Introduction to algebraic operads

Homework 1

Due 11:59PM December 21. *You can either give me your copy by the last course on December 15 or send a scanned version in a* unique and well readable *PDF file to thibaut.mazuir@hu-berlin.de .*

Difficult questions are indicated by a ($\star$).

Problem 1: Enriched categories (10 pt)

2 pt **1.** Prove that the category $\mathtt{Cat}$ endowed with the product of categories $\times$ is a closed symmetric monoidal category.

Let $(\mathtt{D}, \boxtimes, I, \alpha, \lambda, \rho)$ *be a monoidal category. We define a* $\mathtt{D}$-enriched category $\mathtt{C}$ *to be the data of*

(1) A class of objects $\mathrm{Ob}(\mathtt{C})$.
(2) For every $A, B \in \mathtt{C}$ *an object* $\mathtt{C}(A, B) \in \mathtt{D}$.
(3) For every $A, B, C \in \mathtt{C}$ *a morphism* $c_{A,B,C} : \mathtt{C}(B, C) \boxtimes \mathtt{C}(A, B) \to \mathtt{C}(A, C)$.
(4) For every A *a morphism* $u_A : I \to \mathtt{C}(A, A)$.

These data have to satisfy the following axioms

(1) *Associativity: for every* $A, B, C, D \in \mathtt{C}$ *the following diagram is commutative*

$$\begin{array}{ccccc}
(\mathtt{C}(C,D) \boxtimes \mathtt{C}(B,C)) \boxtimes \mathtt{C}(A,B) & & \xrightarrow{c_{B,C,D} \boxtimes \mathrm{id}_{\mathtt{C}(A,B)}} & & \mathtt{C}(B,D) \boxtimes \mathtt{C}(A,B) \\
\downarrow \alpha_{\mathtt{C}(C,D),\mathtt{C}(B,C),\mathtt{C}(A,B)} & & & & \downarrow c_{A,B,D} \\
\mathtt{C}(C,D) \boxtimes (\mathtt{C}(B,C) \boxtimes \mathtt{C}(A,B)) & \xrightarrow[\mathrm{id}_{\mathtt{C}(C,D)} \boxtimes c_{A,B,C}]{} & \mathtt{C}(C,D) \boxtimes \mathtt{C}(A,C) & \xrightarrow[c_{A,C,D}]{} & \mathtt{C}(A,D) \ .
\end{array}$$

(2) *Identity: for every* $A, B \in \mathtt{C}$ *the following diagrams are commutative*

$$\begin{array}{ccc}
I \boxtimes \mathtt{C}(A,B) & \xrightarrow{u_B \boxtimes \mathrm{id}_{\mathtt{C}(A,B)}} & \mathtt{C}(B,B) \boxtimes \mathtt{C}(A,B) \\
& \searrow^{\lambda_{\mathtt{C}(A,B)}} & \downarrow c_{A,B,B} \\
& & \mathtt{C}(A,B)
\end{array}
\qquad
\begin{array}{ccc}
\mathtt{C}(A,B) \boxtimes I & \xrightarrow{\mathrm{id}_{\mathtt{C}(A,B)} \boxtimes u_A} & \mathtt{C}(A,B) \boxtimes \mathtt{C}(A,A) \\
& \searrow^{\rho_{\mathtt{C}(A,B)}} & \downarrow c_{A,A,B} \\
& & \mathtt{C}(A,B) \ .
\end{array}$$

8 pt **2.** Let $\mathtt{C}$ be a closed symmetric monoidal category. Prove that the internal hom $\underline{\mathrm{Hom}}_{\mathtt{C}}$ defines a $\mathtt{C}$-enriched category structure on $\mathtt{C}$.

Problem 2: Cylinder (16 pt)

Let (A, ∂^A), (B, ∂^B) *and* (C, ∂^C) *be chain complexes and* $f : A \to B$, $g : A \to C$ *be chain maps. Define the* cylinder of f and g *to be the chain complex* $\mathrm{Cyl}(f, g)$ *with*

$$\mathrm{Cyl}(f, g)_n = B_n \oplus A_{n-1} \oplus C_n$$

and with differential

$$\partial^{\mathrm{Cyl}}(f,g)(b,a,c) = (\partial^B b - f(a), -\partial^A a, \partial^C c + g(a)) \ .$$

1. Prove that this map defines indeed a differential on $\mathrm{Cyl}(f,g)$. 1 pt

We introduce the maps

$$\iota^B : b \in B \mapsto (b,0,0) \in \mathrm{Cyl}(f,g)$$
$$\iota^C : c \in C \mapsto (0,0,c) \in \mathrm{Cyl}(f,g)$$
$$\pi^A : (b,a,c) \in \mathrm{Cyl}(f,g) \mapsto a \in sA \ .$$

2. Prove that the maps ι^B, ι^C and π^A are chain maps that fit into a short exact sequence 1 pt

$$0 \to B \oplus C \xrightarrow{\iota^B \oplus \iota^C} \mathrm{Cyl}(f,g) \xrightarrow{\pi^A} sA \to 0 \ ,$$

where we recall that $(sA)_n = A_{n-1}$ and $\partial^{sA} = -\partial^A$.

3. Prove that the connecting morphism in the induced long exact sequence 2 pt

$$\cdots \to H_n(A) \xrightarrow{\delta_n} H_n(B) \oplus H_n(C) \to H_n(\mathrm{Cyl}(f,g)) \to H_{n-1}(A) \to \cdots \ .$$

is $\delta_n = (-H_n(f), H_n(g))$.

4. We assume that f is chain homotopic to a chain map $f' : A \to B$ and denote $h_n : A_n \to B_{n+1}$ the chain homotopy from f to f'. Prove that h induces an isomorphism of chain complexes $\phi^h : \mathrm{Cyl}(f,g) \to \mathrm{Cyl}(f',g)$. 3 pt

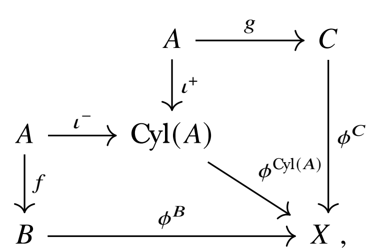
We define the cylinder of A *as* $\mathrm{Cyl}(A) := \mathrm{Cyl}(\mathrm{id}_A, \mathrm{id}_A)$. *We then set*

$$\iota^- : a \in A \mapsto (a,0,0) \in \mathrm{Cyl}(A)$$
$$\iota^+ : a \in A \mapsto (0,0,a) \in \mathrm{Cyl}(A) \ .$$

5. Prove that the following map is a chain map: 1 pt

$$\iota^{\mathrm{Cyl}(A)} : (a_1,a_2,a_3) \in \mathrm{Cyl}(A) \mapsto (f(a_1), a_2, g(a_3)) \in \mathrm{Cyl}(f,g) \ .$$

6. Prove that for every commutative diagram of chain maps of the form 3 pt



there exists a unique chain map $\Phi : \mathrm{Cyl}(f,g) \to X$ such that

$$\Phi\iota^B = \phi^B \qquad \Phi\iota^C = \phi^C \qquad \Phi\iota^{\mathrm{Cyl}(A)} = \phi^{\mathrm{Cyl}(A)} \ .$$

3 pt **7**. Using question **6**, prove that for every diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \phi^B} \\ C & \xrightarrow{\phi^C} & Y \end{array} .$$

which commutes up to homotopy i.e. such that $\phi^B f$ is chain homotopic to $\phi^C g$, there exists a chain map $\Phi : \mathrm{Cyl}(f, g) \to D$ such that $\Phi\iota^B = \phi^B$ and $\Phi\iota^C = \phi^C$.

2 pt **8**. Prove conversely that every chain map $\mathrm{Cyl}(f, g) \to Y$ gives rise to a homotopy-commutative diagram as in question **7**.

Problem 3: Shuffle bialgebras (12 pt)

We define a (p, q)-shuffle *to be a permutation* $\sigma \in \mathfrak{S}_{p+q}$ *such that*

$$\sigma(1) < \cdots < \sigma(p) \qquad \sigma(p+1) < \cdots < \sigma(p+q) .$$

The set of (p, q)-shuffles *is then denoted* $\mathrm{Sh}(p, q) \subset \mathfrak{S}_{p+q}$.

2 pt **1**. Prove that for every permutation $\sigma \in \mathfrak{S}_{p+q}$ there exist unique permutations $\omega \in \mathrm{Sh}(p, q)$, $\alpha \in \mathfrak{S}_p$ and $\beta \in \mathfrak{S}_q$ such that

$$\sigma = \omega(\alpha \times \beta) .$$

Let V *be a vector space.*

3 pt **2**. Prove that there exists a unique morphism of unital associative algebras $\Delta' : T(V) \to T(V) \otimes T(V)$ such that $\Delta'(v) = v \otimes 1 + 1 \otimes V$ for $v \in V$. Prove that it is then given by the formula

$$\Delta'(v_1 \ldots v_n) = \sum_{\substack{p+q=n \\ \sigma \in \mathrm{Sh}(p,q)}} v_{\sigma(1)} \ldots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \ldots v_{\sigma(p+q)} .$$

2 pt **3**. Prove that the coproduct Δ' defines a conilpotent bialgebra structure on $T(V)$, and that it is moreover cocommutative.

3 pt **4**. Prove that there exists a unique morphism of conilpotent coalgebras $\mu' : T^c(V) \otimes T^c(V) \to T^c(V)$ whose projection onto V is 0 except on $(V \otimes \mathbb{K}) \oplus (\mathbb{K} \otimes V)$ where $\mu'(1 \otimes v) = v = \mu'(v \otimes 1)$. Prove that it is then given by the formula

$$\mu'(v_1 \ldots v_p \otimes v_{p+1} \ldots v_{p+q}) = \sum_{\sigma \in \mathrm{Sh}(p,q)} v_{\sigma^{-1}(1)} \ldots v_{\sigma^{-1}(p+q)} .$$

2 pt **5**. Prove that the product μ' defines a conilpotent bialgebra structure on $T^c(V)$, and that it is moreover commutative.

Problem 4: Morita equivalence (18 pt)

Let A and B be two unital algebras. We say that A and B are Morita equivalent *if there exists a (A, B)-bimodule M, a (B, A)-bimodule N and isomorphisms of bimodules*

$$u : M \otimes_B N \simeq A \qquad v : N \otimes_A M \simeq B \ .$$

1. Prove that the following functors are equivalences of categories — 2 pt

$$N \otimes_A - : \texttt{left } A\texttt{-mod} \longrightarrow \texttt{left } B\texttt{-mod}$$
$$- \otimes_A M : \texttt{right } A\texttt{-mod} \longrightarrow \texttt{right } B\texttt{-mod}$$
$$N \otimes_A - \otimes_A M : A\texttt{-bimod} \longrightarrow B\texttt{-bimod} \ .$$

*The goal of questions **2** to **4** is to prove the following theorem:*

Theorem. *If A and B are Morita equivalent and P is an A-bimodule, then there is an isomorphism*

$$H_*(A, P) \simeq H_*(B, N \otimes_A P \otimes_A M) \ .$$

Here $H_*(A, P)$ and $H_*(B, N \otimes_A P \otimes_A M)$ denote Hochschild homologies as defined in Exercise sheet 2.

2. ($\star$) Prove that one can assume without loss of generality that — 5 pt

$$n \cdot u(m \otimes n') = v(n \otimes m) \cdot m'$$
$$m \cdot v(n \otimes m') = u(m \otimes n) \cdot m'$$

for all $m, m' \in M$ and $n, n' \in N$.

*We work in questions **3** and **4** under the assumption of question **2**. As u and v are isomorphisms, there exist elements $m_1, \ldots, m_s \in M$ and $n_1, \ldots, n_s \in N$ such that*

$$u\Big(\sum_{1 \leqslant r \leqslant s} m_r \otimes n_r\Big) = 1_A \ ,$$

and elements $m'_1, \ldots, m'_t \in M$ and $n'_1, \ldots, n'_t \in N$ such that

$$v\Big(\sum_{1 \leqslant r \leqslant t} n'_r \otimes m'_r\Big) = 1_B \ .$$

3. Prove that the map $\phi_* : C_*(A, P) \to C_*(B, N \otimes_A P \otimes_A M)$ defined as — 3 pt

$$\phi_n(p|a_1| \ldots |a_n) = \sum_{1 \leqslant k_i \leqslant s} n_{k_0} \otimes p \otimes m_{k_1} | v(n_{k_1} \otimes a_1 m_{k_2})| \ldots | v(n_{k_n} \otimes a_n m_{k_0}) \ ,$$

and that the map $\psi_* : C_*(B, N \otimes_A P \otimes_A M) \to C_*(A, P)$ defined as

$$\psi_n(n \otimes p \otimes m | b_1 | \ldots | b_n) = \sum_{1 \leqslant k_i \leqslant s} u(m'_{k_0} \otimes n) \cdot p \cdot u(m \otimes n'_{k_1}) | u(m'_{k_1} \otimes b_1 n'_{k_2}) | \ldots | u(m'_{k_n} \otimes b_n n'_{k_0})$$

are chain maps.

4. Prove that $\phi\psi$ is chain homotopic to the identity of $C_*(B, N \otimes_A P \otimes_A M)$ and that $\psi\phi$ is chain homotopic to the identity of $C_*(A, P)$. Conclude the proof of the theorem. — 4 pt

2 pt **5.** Let A be a unital algebra. Let $e \in A$ be an idempotent i.e. an element such that $e^2 = e$, which is such that $A = AeA$. Prove that $B := eAe$ is then a unital algebra which is Morita equivalent to A.

Let A be a be a unital algebra and M an A-bimodule. Then the set $\mathcal{M}_n(M)$ of square matrices of order n with coefficients in M is an $\mathcal{M}_n(A)$-bimodule.

2 pt **6.** Prove that $H_*(A, M) \simeq H_*(\mathcal{M}_n(A), \mathcal{M}_n(M))$.

Problem 5: Sullivan models and homotopy (16 pt)

We work with cohomological conventions in this problem.

3 pt **1.** Consider a Sullivan algebra $(\Lambda V, d)$, two unital dgc algebras A and B, and two morphisms of unital dgc algebras

$$\begin{array}{ccc} & & B \\ & & \big\downarrow \eta \\ (\Lambda V, d) & \xrightarrow{\phi} & A \end{array}$$

where η is a surjective quasi-isomorphism. Prove that there exists a morphism of unital dgc algebras $\Phi : (\Lambda V, d) \to B$ such that $\eta\Phi = \phi$.

Let $\Lambda(t, dt)$ be the free graded commutative algebra generated by a symbol t of degree 0 and a symbol dt of degree 1. We endow it with the differential d defined on the generating elements as $d(t) = dt$ and $d(dt) = 0$.

2 pt **2.** Prove that there is an isomorphism of unital dgc algebras

$$\Lambda(t, dt) \simeq \Lambda(t_0, t_1, dt_0, dt_1)/\langle t_0 + t_1 - 1, dt_0 + dt_1 \rangle$$

where $\Lambda(t_0, t_1, dt_0, dt_1)$ is defined in a similar fashion to $\Lambda(t, dt)$ and $\langle t_0 + t_1 - 1, dt_0 + dt_1 \rangle$ denotes the ideal generated by $t_0 + t_1 - 1$ and $dt_0 + dt_1$.

We define two morphisms of unital dgc coalgebras $\varepsilon_0, \varepsilon_1 : \Lambda(t, dt) \to \mathbb{K}$ by setting $\varepsilon_0(t) = 0$ and $\varepsilon_1(t) = 1$. Two morphisms $\phi_0, \phi_1 : (\Lambda V, d) \to A$ from a Sullivan algebra $(\Lambda V, d)$ to a unital dgc algebra A are then said to be homotopic *if there exists a morphism of unital dgc algebras*

$$\Phi : (\Lambda V, d) \to A \otimes \Lambda(t, dt)$$

such that $(\mathrm{id}_A \otimes \varepsilon_i)\Phi = \phi_i$ for $i = 0, 1$.

5 pt **3.** ($\star$) Prove that "being homotopic" is an equivalence relation on the set of morphisms of unital dgc algebras $(\Lambda V, d) \to A$.

3 pt **4.** ($\star$) Prove that two homotopic morphisms $(\Lambda V, d) \to A$ induce the same map in homology.

Let $(A, 0)$ be a unital graded commutative algebra with zero differential and $(\Lambda V, d)$ a minimal Sullivan model. We define the constant morphism $\varepsilon : (\Lambda V, d) \to (A, 0)$ *by $\varepsilon(V) = 0$.*

3 pt **5.** ($\star$) Prove that if a morphism $\phi : (\Lambda V, d) \to (A, 0)$ is homotopic to ε then $\phi = \varepsilon$.