Higher algebra of A_∞-algebras in Morse theory

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The results presented in this talk are taken from my two recent papers : *Higher algebra of A_∞ and ΩBAs-algebras in Morse theory I* (arXiv:2102.06654) and *Higher algebra of A_∞ and ΩBAs-algebras in Morse theory II* (arxiv:2102.08996).

Definition

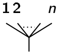
Let A be a dg-module with differential m_1. An A_∞-algebra structure on A is the data of a collection of maps of degree $2-n$

$$m_n : A^{\otimes n} \longrightarrow A \ , \ \ n \geqslant 1,$$

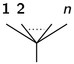
extending m_1 and which satisfy

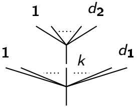
$$[m_1, m_n] = \sum_{\substack{i_1+i_2+i_3=n \\ 2 \leqslant i_2 \leqslant n-1}} \pm m_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}).$$

These equations are called the A_∞-equations.

Representing m_n as 

, these equations can be written as

$[m_1,$ 

$] = \displaystyle\sum_{\substack{h+k=n+1\\ 2\leqslant h\leqslant n-1\\ 1\leqslant i\leqslant k}} \pm$ 

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In particular,

$$[m_1, m_2] = 0 \ ,$$
$$[m_1, m_3] = m_2(\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) \ ,$$

implying that m_2 descends to an associative product on $H^*(A)$. An A_∞-algebra is thus simply a correct notion of a dg-algebra whose product is associative up to homotopy.

The operations m_n are the higher coherent homotopies which keep track of the fact that the product is associative up to homotopy.

Theorem (Homotopy transfer theorem)

Let (A, ∂_A) and (H, ∂_H) be two cochain complexes. Suppose that H is a deformation retract of A, that is that they fit into a diagram

$$\begin{tikzcd} (A, \partial_A) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (H, \partial_H) \arrow[l, shift left, "i"] \end{tikzcd} ,$$

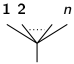
where $\mathrm{id}_A - ip = [\partial, h]$. Then if (A, ∂_A) is endowed with an A_∞-algebra structure, H can be made into an A_∞-algebra such that i and p extend to A_∞-morphisms.

There exists a collection of polytopes, called the *associahedra* and denoted $\{K_n\}$, which encode the A_∞-equations between A_∞-algebras. This means that K_n has a unique cell $[K_n]$ of dimension $n-2$ and that its boundary reads as

$$\partial K_n = \bigcup_{\substack{h+k=n+1 \\ 2\leqslant h\leqslant n-1}} \bigcup_{1\leqslant i\leqslant k} K_k \times_i K_h \ ,$$

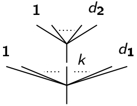
where $\times_i$ is in fact the standard $\times$ cartesian product.

Recall that the A_∞-equations read as

$\partial($ 

$) =$

$$\sum_{\substack{h+k=n+1 \\ 2 \leqslant h \leqslant n-1 \\ 1 \leqslant i \leqslant k}} \pm$$



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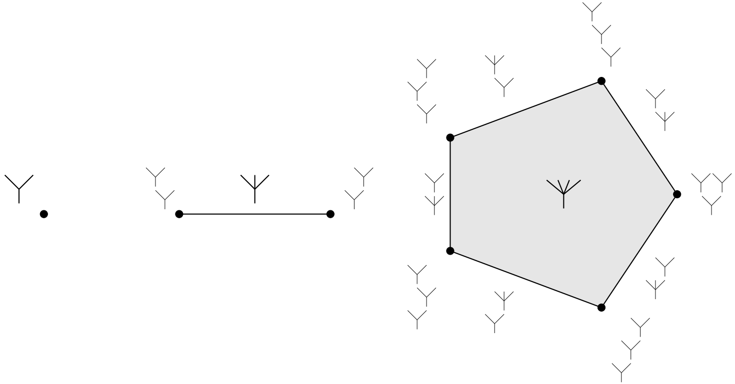


Figure: The associahedra K_2, K_3 and K_4, with cells labeled by the operations they define

Let M be an oriented closed Riemannian manifold endowed with a Morse function f together with a Morse-Smale metric. The Morse cochains $C^*(f)$ form a deformation retract of the singular cochains $C^*_{sing}(M)$ as shown in [Hut08].

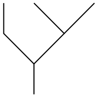
$$\begin{tikzcd} (C^*_{sing}, \partial_{sing}) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (C^*(f), \partial_{Morse}) \arrow[l, shift left, "i"] \end{tikzcd} .$$

The cup product naturally endows the singular cochains $C^*_{sing}(M)$ with a dg-algebra structure. The homotopy transfer theorem ensures that it can be transferred to an A_∞-algebra structure on the Morse cochains $C^*(f)$.

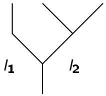
The differential on the Morse cochains is defined by a count of moduli spaces of gradient trajectories. Is it then possible to define higher multiplications m_n on $C^*(f)$ by a count of moduli spaces such that they fit in a structure of A_∞-algebra ?

Question solved for the first time by Abouzaid in [Abo11], drawing from earlier works by Fukaya ([Fuk97] for instance). See also [Mes18] and [AL18].

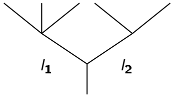
Terminology :



A ribbon tree



A metric ribbon tree



A stable metric ribbon tree

Definition

Define $\mathcal{T}_n$ to be *moduli space of stable metric ribbon trees with n incoming edges*. For each stable ribbon tree type t, we define moreover $\mathcal{T}_n(t) \subset \mathcal{T}_n$ to be the moduli space

$$\mathcal{T}_n(t) := \{\text{stable metric ribbon trees of type } t\} \ .$$

We then have the following cell decomposition

$$\mathcal{T}_n = \bigcup_{t \in SRT_n} \mathcal{T}_n(t) \ .$$

Allowing lengths of internal edges to go to $+\infty$, this moduli space can be compactified into a $(n-2)$-dimensional CW-complex $\overline{\mathcal{T}}_n$, where $\mathcal{T}_n$ is seen as its unique $(n-2)$-dimensional stratum.

Theorem

The compactified moduli space $\overline{\mathcal{T}}_n$ is isomorphic as a CW-complex to the associahedron K_n.

This was first noticed in section 1.4. of Boardman-Vogt [BV73].

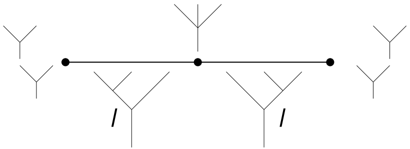


Figure: The compactified moduli space $\overline{\mathcal{T}}_3$

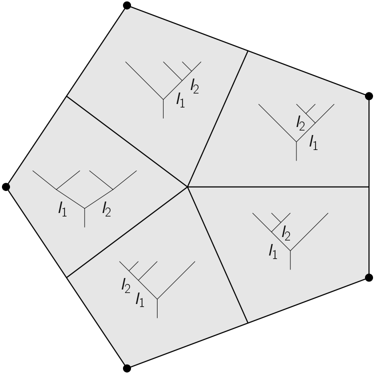
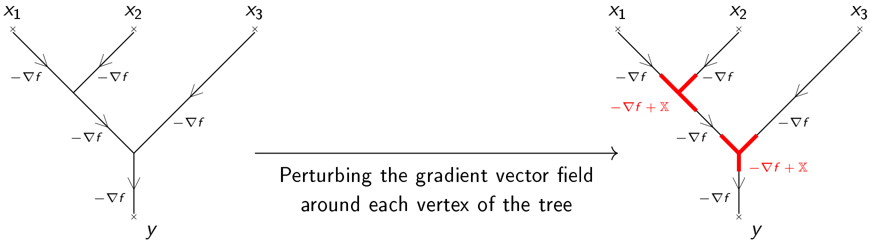


Figure: The compactified moduli space $\overline{\mathcal{T}}_4$

The goal is now to realize these moduli spaces of stable metric ribbon trees in Morse theory.



Definition

$T := (t, \{l_e\}_{e \in E(t)})$ where $\{l_e\}_{e \in E(t)}$ are the lengths of its internal edges of the tree t. *Choice of perturbation data* on T consists of the following data :

(i) a vector field $[0, l_e] \times M \underset{\mathbb{X}_e}{\longrightarrow} TM$, that vanishes on $[1, l_e - 1]$, for every internal edge e of t ;

(ii) a vector field $[0, +\infty[\times M \underset{\mathbb{X}_{e_0}}{\longrightarrow} TM$, that vanishes away from $[0, 1]$, for the outgoing edge e_0 of t ;

(iii) a vector field $]-\infty, 0] \times M \underset{\mathbb{X}_{e_i}}{\longrightarrow} TM$, that vanishes away from $[-1, 0]$, for every incoming edge e_i $(1 \leqslant i \leqslant n)$ of t.

We will write D_e for all segments $[0, l_e]$ as well as for all semi-infinite segments $]-\infty, 0]$ and $[0, +\infty[$ in the rest of the talk.

Definition ([Abo11])

A *perturbed Morse gradient tree* T^{Morse} associated to $(T, \mathbb{X})$ is the data for each edge e of t of a smooth map $\gamma_e : D_e \to M$ such that γ_e is a trajectory of the perturbed negative gradient $-\nabla f + \mathbb{X}_e$, i.e.

$$\dot{\gamma}_e(s) = -\nabla f(\gamma_e(s)) + \mathbb{X}_e(s, \gamma_e(s)) \ ,$$

and such that the endpoints of these trajectories coincide as prescribed by the edges of the tree T.

Definition

Let $\mathbb{X}_n$ be a smooth choice of perturbation data on $\mathcal{T}_n$. For critical points y and $x_1, \dots, x_n$, we define the moduli space

$$\mathcal{T}_n^{\mathbb{X}_n}(y; x_1, \dots, x_n) := \left\{ \begin{array}{c} \text{perturbed Morse gradient trees associated to } (T, \mathbb{X}_T) \\ \text{and connecting } x_1, \dots, x_n \text{ to } y, \text{ for } T \in \mathcal{T}_n \end{array} \right\} .$$

Proposition

Given a generic choice of perturbation data $\mathbb{X}_n$, the moduli space $\mathcal{T}_n^{\mathbb{X}_n}(y; x_1, \ldots, x_n)$ is an orientable manifold of dimension

$$\dim\left(\mathcal{T}_n(y; x_1, \ldots, x_n)\right) = n - 2 + |y| - \sum_{i=1}^{n} |x_i| \ ,$$

where $|x| := \mathrm{dim}(W^S(x))$.

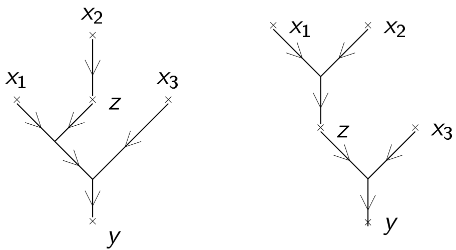
Choose perturbation data $\mathbb{X}_n$ on each moduli space $\mathcal{T}_n$ for $n \geqslant 2$. By assuming some gluing-compatibility conditions on $(\mathbb{X}_n)_{n\geqslant 2}$, the 1-dimensional moduli spaces $\mathcal{T}_n(y; x_1, \dots, x_n)$ can be compactified to manifolds with boundary whose boundary is given by the spaces

(i) corresponding to an internal edge breaking :

$$\mathcal{T}_{i_1+1+i_3}^{\mathbb{X}_{i_1+1+i_3}}(y; x_1, \dots, x_{i_1}, z, x_{i_1+i_2+1}, \dots, x_n) \times \mathcal{T}_{i_2}^{\mathbb{X}_{i_2}}(z; x_{i_1+1}, \dots, x_{i_1+i_2});$$

(ii) corresponding to an external edge breaking :

$$\mathcal{T}(y; z) \times \mathcal{T}_n^{\mathbb{X}_n}(z; x_1, \dots, x_n) \text{ and } \mathcal{T}_n^{\mathbb{X}_n}(y; x_1, \dots, z, \dots, x_n) \times \mathcal{T}(z; x_i)$$



Two examples of perturbed Morse gradient trees breaking at a critical point

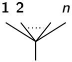
Theorem ([Abo11])

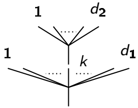
For an admissible choice of perturbation data $\mathbb{X} := (\mathbb{X}_n)_{n \geqslant 2}$, defining for every n the operation m_n as

$$\begin{aligned} m_n : C^*(f) \otimes \cdots \otimes C^*(f) &\longrightarrow C^*(f) \\ x_1 \otimes \cdots \otimes x_n &\longmapsto \sum_{|y| = \sum_{i=1}^n |x_i| + 2 - n} \# \mathcal{T}_n^{\mathbb{X}}(y; x_1, \cdots, x_n) \cdot y \ , \end{aligned}$$

they endow the Morse cochains $C^*(f)$ with an A_∞-algebra structure.

Indeed, the boundary of the previous compactification is modeled on the A_∞-equations for A_∞-algebras :

$[\partial_{Morse},$ 

$] = \sum_{\substack{h+k=n+1 \\ 2 \leqslant h \leqslant n-1 \\ 1 \leqslant i \leqslant k}} \pm$ 

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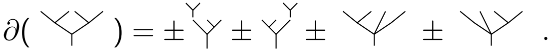
In fact, we construct in [Maz21a] a thinner algebraic structure on $C^*(f)$, called an ΩBAs-algebra structure. It corresponds to associating to each stable ribbon tree t of arity n an operation $C^*(f)^{\otimes n} \to C^*(f)$, and the differential for such an operation is then encoded by the codimension 1 boundary of the corresponding cell in $\mathcal{T}_n$. For instance,



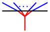
The A_∞-algebra structure on the Morse cochains then stems from this ΩBAs-algebra structure by purely algebraic arguments.

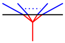
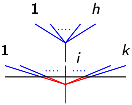
Working on the ΩBAs and not on the A_∞ level is also more rigorous for the analysis involved in these constructions.

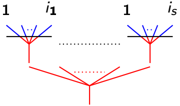
Definition

An A_∞-*morphism* between two A_∞-algebras A and B is a family of maps $f_n : A^{\otimes n} \to B$ of degree $1 - n$ satisfying

$$[m_1, f_n] = \sum_{\substack{i_1+i_2+i_3=n \\ i_2 \geqslant 2}} \pm f_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) + \sum_{\substack{i_1+\cdots+i_s=n \\ s \geqslant 2}} \pm m_s(f_{i_1} \otimes \cdots \otimes f_{i_s}) .$$

Representing the operations f_n as , the operations m_n^B in red and the operations m_n^A in blue, these equations read as

$\left[\partial, \right.$  $\left.\right] = \sum_{\substack{h+k=n+1 \\ 1\leqslant i\leqslant k \\ h\geqslant 2}} \pm$ 

$+ \sum_{\substack{i_1+\cdots+i_s=n \\ s\geqslant 2}} \pm$  .

We check that $[\partial, f_2] = f_1 m_2^A - m_2^B(f_1 \otimes f_1)$.

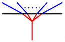
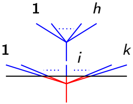
An A_∞-morphism between A_∞-algebras induces a morphism of associative algebras on the level of cohomology, and is a correct notion of morphism which preserves the product up to homotopy.

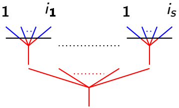
There exists a collection of polytopes, called the *multiplihedra* and denoted $\{J_n\}$, which encode the A_∞-equations for A_∞-morphisms. Again, J_n has a unique $n-1$-dimensional cell $[J_n]$ and the boundary of J_n is exactly

$$\partial J_n = \bigcup_{\substack{h+k=n+1 \\ h \geqslant 2}} \bigcup_{1 \leqslant i \leqslant k} J_k \times_i K_h \cup \bigcup_{\substack{i_1+\cdots+i_s=n \\ s \geqslant 2}} K_s \times J_{i_1} \times \cdots \times J_{i_s} \ ,$$

where $\times_k$ is the standard cartesian product $\times$.

Recall that the A_∞-equations for A_∞-morphisms are

$\partial\Big($  $\Big) = \displaystyle\sum_{\substack{h+k=n+1 \\ 1\leqslant i\leqslant k \\ h\geqslant 2}} \pm$ 

$+ \displaystyle\sum_{\substack{i_1+\cdots+i_s=n \\ s\geqslant 2}} \pm$ 

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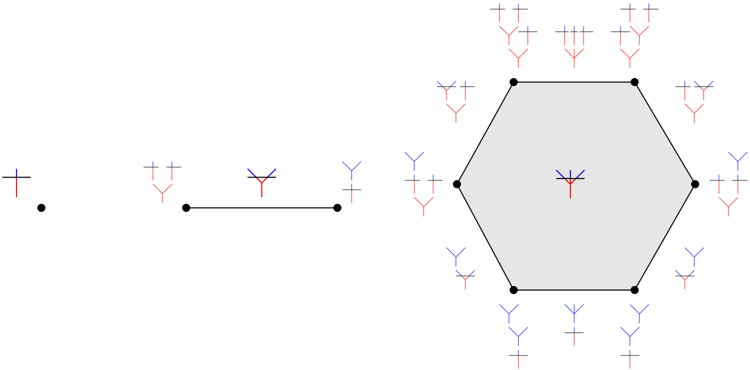


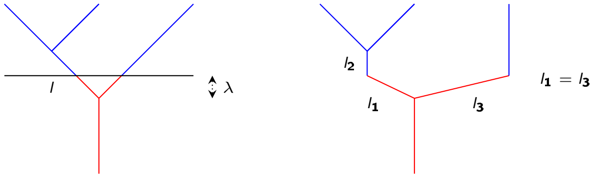
Figure: The multiplihedra J_1, J_2 and J_3 with cells labeled by the operations they define in $A_\infty - \mathrm{Morph}$

Consider an additional Morse function g on the manifold M.

Our goal is now to construct an A_∞-morphism from the Morse cochains $C^*(f)$ to the Morse cochains $C^*(g)$, through a count of moduli spaces of perturbed Morse trees.

Definition

A *stable two-colored metric ribbon tree* or *stable gauged metric ribbon tree* is defined to be a stable metric ribbon tree together with a length $\lambda \in \mathbb{R}$, which is to be thought of as a gauge drawn over the metric tree, at distance λ from its root, where the positive direction is pointing down.

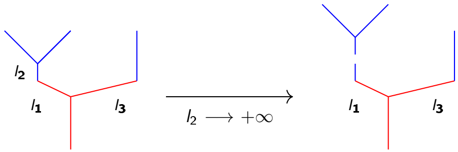


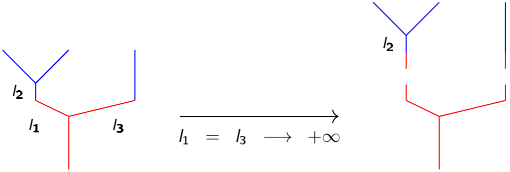
Definition

For $n \geqslant 1$, $\mathcal{CT}_n$ is the *moduli space of stable two-colored metric ribbon trees*. It has a cell decomposition by stable two-colored ribbon tree type,

$$\mathcal{CT}_n = \bigcup_{t_c \in SCRT_n} \mathcal{CT}_n(t_c) .$$

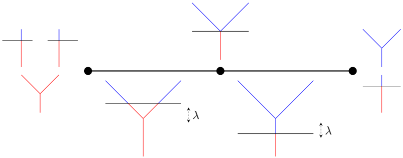
Allowing again internal edges of metric trees to go to $+\infty$, this moduli space $\mathcal{CT}_n$ can be compactified into a $(n-1)$-dimensional CW-complex $\overline{\mathcal{CT}}_n$.



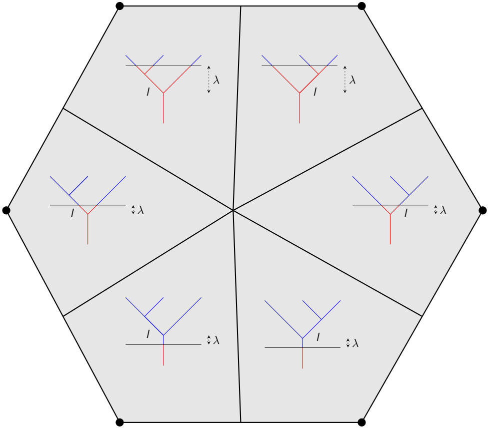


Theorem ([MW10])

The compactified moduli space $\overline{\mathcal{CT}}_n$ is isomorphic as a CW-complex to the multiplihedron J_n.



The compactified moduli space $\overline{\mathcal{CT}}_2$ with its cell decomposition by stable two-colored ribbon tree type



The compactified moduli space $\overline{\mathcal{CT}}_3$ with its cell decomposition by stable two-colored ribbon tree type

Definition

A *two-colored perturbed Morse gradient tree* T_g^{Morse} associated to a pair two-colored metric ribbon tree and perturbation data $(T_g, \mathbb{Y})$ is the data

(i) for each edge f_c of t_c which is above the gauge, of a smooth map

$$D_{f_c} \xrightarrow[\gamma_{f_c}]{} M \ ,$$

such that γ_{f_c} is a trajectory of the perturbed negative gradient $-\nabla f + \mathbb{Y}_{f_c}$,

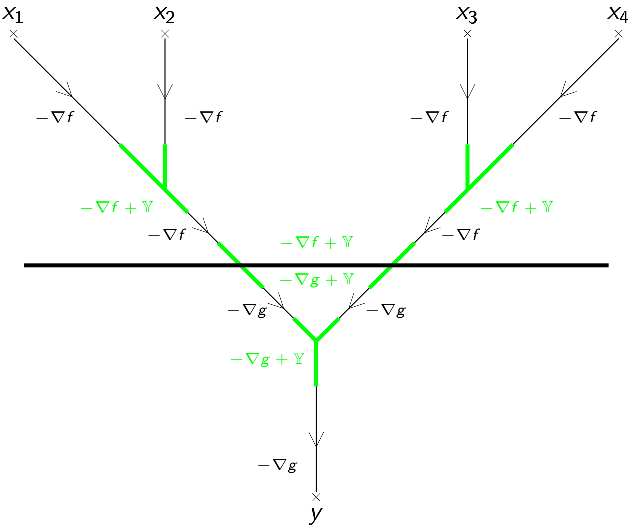
Definition

(ii) for each edge f_c of t_c which is below the gauge, of a smooth map

$$D_{f_c} \xrightarrow[\gamma_{f_c}]{} M \ ,$$

such that γ_{f_c} is a trajectory of the perturbed negative gradient $-\nabla g + \mathbb{Y}_{f_c}$,

and such that the endpoints of these trajectories coincide as prescribed by the edges of the two-colored tree type.


x_1
x_2
x_3
x_4
$-\nabla f$
$-\nabla f$
$-\nabla f$
$-\nabla f$
$-\nabla f + \mathbb{Y}$
$-\nabla f + \mathbb{Y}$
$-\nabla f$
$-\nabla f$
$-\nabla f + \mathbb{Y}$
$-\nabla g + \mathbb{Y}$
$-\nabla g$
$-\nabla g$
$-\nabla g + \mathbb{Y}$
$-\nabla g$
y

Definition

Let $\mathbb{Y}_n$ be a smooth choice of perturbation data on the moduli space $\mathcal{CT}_n$. Given $y \in \mathrm{Crit}(g)$ and $x_1, \dots, x_n \in \mathrm{Crit}(f)$, we define the moduli spaces

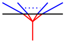
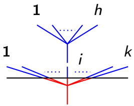
$$\mathcal{CT}_n^{\mathbb{Y}_n}(y; x_1, \dots, x_n) :=$$
$$\left\{ \begin{array}{c} \text{two-colored perturbed Morse gradient trees associated to} \\ (T_g, \mathbb{Y}_{T_g}) \text{ and connecting } x_1, \dots, x_n \text{ to } y \text{ for } T_g \in \mathcal{CT}_n \end{array} \right\} .$$

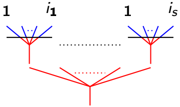
Proposition

Given a generic choice of perturbation data $\mathbb{Y}_n$, the moduli spaces $\mathcal{CT}_n^{\mathbb{Y}_n}(y; x_1, \ldots, x_n)$ are orientable manifolds of dimension

$$\dim\left(\mathcal{CT}_n(y; x_1, \ldots, x_n)\right) = |y| - \sum_{i=1}^{n} |x_i| + n - 1 \ .$$

Given perturbation data $\mathbb{X}^f$ and $\mathbb{X}^g$ for the functions f and g, by assuming some gluing-compatibility conditions for a choice of perturbation data $\mathbb{Y}_n$ for all $n \geqslant 1$, the 1-dimensional moduli spaces $\mathcal{CT}_n^{\mathbb{Y}_n}(y; x_1, \dots, x_n)$ can be compactified into manifolds with boundary whose boundary is modeled on the A_∞-equations for A_∞-morphisms :

$\left[\partial_{Morse} , \right.$  $\left. \right] = \sum_{\substack{h+k=n+1 \\ 1 \leqslant i \leqslant k \\ h \geqslant 2}} \pm$ 

$+ \sum_{\substack{i_1+\cdots+i_s=n \\ s \geqslant 2}} \pm$ 

.

Theorem ([Maz21a])

Let $\mathbb{X}^f$, $\mathbb{X}^g$ and $(\mathbb{Y}_n)_{n \geqslant 1}$ be admissible choices of perturbation data. Defining for every n the operation μ_n as

$$\begin{aligned} \mu_n^{\mathbb{Y}} : C^*(f) \otimes \cdots \otimes C^*(f) &\longrightarrow C^*(g) \\ x_1 \otimes \cdots \otimes x_n &\longmapsto \\ &\sum_{|y| = \sum_{i=1}^n |x_i| + 1 - n} \# \mathcal{CT}_n^{\mathbb{Y}}(y; x_1, \cdots, x_n) \cdot y \ . \end{aligned}$$

they fit into an A_∞-morphism $\mu^{\mathbb{Y}} : (C^*(f), m_n^{\mathbb{X}^f}) \to (C^*(g), m_n^{\mathbb{X}^g})$.

Again, we prove in [Maz21a] that this A_∞-morphism actually stems from an ΩBAs-morphism between the ΩBAs-algebras $C^*(f)$ and $C^*(g)$.

We can moreover prove that this A_∞-morphism induces an isomorphism between the Morse cohomologies.

Considering two A_∞-morphisms F, G, we would like first to determine a notion giving a satisfactory meaning to the sentence "F and G are homotopic". Then, A_∞-homotopies being defined, what is now a good notion of a homotopy between homotopies ? And of a homotopy between two homotopies between homotopies ? And so on.

Definition

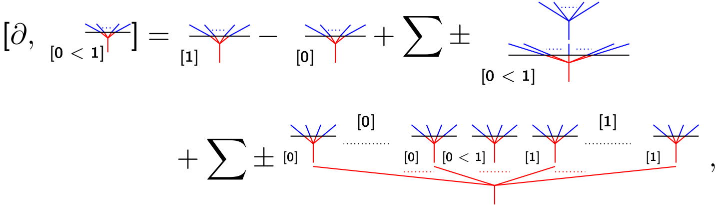
An A_∞-*homotopy* between two A_∞-morphisms $(f_n)_{n\geqslant 1}$ and $(g_n)_{n\geqslant 1}$ is a collection of maps

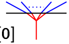
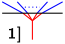
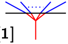
$$h_n : A^{\otimes n} \longrightarrow B \ ,$$

of degree $-n$, satisfying

$$\begin{aligned}[\partial, h_n] =& g_n - f_n + \sum_{\substack{i_1+i_2+i_3=m \\ i_2\geqslant 2}} \pm h_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) \\ & + \sum_{\substack{i_1+\cdots+i_s+l \\ +j_1+\cdots+j_t=n \\ s+1+t\geqslant 2}} \pm m_{s+1+t}(f_{i_1} \otimes \cdots \otimes f_{i_s} \otimes h_l \otimes g_{j_1} \otimes \cdots \otimes g_{j_t}) \ .\end{aligned}$$

In symbolic formalism,



where we denote $[0]$ , $[0<1]$  and $[1]$  respectively for the f_n, the h_n and the g_n.

Notation : the top-dimensional face of the n-simplex Δ^n will be written as $[0 < \cdots < n]$ and its subfaces $I \subset \Delta^n$ as $[i_1 < \cdots < i_k]$.

In that language, the standard Alexander-Whitney coproduct on the dg-module

$$\boldsymbol{\Delta}^n = \bigoplus_{0 \leqslant i_1 < \cdots < i_k \leqslant n} \mathbb{Z}[i_1 < \cdots < i_k] ,$$

can then be defined as

$$\Delta_{\boldsymbol{\Delta}^n}([i_1 < \cdots < i_k]) := \sum_{j=1}^{k} [i_1 < \cdots < i_j] \otimes [i_j < \cdots < i_k] .$$

Definition ([MS03])

Let I be a face of Δ^n. An *overlapping partition* of I is a sequence of faces $(I_l)_{1 \leqslant \ell \leqslant s}$ of I such that

(i) the union of this sequence of faces is I, i.e. $\cup_{1 \leqslant \ell \leqslant s} I_l = I$;

(ii) for all $1 \leqslant \ell < s$, $\max(I_\ell) = \min(I_{\ell+1})$.

An overlapping 6-partition for $[0 < 1 < 2]$ is for instance

$$[0 < 1 < 2] = [0] \cup [0] \cup [0 < 1] \cup [1] \cup [1 < 2] \cup [2] \ .$$

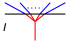
Overlapping partitions are the collection of faces which naturally arise in the Alexander-Whitney coproduct.

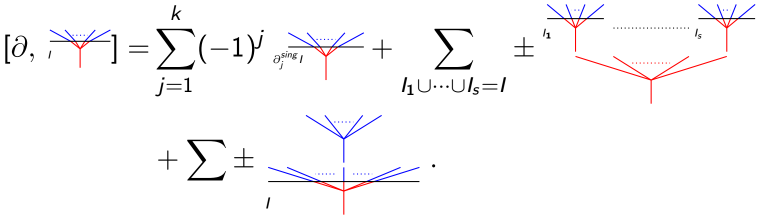
The element $\Delta_{\mathbf{\Delta}^n}(I)$ corresponds to the sum of all overlapping 2-partitions of I. Iterating s times $\Delta_{\mathbf{\Delta}^n}$ yields the sum of all overlapping $(s+1)$-partitions of I.

Definition ([Maz21b])

A *n-morphism* from A to B is defined to be a collection of maps $f_I^{(m)} : A^{\otimes m} \longrightarrow B$ of degree $1 - m + |I|$ for $I \subset \Delta^n$ and $m \geqslant 1$, that satisfy

$$\left[\partial, f_I^{(m)}\right] = \sum_{j=0}^{\dim(I)} (-1)^j f_{\partial_j I}^{(m)} + \sum_{\substack{i_1+\cdots+i_s=m \\ I_1\cup\cdots\cup I_s=I \\ s\geqslant 2}} \pm m_s(f_{I_1}^{(i_1)} \otimes \cdots \otimes f_{I_s}^{(i_s)})$$
$$+ (-1)^{|I|} \sum_{\substack{i_1+i_2+i_3=m \\ i_2\geqslant 2}} \pm f_I^{(i_1+1+i_3)}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) .$$

Equivalently and more visually, a collection of maps  satisfying



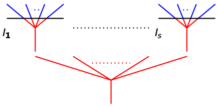
It is straightforward that 0-morphisms then correspond to A_∞-morphisms and 1-morphisms correspond to A_∞-homotopies.

We point out that these equations naturally stem from the shifted bar construction viewpoint. An A_∞-algebra structure on A is equivalent to a coderivation D_A on $\overline{T}(sA)$ such that $D_A^2 = 0$. A n-morphism between A_∞-algebras can then be defined as a morphism of dg-coalgebras $\mathbf{\Delta}^n \otimes \overline{T}(sA) \to \overline{T}(sB)$, where $\mathbf{\Delta}^n$ is a dg-coalgebra model for the n-simplex Δ^n.

We would like to define a family of polytopes encoding n-morphisms between A_∞-algebras. These polytopes will then be called *n-multiplihedra*.

We have seen that A_∞-morphisms are encoded by the multiplihedra. A natural candidate for n-morphisms would thus be $\{\Delta^n \times J_m\}_{m \geqslant 1}$.

However, $\Delta^n \times J_m$ does not fulfill that property as it is. Faces correspond to the data of a face of $I \subset \Delta^n$, and of a broken two-colored tree labeling a face of J_m. This labeling is too coarse, as it does not contain the trees



,

that appear in the A_∞-equations for n-morphisms.

We prove in [Maz21b] that there exists a thinner polytopal subdivision of $\Delta^n \times J_m$ encoding the A_∞-equations for n-morphisms between A_∞-algebras. We define the *n-multiplihedra* to be the polytopes $\Delta^n \times J_m$ endowed with this polytopal subdivision and denote them $n - J_m$.

This thinner polytopal subdivision is obtained by lifting the combinatorics of overlapping partitions to the level of the polytopes Δ^n.

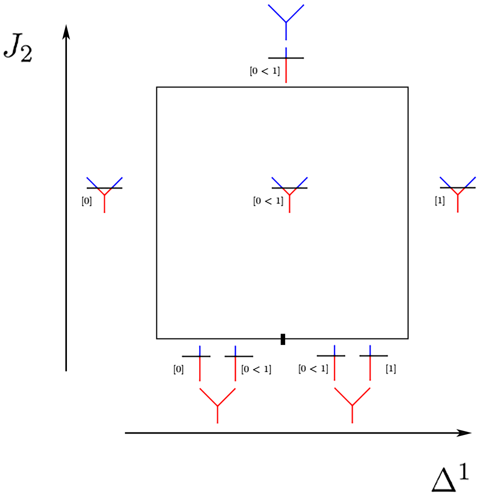


Figure: The 1-multiplihedron $\Delta^1 \times J_2$

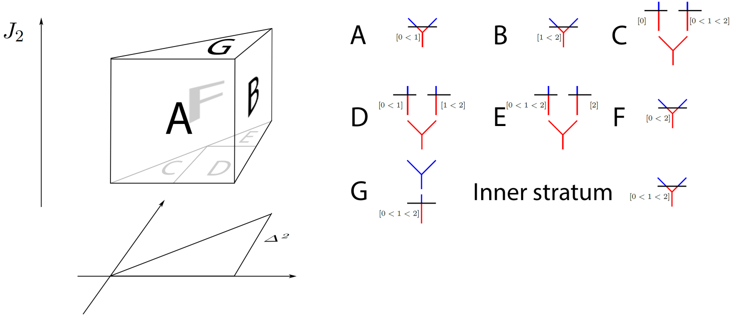


Figure: The 2-multiplihedron $\Delta^2 \times J_2$

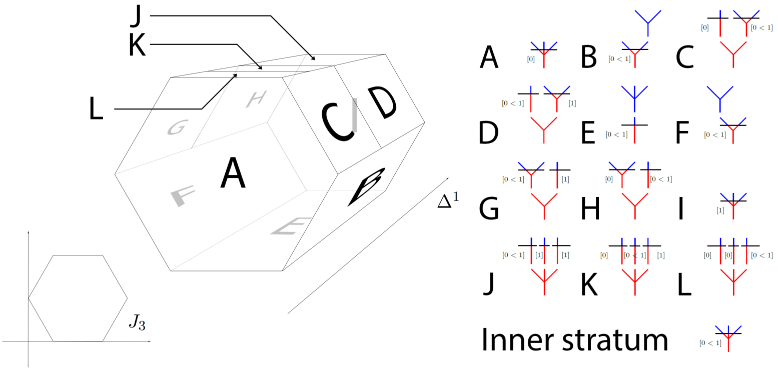


Figure: The 1-multiplihedron $\Delta^1 \times J_3$

Given two Morse functions f, g, choices of perturbation data $\mathbb{X}^f$ and $\mathbb{X}^g$, and choices of perturbation data $\mathbb{Y}$ and $\mathbb{Y}'$, is $\mu^{\mathbb{Y}}$ always A_∞-homotopic to $\mu^{\mathbb{Y}'}$? I.e., when can the following diagram be filled in the A_∞ world

$$\begin{tikzcd} C^*(f) \arrow[r, bend left, "\mu^{\mathbb{Y}}"] \arrow[r, bend right, "\mu^{\mathbb{Y}'}"'] \arrow[r, phantom, "\Downarrow"] & C^*(g) \end{tikzcd} \quad ?$$

While the spaces parametrizing the perturbation data were the $\mathcal{T}_m$ (a model for the associahedra K_m) and the $\mathcal{CT}_m$ (a model for the multiplihedra J_m), perturbation data will now be parametrized by the n-multiplihedra $\Delta^n \times \mathcal{CT}_m$.

The previous pattern can then be repeated. We consider a n-simplex of perturbation data $\mathbb{Y}_{\Delta^n,m} = \{\mathbb{Y}_{\delta,m}\}_{\delta \in \mathring{\Delta}^n}$ on $\mathcal{CT}_m$. Given $y \in \mathrm{Crit}(g)$ and $x_1, \ldots, x_m \in \mathrm{Crit}(f)$, we define the moduli spaces

$$\mathcal{CT}_{\Delta^n,m}^{\mathbb{Y}_{\Delta^n,m}}(y; x_1, \ldots, x_m) := \bigcup_{\delta \in \mathring{\Delta}^n} \mathcal{CT}_m^{\mathbb{Y}_{\delta,m}}(y; x_1, \ldots, x_m) \ .$$

Under some generic assumptions on $\mathbb{Y}_{\Delta^n,m}$, the moduli space $\mathcal{CT}_{\Delta^n,m}(y;x_1,\ldots,x_m)$ is then an orientable manifold of dimension

$$\dim\left(\mathcal{CT}_{\Delta^n,m}(y;x_1,\ldots,x_m)\right) = n+m-1+|y|-\sum_{i=1}^{m}|x_i| \ .$$

Choose perturbation data $\mathbb{X}^f$ and $\mathbb{X}^g$ for the functions f and g together with perturbation data $(\mathbb{Y}_{I,m})^{m \geqslant 1}_{I \subset \Delta^n}$. By assuming some gluing-compatibility conditions on $(\mathbb{Y}_{I,m})^{m \geqslant 1}_{I \subset \Delta^n}$ *modeling the combinatorics of overlapping partitions*, the 1-dimensional moduli spaces $\mathcal{CT}^{\mathbb{Y}_{I,m}}_{I,m}(y; x_1, \dots, x_m)$ can be compactified into manifolds with boundary whose boundary is modeled on the A_∞-equations for n-morphisms.

Theorem ([Maz21b])

Let $\mathbb{X}^f$, $\mathbb{X}^g$ and $(\mathbb{Y}_{I,m})^{m\geqslant 1}_{I\subset\Delta^n}$ be generic choices of perturbation data. Defining for every m the operation $\mu_I^{(m)}$ as

$$\begin{aligned} C^*(f)\otimes\cdots\otimes C^*(f) &\xrightarrow{\mu_I^{(m)}} C^*(g) \\ x_1\otimes\cdots\otimes x_m &\longmapsto \sum_{|y|=\sum_{i=1}^m |x_i|+1-m+|I|} \#\mathcal{CT}_{I,m}^{\mathbb{Y}_{I,m}}(y;x_1,\cdots,x_m)\cdot y\,, \end{aligned}$$

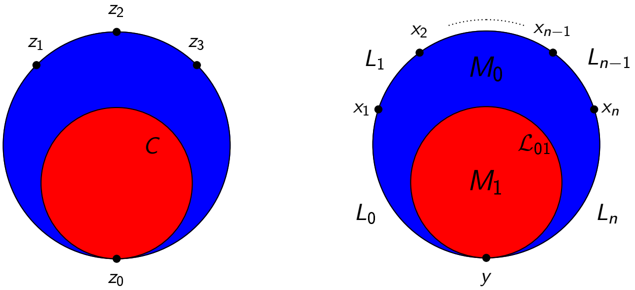
they fit into a n-morphism $\mu_I^{\mathbb{Y}} : (C^*(f), m_n^{\mathbb{X}^f}) \to (C^*(g), m_n^{\mathbb{X}^g})$, $I\subset\Delta^n$.

Corollary ([Maz21b])

Let $\mathbb{Y}$ and $\mathbb{Y}'$ be two admissible choices of perturbation data on the moduli spaces $\mathcal{CT}_m$. The A_∞-morphisms $\mu^{\mathbb{Y}}$ and $\mu^{\mathbb{Y}'}$ are then A_∞-homotopic

$$\begin{tikzcd} C^*(f) \arrow[r, bend left, "\mu^{\mathbb{Y}}"] \arrow[r, bend right, "\mu^{\mathbb{Y}'}"'] \arrow[r, phantom, "\Downarrow"] & C^*(g) \end{tikzcd} .$$

1. It is quite clear that given two compact symplectic manifolds M and N, one should be able to construct n-morphisms between their Fukaya categories $\mathrm{Fuk}(M)$ and $\mathrm{Fuk}(N)$ through counts of moduli spaces of quilted disks (see [MWW18] for the $n = 0$ case).



2. Given three Morse functions f_0, f_1, f_2, choices of perturbation data $\mathbb{X}^i$, and choices of perturbation data $\mathbb{Y}^{ij}$ defining morphisms

$$\begin{aligned}
\mu^{\mathbb{Y}^{01}} &: (C^*(f_0), m_n^{\mathbb{X}^0}) \longrightarrow (C^*(f_1), m_n^{\mathbb{X}^1}) \ , \\
\mu^{\mathbb{Y}^{12}} &: (C^*(f_1), m_n^{\mathbb{X}^1}) \longrightarrow (C^*(f_2), m_n^{\mathbb{X}^2}) \ , \\
\mu^{\mathbb{Y}^{02}} &: (C^*(f_0), m_n^{\mathbb{X}^0}) \longrightarrow (C^*(f_2), m_n^{\mathbb{X}^2}) \ ,
\end{aligned}$$

can we construct an A_∞-homotopy such that $\mu^{\mathbb{Y}^{12}} \circ \mu^{\mathbb{Y}^{01}} \simeq \mu^{\mathbb{Y}^{02}}$ through this homotopy ?

That is, can the following diagram be filled in the A_∞ realm

$$\begin{tikzcd} C^*(f_0) \arrow[r, "\mu^{\mathbb{Y}^{01}}"] \arrow[dr, "\mu^{\mathbb{Y}^{02}}"'] & C^*(f_1) \arrow[d, "\mu^{\mathbb{Y}^{12}}"] \arrow[dl, phantom, "\Leftarrow"] \\ & C^*(f_2) \end{tikzcd} \quad ?$$

Work in progress, see also [MWW18] for a similar question.

3. Links between the n-multiplihedra and the 2-associahedra of Bottman (see [Bot19a] and [Bot19b] for instance) ? We may inspect this matter in a near future with Nate Bottman.

Thanks for your attention !

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