1, z_2, \ldots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form x + y*i.) metadata

path = 1/4+1/4*i precision = 80 Submit Query

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me jsMath

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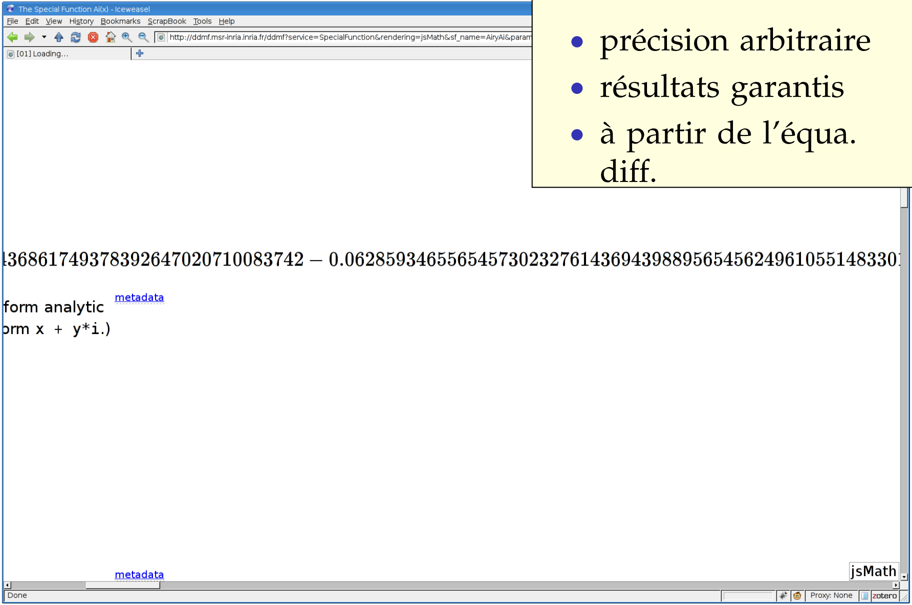

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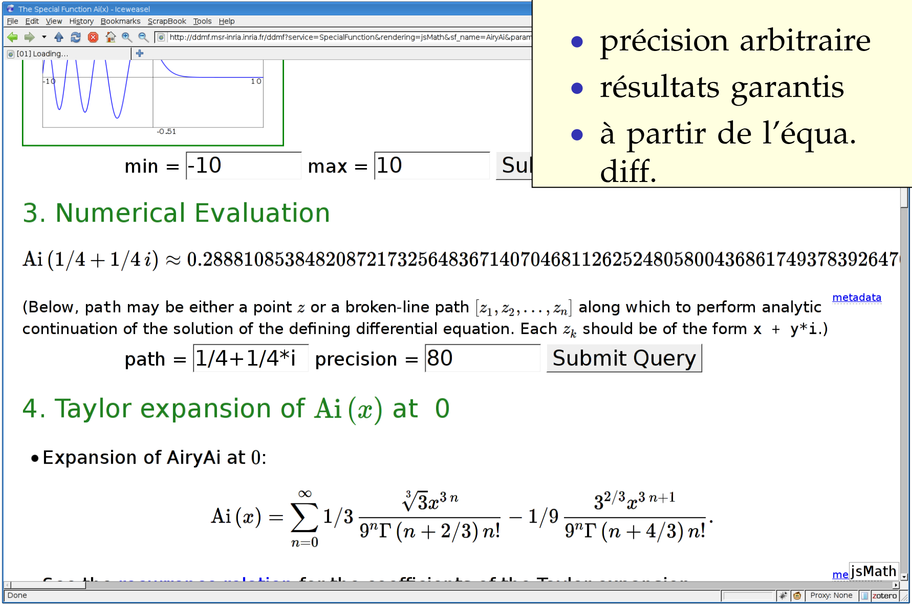

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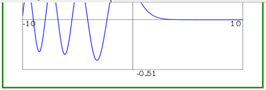

min = -10 max = 10 Su

3. Numerical Evaluation

$\mathrm{Ai}\,(1/4 + 1/4\,i) \approx 0.28881085384820872173256483671407046811262524805800436861749378392647$

(Below, path may be either a point z or a broken-line path $[z_1, z_2, \ldots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form x + y*i.) metadata

path = 1/4+1/4*i precision = 80 Submit Query

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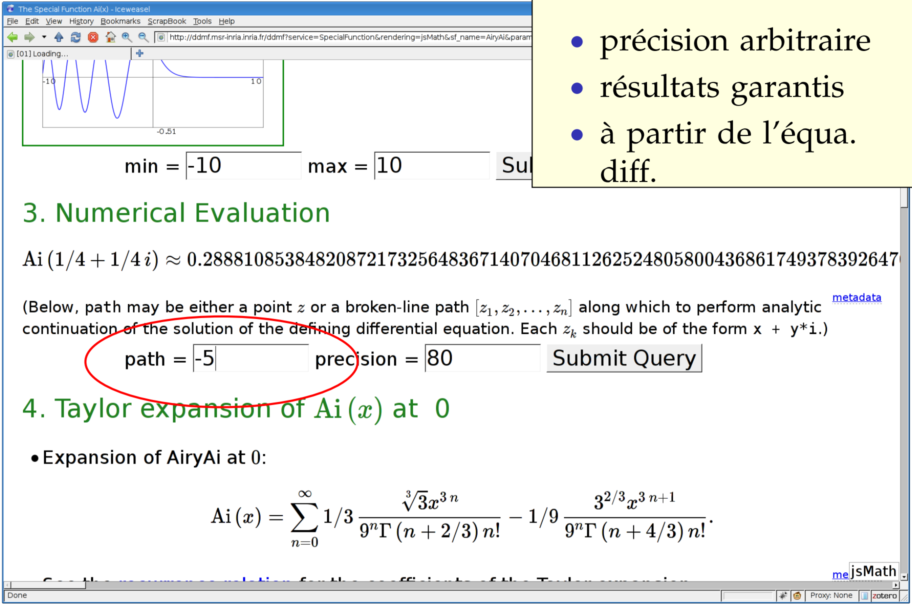

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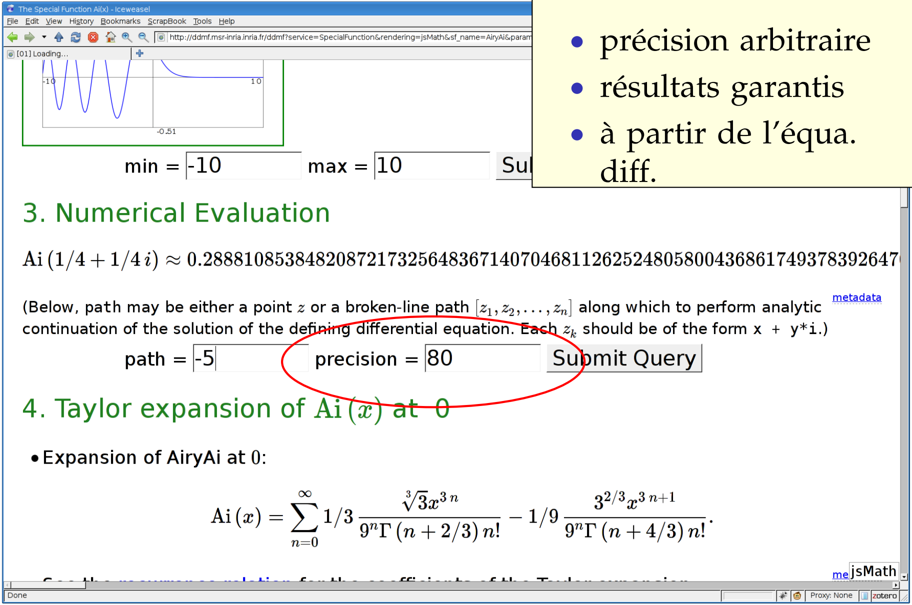

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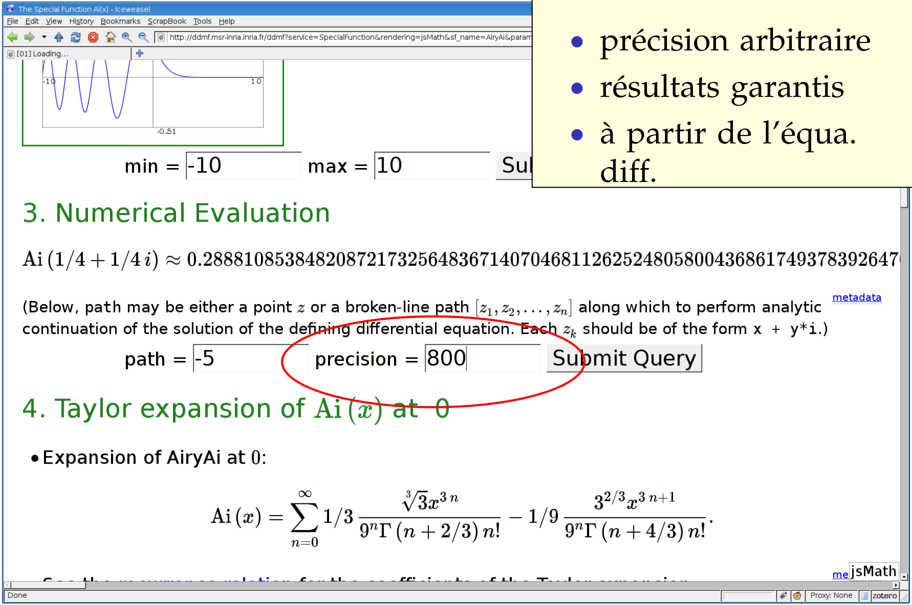

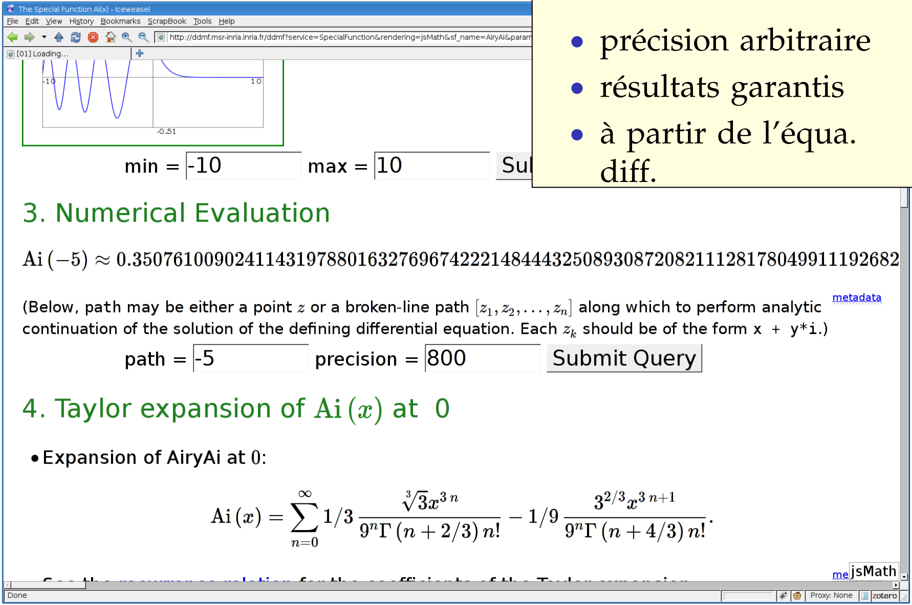
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[01] Loading...

min = -10 max = 10 Su

3. Numerical Evaluation

$$\mathrm{Ai}\,(-5) \approx 0.350761009024114319788016327696742221484443250893087208211128178049911192682$$

(Below, path may be either a point z or a broken-line path $[z_1, z_2, \ldots, z_n]$ along which to perform analytic continuation of the solution of the defining differential equation. Each z_k should be of the form x + y*i.) metadata

path = -5 precision = 800 Submit Query

4. Taylor expansion of $\mathrm{Ai}\,(x)$ at 0

- Expansion of AiryAi at 0:

$$\mathrm{Ai}\,(x) = \sum_{n=0}^{\infty} 1/3\,\frac{\sqrt[3]{3}x^{3\,n}}{9^n\Gamma\,(n+2/3)\,n!} - 1/9\,\frac{3^{2/3}x^{3\,n+1}}{9^n\Gamma\,(n+4/3)\,n!}.$$

me jsMath

Done Proxy: None zotero

- précision arbitraire
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This is AsyRec, A Maple package
accompanying Doron Zeilberger's article:

It finds the asymptotics of solutions of (homog.) linear recurrence equations with polynomial coefficients, using the Birkhoff-Trjitzinsky method .

```
> recop := (n+2)^2*N^2-(7*n^2+21*n+16)*N-8*(n+1)^2;
```

$$recop := (n+2)^2 N^2 - \left(7 n^2 + 21 n + 16\right) N - 8 (n+1)^2$$

```
> AsyC(recop, n, N, 5, [2, 10], 1000);
```

0.36755259694786136634,

$$\frac{8^n \left(1 - \frac{1}{3 n} + \frac{1}{27 n^2} + \frac{1}{81 n^3} + \frac{1}{243 n^4} + \frac{11}{2187 n^5}\right)}{n}$$

(Wimp & Zeilberger 1985, Zeilberger 2008-2009)

/home/mezzarob/docs/recherche/exposes/2010-07-issac/maple1.mw* - [Server 2] - Maple 13

File Edit View Insert Format Table Drawing Plot Spreadsheet Tools Window Help

```
> with(gfun):
> with(NumGfun);
```

[*abs_with_RootOf*, *analytic_continuation*,
bound_diffeq, *bound_diffeq_tail*,
bound_ratpoly, *bound_rec*, *bound_rec_tail*,
diffeqtoproc, *dominant_root*, *evaldiffeq*,
fnth_term, *make_waksman_proc*,
needed_terms, *transition_matrix*]

```
> evaldiffeq(diff(y(z),z)=y(z), y
  (z), 1, 10000);
```

2.7182818284590452353602874713526624977
572470936999595749669676277240766303
535475945713821785251664274274663919
320030599218174135966290435729003342
952605956307381323286279434907632338
298807531952510190115738341879307021
540891499348841675092447614606680822
648001684774118537423454424371075390
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830007520449338265602976067371132007

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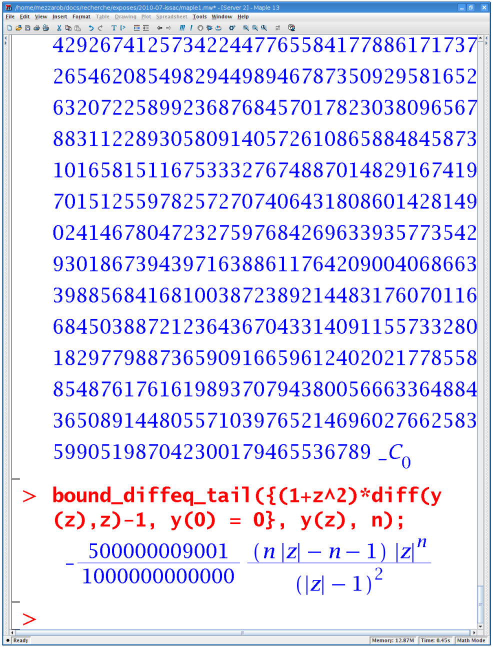
/home/mezzarob/docs/recherche/exposes/2010-07-issac/maple1.mw* - [Server 2] - Maple 13
File Edit View Insert Format Table Drawing Plot Spreadsheet Tools Window Help
4292674125734224477655841778861717 37
2654620854982944989467873509295816 52
6320722589923687684570178230380965 67
8831122893058091405726108658848458 73
1016581511675333276748870148291674 19
7015125597825727074064318086014281 49
0241467804723275976842696339357735 42
9301867394397163886117642090040686 63
3988568416810038723892144831760701 16
6845038872123643670433140911557332 80
1829779887365909166596124020217785 58
8548761761619893707943800566633648 84
3650891448055710397652146960276625 83
5990519870423001794655367 89 _C0
> bound_diffeq_tail({(1+z^2)*diff(y(z),z)-1, y(0) = 0}, y(z), n);
- 500000009001 / 1000000000000 (n |z| - n - 1) |z|^n / (|z| - 1)^2
>
Ready
Memory: 12.87M
Time: 0.45s
Math Mode

http://algo.inria.fr/libraries/ (LGPL)

B. Salvy and P. Zimmermann. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. 1994.


© Bruce Marlin

```
> diffeq := random_diffeq(3, 2);
```

```
> diffeq := random_diffeq(3, 2);
```

$$diffeq := \left\{ \left(\frac{13}{30} + \frac{8}{15} z + \frac{7}{30} z^2\right) y(z) + \left(-\frac{9}{20} + \frac{29}{30} z - \frac{1}{12} z^2\right) \left(\frac{d}{dz} y(z)\right) + \left(-\frac{43}{60} + \frac{49}{60} z + \frac{11}{30} z^2\right) \left(\frac{d^2}{dz^2} y(z)\right) + \left(-\frac{7}{12} + \frac{17}{30} z - \frac{3}{5} z^2\right) \left(\frac{d^3}{dz^3} y(z)\right), y(0) = 0, D(y)(0) = \frac{7}{30}, D^{(2)}(y)(0) = -\frac{43}{60} \right\}$$

```
> diffeq := random_diffeq(3, 2);
```

$$diffeq := \left\{ \left(\frac{13}{30} + \frac{8}{15} z + \frac{7}{30} z^2 \right) y(z) + \left(-\frac{9}{20} + \frac{29}{30} z - \frac{1}{12} z^2 \right) \left(\frac{d}{dz} y(z) \right) + \left(-\frac{43}{60} + \frac{49}{60} z + \frac{11}{30} z^2 \right) \left(\frac{d^2}{dz^2} y(z) \right) + \left(-\frac{7}{12} + \frac{17}{30} z - \frac{3}{5} z^2 \right) \left(\frac{d^3}{dz^3} y(z) \right), y(0) = 0, D(y)(0) = \frac{7}{30}, D^{(2)}(y)(0) = -\frac{43}{60} \right\}$$

```
> evaldiffeq(diffeq, y(z), (1+I)/5, 40);
```

```
> diffeq := random_diffeq(3, 2);
```

$$diffeq := \left\{ \left(\frac{13}{30} + \frac{8}{15} z + \frac{7}{30} z^2\right) y(z) + \left(-\frac{9}{20} + \frac{29}{30} z - \frac{1}{12} z^2\right) \left(\frac{d}{dz} y(z)\right) + \left(-\frac{43}{60} + \frac{49}{60} z + \frac{11}{30} z^2\right) \left(\frac{d^2}{dz^2} y(z)\right) + \left(-\frac{7}{12} + \frac{17}{30} z - \frac{3}{5} z^2\right) \left(\frac{d^3}{dz^3} y(z)\right), y(0) = 0, D(y)(0) = \frac{7}{30}, D^{(2)}(y)(0) = -\frac{43}{60} \right\}$$

```
> evaldiffeq(diffeq, y(z), (1+I)/5, 40);
```

$$0.04485557487767843131893308147593115486663 + 0.01990489830212805305047897725810997882882\, I$$

```
> evaldiffeq(diffeq, y(z), 1/5, 1000000);
```

```
> evaldiffeq(diffeq, y(z), 1/5, 1000000);
```

(29 min plus tard...)

```
> evaldiffeq(diffeq, y(z), 1/5, 1000000);
0.033253281257567506772459381920024394391065961347292863\
    1361178559307565437161078471985962090680571076277606\
    6599384479391829794197618862065053669108217914960590\
    3108048298855823993517550511176819489159174044677130\
    7473025189635972756153431009580734363927305651896233\
    9721759513884230988401642563243102957713043147210864\
    9548515476762402429734385158441412605623777191148968\
. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
    9793325825997236646657321960250165021813974778115734\
    7832262865574719581820528242814824080037691356145556\
    2959879449123182803958425643066993236588095610171972\
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    7129257409846695090670592759003049446015009928821012\
    868701569
```
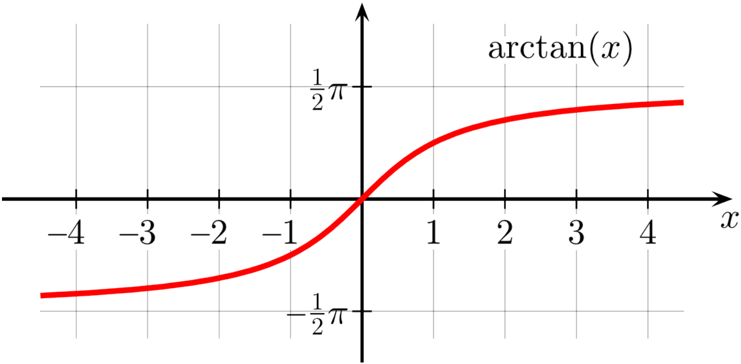
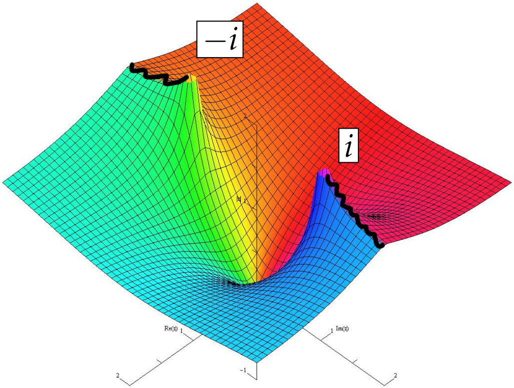
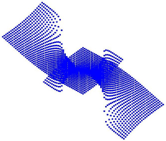

Évaluation numérique

- générale
- automatique
- garantie
- asymptotiquement rapide

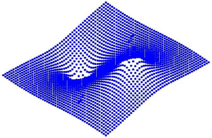


User:Geek3

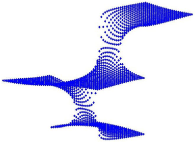




(Re)

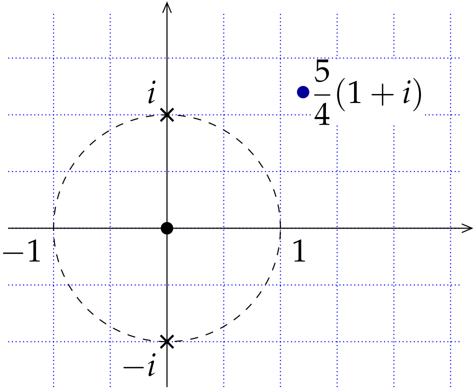


(Im)

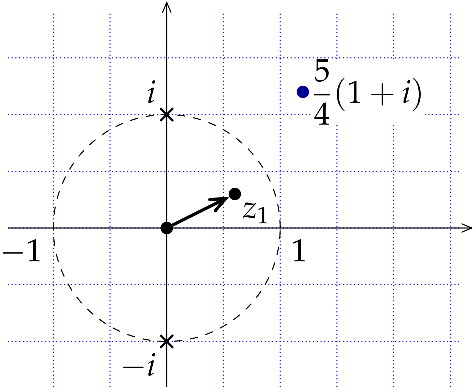


(arg)

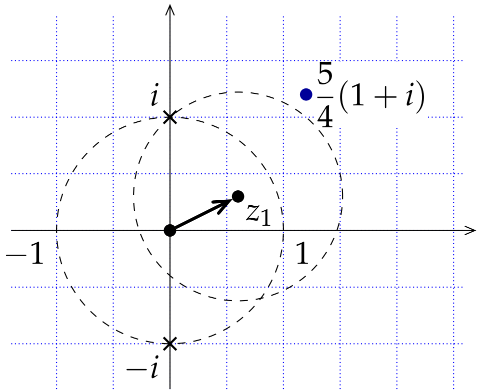
$$(1+z^2)\,y''(z)+2z\,y'(z)=0,$$
$$y(0)=0,\quad y'(0)=1$$



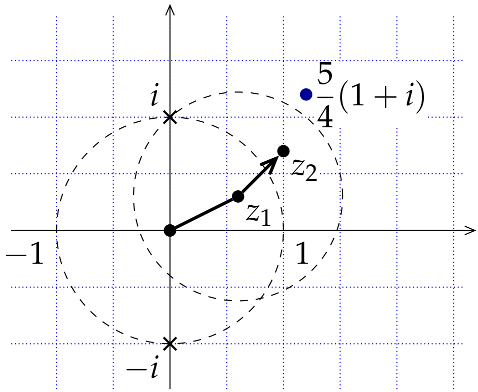
$\frac{5}{4}(1+i)$
i
-1
1
$-i$



$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0,5705170238\cdots + 0,2200896807\ldots i \\ 0 & 0,7288378766\cdots - 0,2065997130\ldots i \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

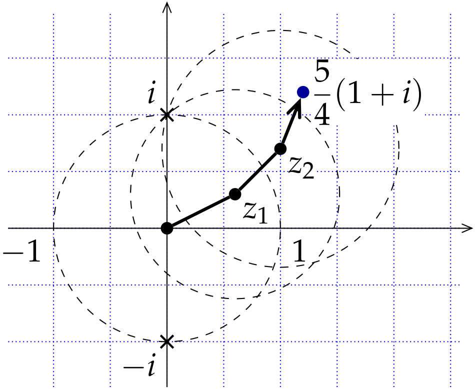


$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0,5705170238\cdots + 0,2200896807\ldots i \\ 0 & 0,7288378766\cdots - 0,2065997130\ldots i \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$



$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0,5705170238\cdots + 0,2200896807\ldots i \\ 0 & 0,7288378766\cdots - 0,2065997130\ldots i \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

$$\begin{bmatrix} y(z_2) \\ y'(z_2) \end{bmatrix} = \begin{bmatrix} 1 & 0,3656231471\cdots + 0,3290407483\ldots i \\ 0 & 0,7515011402\cdots - 0,0792619810\ldots i \end{bmatrix} \begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix}$$



$$\begin{bmatrix} y(z_1) \\ y'(z_1) \end{bmatrix} = \begin{bmatrix} 1 & 0,5705170238\cdots + 0,2200896807\ldots i \\ 0 & 0,7288378766\cdots - 0,2065997130\ldots i \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

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- Schroeppel (1972) – Points particuliers
- Brent (1976) – Fonctions particulières, points quelconques
- Chudnovsky & Chudnovsky (1986-1988) – Méthode générale, esquisse points singuliers réguliers
- van der Hoeven (1999, 2001) – Algorithme complet avec bornes

Théorème (Chudnovsky^2)

Soit y solution d'une équation différentielle linéaire à coefficients polynomiaux. Soit z un point de la surface de Riemann de y.

On peut calculer $y(z)$ à 2^{-n} près en

$$O\Big(M\Big(n \cdot (\log n)^3\Big)\Big)$$

opérations binaires.

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opérations binaires.

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Soit y solution d'une équation différentielle linéaire à coefficients polynomiaux. Soit z un point de la surface de Riemann de y.

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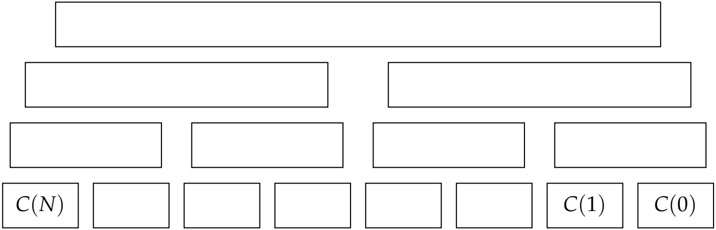
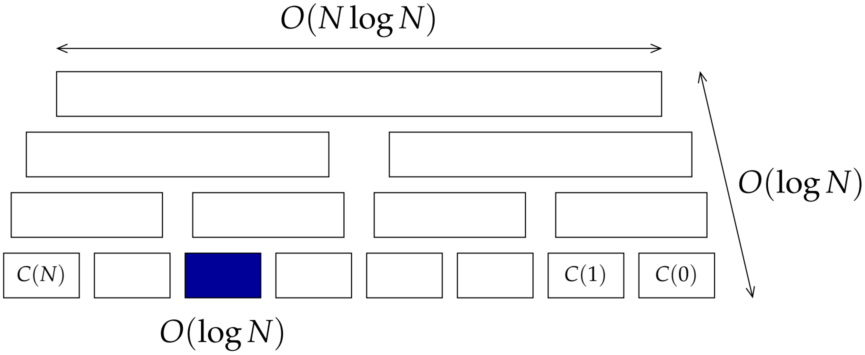
opérations binaires.


$C(N)$
$C(1)$
$C(0)$



$O(N \log N)$
$O(\log N)$
$C(N)$
$C(1)$
$C(0)$
$O(\log N)$

Paramètres

$\kappa, \alpha, \ldots \in \mathbb{Q}$ ou $\bar{\mathbb{Q}}$ t.q.

$$|y_n| \leqslant n!^{\kappa} \cdot \alpha^n \cdot \varphi(n)$$

Outils : méthode des séries majorantes + analyse asymptotique élémentaire (M. & Salvy 2010)

Bornes symboliques

- Lisibles (presque !)
- Asymptotiquement fines

Bornes numériques

- Approx. sûres des paramètres
- Plus rapide (pas d'algébriques)

Idée: Replacer y par une fonction simple qui la "domine"

$$z^2\, y''(z) + z\, y'(z) + (z^2 - \nu^2)\, y(z)$$

0 point singulier

régulier

pour toute solution y,

$\exists N$ tq $y(z) = O(1/|z|^N)$

quand $z \to 0$

ex. : $y(z) = z^{\sqrt{2}}$, $y(z) = \dfrac{\log z}{z}$

irrégulier

croissance non-poly.

en $1/|z|$ possible

quand $z \to 0$

ex. : $y(z) = e^{1/z}$

Théorème (Fuchs, 1866)

Si 0 est un point singulier régulier d'une équation différentielle linéaire à coefficients analytiques, celle-ci admet pour un certain voisinage D de 0 une base de solutions de la forme

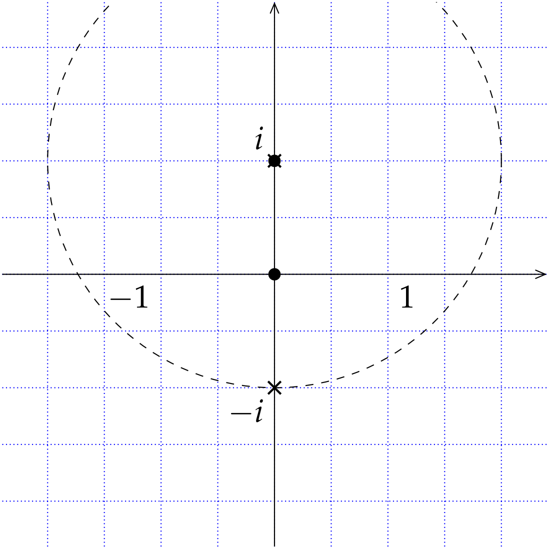
$$z^\lambda\big(y_0(z) + y_1(z)\log z + \cdots + y_t(z)\log^t z\big), \quad z \in D \setminus \{0\}$$

où $\lambda \in \mathbb{C}$ et les y_i sont analytiques sur D.

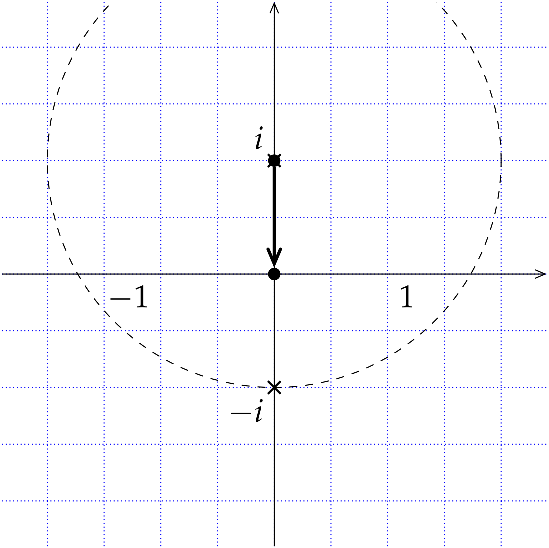
$$L\left(z, z\frac{\mathrm{d}}{\mathrm{d}z}\right) \cdot y(z) = 0$$

$$y(z) = \sum_{n\in\mathbb{Z}} y_n\, z^n \qquad\qquad L(S_n^{-1}, n) \cdot (y_n) = 0$$

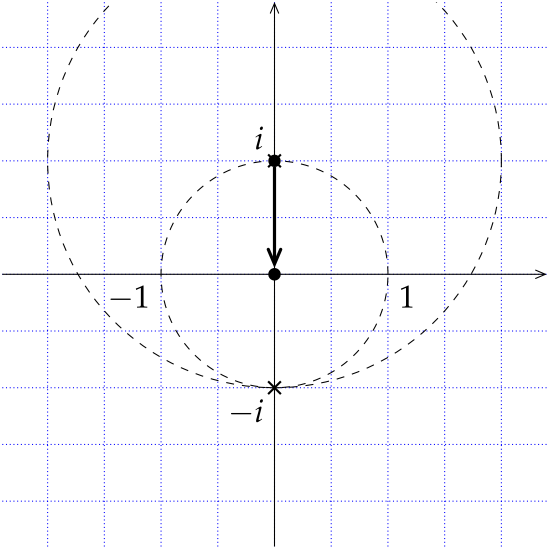
$$y(z) = \sum_{n\in\lambda+\mathbb{Z}} \sum_{k\geqslant 0}^{\text{(finie)}} y_n\, \frac{\log^k z}{k!}\, z^n \qquad\qquad L(S_n^{-1}, n + S_k) \cdot (y_{n,k}) = 0$$



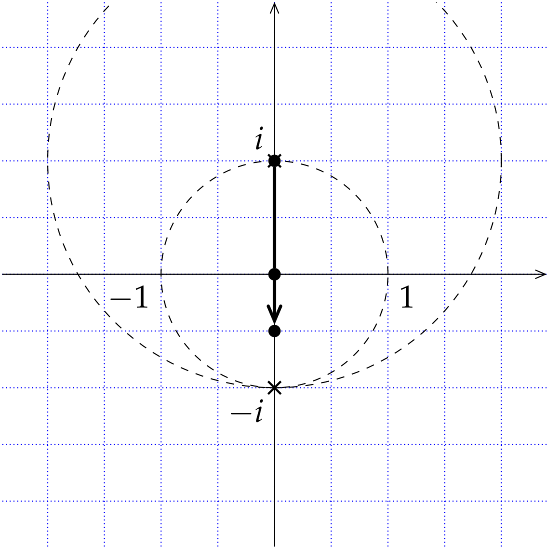
i
-1
1
$-i$



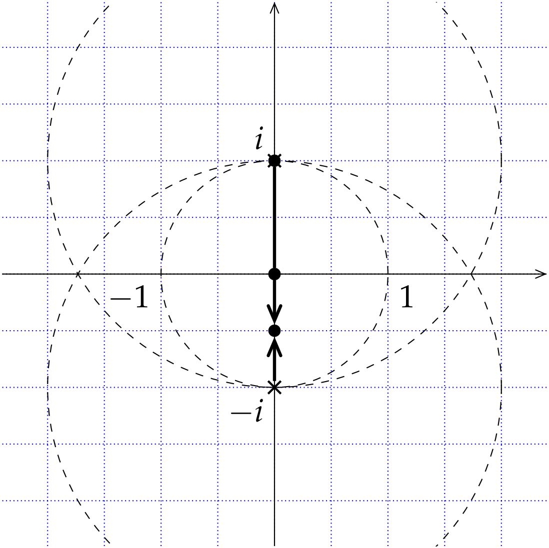
i
−1
1
−i



i
−1
1
−i



i
-1
1
$-i$



i
−1
1
−i

- Wimp et Zeilberger (1985), Zeilberger (2008) – Méthode de Birkhoff-Trjitzinsky (heuristique)
- Flajolet et Puech (1986) – Prolongement analytique numérique pour l'asymptotique
- Banderier, Chern et Hwang (WIP ?) – Calcul de constantes de connection par resommation

Analyse de singularité (Flajolet, Odlyzko)

asymptotique de $y(z) = \sum_n y_n z^n$ en ses singularités

↓ transfert mécanique

asymptotique de (y_n) à l'infini

This is AsyRec, A Maple package
accompanying Doron Zeilberger's article:

It finds the asymptotics of solutions of (homog.) linear recurrence equations with polynomial coefficients, using the Birkhoff-Trjitzinsky method .

```
> recop := (n+2)^2*N^2-(7*n^2+21*n+16)*N-8*(n+1)^2;
```

$$recop := (n+2)^2 N^2 - \left(7n^2 + 21n + 16\right) N - 8(n+1)^2$$

```
> AsyC(recop, n, N, 5, [2, 10], 1000);
```

0.36755259694786136634,

$$\frac{8^n \left(1 - \frac{1}{3n} + \frac{1}{27n^2} + \frac{1}{81n^3} + \frac{1}{243n^4} + \frac{11}{2187n^5}\right)}{n}$$

(Wimp & Zeilberger 1985, Zeilberger 2008-2009)

NumGfun en bref

- Prolongement analytique numérique multiprécision
  général – garanti – automatique – rapide
- Bornes fines
  suites – séries majorantes – restes de séries

Code disponible

`http://algo.inria.fr/libraries/` (GNU LGPL)

Perspectives

- Points singuliers réguliers avec garanties
- Asymptotique automatique
- Aller (plus) vite
- Moins de dépendance à Maple

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Merci !

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