Spectra of large diluted but bushy random graphs

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Erdős-Rényi random graphs

$G(n, p)$

- vertex set $\{1, \ldots, n\}$
- vertices linked by an edge independently with probability $p$
Erdős-Rényi random graphs

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Adjacency matrix $A$

- symmetric
- if $i \neq j$, $P(A_{i,j} = 1) = 1 - P(A_{i,j} = 0) = p$
- for every $i$, $A_{i,i} = 0$
Erdős-Rényi random graphs

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What does the spectrum of $A$ look like?

- if $np \to 0$, single atom mass at 0
- if $np \to \infty$, semi circle law
- if $np \to c > 0$, not much is known...
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 0.5 \]
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 0.5 \text{ (zoomed in)} \]
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 1 \]
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 1 \text{ (zoomed in)} \]
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 1, \, 5 \]
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 1, 5 \text{ (zoomed in)} \]
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 2 \]
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 2 \text{ (zoomed in)} \]
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 2, 5 \]
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 2, 5 \text{ (zoomed in)} \]
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 2, 8 \]
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 2, 8 \text{ (zoomed in)} \]
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 3 \]
Numerical simulations on diluted graphs with 5000 vertices

c = 3 (zoomed in)
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 4 \]
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 5 \]
Numerical simulations on diluted graphs with 5000 vertices

$c = 10$
Numerical simulations on diluted graphs with 5000 vertices

\[ c = 20 \]
\[ \mu_n^c = \frac{1}{n} \sum_{\lambda \in \text{Sp}(c^{-1/2} A)} \delta_\lambda : \text{empirical spectral distribution of } G(n, c/n) \]
État de l’art

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Fact : as \( n \to \infty \), \( \mu_n^c \) converges weakly to a probability measure \( \mu^c \)
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Known properties of \( \mu^c \):

- if \( c \to \infty \), \( \mu^c \) converges weakly to Wigner semi-circle law
- unbounded support
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- dense set of atoms
- \( \mu^c (\{0\}) \) known explicitly \[\text{[Bordenave, Lelarge, Salez 2011]}\]
- \( \mu^c \) is not purely atomic \( iif \) \( c > 1 \) \[\text{[Bordenave, Sen, Virág 2013]}\]
Asymptotic expansion of the spectrum
Asymptotic expansion of the spectrum

If $\mu$ is a (signed) measure and $\int |x|^k |d\mu(x)| < \infty$, denote $m_k(\mu) = \int x^k d\mu(x)$
Asymptotic expansion of the spectrum

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**Theorem:** For every $k \geq 0$ and as $c \to \infty$

$$m_k(\mu^c) = m_k(\sigma) + \frac{1}{c} m_k(\sigma^{\{1\}}) + o\left(\frac{1}{c}\right)$$

where $\sigma$ is the semi-circle law having density

$$\frac{1}{2\pi} \sqrt{4 - x^2} 1_{|x|<2}$$

and $\sigma^{\{1\}}$ is a measure with total mass 0 and density

$$\frac{1}{2\pi} \frac{x^4 - 4x^2 + 2}{\sqrt{4 - x^2}} 1_{|x|<2}.$$
Asymptotic expansion of the spectrum – numerical simulations

100 matrices of size 10000 with $c = 20$

Histogram of $\mu_c^\infty$

Density of $\sigma$
Asymptotic expansion of the spectrum – numerical simulations

100 matrices of size 10000 with $c = 20$

Histogram of $c \left( \mu_n^c - \sigma \right)$

Density of $\sigma^{(1)}$
Proposition: For every \( k \geq 0 \) we have the following asymptotic expansion in \( c \):

\[
m_k(\mu^c) = m_k(\sigma) + \frac{1}{c} m_k(\sigma^{\{1\}}) + \frac{1}{c^2} d_k + o\left(\frac{1}{c^2}\right)
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where the numbers \( d_k \) are NOT the moments of a measure!
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\( \rightarrow \) The asymptotic expansion must take into account the fact that

\[
\mu^c \left( \mathbb{R} \setminus [-2; 2] \right) = O\left( \frac{1}{c^2} \right).
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Asymptotic expansion of the spectrum: second order (I)

**Proposition:** For every \( k \geq 0 \) we have the following asymptotic expansion in \( c \):

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Dilation operator \( \Lambda_\alpha \) for measures defined by \( \Lambda_\alpha(\mu)(A) = \mu(A/\alpha) \) for a measure \( \mu \) and a Borel set \( A \).
**Proposition:** For every $k \geq 0$ we have the following asymptotic expansion in $c$:

$$m_k(\mu^c) = m_k(\sigma) + \frac{1}{c} m_k(\sigma^{\{1\}}) + \frac{1}{c^2} d_k + o\left(\frac{1}{c^2}\right)$$

where the numbers $d_k$ are **NOT** the moments of a measure!

→ *The asymptotic expansion must take into account the fact that*

$$\mu^c (\mathbb{R} \setminus [-2; 2]) = O\left(\frac{1}{c^2}\right).$$

Dilation operator $\Lambda_\alpha$ for measures defined by $\Lambda_\alpha(\mu)(A) = \mu(A/\alpha)$ for a measure $\mu$ and a Borel set $A$.

For example, $\Lambda_\alpha(\sigma)$ is supported on $[-2\alpha; 2\alpha]$. 
**Theorem:** For every $k \geq 0$ and as $c \to \infty$

$$m_k(\mu^c) = m_k \left( \Lambda_{1+\frac{1}{2c}} \left( \sigma + \frac{1}{c} \hat{\sigma}^{\{1\}} + \frac{1}{c^2} \hat{\sigma}^{\{2\}} \right) \right) + o \left( \frac{1}{c^2} \right)$$

where $\hat{\sigma}^{\{1\}}$ is a measure with null total mass and density

$$-\frac{x^4 - 5x^2 + 4}{2\pi \sqrt{4 - x^2}} 1_{|x| < 2}$$

and where $\hat{\sigma}^{\{2\}}$ is a measure with null total mass and density

$$-\frac{2x^8 - 17x^6 + 46x^4 - \frac{325}{8} x^2 + \frac{21}{4}}{\pi \sqrt{4 - x^2}} 1_{|x| < 2}.$$
Second order – numerical simulations

100 matrices of size 10000 with $c = 20$

Density of $\Lambda_{1+\frac{1}{2c}} \left( \hat{\sigma} \{1\} \right)$

Histogram of $c \left( \mu_n^c - \Lambda_{1+\frac{1}{2c}} \left( \sigma \right) \right)$
Second order – numerical simulations

100 matrices of size 10000 with $c = 20$

Histogram of $c^2 \left( \mu_n^c - \Lambda_{1+\frac{1}{2c}} \left( \sigma + \frac{1}{c} \hat{\sigma}^{(1)} \right) \right)$

Density of $\Lambda_{1+\frac{1}{2c}} \left( \hat{\sigma}^{(2)} \right)$
Edge of the Spectrum

\[ m_k(\mu^c) = m_k \left( \Lambda_{1+\frac{1}{2c}} \left( \sigma + \frac{1}{c} \hat{\sigma}^{\{1\}} + \frac{1}{c^2} \hat{\sigma}^{\{2\}} \right) \right) + o \left( \frac{1}{c^2} \right) \]

The measure on the right hand side is supported on \([-2 - 1/c; 2 + 1/c]\).
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This suggests that for $\varepsilon > 0$, as $c \to \infty$,

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\mu^c \left( \left[ -\infty; -2 - \frac{1 + \varepsilon}{c} \right] \cup \left[ 2 + \frac{1 + \varepsilon}{c}; +\infty \right] \right) = o \left( \frac{1}{c^2} \right).
$$
Edge of the Spectrum

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Thank you!