1. Lévy's criterion of convergence in law

(1) For every $t \in \mathbb{R}$, $\omega \in \Omega \mapsto \mathrm{e}^{\mathrm{i}tX(\omega)}$ is measurable and bounded in module by 1, whence integrable, so $\phi(t) = \mathbb{E}[\mathrm{e}^{\mathrm{i}tX(\omega)}]$ is well-defined. Then, for every t_0 fixed in $\mathbb{R}$,

$$\lim_{t \to t_0} \mathrm{e}^{\mathrm{i}tX} = \mathrm{e}^{\mathrm{i}t_0 X}$$

almost surely, and all these random variables have real and imaginary part uniformly bounded by 1. By Lebesgue dominated convergence,

$$\lim_{t \to t_0} \phi(t) = \lim_{t \to t_0} \mathbb{E}[\mathrm{e}^{\mathrm{i}tX}] = \mathbb{E}[\mathrm{e}^{\mathrm{i}t_0 X}] = \phi(t_0),$$

so ϕ is continuous on $\mathbb{R}$.

(2) All the functions considered are measurable on $\mathbb{R} \times [-\varepsilon, \varepsilon]$ and uniformly bounded, and integrated against a finite measure, namely, $\mu \otimes dt$. So, one can use freely Fubini's theorem to compute:

$$\begin{aligned} I_\varepsilon &= \int_{-\varepsilon}^{\varepsilon} (1 - \phi(t))\, dt = \iint_{\mathbb{R} \times [-\varepsilon,\varepsilon]} (1 - \mathrm{e}^{\mathrm{i}tx})\, \mu(dx)\, dt \\ &= \int_{\mathbb{R}} \left(2\varepsilon - \frac{2 \sin(\varepsilon x)}{x} \right) \mu(dx) = 2\varepsilon \left(\int_{\mathbb{R}} \left(1 - \frac{\sin(\varepsilon x)}{\varepsilon x} \right) \mu(dx) \right). \end{aligned}$$

Set $y = \varepsilon x$; for $|y| \geq 2$, one has indeed

$$\frac{\sin y}{y} \leq \frac{|\sin y|}{2} \leq \frac{1}{2} \qquad ; \qquad 1 - \frac{\sin(\varepsilon x)}{\varepsilon x} \geq \frac{1}{2}.$$

Therefore,

$$\begin{aligned} \frac{I_\varepsilon}{2\varepsilon} &\geq \int_{\mathbb{R}} \left(1 - \frac{\sin(\varepsilon x)}{\varepsilon x} \right) \mathbf{1}_{|\varepsilon x| \geq 2}\, \mu(dx) \geq \frac{1}{2} \int_{\mathbb{R}} \mathbf{1}_{|\varepsilon x| \geq 2}\, \mu(dx) \\ \frac{I_\varepsilon}{\varepsilon} &\geq \mu(\{x \mid |\varepsilon x| \geq 2\}). \end{aligned}$$

(3) Let $\eta > 0$. By continuity at 0 of ϕ, $|1 - \phi(t)| = |\phi(0) - \phi(t)| \leq \frac{\eta}{3}$ for t small enough, say smaller than some $\varepsilon > 0$. Then,

$$\frac{I_\varepsilon}{\varepsilon} \leq \frac{1}{\varepsilon} \int_{-\varepsilon}^{\varepsilon} \frac{\eta}{3}\, dt = \frac{2\eta}{3}.$$

By Lebesgue dominated convergence,

$$\lim_{n \to \infty} \frac{I_{\varepsilon,n}}{\varepsilon} = \frac{I_\varepsilon}{\varepsilon},$$

so for N big enough and all $n \geq N$,

$$\frac{I_{\varepsilon,n}}{\varepsilon} \leq \frac{3\, I_\varepsilon}{2\, \varepsilon} \leq \eta.$$

By Question (2), this implies that for all $n \geq N$,

$$\mu_n \left((-2\varepsilon^{-1}, 2\varepsilon^{-1})^{\mathrm{c}} \right) \leq \eta.$$

By decreasing the value of ε, one can also assume that this is true for the n's smaller than N (there is only a finite number of measures to consider). Thus, for every $\eta > 0$, there is a relatively compact set (bounded interval) in $\mathbb{R}$ with complementary of measure smaller than η for all measure in $(\mu_n)_{n \in \mathbb{N}}$; *i.e.*, there is tightness.

(4) The sequence of measures $(\mu_n)_{n \in \mathbb{N}}$ is tight, hence relatively compact; let $(\mu_{\psi(n)})_{n \in \mathbb{N}}$ be a convergent subsequence, and μ its limit. For every t, $x \mapsto \mathrm{e}^{\mathrm{i}tx}$ is a continuous bounded function,

$$\mu(\mathrm{e}^{\mathrm{i}tx}) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\, \mu(dx) = \lim_{n \to \infty} \mu_{\psi(n)}(\mathrm{e}^{\mathrm{i}tx}) = \lim_{n \to \infty} \phi_{\psi(n)}(t) = \phi(t).$$

Hence, ϕ is indeed the characteristic function of a probability measure on $\mathbb{R}$.

(5) Again, $\mu \otimes \nu$ is a finite measure on $\mathbb{R} \times \mathbb{R}$, and the functions under consideration are bounded, so one can use Fubini's theorem.

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}tx}\, \phi_\mu(x)\, \nu(dx) &= \iint_{\mathbb{R}\times\mathbb{R}} \mathrm{e}^{-\mathrm{i}tx} \mathrm{e}^{\mathrm{i}yx} \mu(dy)\, \nu(dx) \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}(y-t)x}\, \nu(dx) \right) \mu(dy) \\ &= \int_{\mathbb{R}} \phi_\nu(y-t)\, \mu(dy).\end{aligned}$$

(6) Let f be a bounded non-negative continuous function on $\mathbb{R}$. One has

$$\begin{aligned}\int_{\mathbb{R}} f(t)\, D(\varepsilon,t)\, dt &= \iint_{\mathbb{R}^2} f(t)\, \mathrm{e}^{-\frac{\varepsilon(t-y)^2}{2}}\, \mu(dy)\, dt \\ &= \iint_{\mathbb{R}^2} f(x+y)\, \mathrm{e}^{-\frac{\varepsilon x^2}{2}}\, \mu(dy)\, dx \\ &= \sqrt{2\pi\varepsilon^{-1}} \iint_{\mathbb{R}^2} f(x+y)\, \mu(dy)\, \nu_{\varepsilon^{-1}}(dx) \\ &= \sqrt{2\pi\varepsilon^{-1}}\, \mathbb{E}[f(X+Y_{\varepsilon^{-1}})].\end{aligned}$$

(7) For every interval $(-\eta,\eta)$ around 0,

$$\mathbb{P}[Y_{\varepsilon^{-1}} \in (-\eta,\eta)^{\mathrm{c}}] \leq \frac{\mathbb{E}[(Y_{\varepsilon^{-1}})^2]}{\eta^2} = \frac{1}{(\varepsilon\eta)^2} \to 0,$$

so one has indeed $Y_{\varepsilon^{-1}} \to_{\mathbb{P}} 0$. As a consequence, the pair $(X, Y_{\varepsilon^{-1}})$ converge in probability to $(X,0)$, and also in law. Since convergence in law is compatible with composition by continuous functions,

$$X + Y_{\varepsilon^{-1}} \rightharpoonup X + 0 = X.$$

(8) Suppose ϕ_μ known. Then, for every $\varepsilon > 0$ and every bounded continuous function f,

$$\mathbb{E}[f(X+Y_{\varepsilon^{-1}})] = \frac{1}{\sqrt{2\pi\varepsilon^{-1}}} \int_{\mathbb{R}} f(t)\, D(\varepsilon,t)\, dt = \frac{1}{\sqrt{2\pi\varepsilon^{-1}}} \iint_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}tx}\, \phi_\mu(x)\, \nu_\varepsilon(dx)\, dt$$

is also known, since it can be computed from f and ϕ_μ. Taking the limit as ε goes to infinity, one gets back $\mathbb{E}[f(X)]$ by the previous Question, that is to say $\mu(f)$. So, ϕ_μ determines $\mu(f)$ for every $f \in \mathscr{C}^b(\mathbb{R})$, and therefore μ by classical arguments of measure theory.

(9) We have seen that $(\mu_n)_{n\in\mathbb{N}}$ was tight under the hypotheses of Lévy's criterion, and the limit of a convergent subsequence $(\mu_{\psi(n)})_{n\in\mathbb{N}}$ is by the previous Question the unique probability measure μ such that $\phi(t) = \mu(\mathrm{e}^{\mathrm{i}tx})$. Therefore, $\mu_n \rightharpoonup \mu$.

(10) This is clear since $(t,x) \mapsto \mathrm{e}^{\mathrm{i}tx}$ is a continuous bounded function.

(11) The characteristic function of $Y_k = B_k - \frac{1}{k}$ is

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}tY_k}] = \mathrm{e}^{-\frac{\mathrm{i}t}{k}}\, \mathbb{E}[\mathrm{e}^{\mathrm{i}tB_k}] = \left(1 + \frac{\mathrm{e}^{\mathrm{i}t}-1}{k}\right) \mathrm{e}^{-\frac{\mathrm{i}t}{k}}.$$

Therefore, since the Y_k's are independent,

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}t\widetilde{X}_n}] = \prod_{k=1}^{n} \left(1 + \frac{\mathrm{e}^{\frac{\mathrm{i}t}{\sqrt{\log n}}}-1}{k}\right) \mathrm{e}^{-\frac{\mathrm{i}t}{k\sqrt{\log n}}}$$

The Taylor expansion of each term $T_k(t)$ is

$$\begin{aligned}&\left(1 + \frac{1}{k}\left(\frac{\mathrm{i}t}{\sqrt{\log n}} - \frac{t^2}{2\log n} + O\!\left(\frac{t^3}{(\log n)^{3/2}}\right)\right)\right)\left(1 - \frac{\mathrm{i}t}{k\sqrt{\log n}} + O\!\left(\frac{t^2}{k^2\,\log n}\right)\right) \\ &= 1 - \frac{t^2}{2k\log n} + O\!\left(\frac{t^2}{k^2\,\log n} + \frac{t^3}{k(\log n)^{3/2}}\right),\end{aligned}$$

with a O that is uniform because it comes only from the Taylor expansion around 0 of the exponential. So,

$$\begin{aligned}\log \mathbb{E}[\mathrm{e}^{\mathrm{i}t\widetilde{X}_n}] &= \sum_{k=1}^{n} \log\left(1 - \frac{t^2}{2k\log n} + O\left(\frac{t^2}{k^2\,\log n} + \frac{t^3}{k(\log n)^{3/2}}\right)\right) \\ &= \sum_{k=1}^{n} -\frac{t^2}{2k\log n} + O\left(\frac{t^4}{k^4(\log n)^2} + \frac{t^2}{k^2\,\log n} + \frac{t^3}{k(\log n)^{3/2}}\right) \\ &= -\frac{t^2}{2} + O\left(\frac{t^2}{\log n} + \frac{t^4}{(\log n)^2} + \frac{t^3}{(\log n)^{1/2}}\right) = -\frac{t^2}{2} + o(1)\end{aligned}$$

since $\sum_{k=1}^{n} \frac{1}{k^2}$ and $\sum_{k=1}^{n} \frac{1}{k^4}$ are convergent. So, $\mathbb{E}[\mathrm{e}^{\mathrm{i}t\widetilde{X}_n}]$ converges pointwise to $\mathrm{e}^{-\frac{t^2}{2}}$, and since $X_n - \widetilde{X}_n$ is a deterministic $O((\log n)^{-1/2})$, the same is true for $\mathbb{E}[\mathrm{e}^{\mathrm{i}tX_n}]$. As $\mathrm{e}^{-\frac{t^2}{2}}$ is the characteristic function of a standard Gaussian, the central limit theorem for X_n is proven.

2. Mod-convergence and Berry-Esseen estimates

(1) By hypothesis, for t fixed,

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{\mathrm{i}tX_n}] = \phi_n\left(\frac{t}{\sqrt{\lambda_n}}\right) &= \mathrm{e}^{-\frac{t^2}{2}}\,\psi_n\left(\frac{t}{\sqrt{\lambda_n}}\right) \\ &= \mathrm{e}^{-\frac{t^2}{2}}\left(\psi\left(\frac{t}{\sqrt{\lambda_n}}\right) + o\left(\frac{1}{\sqrt{\lambda_n}}\right)\right) \\ &= \mathrm{e}^{-\frac{t^2}{2}}\left(\psi(0) + o\left(\frac{1}{\sqrt{\lambda_n}}\right)\right) \to \mathrm{e}^{-\frac{t^2}{2}}.\end{aligned}$$

By Lévy criterion of convergence in law, $X_n \rightharpoonup \mathcal{N}_{(0,1)}$.

(2) Suppose $\mu_n \rightharpoonup \mu$. Since μ is supposed absolutely continuous w.r.t. Lebesgue measure, each interval $I = (-\infty, x)$ is a continuity set for μ:

$$\mu(I^0) = \mu(I) \quad \text{as } I^0 = I \qquad ; \qquad \mu(\overline{I}) = \mu(I) + \mu(\{x\}) = \mu(I) \quad \text{since } \mu(\{x\}) = 0.$$

Therefore, by Portmanteau's theorem, $\lim_{n\to\infty} \mu_n(-\infty, x) = \mu(-\infty, x)$, so F_{μ_n} converges pointwise to F_μ. By Dini's theorem, since the cumulative functions are non-decreasing functions bounded from above by 1 and from below by 0, this is in fact a uniform convergence, so

$$d(\mu_n, \mu) = \sup_{x\in\mathbb{R}} |F_{\mu_n}(x) - F_\mu(x)| \to 0.$$

Conversely, suppose $d(\mu_n, \mu) \to 0$, and consider an open set $U \subset \mathbb{R}$. It can be written as a countable union of disjoint open intervals: $U = \bigsqcup_{k\in\mathbb{N}} (a_k, b_k)$. Therefore,

$$\mu(U) = \sum_{k\in\mathbb{N}} \mu(a_k, b_k) = \sum_{k=0}^{\infty} F_\mu(b_k) - F_\mu(a_k),$$

using for the second equality the fact that $\mu(\{a_k\}) = 0$ for all k. Fix $\varepsilon > 0$, and K such that

$$\sum_{k=0}^{K} F_\mu(b_k) - F_\mu(a_k) \geq \mu(U) - \varepsilon.$$

By continuity of F_μ, one can then choose $\eta > 0$ such that

$$\sum_{k=0}^{K} F_\mu(b_k) - F_\mu(a_k + \eta) \geq \mu(U) - 2\varepsilon;$$

notice that $\bigsqcup_{k=0}^{K} [a_k + \eta, b_k) \subset U$. Since F_{μ_n} converges to F_μ, one has then

$$\begin{aligned}\liminf_{n\to\infty} \mu_n(U) &\geq \liminf_{n\to\infty}\left(\sum_{k=0}^{K} \mu_n([a_k + \eta, b_k))\right) = \liminf_{n\to\infty}\left(\sum_{k=0}^{K} F_{\mu_n}(b_k) - F_{\mu_n}(a_k + \eta)\right) \\ &\geq \mu(U) - 2\varepsilon.\end{aligned}$$

It is true for every $\varepsilon > 0$, so $\liminf_{n\to\infty} \mu_n(U) \geq \mu(U)$, and by Portmanteau's theorem $\mu_n \rightharpoonup \mu$.

(3) The positivity of Δ_T is obvious since $1 \geq \cos Tx$. Notice then that the function $x \mapsto \frac{1-\cos Tx}{\pi T x^2}$ is in $\mathscr{L}^1(\mathbb{R}) \cap \mathscr{L}^2(\mathbb{R})$ (it is continuous and bounded by $\frac{1}{x^2}$ at infinity). Therefore, it is the inverse Fourier transform of its Fourier transform. So, it is equivalent to prove

$$\widehat{\Delta_T}(t) = \mathbf{1}_{|t|\leq T}\left(1 - \frac{|t|}{T}\right)$$

and to prove that

$$\Delta_T(x) = \frac{1}{2\pi}\int_{-T}^{T}\left(1 - \frac{|t|}{T}\right)\mathrm{e}^{-\mathrm{i}tx}\,dt.$$

This integral is easily computed:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-T}^{T}\left(1 - \frac{|t|}{T}\right)\mathrm{e}^{-\mathrm{i}tx}\,dt &= \frac{1}{2\pi}\left[\frac{\mathrm{e}^{-\mathrm{i}tx}}{-\mathrm{i}x}\right]_{-T}^{T} + \frac{1}{2\pi T}\int_{-T}^{0} t\,\mathrm{e}^{-\mathrm{i}tx}\,dt - \frac{1}{2\pi T}\int_{0}^{T} t\,\mathrm{e}^{-\mathrm{i}tx}\,dt \\
&= \frac{\sin Tx}{\pi x} + \frac{1}{2\pi T}\left(\left[\frac{t\,\mathrm{e}^{-\mathrm{i}tx}}{-\mathrm{i}x}\right]_{-T}^{0} - \left[\frac{t\,\mathrm{e}^{-\mathrm{i}tx}}{-\mathrm{i}x}\right]_{0}^{T}\right) \\
&\quad + \frac{1}{2\pi T}\left(\int_{-T}^{0}\frac{\mathrm{e}^{-\mathrm{i}tx}}{\mathrm{i}x}\,dt - \int_{0}^{T}\frac{\mathrm{e}^{-\mathrm{i}tx}}{\mathrm{i}x}\,dt\right) \\
&= \frac{1}{2\pi T x^2}\left(\left[\mathrm{e}^{-\mathrm{i}tx}\right]_{-T}^{0} - \left[\mathrm{e}^{-\mathrm{i}tx}\right]_{0}^{T}\right) = \frac{1-\cos Tx}{\pi T x^2}.
\end{aligned}$$

We have then computed the Fourier transform of Δ_T, and in particular,

$$\int_{\mathbb{R}} \Delta_T(x)\,dx = \widehat{\Delta_T}(0) = 1.$$

Finally, set $h > 0$. The mass of $(-h,h)^c$ is smaller than

$$\frac{2}{\pi T}\int_{h}^{\infty}\frac{1-\cos Tx}{x^2}\,dx \leq \frac{2}{\pi T}\int_{h}^{\infty}\frac{2}{x^2}\,dx = \frac{4}{\pi T h}.$$

(4) If $\eta = 0$,then $F = G$, $D = F - G = 0$, and so is its convolution by Δ_T:

$$D_T(x) = \int_{\mathbb{R}} D(x-y)\,\Delta_T(y)\,dy = \int_{\mathbb{R}} 0\,dy = 0 \quad \Rightarrow \quad \eta_T = 0.$$

Suppose $\eta > 0$ and fix x_0 such that $D(x_0) = \eta$. Since F is increasing and G is Lipschitz with constant m, for every $|y| \leq \frac{\eta}{2m}$,

$$\begin{aligned}
D(x-y) &= F(x-y) - G(x-y) \\
&\geq F(x_0) - G(x_0) - m(x-y-x_0) = \eta - mh + my = \frac{\eta}{2} + my.
\end{aligned}$$

Then,

$$\begin{aligned}
\eta_T \geq D_T(x) &= \int_{\mathbb{R}} D(x-y)\,\Delta_T(y)\,dy \\
&\geq -\eta\int_{(-h,h)^c}\Delta_T(y)\,dy + \int_{(-h,h)}\left(\frac{\eta}{2} + my\right)\Delta_T(y)\,dy \\
&\geq -\eta\int_{(-h,h)^c}\Delta_T(y)\,dy + \frac{\eta}{2}\int_{(-h,h)}\Delta_T(y)\,dy \qquad \text{since } y\,\Delta_T(y) \text{ is odd;} \\
&\geq \frac{\eta}{2} - \frac{3\eta}{2}\int_{(-h,h)^c}\Delta_T(y)\,dy \\
&\geq \frac{\eta}{2} - \frac{6\eta}{\pi T h} = \frac{\eta}{2} - \frac{12m}{\pi T}.
\end{aligned}$$

The inequality is also trivially true when $\eta = 0$.

(5) Recall that the Fourier transform of a convolution is the product of Fourier transforms:

$$\begin{aligned}\widehat{(a*b)}(t) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} a(x-y)\, b(y)\, dy\right) \mathrm{e}^{\mathrm{i}tx}\, dx \\ &= \iint_{\mathbb{R}\times\mathbb{R}} (a(x-y)\, \mathrm{e}^{\mathrm{i}t(x-y)}\, dx)\, (b(y)\, \mathrm{e}^{\mathrm{i}ty}\, dy) = \widehat{a}(t)\, \widehat{b}(t)\end{aligned}$$

We use this result with $a = D$ and $b = \Delta_T$, after a Fourier inversion of D_T (we have the right to do so, since a and b are both in $\mathscr{L}^1(\mathbb{R}) \cap \mathscr{L}^2(\mathbb{R})$):

$$\begin{aligned}D_T(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{D_T}(t)\, \mathrm{e}^{-\mathrm{i}tx}\, dt \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{D}(t)\, \widehat{\Delta_T}(t)\, \mathrm{e}^{-\mathrm{i}tx}\, dt \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\widehat{F}(t) - \widehat{G}(t)\right) \widehat{\Delta_T}(t)\, \mathrm{e}^{-\mathrm{i}tx}\, dt.\end{aligned}$$

Notice then that $\widehat{a'}(t) = -\mathrm{i}t\, \widehat{a}(t)$ by integration by parts:

$$\widehat{a'}(t) = \int_{\mathbb{R}} a'(x)\, \mathrm{e}^{\mathrm{i}tx}\, dx = -\mathrm{i}t \int_{\mathbb{R}} a(x)\, \mathrm{e}^{\mathrm{i}tx}\, dx = -\mathrm{i}t\, \widehat{a}(t).$$

Therefore,

$$\begin{aligned}\widehat{F}(t) - \widehat{G}(t) &= \frac{\widehat{\mu}(t) - \widehat{g}(t)}{-\mathrm{i}t} = \frac{\phi_\mu(t) - \phi_G(t)}{-\mathrm{i}t}; \\ D_T(x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\Delta_T}(t) \left(\frac{\phi_\mu(t) - \phi_G(t)}{-\mathrm{i}t}\right) \mathrm{e}^{-\mathrm{i}tx}\, dt \\ &\leq \frac{1}{2\pi} \int_{\mathbb{R}} \left|\widehat{\Delta_T}(t)\right| \left|\frac{\phi_\mu(t) - \phi_G(t)}{t}\right| dt \\ &\leq \frac{1}{2\pi} \int_{-T}^{T} \left|\frac{\phi_\mu(t) - \phi_G(t)}{t}\right| dt.\end{aligned}$$

This is true for every x, so η_T has the same bound, and then

$$\eta \leq 2\eta_T + \frac{24m}{\pi T} \leq \frac{1}{\pi} \int_{-T}^{T} \left|\frac{\phi_\mu(t) - \phi_G(t)}{t}\right| dt \;\; + \frac{24m}{\pi T}.$$

(6) We calculate

$$\begin{aligned}\phi_{\mu_n}(t) = \mathbb{E}[\mathrm{e}^{\mathrm{i}tX_n}] = \mathbb{E}[\mathrm{e}^{\mathrm{i}tY_n/\sqrt{\lambda_n}}] &= \mathrm{e}^{-\frac{t^2}{2}}\, \psi_n\!\left(\frac{t}{\sqrt{\lambda_n}}\right) \\ &= \mathrm{e}^{-\frac{t^2}{2}} \left(\psi\!\left(\frac{t}{\sqrt{\lambda_n}}\right) + O\!\left(\frac{t}{\sqrt{\lambda_n}}\right) o\!\left(\frac{1}{\sqrt{\lambda_n}}\right)\right)\end{aligned}$$

since one has uniform convergence $\psi_n \to \psi$ at speed $o(\sqrt{\lambda_n}^{-1})$ if the argument $t/\sqrt{\lambda_n}$ stays bounded. The remainder is a $o\!\left(\frac{t}{\sqrt{\lambda_n}}\right)$; then we can expand ψ in Taylor series to get

$$\phi_{\mu_n}(t) = \mathrm{e}^{-\frac{t^2}{2}} \left(1 + \frac{\psi'(0)\, t}{\sqrt{\lambda_n}} + o\!\left(\frac{t}{\sqrt{\lambda_n}}\right)\right).$$

Then,

$$\begin{aligned}\int_{\mathbb{R}} g_n(x)\, \mathrm{e}^{\mathrm{i}tx}\, dx &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(1 - \frac{\mathrm{i}\psi'(0)\, x}{\sqrt{\lambda_n}}\right) \mathrm{e}^{\mathrm{i}tx - \frac{x^2}{2}}\, dx \\ &= \left(1 - \frac{\psi'(0)}{\sqrt{\lambda_n}} \frac{\partial}{\partial t}\right) \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx - \frac{x^2}{2}}\, dx\right) \\ &= \left(1 - \frac{\psi'(0)}{\sqrt{\lambda_n}} \frac{\partial}{\partial t}\right) \mathrm{e}^{-\frac{t^2}{2}} \\ &= \mathrm{e}^{-\frac{t^2}{2}} \left(1 + \frac{\psi'(0)\, t}{\sqrt{\lambda_n}}\right).\end{aligned}$$

This function is bounded, so the integral $G_n(x)$ is indeed m-Lipschitz for a certain constant m. Moreover, one has indeed $G_n(+\infty) = 1$ and $G_n(-\infty) = 0$, so Berry's lemma will apply.

(7) Berry's lemma ensures that

$$\begin{aligned}\eta_n &= \sup_{x\in\mathbb{R}} |F_{\mu_n}(x) - G_n(x)| \\ &\leq \frac{1}{\pi}\int_{-K\sqrt{\lambda_n}}^{K\sqrt{\lambda_n}} \left|\frac{\phi_{\mu_n}(t) - \widehat{g_n}(t)}{t}\right| dt + \frac{24m}{\pi K\sqrt{\lambda_n}} \\ &\leq \frac{1}{\pi\sqrt{\lambda_n}}\left(\int_{(-K\sqrt{\lambda_n},K\sqrt{\lambda_n})\setminus(-\varepsilon\sqrt{\lambda_n},\varepsilon\sqrt{\lambda_n})} + \int_{(-\varepsilon\sqrt{\lambda_n},\varepsilon\sqrt{\lambda_n})}\right)\left|\mathrm{e}^{-\frac{t^2}{2}}\,\upsilon\!\left(\frac{t}{\sqrt{\lambda_n}}\right)\right| dt + \frac{24m}{\pi K\sqrt{\lambda_n}}\end{aligned}$$

where $\upsilon(\cdot)$ is a function going to zero at zero. Fix $\theta > 0$, and ε such that $|\upsilon(\cdot)|$ is smaller than θ on $(-\varepsilon,\varepsilon)$. The integral on $(-\varepsilon\sqrt{\lambda_n},\varepsilon\sqrt{\lambda_n})$ is then smaller that

$$\frac{\theta}{\pi\sqrt{\lambda_n}}\int_{\mathbb{R}} \mathrm{e}^{-\frac{t^2}{2}}\,dt \leq \frac{\theta}{\sqrt{\lambda_n}}.$$

Fix then K such that $\frac{24m}{\pi K} \leq \theta$. The third summand in the upper bound is then also smaller than $\frac{\theta}{\sqrt{\lambda_n}}$, and on the other hand, $\upsilon(\cdot)$ is bounded by some constant $C(K)$ on $(-K,K)$. This leads to

$$\begin{aligned}\eta_n &\leq \frac{2\theta}{\sqrt{\lambda_n}} + \frac{1}{\pi\sqrt{\lambda_n}}\int_{(-K\sqrt{\lambda_n},K\sqrt{\lambda_n})\setminus(-\varepsilon\sqrt{\lambda_n},\varepsilon\sqrt{\lambda_n})} C(K)\,\mathrm{e}^{-\frac{t^2}{2}}\,dt \\ &\leq \frac{2\theta}{\sqrt{\lambda_n}} + \frac{C(K)}{\pi\sqrt{\lambda_n}}\int_{\mathbb{R}\setminus(-\varepsilon\sqrt{\lambda_n},\varepsilon\sqrt{\lambda_n})} \mathrm{e}^{-\frac{t^2}{2}}\,dt \\ &\leq \frac{2\theta}{\sqrt{\lambda_n}} + \frac{C(K)\,\mathrm{e}^{-\frac{\varepsilon^2\lambda_n}{2}}}{\pi\varepsilon\lambda_n} \leq \frac{3\theta}{\sqrt{\lambda_n}} \quad \text{for } n \text{ big enough.}\end{aligned}$$

Since this is true for every θ, the estimate $o(1/\sqrt{\lambda_n})$ is shown.

(8) The previous Question proves that up to a uniform $o(1/\sqrt{\lambda_n})$,

$$F_{\mu_n}(x) \simeq F_{\mathcal{N}_{(0,1)}}(x) + \int_{-\infty}^{x} \frac{-\mathrm{i}\psi'(0)\,y}{\sqrt{\lambda_n}}\,\frac{\mathrm{e}^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy = F_{\mathcal{N}_{(0,1)}}(x) + \frac{\mathrm{i}\psi'(0)}{\sqrt{2\pi\lambda_n}}\,\mathrm{e}^{-\frac{x^2}{2}},$$

so

$$d(\mu_{X_n},\mathcal{N}_{(0,1)}) \simeq \frac{|\mathrm{Im}(\psi'(0))|}{\sqrt{2\pi\lambda_n}}\,\sup_{x\in\mathbb{R}}\left(\mathrm{e}^{-\frac{x^2}{2}}\right) \simeq \frac{|\mathrm{Im}(\psi'(0))|}{\sqrt{2\pi\lambda_n}}.$$