Higher algebra of A_∞-algebras and the n-multiplihedra

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The results presented in this talk are taken from my two recent papers : *Higher algebra of A_∞ and ΩBAs-algebras in Morse theory I* (arXiv:2102.06654) and *Higher algebra of A_∞ and ΩBAs-algebras in Morse theory II* (arxiv:2102.08996).

1. A_∞-algebras and A_∞-morphisms
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3. The n-multiplihedra
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Suspension : Let A be a graded module (over the ring $\mathbb{Z}$). We denote sA, the *suspension of* A to be the graded module defined by $(sA)^i := A^{i-1}$.

Cohomological conventions : differentials will have degree $+1$.

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Definition

Let A be a dg-module with differential m_1. An A_∞-algebra structure on A is the data of a collection of maps of degree $2-n$

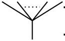
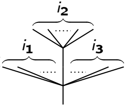
$$m_n : A^{\otimes n} \longrightarrow A \ , \ \ n \geqslant 1,$$

extending m_1 and which satisfy

$$[m_1, m_n] = \sum_{\substack{i_1+i_2+i_3=n \\ 2\leqslant i_2 \leqslant n-1}} \pm m_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}).$$

These equations are called the A_∞-equations.

Representing m_n as  a corolla of arity n, these equations can be written as

$[m_1,$  $] = \sum_{\substack{i_1+i_2+i_3=n \\ 2\leq i_2\leq n-1}} \pm$  .

In particular,

$$[m_1, m_2] = 0 \ ,$$
$$[m_1, m_3] = m_2(\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) \ ,$$

implying that m_2 descends to an associative product on $H^*(A)$. An A_∞-algebra is thus simply a correct notion of a dg-algebra whose product is associative up to homotopy.

The operations m_n are the higher coherent homotopies which keep track of the fact that the product is associative up to homotopy.

Define the reduced tensor coalgebra of a graded module V to be

$$\overline{T}V := V \oplus V^{\otimes 2} \oplus \cdots$$

endowed with the coassociative comultiplication

$$\Delta_{\overline{T}V}(v_1 \dots v_n) := \sum_{i=1}^{n-1} v_1 \dots v_i \otimes v_{i+1} \dots v_n \ .$$

Using the universal property of the bar construction, we have the following one-to-one correspondence

$$\left\{ \begin{array}{c} \text{collections of morphisms of degree } 2-n \\ m_n : A^{\otimes n} \to A \ , \ n \geqslant 1, \\ \text{satisfying the } A_\infty\text{-equations} \end{array} \right\}$$
$$\longleftrightarrow \left\{ \begin{array}{c} \text{coderivations } D \text{ of degree } +1 \text{ of } \overline{T}(sA) \\ \text{such that } D^2 = 0 \end{array} \right\} .$$

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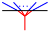
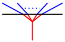
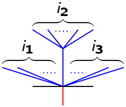
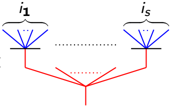
Definition

An A_∞-*morphism* between two A_∞-algebras A and B is a family of maps $f_n : A^{\otimes n} \to B$ of degree $1-n$ satisfying

$$[m_1, f_n] = \sum_{\substack{i_1+i_2+i_3=n \\ i_2 \geqslant 2}} \pm f_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) + \sum_{\substack{i_1+\cdots+i_s=n \\ s \geqslant 2}} \pm m_s(f_{i_1} \otimes \cdots \otimes f_{i_s}) \ .$$

Representing the operations f_n as , the operations m_n^A in red and the operations m_n^B in blue, these equations read as

$\left[\partial,\right.$  $\left.\right] = \sum_{\substack{i_1+i_2+i_3=n \\ i_2 \geq 2}} \pm$  $+ \sum_{\substack{i_1+\cdots+i_s=n \\ s \geq 2}} \pm$  .

We check that $[\partial, f_2] = f_1 m_2^A - m_2^B(f_1 \otimes f_1)$.

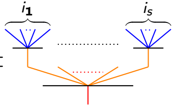
An A_∞-morphism between A_∞-algebras induces a morphism of associative algebras on the level of cohomology, and is a correct notion of morphism which preserves the product up to homotopy.

Using the universal property of the bar construction, an A_∞-*morphism* between two A_∞-algebras A and B can be equivalently defined as a dg-coalgebra morphism $F : (\overline{T}(sA), D_A) \to (\overline{T}(sB), D_B)$ between their shifted bar constructions.

Given two coalgebra morphisms $F : \overline{T}V \to \overline{T}W$ and $G : \overline{T}W \to \overline{T}Z$, the family of morphisms associated to $G \circ F$ is given by

$$(G \circ F)_n = \sum_{i_1+\cdots+i_s=n} \pm g_s(f_{i_1} \otimes \cdots \otimes f_{i_s}) \ .$$

Equivalently,

$$(G \circ F)_n = \sum_{i_1+\cdots+i_s=n} \pm$$ 

. (1)

A_∞-algebras together with A_∞-morphisms form a category, denoted $\mathtt{A}_\infty - \mathtt{alg}$, which can be seen as a full subcategory of $\mathtt{dg} - \mathtt{Cogc}$ of cocomplete dg-coalgebras, using the shifted bar construction viewpoint.

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The category $\mathtt{A}_\infty - \mathtt{alg}$ provides a framework that behaves well with respect to homotopy-theoretic constructions, when studying homotopy theory of associative algebras. See for instance [LH02] and [Val20].

It is because this category is encoded by the two-colored operad

It is a quasi-free object in the model category of two-colored operads in dg-modules and a fibrant-cofibrant replacement of the two-colored operad As^2, which encodes associative algebras with morphisms of algebras,

$$A_\infty^2 \xrightarrow{\sim} As^2 \ .$$

Theorem (Homotopy transfer theorem)

Let (A, ∂_A) and (H, ∂_H) be two cochain complexes. Suppose that H is a deformation retract of A, that is that they fit into a diagram

$$h \circlearrowright (A, \partial_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, \partial_H) ,$$

where $\mathrm{id}_A - ip = [\partial, h]$. Then if (A, ∂_A) is endowed with an A_∞-algebra structure, H can be made into an A_∞-algebra such that i and p extend to A_∞-morphisms.

1 A_∞-algebras and A_∞-morphisms

2 Higher algebra of A_∞-algebras

3 The n-multiplihedra

4 Higher morphisms in Morse theory

Our goal now : study the *higher algebra of A_∞-algebras*.

Considering two A_∞-morphisms F, G, we would like first to determine a notion giving a satisfactory meaning to the sentence "F and G are homotopic". Then, A_∞-homotopies being defined, what is now a good notion of a homotopy between homotopies ? And of a homotopy between two homotopies between homotopies ? And so on.

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Start with a notion of homotopy. Drawn from [LH02].

Take C and C' two dg-coalgebras, F and G morphisms $C \to C'$ of dg-coalgebras. A (F, G)-*coderivation* is a map $H : C \to C'$ such that

$$\Delta_{C'} H = (F \otimes H + H \otimes G)\Delta_C \ .$$

The morphisms F and G are then said to be *homotopic* if there exists a (F, G)-coderivation H of degree -1 such that

$$[\partial, H] = G - F \ .$$

Define

$$\boldsymbol{\Delta}^1 := \mathbb{Z}[0] \oplus \mathbb{Z}[1] \oplus \mathbb{Z}[0<1] \ ,$$

with differential ∂^{sing}

$$\partial^{sing}([0<1]) = [1]-[0] \qquad \partial^{sing}([0]) = 0 \qquad \partial^{sing}([1]) = 0 \ ,$$

and coproduct the Alexander-Whitney coproduct

$$\begin{aligned} \Delta_{\boldsymbol{\Delta}^1}([0<1]) &= [0]\otimes[0<1] + [0<1]\otimes[1] \\ \Delta_{\boldsymbol{\Delta}^1}([0]) &= [0]\otimes[0] \\ \Delta_{\boldsymbol{\Delta}^1}([1]) &= [1]\otimes[1] \ . \end{aligned}$$

The elements $[0]$ and $[1]$ have degree 0, and the element $[0<1]$ has degree -1.

We check that there is a one-to-one correspondence between (F, G)-coderivations and morphisms of dg-coalgebras $\mathbf{\Delta}^1 \otimes C \longrightarrow C'$.

Definition

For two A_∞-algebras $(\overline{T}(sA), D_A)$ and $(\overline{T}(sB), D_B)$ and two A_∞-morphisms $F, G : (\overline{T}(sA), D_A) \to (\overline{T}(sB), D_B)$, an *$A_\infty$-homotopy from F to G* is defined to be a morphism of dg-coalgebras

$$H : \mathbf{\Delta}^1 \otimes \overline{T}(sA) \longrightarrow \overline{T}(sB) ,$$

whose restriction to the [0] summand is F and whose restriction to the [1] summand is G.

Definition

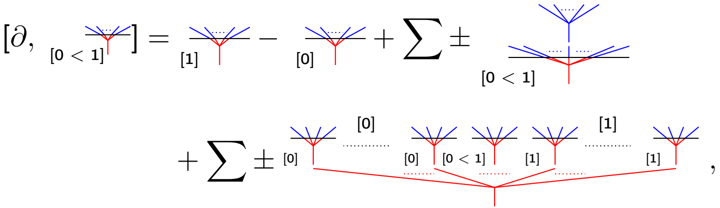
An A_∞*-homotopy* between two A_∞-morphisms $(f_n)_{n\geqslant 1}$ and $(g_n)_{n\geqslant 1}$ is a collection of maps

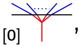
$$h_n : A^{\otimes n} \longrightarrow B \ ,$$

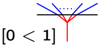
of degree $-n$, satisfying

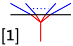
$$[\partial, h_n] = g_n - f_n + \sum_{\substack{i_1+i_2+i_3=m \\ i_2\geqslant 2}} \pm h_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3})$$
$$+ \sum_{\substack{i_1+\cdots+i_s+l \\ +j_1+\cdots+j_t=n \\ s+1+t\geqslant 2}} \pm m_{s+1+t}(f_{i_1} \otimes \cdots \otimes f_{i_s} \otimes h_l \otimes g_{j_1} \otimes \cdots \otimes g_{j_t}) \ .$$

In symbolic formalism,



where we denote 

, 

and 

respectively for the f_n, the h_n and the g_n.

The relation *being A_∞-homotopic* on the class of A_∞-morphisms is an equivalence relation. It is moreover stable under composition.

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Define $\mathbf{\Delta}^n$ the graded module generated by the faces of the standard n-simplex Δ^n,

$$\mathbf{\Delta}^n = \bigoplus_{0 \leqslant i_1 < \cdots < i_k \leqslant n} \mathbb{Z}[i_1 < \cdots < i_k] \ .$$

The grading is $|I| := -\mathrm{dim}(I)$ for $I \subset \Delta^n$.

It has a dg-coalgebra structure, with differential

$$\partial_{\mathbf{\Delta}^n}([i_1 < \cdots < i_k]) := \sum_{j=1}^{k} (-1)^j [i_1 < \cdots < \widehat{i_j} < \cdots < i_k] \ ,$$

and coproduct the Alexander-Whitney coproduct

$$\Delta_{\mathbf{\Delta}^n}([i_1 < \cdots < i_k]) := \sum_{j=1}^{k} [i_1 < \cdots < i_j] \otimes [i_j < \cdots < i_k] \ .$$

Definition ([MS03])

Let I be a face of Δ^n. An *overlapping partition* of I to be a sequence of faces $(I_l)_{1 \leqslant \ell \leqslant s}$ of I such that

(i) the union of this sequence of faces is I, i.e. $\cup_{1 \leqslant \ell \leqslant s} I_l = I$;

(ii) for all $1 \leqslant \ell < s$, $\max(I_\ell) = \min(I_{\ell+1})$.

An overlapping 6-partition for $[0 < 1 < 2]$ is for instance

$$[0 < 1 < 2] = [0] \cup [0] \cup [0 < 1] \cup [1] \cup [1 < 2] \cup [2] \ .$$

Overlapping partitions are the collection of faces which naturally arise in the Alexander-Whitney coproduct.

The element $\Delta_{\mathbf{\Delta}^n}(I)$ corresponds to the sum of all overlapping 2-partitions of I. Iterating s times $\Delta_{\mathbf{\Delta}^n}$ yields the sum of all overlapping $(s+1)$-partitions of I.

We have seen that A_∞-morphisms correspond to the set

$$\mathrm{Hom}_{\mathrm{dg-Cogc}}(\overline{T}(sA), \overline{T}(sB))$$

and A_∞-homotopies correspond to the set

$$\mathrm{Hom}_{\mathrm{dg-Cogc}}(\mathbf{\Delta}^1 \otimes \overline{T}(sA), \overline{T}(sB)) \ ,$$

Definition ([Maz21b])

We define the set of n-morphisms between A and B as

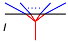
$$\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A, B)_n := \mathrm{Hom}_{\mathrm{dg-Cogc}}(\mathbf{\Delta}^n \otimes \overline{T}(sA), \overline{T}(sB)) \ .$$

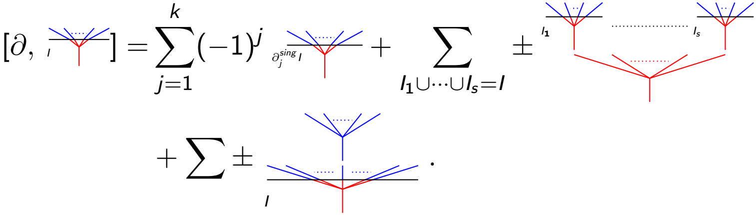
Definition ([Maz21b])

A *n-morphism* from A to B is defined to be a collection of maps $f_I^{(m)} : A^{\otimes m} \longrightarrow B$ of degree $1 - m + |I|$ for $I \subset \Delta^n$ and $m \geqslant 1$, that satisfy

$$\left[\partial, f_I^{(m)}\right] = \sum_{j=0}^{\dim(I)} (-1)^j f_{\partial_j I}^{(m)} + \sum_{\substack{i_1+\cdots+i_s=m \\ I_1\cup\cdots\cup I_s=I \\ s\geqslant 2}} \pm m_s(f_{I_1}^{(i_1)} \otimes \cdots \otimes f_{I_s}^{(i_s)})$$

$$+ (-1)^{|I|} \sum_{\substack{i_1+i_2+i_3=m \\ i_2\geqslant 2}} \pm f_I^{(i_1+1+i_3)}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) .$$

Equivalently and more visually, a collection of maps  satisfying



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The dg-coalgebras $\mathbf{\Delta}^\bullet := \{\mathbf{\Delta}^n\}_{n\geqslant 0}$ naturally form a cosimplicial dg-coalgebra.

The sets $\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A,B)_n$ then fit into a HOM-simplicial set $\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A,B)_\bullet$. This HOM-simplicial set provides a satisfactory framework to study the higher algebra of A_∞-algebras.

Theorem ([Maz21b])

For A and B two A_∞-algebras, the simplicial set $\mathrm{HOM}_{A_\infty}(A, B)_\bullet$ is a Kan complex.

Proof.

This theorem stems from the fact that the cosimplicial cocomplete dg-coalgebra $\boldsymbol{C} := \{\boldsymbol{\Delta}^n \otimes \overline{T}(sA)\}_{n\geqslant 0}$ is a cosimplicial replacement of $\overline{T}(sA)$ in the model category $dg - Cogc$ defined in [LH02]. □

Proposition

For every inner horn $\Lambda_n^k \subset \Delta^n$, there is a one-to-one correspondence

$$\left\{ \textit{fillers} \quad \begin{array}{ccc} \Lambda_n^k & \longrightarrow & \mathrm{HOM}_{A_\infty}(A, B)_\bullet \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array} \right\}$$

$$\longleftrightarrow \left\{ \begin{array}{c} \textit{families of maps of degree } 1 - m - n \\ f_{\Delta^n}^{(m)} : A^{\otimes m} \to B, \ m \geqslant 1 \end{array} \right\} .$$

An inner horn $\Lambda_n^k \to \mathrm{HOM}_{A_\infty}(A,B)_\bullet$ corresponds to a collection of degree $1-m-\dim(I)$ morphisms $f_I^{(m)} : A^{\otimes m} \longrightarrow B$ for $I \subset \Lambda_n^k$ which satisfy the A_∞-equations for higher morphisms.

The previous proposition then states that filling the horn $\Lambda_n^k \subset \Delta^n$ amounts to choosing an arbitrary collection of degree $1-m-n$ morphisms $f_{\Delta^n}^{(m)} : A^{\otimes m} \to B$ and that they completely determine the collection of morphisms for the missing face $f^{(m)}_{[0<\cdots<\hat{k}<\cdots<n]}$.

The simplicial homotopy groups of the Kan complex $\mathrm{HOM}_{A_\infty}(A, B)_\bullet$ can moreover be explicitly computed.

Beware that the points of these Kan complexes are the A_∞-morphisms, and the arrows between them are the A_∞-homotopies. This can be misleading at first sight, but *the points are the morphisms and NOT the algebras* and *the arrows are the homotopies and NOT the morphisms*.

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We would like to see the simplicial sets $\mathrm{HOM}_{\mathtt{A}_\infty - \mathtt{alg}}(A,B)_\bullet$ as part of a simplicial enrichment of the category $\mathtt{A}_\infty - \mathtt{alg}$. In other words, we would like to define simplicial maps

$$\mathrm{HOM}_{\mathtt{A}_\infty - \mathtt{alg}}(A,B)_n \times \mathrm{HOM}_{\mathtt{A}_\infty - \mathtt{alg}}(B,C)_n \longrightarrow \mathrm{HOM}_{\mathtt{A}_\infty - \mathtt{alg}}(A,C)_n ,$$

lifting the composition on the $\mathrm{HOM}_0 = \mathrm{Hom}$.

This would then endow $\mathtt{A}_\infty - \mathtt{alg}$ with a structure of ∞-category.

All the natural approaches to lift the composition in $\mathtt{A}_\infty - \mathtt{alg}$ to $\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A,B)_\bullet$ fail to work. Hence, it is still an open question to know whether these HOM-simplicial sets could fit into a simplicial enrichment of the category $\mathtt{A}_\infty - \mathtt{alg}$.

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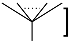
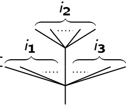
4 Higher morphisms in Morse theory

There exists a collection of polytopes, called the *associahedra* and denoted $\{K_n\}$, which encode the A_∞-equations between A_∞-algebras. This means that K_n has a unique cell $[K_n]$ of dimension $n-2$ and that its boundary reads as

$$\partial K_n = \bigcup_{\substack{i_1+i_2+i_3=n \\ 2\leqslant i_2\leqslant n-1}} K_{i_1+1+i_3} \times K_{i_2} \ ,$$

where $\times$ is the standard cartesian product.

Recall that the A_∞-equations read as

$$[m_1, \;\; $$  $$] = \sum_{\substack{i_1+i_2+i_3=n \\ 2\leqslant i_2 \leqslant n-1}} \pm$$ 

$$.$$

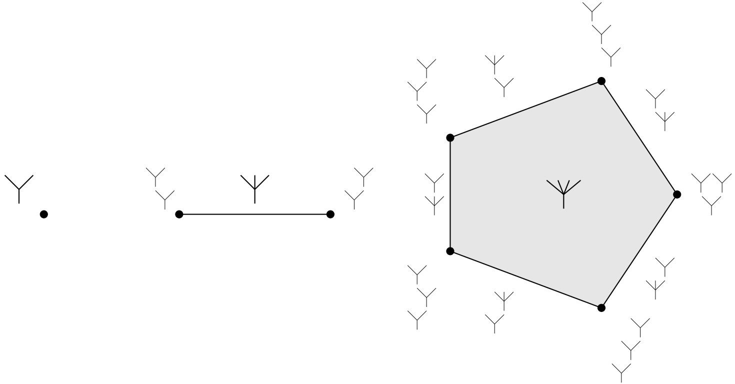


Figure: The associahedra K_2, K_3 and K_4, with cells labeled by the operations they define

The polytopes K_n fit in fact into an operad in polytopes, whose image under the cellular chains functor yields the operad A_∞, as proven in [MTTV19].

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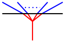
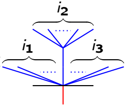
4 Higher morphisms in Morse theory

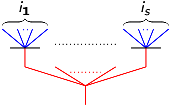
There exists a collection of polytopes, called the *multiplihedra* and denoted $\{J_n\}$, which encode the A_∞-equations for A_∞-morphisms. Again, J_n has a unique $n-1$-dimensional cell $[J_n]$ and the boundary of J_n is exactly

$$\partial J_n = \bigcup_{\substack{i_1+i_2+i_3=n \\ i_2 \geqslant 2}} J_{i_1+1+i_3} \times K_{i_2} \cup \bigcup_{\substack{i_1+\cdots+i_s=n \\ s \geqslant 2}} K_s \times J_{i_1} \times \cdots \times J_{i_s} \ ,$$

where $\times$ is the standard cartesian product $\times$.

Recall that the A_∞-equations for A_∞-morphisms are

$\left[\partial,\right.$  $\left.\right] = \sum_{\substack{i_1+i_2+i_3=n \\ i_2\geqslant 2}} \pm$ 

$+ \sum_{\substack{i_1+\cdots+i_s=n \\ s\geqslant 2}} \pm$ 

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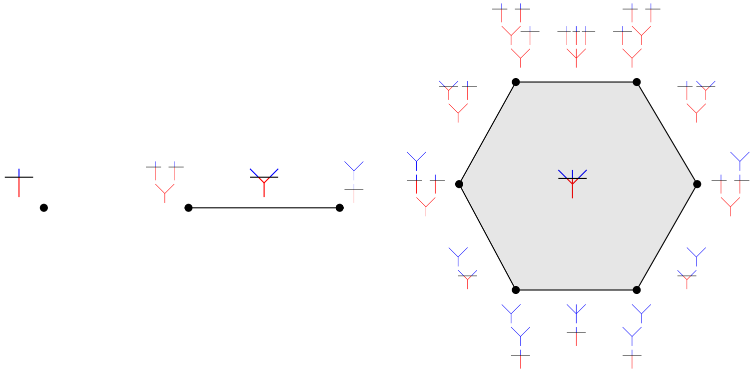


Figure: The multiplihedra J_1, J_2 and J_3 with cells labeled by the operations they define in $A_\infty - \mathrm{Morph}$

The polytopes J_n fit in fact into an operadic bimodule in polytopes, whose image under the cellular chains functor yields the operad M_∞ encoding A_∞-morphisms, as proven in [MLA].

1. A_∞-algebras and A_∞-morphisms

2. Higher algebra of A_∞-algebras

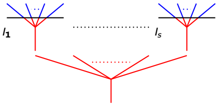
3. The n-multiplihedra
    - The associahedra
    - The multiplihedra
    - The n-multiplihedra

4. Higher morphisms in Morse theory

We would like to define a family of polytopes encoding n-morphisms between A_∞-algebras. These polytopes will then be called *n-multiplihedra*.

We have seen that A_∞-morphisms $\overline{T}(sA) \to \overline{T}(sB)$ are encoded by the multiplihedra. n-morphisms being defined as the set of morphisms $\mathbf{\Delta}^n \otimes \overline{T}(sA) \to \overline{T}(sB)$, a natural candidate would thus be $\{\Delta^n \times J_m\}_{m \geqslant 1}$.

However, $\Delta^n \times J_m$ does not fulfill that property as it is. Faces correspond to the data of a face of $I \subset \Delta^n$, and of a broken two-colored tree labeling a face of J_m. This labeling is too coarse, as it does not contain the trees



,

that appear in the A_∞-equations for n-morphisms.

We thus want to lift the combinatorics of overlapping partitions to the level of the n-simplices Δ^n.

Proposition ([Maz21b])

For each $s \geqslant 1$, there exists a polytopal subdivision of the standard n-simplex Δ^n whose top-dimensional cells are in one-to-one correspondence with all s-overlapping partitions of Δ^n.

Taking the realizations

$$\begin{aligned}\Delta^n &:= \mathrm{conv}\{(1,\ldots,1,0,\ldots,0) \in \mathbb{R}^n\} \\ &= \{(z_1,\ldots,z_n) \in \mathbb{R}^n | 1 \geqslant z_1 \geqslant \cdots \geqslant z_n \geqslant 0\} \ ,\end{aligned}$$

this polytopal subdivision can be realized as the subdivision obtained after dividing Δ^n by all hyperplanes $z_i = (1/2)^k$, for $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant s$.

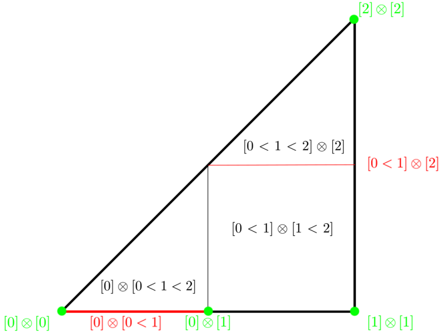


Figure: The subdivision of Δ^2 by overlapping 2-partitions

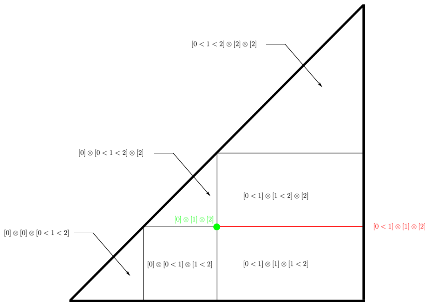


Figure: The subdivision of Δ^2 by overlapping 3-partitions

The previous issue can then be solved by constructing a thinner polytopal subdivision of $\Delta^n \times J_m$.

Consider a face F of J_m, with exactly s unbroken two-colored trees appearing in the two-colored broken tree labeling it. We refine the polytopal subdivision of $\Delta^n \times F$ into $\Delta^n_s \times F$, where Δ^n_s denotes Δ^n endowed with the subdivision encoding s-overlapping partitions.

This refinement process can be done consistently for each face F of J_m, in order to obtain a new polytopal subdivision of $\Delta^n \times J_m$.

Definition ([Maz21b])

The *n-multiplihedra* are defined to be the polytopes $\Delta^n \times J_m$ endowed with the previous polytopal subdivision. We denote them $n - J_m$.

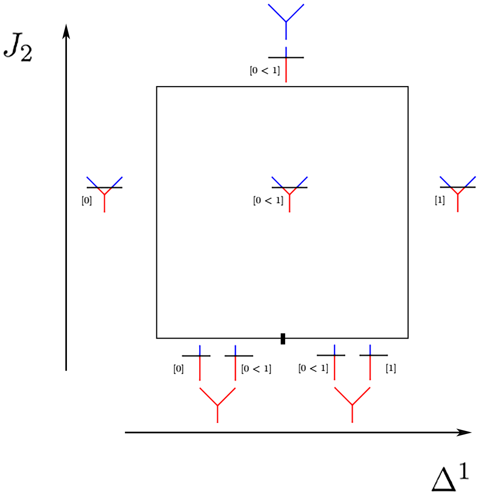


Figure: The 1-multiplihedron $\Delta^1 \times J_2$

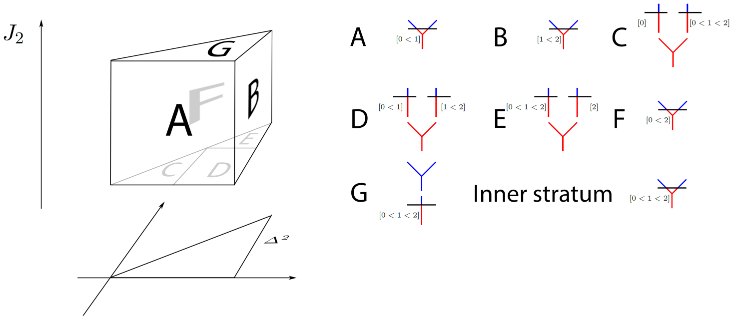


Figure: The 2-multiplihedron $\Delta^2 \times J_2$

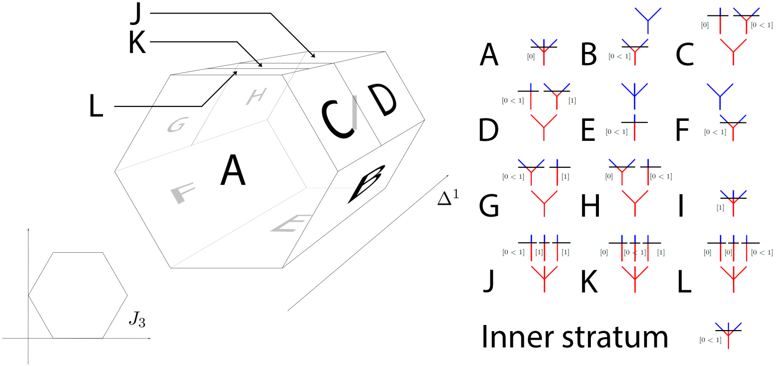


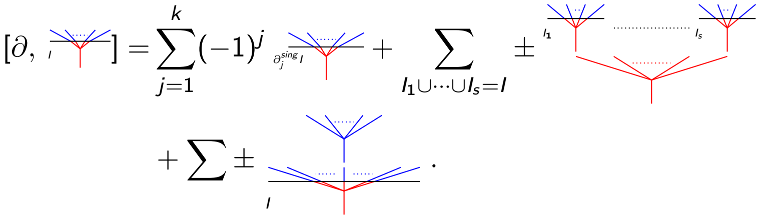
Figure: The 1-multiplihedron $\Delta^1 \times J_3$

By construction, the boundary of the cell $[n - J_m]$ is given by

$$\partial^{sing}[n - J_m] \cup \bigcup_{\substack{h+k=m+1 \\ 1 \leqslant i \leqslant k \\ h \geqslant 2}} [n - J_k] \times_i [K_h] \cup \bigcup_{\substack{i_1+\cdots+i_s=m \\ I_1 \cup \cdots \cup I_s = \Delta^n \\ s \geqslant 2}} [K_s] \times [\mathrm{dim}(I_1) - J_{i_1}] \times \cdots \times [\mathrm{dim}(I_s) - J_{i_s}] \ ,$$

where $I_1 \cup \cdots \cup I_s = \Delta^n$ is an overlapping partition of Δ^n.

Recall that the $n - A_\infty$-equations read as



In other words, the n-multiplihedra encode n-morphisms between A_∞-algebras.

1. A_∞-algebras and A_∞-morphisms
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Let M be an oriented closed Riemannian manifold endowed with a Morse function f together with a Morse-Smale metric. The Morse cochains $C^*(f)$ form a deformation retract of the singular cochains $C^*_{sing}(M)$ as shown in [Hut08].

$$h \circlearrowright (C^*_{sing}, \partial_{sing}) \underset{i}{\overset{p}{\rightleftarrows}} (C^*(f), \partial_{Morse}) .$$

The cup product naturally endows the singular cochains $C^*_{sing}(M)$ with a dg-algebra structure. The homotopy transfer theorem ensures that it can be transferred to an A_∞-algebra structure on the Morse cochains $C^*(f)$.

The differential on the Morse cochains is defined by a count of moduli spaces of gradient trajectories. Is it then possible to define higher multiplications m_n on $C^*(f)$ by a count of moduli spaces such that they fit in a structure of A_∞-algebra ?

Question solved for the first time by Abouzaid in [Abo11], drawing from earlier works by Fukaya ([Fuk97] for instance). See also [Mes18] and [AL18].

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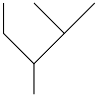
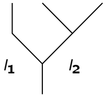
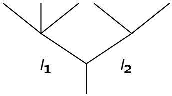
Terminology :



A ribbon tree



A metric ribbon tree



A stable metric ribbon tree

Define $\mathcal{T}_n$ to be *moduli space of stable metric ribbon trees with n incoming edges*.

Allowing lengths of internal edges to go to $+\infty$, this moduli space can be compactified into a $(n-2)$-dimensional CW-complex $\overline{\mathcal{T}}_n$, where $\mathcal{T}_n$ is seen as its unique $(n-2)$-dimensional stratum.

Theorem

The compactified moduli space $\overline{\mathcal{T}}_n$ is isomorphic as a CW-complex to the associahedron K_n.

This was first noticed in section 1.4. of Boardman-Vogt [BV73].

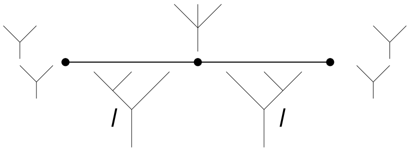


Figure: The compactified moduli space $\overline{\mathcal{T}}_3$

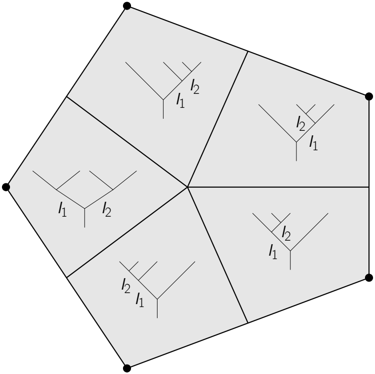
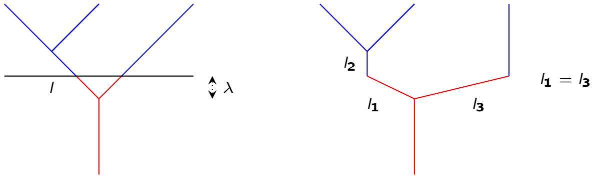


Figure: The compactified moduli space $\overline{\mathcal{T}}_4$

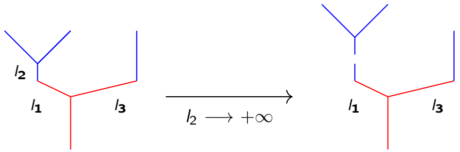
Definition

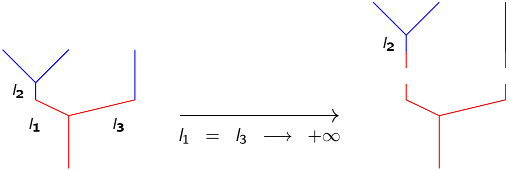
A *stable two-colored metric ribbon tree* or *stable gauged metric ribbon tree* is defined to be a stable metric ribbon tree together with a length $\lambda \in \mathbb{R}$, which is to be thought of as a gauge drawn over the metric tree, at distance λ from its root, where the positive direction is pointing down.



For $n \geqslant 1$, denote $\mathcal{CT}_n$ the *moduli space of stable two-colored metric ribbon trees*.

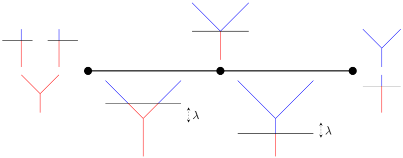
Allowing again internal edges of metric trees to go to $+\infty$, this moduli space $\mathcal{CT}_n$ can be compactified into a $(n-1)$-dimensional CW-complex $\overline{\mathcal{CT}}_n$.



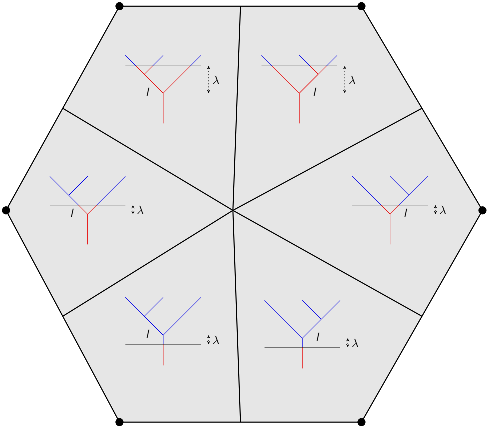


Theorem ([MW10])

The compactified moduli space $\overline{\mathcal{CT}}_n$ is isomorphic as a CW-complex to the multiplihedron J_n.



The compactified moduli space $\overline{\mathcal{CT}}_2$ with its cell decomposition by stable two-colored ribbon tree type



The compactified moduli space $\overline{\mathcal{CT}}_3$ with its cell decomposition by stable two-colored ribbon tree type

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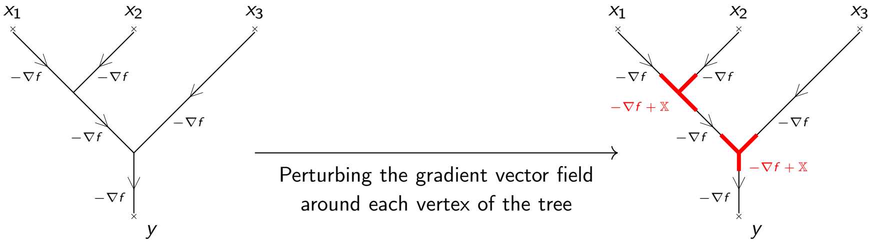
These moduli spaces of metric trees can be realized in Morse theory, as moduli spaces of perturbed Morse gradient trees.



A generic choice of perturbation data on the moduli spaces $\mathcal{T}_m$ defines an A_∞-algebra structure on the Morse cochains $C^*(f)$, whose operation of arity n is defined by counting the points of 0-dimensional moduli spaces of perturbed Morse trees of arity n. ([Maz21a])

In a similar fashion, a generic choice of perturbation data on the moduli spaces $\mathcal{CT}_m$ defines an A_∞-morphism between the Morse cochains $C^*(f)$ and $C^*(g)$, whose operations are defined by counting the points of 0-dimensional moduli spaces of perturbed 2-colored Morse trees. ([Maz21a])

Finally, a generic choice of perturbation data on $\Delta^n \times \mathcal{CT}_m$, i.e. a n-simplex of perturbation data on the moduli spaces $\mathcal{CT}_m$, defines a n-morphism between the Morse cochains $C^*(f)$ and $C^*(g)$, whose operations are again defined by counting perturbed Morse trees. ([Maz21b])

These higher morphisms between Morse cochain complexes will be called *geometric*.

Theorem ([Maz21b])

The geometric n-morphisms fit into a simplicial set

$$\mathrm{HOM}^{geom}_{A_\infty}(C^*(f), C^*(g))_\bullet \subset \mathrm{HOM}_{A_\infty}(C^*(f), C^*(g))_\bullet \ ,$$

which is a Kan complex and is moreover contractible.

Corollary ([Maz21b])

Two geometric A_∞-morphisms between Morse cochain complexes are always A_∞-homotopic.

The previous theorem gives in fact a higher categorical meaning to the fact that continuation morphisms in Morse theory are well-defined up to homotopy at chain level.

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1. It is also quite clear that given two compact symplectic manifolds M and N, one should be able to construct n-morphisms between their Fukaya categories $\mathrm{Fuk}(M)$ and $\mathrm{Fuk}(N)$ through counts of moduli spaces of quilted disks (under the correct technical assumptions).

2. Another interesting question would be to know which higher algebra arises from realizing moduli spaces of multigauged metric trees in Morse theory.

This question might in fact exhibit some links between the n-multiplihedra and the 2-associahedra of Bottman (see [Bot19a] and [Bot19b] for instance).

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