Random triangulations
coupled with an Ising model

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1. Introduction: 2DQG and planar maps
2. Local weak topology
3. Adding matter: Ising model
4. Combinatorics of triangulations with spins
5. Local limit of triangulations with spins
"We have to develop an art of handling sums over random surfaces. These sums replace the old fashioned (and extremely useful) sums over random paths."

[Polyakov 81]
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Sums over random paths: Feynman path integrals.

Well understood question:
Pick \( a, b \in \mathbb{R}^2 \), what does a random path \( \gamma : [0,1] \rightarrow \mathbb{R}^2 \) chosen "uniformly at random" between all paths from \( a \) to \( b \) look like?
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**Not so well understood question:**
What does a random metric on \(S^2\) distributed "uniformly" look like?

Brownian surface?
2D Quantum Gravity?

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First idea: try discrete metric spaces (Donsker)
Definition:
A planar map is a proper embedding of a finite connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).
Planar Maps as discrete planar metric spaces

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- **faces:** connected components of the complement of edges

- **$p$-angulation:** each face is bounded by $p$ edges
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This is a triangulation
Planar Maps as discrete planar metric spaces

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M Planar Map:
- $V(M) :=$ set of vertices of $M$
- $d_{gr} :=$ graph distance on $V(M)$
- $(V(M), d_{gr})$ is a (finite) metric space
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**M Planar Map:**
- Mark an oriented edge of the map

**Rooted map:** mark an oriented edge of the map
"Classical" large random triangulations

Euler relation in a triangulation: number of edges / vertices / faces linked

Take a triangulation of size $n$ uniformly at random. What does it look like if $n$ is large?

Two points of view: global/local, continuous/discrete
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**Global:**
Rescale distances to keep diameter bounded

[Le Gall 13, Miermont 13]: converges to the **Brownian map**.

- Gromov-Hausdorff topology
- Continuous metric space
- Homeomorphic to the sphere
- Hausdorff dimension 4
- **Universality**
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**Local:**
Don’t rescale distances and look at neighborhoods of the root
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Local:
Don’t rescale distances and look at neighborhoods of the root

[Angel – Schramm 03, Krikun 05]: Converges to the Uniform Infinite Planar Triangulation

- Local topology
- Metric balls of radius $R$ grow like $R^4$
- "Universality" of the exponent 4.
Local Topology for planar maps

\[ M_f := \{\text{finite rooted planar maps}\}. \]

**Definition:**
The local topology on \( M_f \) is induced by the distance:

\[
d_{loc}(m, m') := (1 + \max\{r \geq 0 : B_r(m) = B_r(m')\})^{-1}
\]

where \( B_r(m) \) is the graph made of all the vertices and edges of \( m \) which are within distance \( r \) from the root.
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- \((\mathcal{M}, d_{loc})\): closure of \((\mathcal{M}_f, d_{loc})\). It is a **Polish** space (complete and separable).
- \( \mathcal{M}_\infty := \mathcal{M} \setminus \mathcal{M}_f \) set of infinite planar maps.
Local convergence: simple examples

Root = 0
Local convergence: simple examples

\[ \lim_{n \to (\mathbb{Z}_+, 0)} \]

Root = 0
Local convergence: simple examples

\[ \rightarrow (\mathbb{Z}_+, 0) \]

Root = 0

Uniformly chosen root
Local convergence: simple examples

\[ n \xrightarrow{} (\mathbb{Z}_+, 0) \]

Root = 0

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Local convergence: simple examples

\[ 0 \quad 1 \quad 2 \quad n \quad \rightarrow \quad (\mathbb{Z}_+, 0) \]

Root = 0

\[ 0 \quad 1 \quad 2 \quad n \quad \rightarrow \quad (\mathbb{Z}, 0) \]

Uniformly chosen root
Local convergence: simple examples

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Root does not matter
Local convergence: simple examples

\[
\begin{align*}
0 & \quad 1 & \quad 2 & \quad n & \rightarrow & (\mathbb{Z}_+, 0) \\
\text{Root } &= 0
\end{align*}
\]

\[
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0 & \quad 1 & \quad 2 & \quad n & \rightarrow & (\mathbb{Z}, 0) \\
\text{Uniformly chosen root}
\end{align*}
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$$0 \ 1 \ 2 \ n \rightarrow (\mathbb{Z}_+, 0)$$

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$$n \rightarrow (\mathbb{Z}, 0)$$

Root does not matter

$$n \rightarrow (\mathbb{Z}^2, 0)$$

Uniformly chosen root
Local convergence: more complicated examples

Uniform plane rooted trees with $n$ vertices:

$n = 1$  

$n = 2$  

$n = 3$  

$n = 4$  

$1/2$  

$1/2$  

$1/5$  

$1/5$  

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Local convergence: more complicated examples

Uniform plane rooted trees with $n$ vertices:

- $n = 1$
- $n = 2$
- $n = 3$
- $n = 4$
- $n = 500$
- $n = 1000$
Local convergence: more complicated examples

Uniform plane rooted trees with $n$ vertices:

$$n = 1$$

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$$n = 3$$

$$n = 4$$

$$n = 500$$

$$n = 1000$$

The limit is a **probability distribution** on infinite trees with **one** infinite branch.
Local convergence of uniform triangulations

**Theorem** [Angel – Schramm, ’03]

As \( n \to \infty \), the uniform distribution on triangulations of size \( n \) converges weakly to a probability measure called the Uniform Infinite Planar Triangulation (or \text{UIPT}) for the **local topology**.

Courtesy of Igor Kortchemski

Courtesy of Timothy Budd
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Some properties of the **UIPT**:
- The UIPT has almost surely one end [Angel – Schramm, ’03]
- Volume (nb. of vertices) and perimeters of balls known to some extent.
  
  For example \( \mathbb{E} [ |B_r(T_\infty)| ] \sim \frac{2}{7} r^4 \) [Angel ’04, Curien – Le Gall ’12]
- Volume of hulls explicit [M. 16]
- ”Uniqueness” of geodesic rays and horofunctions [Curien – M. 18]
- Bond and site percolation well understood [Angel, Angel–Curien, M.–Nolin]
- Simple random Walk is recurrent [Gurel-Gurevich and Nachmias ’13]
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**Universality**: we expect the same behavior for slightly different models (e.g. quadrangulations, triangulations without loops, ...).
Adding matter: Ising model on triangulations

How does Ising model influence the underlying map?
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How does Ising model influence the underlying map?

First, Ising model on a finite deterministic graph:

\[ G = (V, E) \text{ finite graph} \]

**Spin configuration** on \( G \):

\[ \sigma : V \to \{-1, +1\}. \]
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**Ising model** on \( G \): take a random spin configuration with probability
\[
P(\sigma) \propto e^{-\frac{\beta}{2} \sum_{v \sim v'} \mathbf{1}_{\{\sigma(v) \neq \sigma(v')\}}} \]
\[ \beta > 0: \text{ inverse temperature.} \]
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Combinatorial formulation: \( P(\sigma) \propto \nu^{m(\sigma)} \)
with \( m(\sigma) = \) number of monochromatic edges and \( \nu = e^\beta \).
Adding matter: Ising model on triangulations

\[ \mathcal{T}_n = \{ \text{rooted planar triangulations with } 3n \text{ edges} \}. \]
Random triangulation in \( \mathcal{T}_n \) with probability \( \propto \nu^m(T,\sigma) \)?
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Random triangulation in \( \mathcal{T}_n \) with probability \( \propto \nu^m(T,\sigma) \)?

Generating series of \textbf{Ising-weighted triangulations}:

\[
Q(\nu, t) = \sum_{T \in \mathcal{T}_f} \sum_{\sigma : V(T) \to \{ -1, +1 \}} \nu^m(T,\sigma) t^{e(T)}.
\]
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\]

**Theorem** [Bernardi – Bousquet-Mélou 11]

For every \( \nu \) the series \( Q(\nu, t) \) is algebraic, has \( \rho_\nu > 0 \) as unique dominant singularity and satisfies

\[
[t^{3n}]Q(\nu, t) \sim \begin{cases} 
\kappa \rho_\nu^{-n} n^{-7/3} & \text{if } \nu = \nu_c := 1 + \frac{1}{\sqrt{7}}, \\
\kappa \rho_\nu^{-n} n^{-5/2} & \text{if } \nu \neq \nu_c.
\end{cases}
\]

This suggests an unusual behavior of the underlying maps for \( \nu = \nu_c \).

See also [Boulatov – Kazakov 1987], [Bousquet-Mélou – Schaeffer 03] and [Bouttier – Di Francesco – Guitter 04].
Adding matter: the model and Watabiki’s predictions

Probability measure on triangulations of $\mathcal{T}_n$ with a spin configuration:

$$P_n^\nu \left( \{(T, \sigma)\} \right) = \frac{\nu^m(T, \sigma)}{[t^{3n}]Q(\nu, t)}.$$
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**Counting exponent:**

coeff $[t^n]$ of generating series of (decorated) maps $\sim \kappa \rho^{-n} n^{-\alpha}$

**Central charge $c$:**

\[
\alpha = \frac{25 - c + \sqrt{(1 - c)(25 - c)}}{12}
\]

**Hausdorff dimension:** [Watabiki 93]

\[
D_H = 2 \frac{\sqrt{25 - c} + \sqrt{49 - c}}{\sqrt{25 - c} + \sqrt{1 - c}}
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- $\alpha = 5/2$ gives $D_H = 4$
- $\alpha = 7/3$ gives $D_H = \frac{7 + \sqrt{97}}{4} \approx 4.21$

Hausdorff dimension: [Watabiki 93]

$$D_H = 2 \frac{\sqrt{25 - c} + \sqrt{49 - c}}{\sqrt{25 - c} + \sqrt{1 - c}}$$
**Definition:**
The **local topology** on $\mathcal{T}_f$ is induced by the distance:

$$d_{loc}(T, T') := (1 + \max\{r \geq 0 : B_r(T) = B_r(T')\})^{-1}$$

where $B_r(T)$ is the submap (with spins) of $T$ composed by the **faces** of $T$ with a vertex at distance $< r$ from the root.
Local Topology for planar maps: balls

**Definition:**
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- $(\mathcal{T}, d_{loc})$: closure of $(\mathcal{T}_f, d_{loc})$. It is a **Polish** space.
- $\mathcal{T}_\infty := \mathcal{T} \setminus \mathcal{T}_f$ set of **infinite** planar triangulations.
Weak convergence for the local topology

**Portemanteau theorem + Levy – Prokhorov metric:**
A sequence of measures measures \((P_n)\) on \(\mathcal{T}_f\) converge weakly to a measure \(P\) on \(\mathcal{T}_\infty\) if:

1. For every \(r > 0\) and every possible \(r\)-ball \(\Delta\)

\[
P_n\left(\left\{(T,v) \in \mathcal{T}_f : B_r(T,v) = \Delta\right\}\right) \rightarrow_{n \to \infty} P\left(\left\{T \in \mathcal{T}_\infty : B_r(T) = \Delta\right\}\right).
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\]

**Problem:** not sufficient since the space \((T, d_{loc})\) is not compact!

Ex:

```
\begin{tikzpicture}
  \node at (0,0) [inner sep=0pt, outer sep=0pt] {\textbullet};
  \node at (1,1) [inner sep=0pt, outer sep=0pt] {\textbullet};
  \node at (2,0) [inner sep=0pt, outer sep=0pt] {\textbullet};
  \node at (1,-1) [inner sep=0pt, outer sep=0pt] {\textbullet};
  \node at (0,-2) [inner sep=0pt, outer sep=0pt] {\textbullet};
  \node at (-1,-1) [inner sep=0pt, outer sep=0pt] {\textbullet};
  \node at (-2,0) [inner sep=0pt, outer sep=0pt] {\textbullet};
  \node at (-1,1) [inner sep=0pt, outer sep=0pt] {\textbullet};
  \draw (0,0) -- (1,1) -- (2,0) -- (1,-1) -- (0,-2) -- (-1,-1) -- (-2,0) -- (-1,1) -- cycle;
\end{tikzpicture}
```
Weak convergence for the local topology

**Portemanteau theorem + Levy – Prokhorov metric:**

A sequence of measures measures \((P_n)\) on \(T_f\) converge weakly to a measure \(P\) on \(T_\infty\) if:

1. For every \(r > 0\) and every possible \(r\)-ball \(\Delta\)

\[
P_n \left( \left\{ (T, v) \in T_f : B_r(T, v) = \Delta \right\} \right) \xrightarrow{n \to \infty} P \left( \left\{ T \in T_\infty : B_r(T) = \Delta \right\} \right).
\]

2. No loss of mass at the limit: Tightness of \((P_n)\), or the measure \(P\) defined by the limits in 1. is a probability measure.

- Vertex degrees are tight (at finite distance from the root)

\[
\forall r > 0, \sum_{r \text{-balls } \Delta} P \left( \left\{ T \in T_\infty : B_r(T) = \Delta \right\} \right) = 1.
\]
Local convergence and generating series

Need to evaluate, for every possible ball \( \Delta \)
(here, one boundary to keep it simple)
Local convergence and generating series

Need to evaluate, for every possible ball $\Delta$
(here, one boundary to keep it simple)

Simple (rooted) cycle,
spins given by a word $\omega$
Local convergence and generating series

Need to evaluate, for every possible ball $\Delta$
(here, one boundary to keep it simple)

$P_n(\Delta, ???) = \frac{\nu^m(\Delta) - m(\omega) [t^{3n} - e(\Delta) + |\omega|] Z_\omega(\nu, t)}{[t^{3n}]Q(\nu, t)}$

Simple (rooted) cycle, spins given by a word $\omega$

$Z_\omega(\nu, t) :=$ generating series of triangulations with simple boundary $\omega$
Local convergence and generating series

Need to evaluate, for every possible ball \( \Delta \)
(here, one boundary to keep it simple)

\[
P_n \left( \begin{array}{c} \Delta \\ ??? \end{array} \right) = \frac{\nu^{m(\Delta) - m(\omega)} [t^{3n} - e(\Delta) + |\omega|]}{[t^{3n}] Q(\nu, t)} Z_\omega(\nu, t)
\]

Simple (rooted) cycle,
spins given by a word \( \omega \)

\( Z_\omega(\nu, t) := \) generating series of
triangulations with simple boundary \( \omega \)

**Theorem [Albenque – M. – Schaeffer 18+]**
For every \( \omega \) and \( \nu \), the series \( t^{|\omega|} Z_\omega(\nu, t) \) is algebraic, has \( \rho_\nu = t_\nu^3 \) as unique dominant singularity and satisfies

\[
[t^{3n}] Z_\omega(\nu, t) \sim \begin{cases} 
\kappa_\omega(\nu_c) \rho_{\nu_c}^{-n} n^{-7/3} & \text{if } \nu = \nu_c := 1 + \frac{1}{\sqrt{7}}, \\
\kappa_\omega(\nu) \rho_{\nu}^{-n} n^{-5/2} & \text{if } \nu \neq \nu_c.
\end{cases}
\]
Triangulations with simple boundary

Fix a word $\omega$, with injections from and into triangulations of the sphere:

$$[t^{3n}] t^{|\omega|} Z_\omega = \Theta \left( \rho_\nu^{-n} n^{-\alpha} \right), \text{ with } \alpha = 5/2 \text{ of } 7/3 \text{ depending on } \nu.$$ 

To get exact asymptotics we need, as series in $t^3$,

1. algebraicity,
2. no other dominant singularity than $\rho_\nu.$
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Fix a word $\omega$, with injections from and into triangulations of the sphere:

$$[t^{3n}] t^{|\omega|} Z_\omega = \Theta \left( \rho^{-n} n^{-\alpha} \right), \text{ with } \alpha = 5/2 \text{ of } 7/3 \text{ depending on } \nu.$$  

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Tutte’s equation (or peeling equation, or loop equation... ):

$$Z_\omega = \left( Z_{\oplus \omega} + Z_{\ominus \omega} + \sum_{\omega = \omega_1 \omega_2} Z_{\omega_1} \cdot Z_{\omega_2} \right) \times \nu^{1_{\text{Tw} = \overline{\omega}}} t$$
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Fix a word $\omega$, with injections from and into triangulations of the sphere:

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2. no other dominant singularity than $\rho_{\nu}$.

Tutte’s equation (or peeling equation, or loop equation... ):

\[ Z_\omega = \left( Z_{\ominus \omega} + Z_{\ominus \omega} + \sum_{\omega = \omega_1 \omega_2} Z_{\omega_1} \cdot Z_{\omega_2} \right) \times \nu^{1_{\overset{\rightarrow}{\omega} = \overset{\leftarrow}{\omega}}} t \]

Double induction on $|\omega|$ and number of $\ominus$’s:

enough to prove 1. and 2. for the $t^p Z_{\ominus p}$’s
Positive boundary conditions: two catalytic variables

\[ A(x) := \sum_{p \geq 1} Z_{\oplus p} x^p = \nu t x^2 + \]

\[ + \frac{\nu t}{x} (A(x))^2 \]
Positive boundary conditions: two catalytic variables

\[ A(x) := \sum_{p \geq 1} Z_{\oplus p} x^p = \nu t x^2 + \]

Peeling equation at interface \( \ominus \rightarrow \oplus \):

\[ S(x, y) := \sum_{p, q \geq 1} Z_{\ominus p \oplus q} x^p y^q \]
Positive boundary conditions: two catalytic variables

\[
A(x) := \sum_{p \geq 1} Z_{\oplus p} x^p = \nu t x^2 + \frac{\nu t}{x} \left( A(x) - x Z_{\oplus} \right) + \nu t [y] S(x, y) + \frac{\nu t}{x} (A(x))^2
\]

Peeling equation at interface ⊖–⊕:

\[
S(x, y) := \sum_{p, q \geq 1} Z_{\oplus p \ominus q} x^p y^q
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Positive boundary conditions: two catalytic variables

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\]

Peeling equation at interface \( \ominus - \oplus \):

\[
S(x, y) := \sum_{p, q \geq 1} Z_{\ominus p \ominus q} x^p y^q = t x y + \frac{t}{x} \left( S(x, y) - x [x] S(x, y) \right) + \frac{t}{y} \left( S(x, y) - y [y] S(x, y) \right) + \frac{t}{x} S(x, y) A(x) + \frac{t}{y} S(x, y) A(y)
\]
Kernel method: equation for $S$ reads

$$K(x, y) \cdot S(x, y) = R(x, y)$$

where

$$K(x, y) = 1 - \frac{t}{x} - \frac{t}{y} - \frac{t}{x} A(x) - \frac{t}{y} A(y).$$
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$$K(x, y) = 1 - \frac{t}{x} - \frac{t}{y} - \frac{t}{x} A(x) - \frac{t}{y} A(y).$$

1. Find two series $Y_1$ and $Y_2$ in $\mathbb{Q}(x)[[t]]$ such that $K(x, Y_i/t) = 0$. 

From two catalytic variables to one: Tutte’s invariants
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\]

1. Find **two** series \( Y_1 \) and \( Y_2 \) in \( \mathbb{Q}(x)[[t]] \) such that \( K(x, Y_i/t) = 0 \).

It gives

\[
\frac{1}{Y_1} (A(Y_1/t) + 1) = \frac{1}{Y_2} (A(Y_2/t) + 1).
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From two catalytic variables to one: Tutte’s invariants

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$I(y) := \frac{1}{y} \left( A(y/t) + 1 \right)$ is called an **invariant**.
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**Kernel method:** equation for $S$ reads

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$$K(x, y) = 1 - t^x - t^y - t^x A(x) - t^y A(y).$$

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2. Work a bit with the help of $R(x, Y_i/t) = 0$ to get a **second invariant** $J(y)$ depending only on $t, Z \oplus (t), y$ and $A(y/t)$. 
From two catalytic variables to one: Tutte’s invariants

**Kernel method:** equation for $S$ reads

$$K(x, y) \cdot S(x, y) = R(x, y)$$

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1. Find two series $Y_1$ and $Y_2$ in $\mathbb{Q}(x)[[t]]$ such that $K(x, Y_i/t) = 0$.

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2. Work a bit with the help of $R(x, Y_i/t) = 0$ to get a second invariant $J(y)$ depending only on $t, Z \oplus(t), y$ and $A(y/t)$.

3. Prove that $J(y) = C_0(t) + C_1(t)I(y) + C_2(t)I^2(y)$ with $C_i$’s explicit polynomials in $t, Z \oplus(t)$ and $Z \oplus^2(t)$.

**Equation with one catalytic variable** for $A(y)$ with $Z \oplus$ and $Z \oplus^2$!
Explicit solution for positive boundary conditions

Equation with one catalytic variable reads:

\[ 2t^2\nu(1 - \nu) \left( \frac{A(y)}{y} - Z_\ominus \right) = y \cdot \text{Pol} \left( \nu, \frac{A(y)}{y}, Z_\ominus, Z_\ominus^2, t, y \right) \]

[Bousquet-Mélou – Jehanne 06] gives algebraicity and strategy to solve this kind of equation.
Explicit solution for positive boundary conditions

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Much easier: [Bernardi – Bousquet Mélou 11] gives us \( Z_\oplus \) and \( Z_\oplus^2 \)!
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Maple: rational (and Lagrangian) parametrization!

\[ t^3 = U \frac{P_1(\mu, U)}{4(1 - 2U)^2(1 + \mu)^3} \]
\[ y = V \frac{P_2(\mu, U, V)}{(1 - 2U)(1 + \mu)^2(1 - V)^2} \]
\[ t^3 A(t, ty) = \frac{VP_3(\mu, U, V)}{4(1 - 2U)^2(1 + \mu)^3(1 - V)^3} \]

with \( \nu = \frac{1+\mu}{1-\mu} \) and \( P_i \)'s explicit polynomials.
Going back to local convergence

1. Fix \( r \geq 0 \) and take \( \Delta \) a \( r \)-ball with boundary spins \( \partial \Delta = (\omega_1, \ldots, \omega_k) \):

\[
P_n \left( B_r(T, \nu) = \Delta \right) = \frac{\nu^{m(\Delta) - m(\partial \Delta)} [t^{3n} - e(\Delta) + |\partial \Delta|] \left( \prod_{i=1}^{k} Z_{\omega_i}(\nu, t) \right)}{[t^{3n}] Q(\nu, t)} \left( \prod_{i=1}^{k} Z_{\omega_i}(\nu, \nu) \right) \cdot \sum_{j=1}^{k} \frac{\nu^{m(\Delta) - m(\partial \Delta)} t^{\Delta - |\omega_j|} \kappa_{\omega_j}}{\kappa t^{\omega_j} Z_{\omega_j}(\nu, t)}.
\]

\[ \longrightarrow n \rightarrow \infty \]
2. Remains to prove tightness.

1. Fix $r \geq 0$ and take $\Delta$ a $r$-ball with boundary spins $\partial \Delta = (\omega_1, \ldots, \omega_k)$:

$$\mathbb{P}_n (B_r(T, \nu) = \Delta) = \frac{\nu^m(\Delta) - m(\partial \Delta) [t^{3n-e(\Delta)} + |\partial \Delta|] \left( \prod_{i=1}^k Z_{\omega_i}(\nu, t) \right)}{[t^{3n}] Q(\nu, t)} \to_{n \to \infty} \left( \prod_{i=1}^k Z_{\omega_i}(\nu, t_\nu) \right) \cdot \sum_{j=1}^k \frac{\nu^m(\Delta) - m(\partial \Delta) t_\nu |\Delta| - |\omega|}{\kappa t_\nu |\omega_j| \cdot Z_{\omega_j}(\nu, t_\nu)} \kappa_{\omega_j}.$$
Going back to local convergence

1. Fix $r \geq 0$ and take $\Delta$ a $r$-ball with boundary spins $\partial \Delta = (\omega_1, \ldots, \omega_k)$:

$$
\mathbb{P}_n (B_r(T, \nu) = \Delta) = \frac{\nu^m(\Delta) - m(\partial \Delta)}{[t^{3n}]Q(\nu, t)} \left[ t^{3n} - e(\Delta) + |\partial \Delta | \right] \left( \prod_{i=1}^{k} Z_{\omega_i}(\nu, t) \right) \rightarrow_{n \to \infty} \left( \prod_{i=1}^{k} Z_{\omega_i}(\nu, t) \right) \cdot \sum_{j=1}^{k} \frac{\nu^m(\Delta) - m(\partial \Delta)}{\kappa} \frac{t_{\nu} |\Delta| - |\omega|}{t_{\nu} |\omega_j| Z_{\omega_j}(\nu, t_{\nu})} \kappa_{\omega_j}.
$$

2. Remains to prove tightness.

- Maps are uniformly rooted: tightness of root degree is enough.
Going back to local convergence

1. Fix $r \geq 0$ and take $\Delta$ a $r$-ball with boundary spins $\partial \Delta = (\omega_1, \ldots, \omega_k)$:

$$
\mathbb{P}_n (B_r(T,v) = \Delta) = \frac{\nu^m(\Delta)-m(\partial \Delta) \left[ t^{3n} - e(\Delta) + |\partial \Delta| \right] \left( \prod_{i=1}^k Z_{\omega_i}(\nu, t) \right)}{[t^{3n}]Q(\nu, t)} 
\rightarrow_{n \to \infty} \left( \prod_{i=1}^k Z_{\omega_i}(\nu, t_v) \right) \cdot \sum_{j=1}^k \frac{\nu^m(\Delta)-m(\partial \Delta) t_v |\Delta|-|\omega|}{\kappa t_v |\omega_j|} \cdot Z_{\omega_j}(\nu, t_v).$$

2. Remains to prove tightness.

- Maps are uniformly rooted: tightness of root degree is enough
- We show that expected degree at the root under $\mathbb{P}_n$ is bounded with $n$.
A simple tightness argument

We want to study the degree of the root vertex $\delta$:

Mark a uniform edge conditionally on the triangulation

$$\overline{P}_n (\delta \in e) = \sum_{k=1}^{3n} \overline{P}(\delta \in e | \deg(\delta) = k) \cdot \overline{P}_n (\deg(\delta) = k)$$

$$\geq \sum_{k=1}^{3n} \frac{k}{2 \cdot 3n} \overline{P}_n (\deg(\delta) = k) = \frac{1}{6n} \mathbb{E}_n [\deg(\delta)]$$
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\]

Cut open the marked edge and the root:

\[
P_n(\delta \in e) \leq \max \left\{ \frac{1}{\nu}, 1 \right\}^2 \frac{[t^{3n+2}](Z_4 + Z_2^2 + Z_1^2 + Z_1^2 Z_2 + Z_1 Z_3)}{3n[t^{3n}]Z}
\]
\[
= O(1/n)
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Mark a uniform edge conditionally on the triangulation

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\]

\[
E_n [\text{deg}(\delta)] = O(1).
\]
The story so far

What we know:

• Convergence in law for the local topology.
• The limiting random triangulation has one end $a.s.$
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- Recurrence of SRW (vertex degrees have exponential tails)
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What we would like to know:

- Singularity with respect to the UIPT?
- Volume growth?
The story so far

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- Singularity with respect to the UIPT?
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- At least volume growth $\neq 4$ at $\nu_c$?
Summer school **Random trees and graphs**
July 1 to 5, 2019 in Marseille France
Org. M. Albenque, J. Bettinelli, J. Rué and L. Menard

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Thank you for your attention!

Summer school **Random walks and models of complex networks**
July 8 to 19, 2019 in Nice
Org. B. Reed and D. Mitsche

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Thank you for your attention!