Soutenance de thèse

Morse theory and higher algebra of A_∞-algebras

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IMJ-PRG - Sorbonne Université

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1 Introduction
- Morse theory
- Homotopy transfer theorem
- Main result and plan of the talk

2 The A_∞-algebra structure on the Morse cochains

3 Continuation morphisms

4 Higher morphisms between A_∞-algebras

5 Higher morphisms in Morse theory

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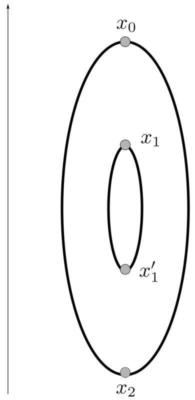

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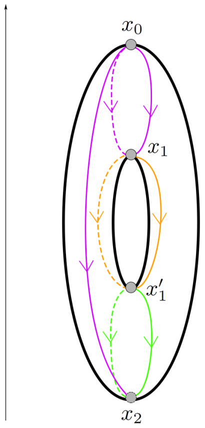
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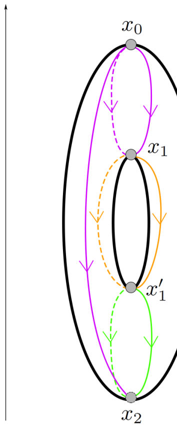
Under some technical assumptions, there exists at most a finite number of trajectories between two critical points x and y of consecutive degrees $|y| = |x| + 1$.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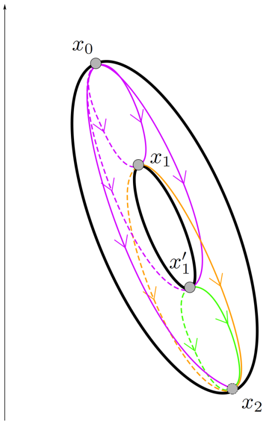
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$$C^k(f) := \bigoplus_{x \in \mathrm{Crit}(f),\ |x|=k} \mathbb{Z} \cdot x .$$

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One can then define maps $\partial^k : C^k(f) \longrightarrow C^{k+1}(f)$ by the formula

$$\partial^k(x) := \sum_{|y|=k+1} \#\mathcal{M}(y;x) \cdot y .$$

These maps satisfy the equation $\partial^{k+1} \circ \partial^k = 0$.

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In other words, they define a cochain complex, called the *Morse cochains*

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The cohomology groups $H^k(f) := \mathrm{Ker}(\partial^k)/\mathrm{Im}(\partial^{k-1})$ are called the *Morse cohomology groups*.

Theorem ([Flo89][1])

The Morse cohomology groups are isomorphic to the singular cohomology groups $H^k_{sing}(M) \simeq H^k(f)$.

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If we interpret symplectic topology as a quantification of differential topology, pseudo-holomorphic curves theory can be interpreted as the quantification of Morse theory. See for instance [Fuk93][2].

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$$h \circlearrowright (C^*_{sing}, \partial_{sing}) \underset{i}{\overset{p}{\rightleftarrows}} (C^*(f), \partial_{Morse}) ,$$

where $\mathrm{id} - ip = \partial_{sing} h + h \partial_{sing}$.

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The cup product endows the singular cochains $C^*_{sing}(M)$ with a (dg) associative algebra structure. Can it be transferred to an associative algebra structure on the Morse cochains $C^*(f)$?

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The answer is yes, if we allow the product on $C^*(f)$ to be associative only up to homotopy.

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Theorem (Homotopy transfer theorem, [Kad80][4])

Let (A, ∂_A) and (H, ∂_H) be two cochain complexes. Suppose that H is a deformation retract of A, that is that they fit into a diagram

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An A_∞-algebra on a cochain complex is an explicit model for an algebra whose product is associative only up to homotopy. We will define it later.

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The differential on the Morse cochains is defined by a count of Morse trajectories. Is it then possible to define a product on the Morse cochains $C^*(f)$ by counting some geometric objects defined using the Morse function f ?

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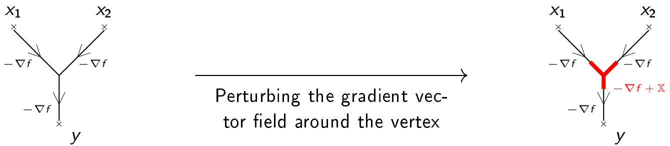

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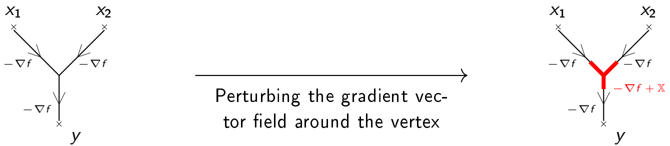

The intuition behind this construction is the intersection product.

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Definition

The *geometric* product on the Morse cochains $C^*(f)$ is defined as

$$\mu(x_1, x_2) := \sum_{|y|=|x_1|+|x_2|} \#\mathcal{M}(y; x_1, x_2) \cdot y \ .$$

Abouzaid proves that this product is homotopy associative and is part of a *geometric* A_∞-algebra structure on the Morse cochains.

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The main result of this thesis is to explain in which sense these geometric A_∞-algebra structures on the Morse cochains are unique up to homotopy:

Consider two Morse functions f and g on the manifold M:

[7] T. Mazuir. *Higher algebra of A_∞ and ΩBAs-algebras in Morse theory I*, 2021.
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Consider two Morse functions f and g on the manifold M:

Theorem ([Maz21a][7], [Maz21b][8])

The collection of geometric higher morphisms between $C^*(f)$ and $C^*(g)$ fit into a simplicial set which is a Kan complex and is moreover contractible.

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The theorem gives a higher categorical meaning to the fact that continuation morphisms in Morse theory are unique up to homotopy at the chain level.

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Definition

Let A be a cochain complex with differential m_1. An A_∞-algebra structure on A is the data of a collection of maps of degree $2-n$

$$m_n : A^{\otimes n} \longrightarrow A \ , \ \ n \geqslant 1,$$

extending m_1 and which satisfy

$$[m_1, m_n] = \sum_{\substack{i_1+i_2+i_3=n \\ 2\leqslant i_2 \leqslant n-1}} \pm m_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}).$$

These equations are called the A_∞-equations.

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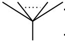
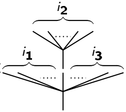
Here $[m_1, m_n] = m_1 m_n - (-1)^n \sum_{i=0}^{n-1} m_n(\mathrm{id}^{\otimes i} \otimes m_1 \otimes \mathrm{id}^{\otimes n-i-1})$.

Representing m_n as  a corolla of arity n, these equations can be written as

$$[m_1, \text{(corolla)}] = \sum_{\substack{i_1+i_2+i_3=n \\ 2 \leqslant i_2 \leqslant n-1}} \pm \text{(tree)} .$$





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In particular,

$$[m_1, m_2] = 0 \ ,$$
$$[m_1, m_3] = m_2(\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) \ ,$$

implying that m_2 descends to an associative product on $H^*(A)$.

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The operations m_n can be interpreted as the higher coherent homotopies keeping track of the fact that the product is associative up to homotopy.

1 Introduction

2 The A_∞-algebra structure on the Morse cochains

- A_∞-algebras
- The associahedra
- Moduli spaces of metric trees
- Perturbed Morse trees and A_∞-structures in Morse theory

3 Continuation morphisms

4 Higher morphisms between A_∞-algebras

5 Higher morphisms in Morse theory

There exists a collection of polytopes, called the *associahedra* and denoted K_n, which encode the A_∞-equations between A_∞-algebras.

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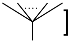
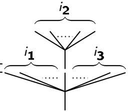
$$\partial K_n = \bigcup_{\substack{i_1+i_2+i_3=n \\ 2 \leqslant i_2 \leqslant n-1}} K_{i_1+1+i_3} \times K_{i_2} \ .$$

Recall that the A_∞-equations read as

$[m_1,$  $] = \sum_{\substack{i_1+i_2+i_3=n \\ 2 \leqslant i_2 \leqslant n-1}} \pm$ 

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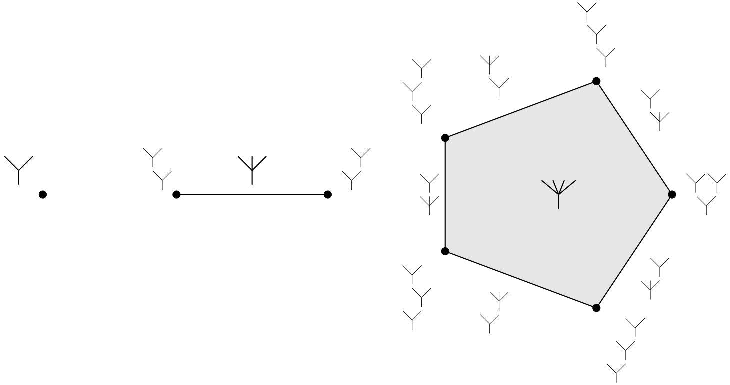


Figure: The associahedra K_2, K_3 and K_4, with cells labeled by the operations they define

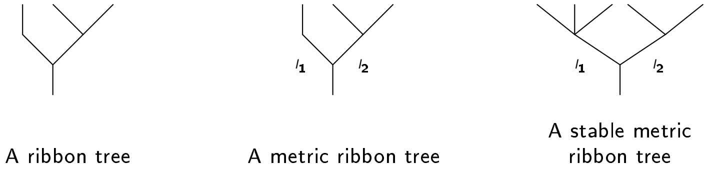
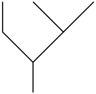
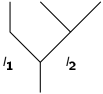


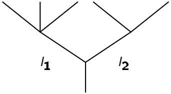
Figure: Terminology



A ribbon tree



A metric ribbon tree



A stable metric ribbon tree

Figure: Terminology

Definition

We denote $\mathcal{T}_n$ the *moduli space of stable metric ribbon trees with n incoming edges*.

Allowing lengths of internal edges to go to $+\infty$, this moduli space can be compactified into a $(n-2)$-dimensional CW-complex $\overline{\mathcal{T}}_n$.

[9] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*, 1973.

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Theorem ([BV73][9])

The compactified moduli space $\overline{\mathcal{T}}_n$ is isomorphic as a CW-complex to the associahedron K_n.

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The compactified moduli spaces of stable metric ribbon trees $\overline{\mathcal{T}}_n$ encode the notion of A_∞-algebra.

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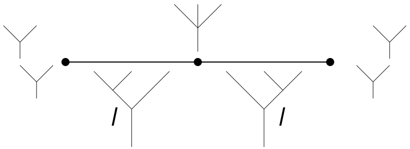

Figure: The compactified moduli space $\overline{\mathcal{T}}_3$

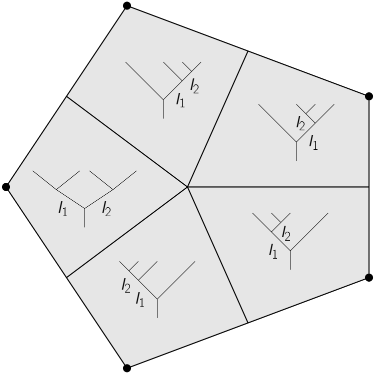

Figure: The compactified moduli space $\overline{\mathcal{T}}_4$

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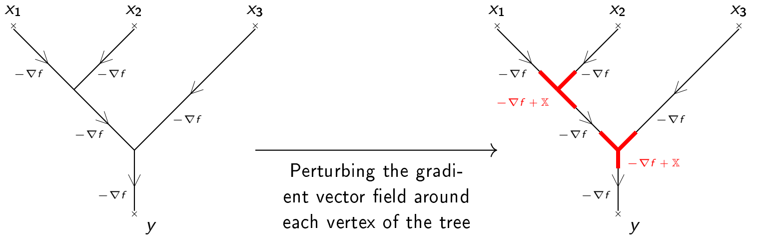

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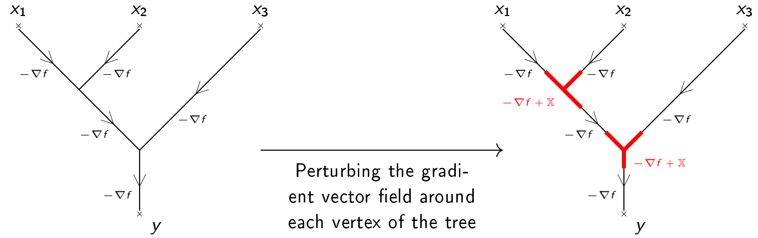

The vector field $\mathbb{X}$ will be called a *perturbation* of the negative gradient $-\nabla f$.

Theorem ([Abo11],[Mes18][10], [Maz21a])

Under an admissible choice of perturbation data $\mathbb{X}$ on the moduli spaces $\mathcal{T}_n$, the moduli spaces of perturbed Morse trees define an A_∞-algebra structure on the Morse cochains $C^*(f)$.

[10] S. Mescher. *Perturbed gradient flow trees and A_∞-algebra structures in Morse cohomology*, 2018.

[11] H. Abbaspour and F. Laudenbach, *Morse complexes and multiplicative structures*, 2022.

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This means that the operations of arity n of the A_∞-algebra structure on $C^*(f)$ is defined by counting perturbed Morse trees of arity n.

[10] S. Mescher. *Perturbed gradient flow trees and A_∞-algebra structures in Morse cohomology*, 2018.

[11] H. Abbaspour and F. Laudenbach, *Morse complexes and multiplicative structures*, 2022.

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Abbaspour and Laudenbach provide an alternative construction of this A_∞-algebra structure in [AL22][11].

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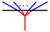
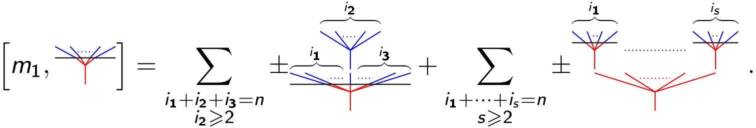
Definition

An A_∞-*morphism* $A \rightsquigarrow B$ between two A_∞-algebras A and B is a family of maps $f_n : A^{\otimes n} \to B$ of arity $n \geqslant 1$ and of degree $1-n$ satisfying

$$[m_1, f_n] = \sum_{\substack{i_1+i_2+i_3=n \\ i_2 \geqslant 2}} \pm f_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2}^A \otimes \mathrm{id}^{\otimes i_3}) + \sum_{\substack{i_1+\cdots+i_s=n \\ s \geqslant 2}} \pm m_s^B(f_{i_1} \otimes \cdots \otimes f_{i_s}) \ .$$

Representing the operations f_n as , the operations m_n^A in blue and the operations m_n^B in red, these equations read as



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A *quasi-isomorphism* is an A_∞-morphism inducing an isomorphism in cohomology.
We will write $A \simeq B$ if there exists a quasi-isomorphism $A \rightsquigarrow B$.

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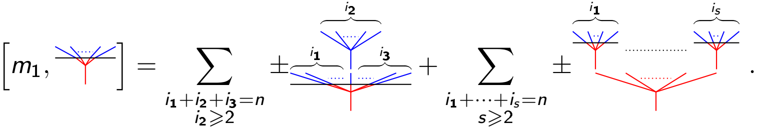
$$\partial J_n = \bigcup_{\substack{i_1+i_2+i_3=n \\ i_2 \geqslant 2}} J_{i_1+1+i_3} \times K_{i_2} \cup \bigcup_{\substack{i_1+\cdots+i_s=n \\ s \geqslant 2}} K_s \times J_{i_1} \times \cdots \times J_{i_s} \ .$$

Recall that the A_∞-equations for A_∞-morphisms are



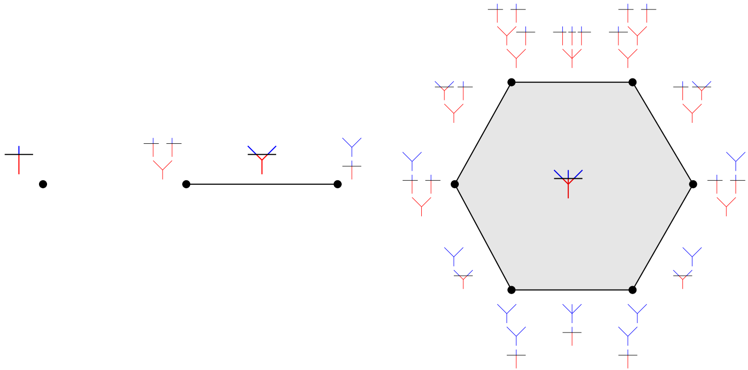
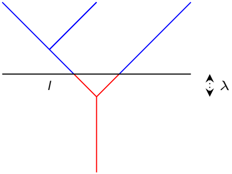


Figure: The multiplihedra J_1, J_2 and J_3 with cells labeled by the operations they define

Definition

A *stable 2-colored metric ribbon tree* is defined to be a stable metric ribbon tree together with a length $\lambda \in \mathbb{R}$, which is to be thought of as a dividing line drawn over the metric tree, at distance λ from its root.



Definition

For $n \geqslant 1$, we denote $\mathcal{CT}_n$ the *moduli space of stable 2-colored metric ribbon trees*.

[12] S. Ma'u and C. Woodward. *Geometric realizations of the multiplihedra*, 2010.

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Allowing again internal edges of metric trees to go to $+\infty$, this moduli space can be compactified into a $(n-1)$-dimensional CW-complex $\overline{\mathcal{CT}}_n$.

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The compactified moduli spaces of stable 2-colored metric ribbon trees $\overline{\mathcal{CT}}_n$ encode the notion of A_∞-morphism.

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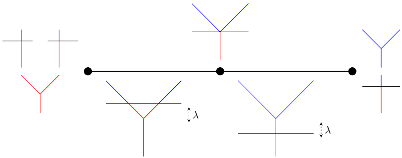

Figure: The compactified moduli space $\overline{\mathcal{CT}}_2$ with its cell decomposition by 2-colored tree type

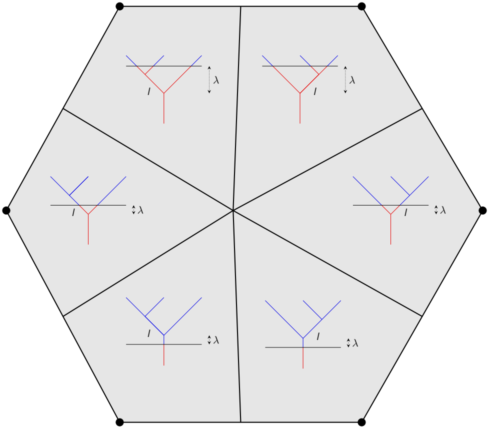

Figure: The compactified moduli space $\overline{\mathcal{CT}}_3$ with its cell decomposition by 2-colored tree type

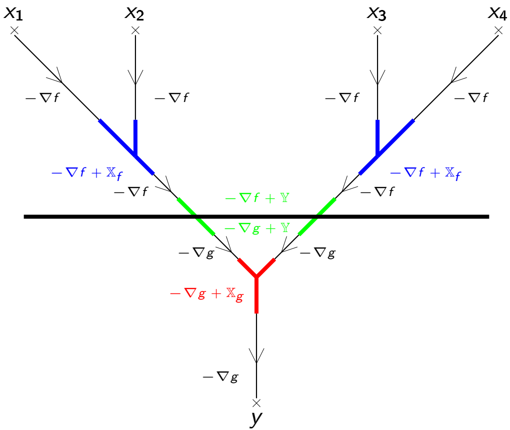

Figure: Perturbed 2-colored Morse tree

Theorem ([Maz21a])

Under an admissible choice of perturbation data $\mathbb{Y}$ on the moduli spaces $\mathcal{CT}_n$, the moduli spaces of perturbed 2-colored Morse trees define an A_∞-morphism between the Morse cochains $C^*(f)$ and $C^*(g)$.

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Proposition ([Maz21a])

Continuation morphisms are quasi-isomorphisms.

Given two continuation morphisms $\mu, \mu' : C^*(f) \rightsquigarrow C^*(g)$, we would now like to know whether they are always homotopic or not.

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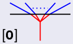
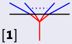
We first have to determine a notion giving a satisfactory meaning to the sentence "μ and μ' are homotopic".

Definition ([LH03][13])

Given two A_∞-morphisms $(f_n)_{n\geqslant 1}$ and $(g_n)_{n\geqslant 1}$, represented respectively by the 2-colored corollae 

and 

, an A_∞-*homotopy* between them corresponds to:

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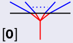
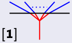
Given two A_∞-morphisms $(f_n)_{n\geqslant 1}$ and $(g_n)_{n\geqslant 1}$, represented respectively by the 2-colored corollae 

and 

, an A_∞-*homotopy* between them corresponds to:

- a collection of maps

$$h_n : A^{\otimes n} \longrightarrow B \ ,$$

of arity $n \geqslant 1$ and degree $-n$.

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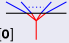
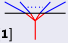
Given two A_∞-morphisms $(f_n)_{n\geqslant 1}$ and $(g_n)_{n\geqslant 1}$, represented respectively by the 2-colored corollae [0]  and [1] , an A_∞-*homotopy* between them corresponds to:

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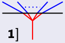
We represent them with 2-colored corollae [0 < 1] .

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Definition ([LH03])

- satisfying the equations

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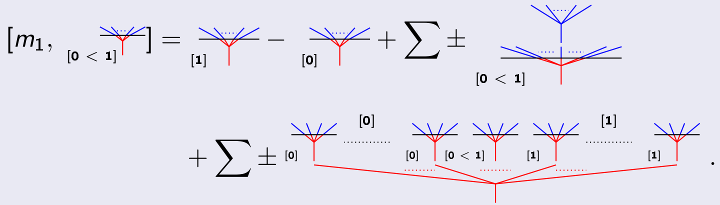
$$+ \sum_{\substack{i_1+\cdots+i_s+l \\ +j_1+\cdots+j_t=n \\ s+1+t \geqslant 2}} \pm m_{s+1+t}(f_{i_1} \otimes \cdots \otimes f_{i_s} \otimes h_l \otimes g_{j_1} \otimes \cdots \otimes g_{j_t})$$



1 Introduction

2 The A_∞-algebra structure on the Morse cochains

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4 Higher morphisms between A_∞-algebras
- A_∞-homotopies
- Higher morphisms between A_∞-algebras
- The HOM-simplicial sets $\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A, B)_\bullet$

5 Higher morphisms in Morse theory

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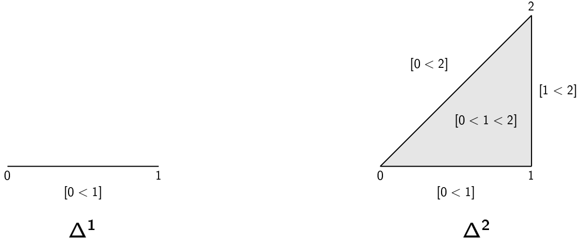

Definition ([MS03][14])

Let I be a face of Δ^n. An *overlapping s-partition* of I is a sequence of subfaces $(I_l)_{1 \leqslant \ell \leqslant s}$ of I such that

(i) the union of this sequence of faces is I, i.e. $\cup_{1 \leqslant \ell \leqslant s} I_l = I$;

(ii) for all $1 \leqslant \ell < s$, $\max(I_\ell) = \min(I_{\ell+1})$.

[14] J. E. McClure and J. H. Smith. *Multivariable cochain operations and little n-cubes*, 2003.

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The symbol $\cup$ denotes the set-theoretic union where a face $[i_1 < \cdots < i_k] \subset \Delta^n$ is seen as the set $\{i_1 < \cdots < i_k\} \subset \mathbb{N}$.

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An overlapping 6-partition for $[0 < 1 < 2]$ is for instance

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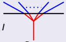
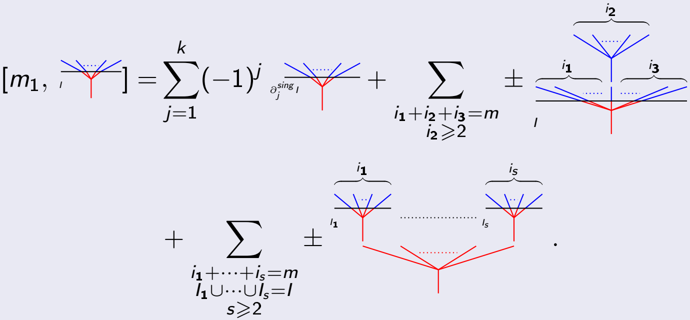
An overlapping 3-partition for $[0 < 1 < 2 < 3 < 4 < 5]$ is for instance

$$[0 < 1 < 2 < 3 < 4 < 5] = [0 < 1] \cup [1 < 2 < 3] \cup [3 < 4 < 5] \ .$$

Definition ([Maz21b])

A *n-morphism* from A to B is defined to be the data for each face $I \subset \Delta^n$ of a collection of maps  $: A^{\otimes m} \longrightarrow B$ of arity $m \geqslant 1$ and of degree $1 - m - \dim(I)$, that satisfy



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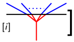
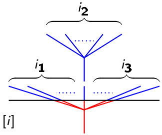
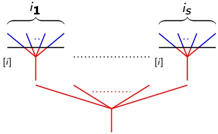
For $i = 0, 1$,

$[m_1,$ 

$] = \sum_{\substack{i_1+i_2+i_3=m \\ i_2 \geqslant 2}} \pm$ 

$+ \sum_{\substack{i_1+\cdots+i_s=m \\ s \geqslant 2}} \pm$ 

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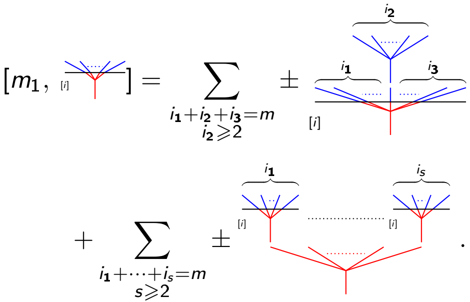
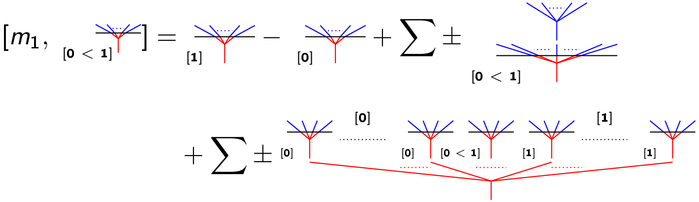


the overlapping partitions of $[i]$ being exactly

$$[i] = [i] \cup \cdots \cup [i] \ ,$$

and



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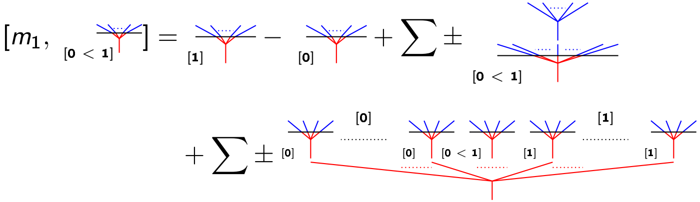

the overlapping partitions of $[0 < 1]$ being exactly

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1 Introduction

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Proposition ([Maz21b])

The sets of n-morphisms between two A_∞-algebras A and B, that we denote $\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A,B)_n$, fit into a simplicial set

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Theorem ([Maz21b])

For A and B two A_∞-algebras, the simplicial set $\mathrm{HOM}_{A_\infty}(A,B)_\bullet$ is a Kan complex.

Write Δ^n the simplicial set realizing the standard n-simplex, and Λ_n^k the simplicial set realizing the simplicial subcomplex obtained from Δ^n by removing the faces $[0 < \cdots < n]$ and $[0 < \cdots < \widehat{k} < \cdots < n]$.

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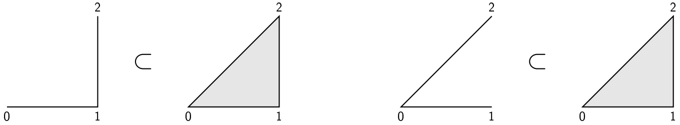

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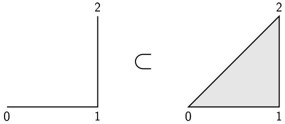
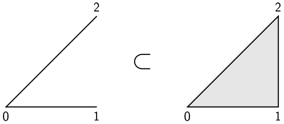


$\Lambda^1_2 \subset \Delta^2$



$\Lambda^0_2 \subset \Delta^2$

The simplicial set Λ^k_n is called a *horn*.

Definition

A *Kan complex*/an ∞-*groupoid* is a simplicial set X which has the left-lifting property with respect to all horn inclusions $\Lambda_n^k \hookrightarrow \Delta^n$.

$$\begin{array}{ccc} \Lambda_n^k & \xrightarrow{\quad u \quad} & X \\ \big\downarrow & \nearrow_{\exists\, \overline{u}} & \\ \Delta^n & & \end{array}$$

Recall that a groupoid is a category all of whose morphisms are invertible.

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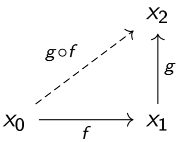

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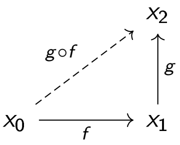

$\Lambda^1_2 \subset \Delta^2$

This composition is moreover associative up to homotopy.

Theorem ([Maz21b])

For A and B two A_∞-algebras, the simplicial set $\mathrm{HOM}_{A_\infty}(A,B)_\bullet$ is a Kan complex.

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The proof relies on the use of cosimplicial resolutions in the model category of conilpotent dg-coalgebras defined in [LH03].

We would like to define a family of polytopes encoding n-morphisms between A_∞-algebras. These polytopes will be called the *n-multiplihedra*.

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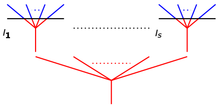
However the faces of $\Delta^n \times J_m$ correspond to the data of a face of $I \subset \Delta^n$, and of a broken 2-colored tree labeling a face of J_m.

This labeling is too coarse, as it does not contain the trees



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that appear in the A_∞-equations for n-morphisms.

We thus want to lift the combinatorics of overlapping partitions to the level of the n-simplices Δ^n.

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Proposition ([Maz21b])

For each $s \geqslant 1$, there exists a polytopal subdivision of the standard n-simplex Δ^n whose top-dimensional cells are in one-to-one correspondence with all overlapping s-partitions of Δ^n.

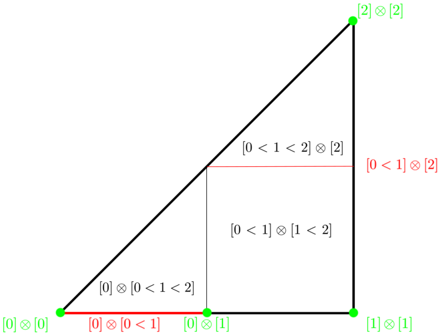
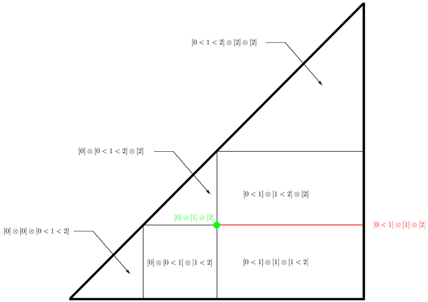


Figure: The subdivision of Δ^2 by overlapping 2-partitions



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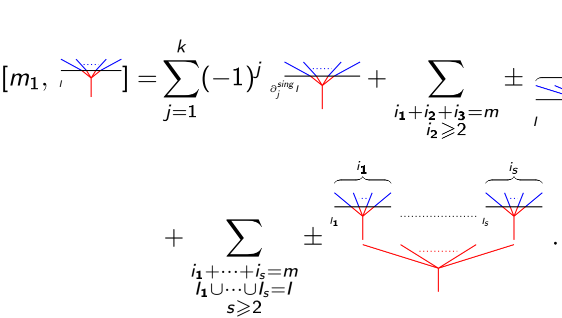
Definition ([Maz21b])

The *n-multiplihedra* are defined to be the polytopes $\Delta^n \times J_m$ endowed with the previous polytopal subdivision. We denote them n-J_m.

Recall that the A_∞-equations for n-morphisms read as



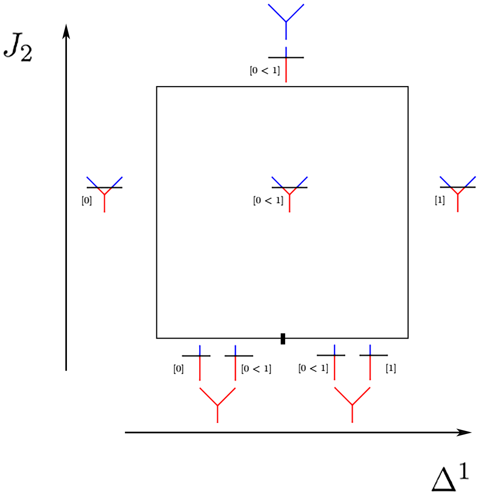


Figure: The 1-multiplihedron 1-J_2

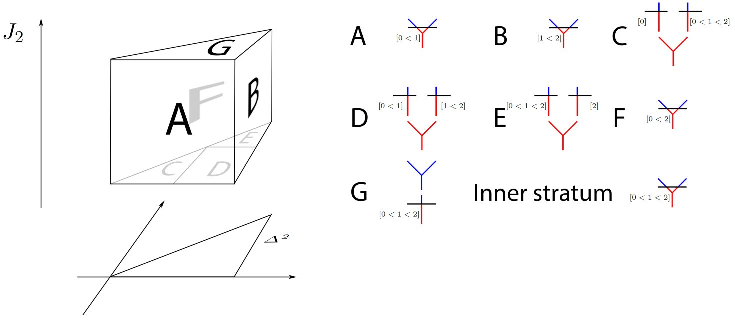


Figure: The 2-multiplihedron 2-J_2

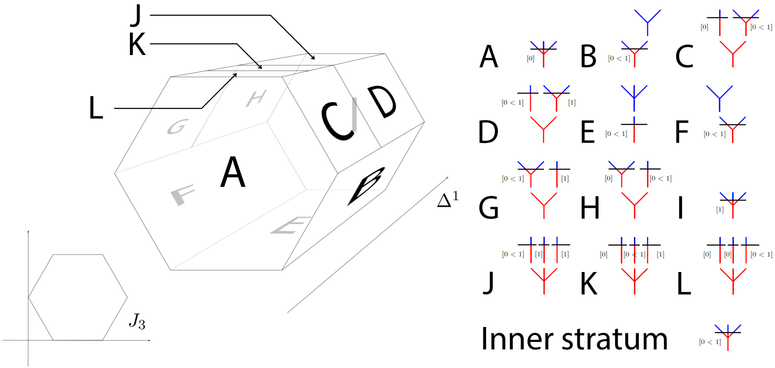


Figure: The 1-multiplihedron 1-J_3

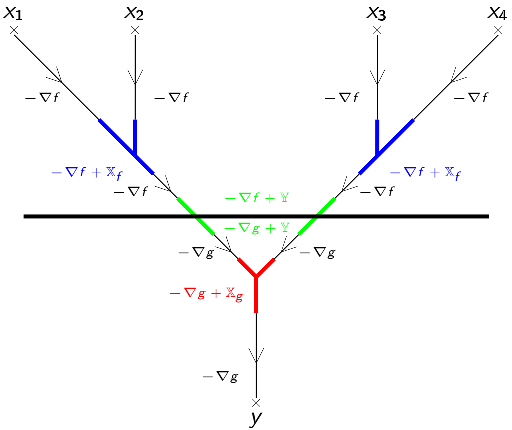


Figure: Perturbed 2-colored Morse tree

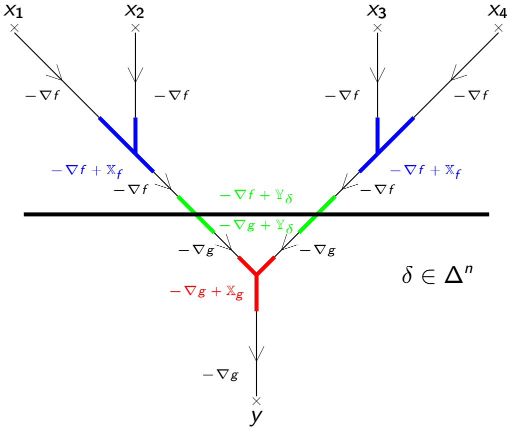


Figure: Perturbed 2-colored Morse tree associated to a n-simplex of perturbation data $(\mathbb{Y}_\delta)_{\delta \in \Delta^n}$

Theorem ([Maz21b])

Under an admissible choice of n-simplex of perturbation data $(\mathbb{Y}_\delta)_{\delta \in \Delta^n}$ *on the moduli spaces* $\mathcal{CT}_m$*, the moduli spaces of perturbed 2-colored Morse trees associated to* $(\mathbb{Y}_\delta)_{\delta \in \Delta^n}$ *define a n-morphism between the Morse cochains* $C^*(f)$ *and* $C^*(g)$*.*

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These higher morphisms between Morse cochains will be called *geometric*.

Theorem ([Maz21b])

The geometric higher morphisms fit into a simplicial set

$$\mathrm{HOM}^{geom}_{A_\infty}(C^*(f), C^*(g))_\bullet \subset \mathrm{HOM}_{A_\infty}(C^*(f), C^*(g))_\bullet ,$$

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- It is contractible, i.e. has trivial homotopy groups.

The proof relies on the fact that every admissible choice of perturbation data parametrized by a simplicial subcomplex of Δ^n extends to an admissible choice of perturbation data on Δ^n.

The fact that $\mathrm{HOM}^{geom}_{A_\infty}(C^*(f), C^*(g))_\bullet$ is contractible implies that two continuation morphisms $C^*(f) \rightsquigarrow C^*(g)$ are always homotopic.

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The theorem gives a higher categorical meaning to the fact that continuation morphisms in Morse theory are unique up to homotopy at the chain level.

1 Introduction

2 The A_∞-algebra structure on the Morse cochains

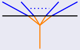
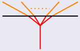
3 Continuation morphisms

4 Higher morphisms between A_∞-algebras

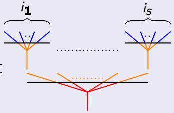
5 Higher morphisms in Morse theory

- The n-multiplihedra
- Higher morphisms in Morse theory
- Further developments

Definition

Given two A_∞-morphisms $A \rightsquigarrow B$ and $B \rightsquigarrow C$ represented by collections of operations  $: A^{\otimes n} \to B$ and  $: B^{\otimes n} \to C$, their *composition* is defined to be the A_∞-morphism $A \rightsquigarrow C$ whose operation of arity n is

$$\sum_{i_1+\cdots+i_s=n} \pm$$



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It would be interesting to know when the diagram of continuation morphisms

$$\begin{array}{ccc} C^*(f_0) & \xrightarrow{\mu_{01}} & C^*(f_1) \\ & \searrow^{\mu_{02}} \;\Leftarrow & \downarrow \mu_{12} \\ & & C^*(f_2) \end{array}$$

is homotopy commutative.

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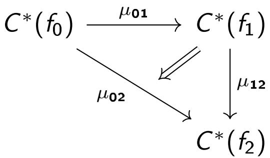

is homotopy commutative.

In other words, when the A_∞-morphisms $\mu_{12} \circ \mu_{01}$ and μ_{02} are homotopic.

The spaces of metric trees defining such an A_∞-homotopy should be the moduli spaces of 3-colored metric trees.

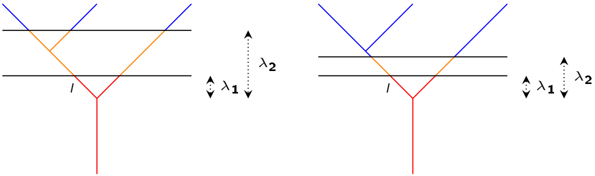

Figure: Two examples of 3-colored metric trees

A similar question was studied and solved in [MWW18][15] for geometric A_∞-functors between Fukaya categories. Their construction should in fact adapt nicely to our present case.

[15] S. Ma'u, K. Wehrheim, and C. Woodward. *A_∞-functors for Lagrangian correspondences.*, 2018.

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In general, it would be interesting to know which higher algebra arises from realizing moduli spaces of multi-colored metric trees in Morse theory.

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In general, it would be interesting to know which higher algebra arises from realizing moduli spaces of multi-colored metric trees in Morse theory.

This question might in fact exhibit some links between the n-multiplihedra and the 2-associahedra of Bottman ([Bot19][16]).

[15] S. Ma'u, K. Wehrheim, and C. Woodward. *A_∞-functors for Lagrangian correspondences.*, 2018.
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It is also quite clear that given two compact symplectic manifolds M and N, one should be able to construct n-morphisms between their Fukaya categories $\mathrm{Fuk}(M)$ and $\mathrm{Fuk}(N)$ through counts of moduli spaces of pseudo-holomorphic quilted disks.

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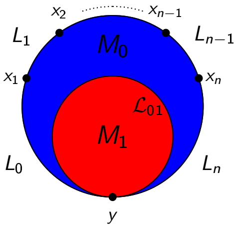

Figure: An example of a pseudo-holomorphic quilted disk

See for instance the paper of Ma'u, Wehrheim and Woodward [MWW18].

Thank you for your attention !

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