Higher algebra of A_∞-algebras and the n-multiplihedra

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The results presented in this talk are taken from my two recent papers : *Higher algebra of* A_∞ *and* ΩBAs*-algebras in Morse theory I* (arXiv:2102.06654) and *Higher algebra of* A_∞ *and* ΩBAs*-algebras in Morse theory II* (arxiv:2102.08996).

The talk will be divided in three parts : recollections on A_∞-algebras and A_∞-morphisms ; definition of *higher morphisms between* A_∞*-algebras*, or $n - A_\infty$*-morphisms*, and their properties ; definition of the *n-multiplihedra*, which are new families of polytopes generalizing the standard multiplihedra and which encode $n - A_\infty$-morphisms between A_∞-algebras.

The material presented in this first part is standard and was drawn from [LV12], [Val14] and [LH02].

Suspension : Let A be a graded $\mathbb{Z}$-module. We denote sA, the *suspension of* A to be the graded $\mathbb{Z}$-module defined by $(sA)^i := A^{i-1}$. In other words, for $a \in A$, $|sa| = |a| - 1$.
For instance, a degree $2-n$ map $A^{\otimes n} \to A$ is equivalent to a degree $+1$ map $(sA)^{\otimes n} \to sA$.

Cohomological conventions : differentials will have degree $+1$.

Definition

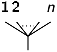
Let A be a dg-$\mathbb{Z}$-module with differential m_1. An A_∞-algebra structure on A is the data of a collection of maps of degree $2-n$

$$m_n : A^{\otimes n} \longrightarrow A \ , \ \ n \geqslant 1,$$

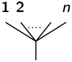
extending m_1 and which satisfy

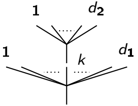
$$[m_1, m_n] = \sum_{\substack{i_1+i_2+i_3=n \\ 2\leqslant i_2 \leqslant n-1}} \pm m_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}).$$

These equations are called the A_∞-equations.

Representing m_n as 

, these equations can be written as

$[m_1,$ 

$] = \sum_{\substack{h+k=n+1 \\ 2 \leqslant h \leqslant n-1 \\ 1 \leqslant i \leqslant k}} \pm$ 

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In particular,

$$[m_1, m_2] = 0 \ ,$$
$$[m_1, m_3] = m_2(\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) \ ,$$

implying that m_2 descends to an associative product on $H^*(A)$. An A_∞-algebra is thus simply a correct notion of a dg-algebra whose product is associative up to homotopy.

The operations m_n are the higher coherent homotopies which keep track of the fact that the product is associative up to homotopy.

A_∞-algebras can also be defined using the shifted bar construction.

The *reduced tensor coalgebra*, or *bar construction*, of a graded $\mathbb{Z}$-module V is defined to be

$$\overline{T}V := V \oplus V^{\otimes 2} \oplus \cdots$$

endowed with the coassociative comultiplication

$$\Delta_{\overline{T}V}(v_1 \dots v_n) := \sum_{i=1}^{n-1} v_1 \dots v_i \otimes v_{i+1} \dots v_n \ .$$

Then for a graded $\mathbb{Z}$-module A, we can check that there is a one-to-one correspondence

$$\left\{ \begin{array}{c} \text{collections of morphisms of degree } 2-n \\ m_n : A^{\otimes n} \to A \ , \ \ n \geqslant 1, \\ \text{satisfying the } A_\infty\text{-equations} \end{array} \right\}$$

$$\longleftrightarrow \left\{ \begin{array}{c} \text{coderivations } D \text{ of degree } +1 \text{ of } \overline{T}(sA) \\ \text{such that } D^2 = 0 \end{array} \right\} .$$

Denoting $b_n : (sA)^{\otimes n} \to sA$ the degree $+1$ maps which determine the coderivation $D : \overline{T}(sA) \to \overline{T}(sA)$ (universal property of the bar construction), check that the restriction of D to the $(sA)^{\otimes n}$ summand is given by

$$\sum_{i_1+i_2+i_3=n} \pm \mathrm{id}^{\otimes i_1} \otimes b_{i_2} \otimes \mathrm{id}^{\otimes i_3} ,$$

and that the equation $D^2 = 0$ is equivalent to the A_∞-equations for the maps b_n.

This one-to-one correspondence yields an equivalent definition for A_∞-algebras.

Definition

An A_∞-*morphism* between two A_∞-algebras A and B is a dg-coalgebra morphism $F : (\overline{T}(sA), D_A) \to (\overline{T}(sB), D_B)$ between their shifted bar constructions.

As previously, one-to-one correspondence

$$\left\{ \begin{array}{c} \text{collections of morphisms of degree } 1-n \\ f_n : A^{\otimes n} \to B \ , \ n \geqslant 1, \end{array} \right\}$$

$$\longleftrightarrow \left\{ \begin{array}{c} \text{morphisms of graded coalgebras} \\ F : \overline{T}(sA) \to \overline{T}(sB) \end{array} \right\} .$$

Component of F mapping $(sA)^{\otimes n}$ to $(sB)^{\otimes s}$ given by

$$\sum_{i_1+\cdots+i_s=n} \pm f_{i_1} \otimes \cdots \otimes f_{i_s} \ .$$

A coalgebra morphism preserves the differential if and only if

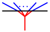
$$\begin{aligned} &\sum_{i_1+i_2+i_3=n} \pm f_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2}^A \otimes \mathrm{id}^{\otimes i_3}) \\ = &\sum_{i_1+\cdots+i_s=n} \pm m_s^B(f_{i_1} \otimes \cdots \otimes f_{i_s}) \ . \end{aligned} \tag{$\star$}$$

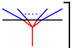
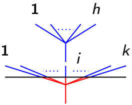
This yields an equivalent definition with operations for A_∞-morphisms :

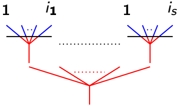
Definition

An A_∞-*morphism* between two A_∞-algebras A and B is a family of maps $f_n : A^{\otimes n} \to B$ of degree $1 - n$ satisfying

$$[m_1, f_n] = \sum_{\substack{i_1+i_2+i_3=n \\ i_2 \geqslant 2}} \pm f_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) + \sum_{\substack{i_1+\cdots+i_s=n \\ s \geqslant 2}} \pm m_s(f_{i_1} \otimes \cdots \otimes f_{i_s}) \ .$$

Representing the operations f_n as , the operations m_n^A in red and the operations m_n^B in blue, these equations read as

$\left[\partial,\right.$  $\left.\right] = \sum_{\substack{h+k=n+1\\ 1\leqslant i\leqslant k\\ h\geqslant 2}} \pm$ 

$+ \sum_{\substack{i_1+\cdots+i_s=n\\ s\geqslant 2}} \pm$  .

We check that $[\partial, f_2] = f_1 m_2^A - m_2^B(f_1 \otimes f_1)$.

An A_∞-morphism between A_∞-algebras induces a morphism of associative algebras on the level of cohomology, and is a correct notion of morphism which preserves the product up to homotopy.

Given two coalgebra morphisms $F : \overline{T}V \to \overline{T}W$ and $G : \overline{T}W \to \overline{T}Z$, the family of morphisms associated to $G \circ F$ is given by

$$(G \circ F)_n := \sum_{i_1 + \cdots + i_s = n} \pm g_s(f_{i_1} \otimes \cdots \otimes f_{i_s}) \ .$$

This formula defines the composition of A_∞-morphisms. Hence, A_∞-algebras together with A_∞-morphisms form a category, denoted $\mathtt{A}_\infty - \mathtt{alg}$. This category can be seen as a full subcategory of $\mathtt{dg} - \mathtt{cog}$ using the shifted bar construction viewpoint.

Let $\mathcal{C}$ be one the two following monoidal categories : the category $(\mathtt{dg}-\mathbb{Z}-\mathtt{mod},\otimes)$ or the category of polytopes $(\mathtt{Poly},\times)$.

Definition

A *$\mathcal{C}$-operad P* is the data of a collection of objects $\{P_n\}_{n\geqslant 1}$ of $\mathcal{C}$ together with a unit element $e\in P_1$ and with compositions

$$P_k\otimes P_{i_1}\otimes\cdots\otimes P_{i_k}\underset{c_{i_1,\dots,i_k}}{\longrightarrow}P_{i_1+\cdots+i_k}$$

which are unital and associative.

The objects P_n are to be thought as spaces encoding arity n operations while the compositions $c_{i_1,\dots,i_k}$ define how to compose these operations together.

Let A be a dg-$\mathbb{Z}$-module and $n \geqslant 1$. The space of graded maps $\mathrm{Hom}(A^{\otimes n}, A)$ is a graded $\mathbb{Z}$-module, and the collection of spaces $\mathrm{Hom}(A) := \mathrm{Hom}(A^{\otimes n}, A)$ can naturally be endowed with an operad structure (compositions are defined as one expects).

For P be a $(\mathtt{dg} - \mathbb{Z} - \mathtt{mod})$-operad, a *structure of P-algebra* on A is defined to be the datum of a morphism of operads

$$P \longrightarrow \mathrm{Hom}(A) \ .$$

In other words, of a way to interpret each operation of P_n in $\mathrm{Hom}(A^{\otimes n}, A)$, such that abstract composition in P coincides with actual composition in $\mathrm{Hom}(A)$.

Consider $\{P_n\}_{n\geqslant 1}$ and $\{Q_n\}_{n\geqslant 1}$ two operads, and $\{R_n\}_{n\geqslant 1}$ a collection of spaces of operations.

Definition

A (P, Q)-*operadic bimodule structure* on R is the data of action-composition maps

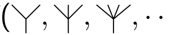
$$R_k \otimes Q_{i_1} \otimes \cdots \otimes Q_{i_k} \underset{\mu_{i_1,\dots,i_k}}{\longrightarrow} R_{i_1+\cdots+i_k} \ ,$$

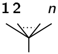
$$P_h \otimes R_{j_1} \otimes \cdots \otimes R_{j_h} \underset{\lambda_{j_1,\dots,j_h}}{\longrightarrow} R_{j_1+\cdots+j_h} \ ,$$

which are compatible with one another, with identities, and with compositions in P and Q.

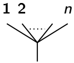
Let A and B be two dg-$\mathbb{Z}$-modules. We have seen that they each determine an operad, $\mathrm{Hom}(A)$ and $\mathrm{Hom}(B)$ respectively. The collection of spaces $\mathrm{Hom}(A, B) := \{\mathrm{Hom}(A^{\otimes n}, B)\}_{n \geqslant 1}$ in dg-$\mathbb{Z}$-modules is then a $(\mathrm{Hom}(B), \mathrm{Hom}(A))$-operadic bimodule where the action-composition maps are defined as one could expect.

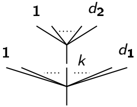
The structure of A_∞-algebra is governed by the operad A_∞ : the quasi-free $\mathtt{dg} - \mathbb{Z} - \mathtt{mod}$-operad

$A_\infty := \mathcal{F}($  $)$

generated in arity n by one operation 

of degree $2 - n$ and whose differential is defined by

$\partial($ 

$) = \sum_{\substack{h+k=n+1 \\ 2 \leqslant h \leqslant n-1 \\ 1 \leqslant i \leqslant k}} \pm$ 

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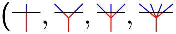
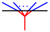
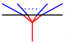
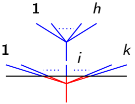
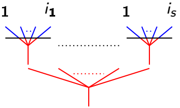
A_∞-morphisms between A_∞-algebras are encoded by the (A_∞,A_∞)-operadic bimodule $A_\infty - \mathrm{Morph}$, defined to be the (A_∞, A_∞)-operadic bimodule

$A_\infty - \mathrm{Morph} = \mathcal{F}^{A_\infty,A_\infty}($  $, \cdots)$,

generated in arity n by one operation  of degree $1 - n$ and ...

... whose differential is defined by

$\partial($  $) = \sum_{\substack{h+k=n+1\\ 1\leqslant i\leqslant k\\ h\geqslant 2}} \pm$ 

$+ \sum_{\substack{i_1+\cdots+i_s=n\\ s\geqslant 2}} \pm$ 

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where the generating operations of the operad A_∞ acting on the right in blue  and the ones of the operad A_∞ acting on the left in red  .

The category $\mathtt{A}_\infty - \mathtt{alg}$ provides a framework that behaves well with respect to homotopy-theoretic constructions, when studying homotopy theory of associative algebras. See for instance [Val20] and [LH02].

It is because this category is encoded by the two-colored operad

$$A_\infty^2 := \mathcal{F}(\textcolor{red}{\curlyvee}, \textcolor{red}{\curlyvee}, \textcolor{red}{\curlyvee}, \cdots, \textcolor{blue}{\curlyvee}, \textcolor{blue}{\curlyvee}, \textcolor{blue}{\curlyvee}, \cdots, \textcolor{red}{+}, \curlyvee, \curlyvee, \curlyvee, \cdots) \ .$$

It is a quasi-free object in the model category of two-colored operads in dg-$\mathbb{Z}$-modules and a fibrant-cofibrant replacement of the two-colored operad As^2, which encodes associative algebras with morphisms of algebras,

$$A_\infty^2 \xrightarrow{\sim} As^2 \ .$$

Theorem (Homotopy transfer theorem)

Let (A, ∂_A) and (H, ∂_H) be two cochain complexes. Suppose that H is a homotopy retract of A, that is that they fit into a diagram Il faut corriger ce théorème

$$\begin{tikzcd} (A, \partial_A) \arrow[loop left, "h"] \arrow[r, shift left, "p"] & (H, \partial_H) \arrow[l, shift left, "i"] \end{tikzcd} ,$$

where $\mathrm{id}_A - ip = [\partial, h]$ and $pi = \mathrm{id}_H$. Then if (A, ∂_A) is endowed with an associative algebra structure, H can be made into an A_∞-algebra such that i and p extend to A_∞-morphisms, that are then A_∞-quasi-isomorphisms.

Our goal now : study the *higher algebra of A_∞-algebras*.

Higher algebra is a general term standing for all problems that involve defining coherent sets of *higher homotopies* (also called *n-morphisms*) when starting from a basic homotopy setting.

Considering two A_∞-morphisms F, G, we would like first to determine a notion giving a satisfactory meaning to the sentence "F and G are homotopic". Then, A_∞-homotopies being defined, what is now a good notion of a homotopy between homotopies ? And of a homotopy between two homotopies between homotopies ? And so on.

Start with a notion of homotopy. Drawn from [LH02]. Pas le premier : Guggenheim ? Munkolm ? cité dans LH ?

Take C and C' two dg-coalgebras, F and G morphisms $C \to C'$ of dg-coalgebras. A (F, G)-*coderivation* is a map $H : C \to C'$ such that

$$\Delta_{C'} H = (F \otimes H + H \otimes G)\Delta_C \ .$$

The morphisms F and G are then said to be *homotopic* if there exists a (F, G)-coderivation H of degree -1 such that

$$[\partial, H] = G - F \ .$$

Define

$$\boldsymbol{\Delta}^1 := \mathbb{Z}[0] \oplus \mathbb{Z}[1] \oplus \mathbb{Z}[0<1] \ ,$$

with differential ∂^{sing}

$$\partial^{sing}([0<1]) = [1]-[0] \qquad \partial^{sing}([0]) = 0 \qquad \partial^{sing}([1]) = 0 \ ,$$

and coproduct the Alexander-Whitney coproduct

$$\begin{aligned} \Delta_{\boldsymbol{\Delta}^1}([0<1]) &= [0]\otimes[0<1] + [0<1]\otimes[1] \\ \Delta_{\boldsymbol{\Delta}^1}([0]) &= [0]\otimes[0] \\ \Delta_{\boldsymbol{\Delta}^1}([1]) &= [1]\otimes[1] \ . \end{aligned}$$

The elements $[0]$ and $[1]$ have degree 0, and the element $[0<1]$ has degree -1.

We check that there is a one-to-one correspondence between (F,G)-coderivations and morphisms of dg-coalgebras $\mathbf{\Delta}^1 \otimes C \longrightarrow C'$.

Definition

For two A_∞-algebras $(\overline{T}(sA), D_A)$ and $(\overline{T}(sB), D_B)$ and two A_∞-morphisms $F, G : (\overline{T}(sA), D_A) \to (\overline{T}(sB), D_B)$, an *$A_\infty$-homotopy from F to G* is defined to be a morphism of dg-coalgebras

$$H : \mathbf{\Delta}^1 \otimes \overline{T}(sA) \longrightarrow \overline{T}(sB) ,$$

whose restriction to the [0] summand is F and whose restriction to the [1] summand is G.

Using the universal property of the bar construction, this definition can be rephrased in terms of operations.

Definition

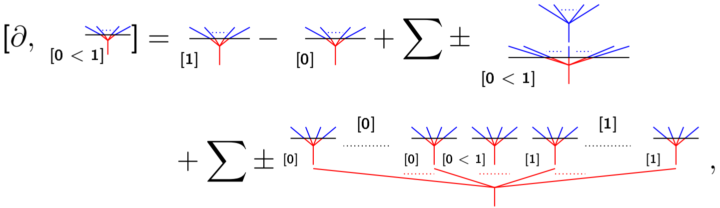
An A_∞-*homotopy* between two A_∞-morphisms $(f_n)_{n\geqslant 1}$ and $(g_n)_{n\geqslant 1}$ is a collection of maps

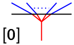
$$h_n : A^{\otimes n} \longrightarrow B \ ,$$

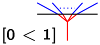
of degree $-n$, satisfying

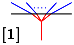
$$[\partial, h_n] = g_n - f_n + \sum_{\substack{i_1+i_2+i_3=m \\ i_2\geqslant 2}} \pm h_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) + \sum_{\substack{i_1+\cdots+i_s+l \\ +j_1+\cdots+j_t=n \\ s+1+t\geqslant 2}} \pm m_{s+1+t}(f_{i_1} \otimes \cdots \otimes f_{i_s} \otimes h_l \otimes g_{j_1} \otimes \cdots \otimes g_{j_t}) \ .$$

In symbolic formalism,



where we denote 

, 

and 

respectively for the f_n, the h_n and the g_n.

The relation *being A_∞-homotopic* on the class of A_∞-morphisms is an equivalence relation. It is moreover stable under composition. These results cannot all be proven using naive algebraic tools, some of them require considerations of model categories.

Move on to n-morphisms between A_∞-algebras.

Define $\mathbf{\Delta}^n$ the graded $\mathbb{Z}$-module generated by the faces of the standard n-simplex Δ^n,

$$\mathbf{\Delta}^n = \bigoplus_{0 \leqslant i_1 < \cdots < i_k \leqslant n} \mathbb{Z}[i_1 < \cdots < i_k] \ .$$

The grading is $|I| := -\dim(I)$ for $I \subset \Delta^n$.

It has a dg-coalgebra structure, with differential

$$\partial_{\mathbf{\Delta}^n}([i_1 < \cdots < i_k]) := \sum_{j=1}^{k} (-1)^j [i_1 < \cdots < \widehat{i_j} < \cdots < i_k] \ ,$$

and coproduct the Alexander-Whitney coproduct

$$\Delta_{\mathbf{\Delta}^n}([i_1 < \cdots < i_k]) := \sum_{j=1}^{k} [i_1 < \cdots < i_j] \otimes [i_j < \cdots < i_k] \ .$$

Definition ([MS03])

Let I be a face of Δ^n. An *overlapping partition* of I to be a sequence of faces $(I_l)_{1\leqslant \ell \leqslant s}$ of I such that

(i) the union of this sequence of faces is I, i.e. $\cup_{1\leqslant \ell \leqslant s} I_l = I$;

(ii) for all $1 \leqslant \ell < s$, $\max(I_\ell) = \min(I_{\ell+1})$.

An overlapping 6-partition for $[0 < 1 < 2]$ is for instance

$$[0 < 1 < 2] = [0] \cup [0] \cup [0 < 1] \cup [1] \cup [1 < 2] \cup [2] \ .$$

Overlapping partitions are the collection of faces which naturally arise in the Alexander-Whitney coproduct.

The element $\Delta_{\mathbf{\Delta}^n}(I)$ corresponds to the sum of all overlapping 2-partitions of I. Iterating s times $\Delta_{\mathbf{\Delta}^n}$ yields the sum of all overlapping $(s+1)$-partitions of I.

We have seen that A_∞-morphisms correspond to the set

$$\mathrm{Hom}_{\mathrm{dg-cog}}(\overline{T}(sA), \overline{T}(sB))$$

and A_∞-homotopies correspond to the set

$$\mathrm{Hom}_{\mathrm{dg-cog}}(\mathbf{\Delta}^1 \otimes \overline{T}(sA), \overline{T}(sB)) \ ,$$

Definition ([Maz21b])

We define the set of n-morphisms between A and B as

$$\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A,B)_n := \mathrm{Hom}_{\mathrm{dg-cog}}(\mathbf{\Delta}^n \otimes \overline{T}(sA), \overline{T}(sB)) \ .$$

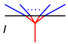
Using the universal property of the bar construction, n-morphisms admit a nice combinatorial description in terms of operations.

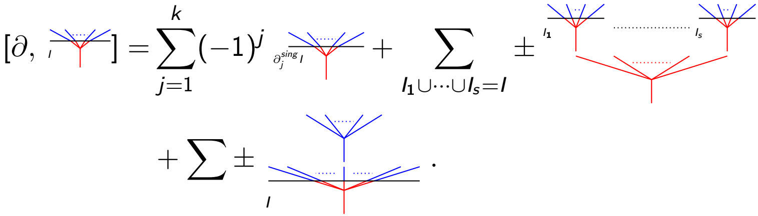
Definition ([Maz21b])

A *n-morphism* from A to B is defined to be a collection of maps $f_I^{(m)} : A^{\otimes m} \longrightarrow B$ of degree $1 - m + |I|$ for $I \subset \Delta^n$ and $m \geqslant 1$, that satisfy

$$\left[\partial, f_I^{(m)}\right] = \sum_{j=0}^{\dim(I)} (-1)^j f_{\partial_j I}^{(m)} + \sum_{\substack{i_1+\cdots+i_s=m \\ I_1\cup\cdots\cup I_s=I \\ s\geqslant 2}} \pm m_s(f_{I_1}^{(i_1)} \otimes \cdots \otimes f_{I_s}^{(i_s)})$$

$$+ (-1)^{|I|} \sum_{\substack{i_1+i_2+i_3=m \\ i_2\geqslant 2}} \pm f_I^{(i_1+1+i_3)}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) \ .$$

Equivalently and more visually, a collection of maps  satisfying



1. A_∞-algebras and A_∞-morphisms
2. Higher algebra of A_∞-algebras
   - A_∞-homotopies
   - Higher morphisms between A_∞-algebras
   - The HOM-simplicial sets $\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A,B)_\bullet$
   - A simplicial enrichment of the category $\mathtt{A}_\infty - \mathtt{alg}$?
3. The n-multiplihedra

The dg-coalgebras $\mathbf{\Delta}^\bullet := \{\mathbf{\Delta}^n\}_{n\geqslant 0}$ naturally form a cosimplicial dg-coalgebra.

The sets $\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A,B)_n$ then fit into a HOM-simplicial set $\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A,B)_\bullet$. This HOM-simplicial set provides a satisfactory framework to study the higher algebra of A_∞-algebras.

Theorem ([Maz21b])

For A and B two A_∞-algebras, the simplicial set $\mathrm{HOM}_{A_\infty}(A,B)_\bullet$ is an ∞-category.

Write Δ^n the simplicial set realizing the standard n-simplex Δ^n, and Λ_n^k the simplicial set realizing the simplicial subcomplex obtained from Δ^n by removing the faces $[0 < \cdots < n]$ and $[0 < \cdots < \widehat{k} < \cdots < n]$. The simplicial set Λ_n^k is called a *horn*, and if $0 < k < n$ it is called an *inner horn*.

An ∞-category is a simplicial set X which has the left-lifting property with respect to all inner horn inclusions $\Lambda^k_n \to \Delta^n$.

$$\begin{tikzcd} \Lambda^k_n \arrow[r, "u"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists\ \overline{u}"'] & \end{tikzcd}$$

The vertices of X are then to be seen as objects, and its edges correspond to morphisms.

Theorem ([Maz21b])

For A and B two A_∞-algebras, the simplicial set $\mathrm{HOM}_{A_\infty}(A,B)_\bullet$ is an ∞-category.

Beware that the points of these ∞-categories are the A_∞-morphisms, and the arrows between them are the A_∞-homotopies. This can be misleading at first sight, but *the points are the morphisms and NOT the algebras* and *the arrows are the homotopies and NOT the morphisms*.

Consider an inner horn $\Lambda_n^k \to \mathrm{HOM}_{A_\infty}(A,B)_\bullet$, where $0 < k < n$.

We want to complete the diagram

$$\begin{tikzcd} \Lambda_n^k \arrow[r] \arrow[d] & \mathrm{HOM}_{A_\infty}(A,B)_\bullet \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

.

This inner horn corresponds to a collection of degree $1 - m + |I|$ morphisms

$$f_I^{(m)} : A^{\otimes m} \longrightarrow B$$

for $I \subset \Lambda_n^k$, which satisfy the A_∞-equations.

Filling this horn amounts then to defining a collection of operations

$$f^{(m)}_{[0<\cdots<\widehat{k}<\cdots<n]} : A^{\otimes m} \longrightarrow B \text{ and } f^{(m)}_{\Delta^n} : A^{\otimes m} \longrightarrow B \ ,$$

of respective degree $1 - m - (n-1)$ and $1 - m - n$, and satisfying the A_∞-equations.

Inspecting the proof in details (which can be reduced to tedious combinatorics) shows that they are in fact *algebraic* ∞*-categories*. See also [RNV20]. Pas les premiers : qui était-ce ?

Proposition ([Maz21b])

There is a natural one-to-one correspondence between

$$\left\{ \textit{fillers} \quad \begin{tikzcd} \Lambda_n^k \arrow[r] \arrow[d] & \mathrm{HOM}_{A_\infty}(A, B)_\bullet \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \right\}$$

and

$$\left\{ \begin{array}{c} \textit{families of maps of degree } 1 - m - n \\ U^{(m)} : A^{\otimes m} \to B, \ m \geqslant 1 \end{array} \right\} .$$

We would like to see the simplicial sets $\mathrm{HOM}_{\mathtt{A}_\infty-\mathtt{alg}}(A,B)_\bullet$ as part of a simplicial enrichment of the category $\mathtt{A}_\infty - \mathtt{alg}$. In other words, we would like to define simplicial maps

$$\mathrm{HOM}_{\mathtt{A}_\infty-\mathtt{alg}}(A,B)_n \times \mathrm{HOM}_{\mathtt{A}_\infty-\mathtt{alg}}(B,C)_n \longrightarrow \mathrm{HOM}_{\mathtt{A}_\infty-\mathtt{alg}}(A,C)_n \ ,$$

lifting the composition on the $\mathrm{HOM}_0 = \mathrm{Hom}$.

This would then endow $\mathtt{A}_\infty - \mathtt{alg}$ with a structure of $(\infty, 2)$-category.

All the natural approaches to lift the composition in $\mathtt{A}_\infty - \mathtt{alg}$ to $\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A,B)_\bullet$ fail to work. Hence, it is still an open question to know whether these HOM-simplicial sets could fit into a simplicial enrichment of the category $\mathtt{A}_\infty - \mathtt{alg}$. In fact, it is unclear to the author why such a statement should be true.

1. A_∞-algebras and A_∞-morphisms
2. Higher algebra of A_∞-algebras
3. The n-multiplihedra
   - Operadic algebra in the category `Poly`
   - The associahedra
   - The multiplihedra
   - The n-multiplihedra
   - Towards Morse and Floer theory

Define a *polytope* to be the convex hull of a finite number of points in a Euclidean space $\mathbb{R}^n$.

Following [MTTV19], polytopes fit into a category $\mathtt{Poly}$. Beware that the morphisms of this category are not the usual affine maps. It forms a monoidal category with product the usual cartesian product, and a monoidal subcategory of $\mathtt{CW}$.

The cellular chain functor $C_*^{cell} : \mathtt{Poly} \to \mathtt{dg} - \mathbb{Z} - \mathtt{mod}$ then satisfies

$$C_*^{cell}(P \times Q) = C_*^{cell}(P) \otimes C_*^{cell}(Q) \ .$$

We will in fact work with the functor

$$C_{-*}^{cell} : \mathtt{CW} \longrightarrow \mathtt{dg} - \mathbb{Z} - \mathtt{mod} \ ,$$

where $C_{-*}^{cell}(P)$ is simply the $\mathbb{Z}$-module $C_*^{cell}(P)$ taken with its opposite grading.

In particular the functor C_{-*}^{cell} takes operads and operadic bimodules in $\mathtt{Poly}$ to operads and operadic bimodules in $\mathtt{dg} - \mathbb{Z} - \mathtt{mod}$.

1. A_∞-algebras and A_∞-morphisms
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   - The multiplihedra
   - The n-multiplihedra
   - Towards Morse and Floer theory

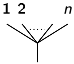
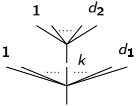
The $\mathtt{dg} - \mathbb{Z} - \mathtt{mod}$-operad A_∞ actually stems from a Poly-operad. This was fully proven in [MTTV19].

There exists a collection of polytopes, called the *associahedra* and denoted $\{K_n\}$, endowed with a structure of operad in the category Poly and whose image under the functor C_{-*}^{cell} yields the operad A_∞.

In particular K_n has a unique cell $[K_n]$ of dimension $n-2$ whose image under ∂_{cell} is the A_∞-equation, that is such that

$$\partial_{cell}[K_n] = \sum \pm \circ_i ([K_k] \otimes [K_h]) \ .$$

Recall that the A_∞-equations read as

$\partial($  $) = \sum_{\substack{h+k=n+1 \\ 2 \leqslant h \leqslant n-1 \\ 1 \leqslant i \leqslant k}} \pm$  .

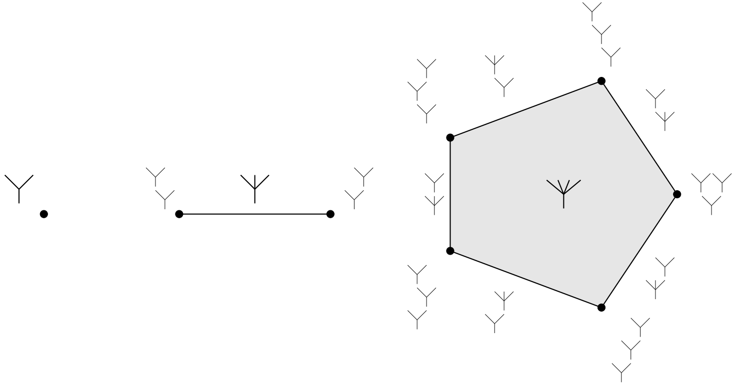


Figure: The associahedra K_2, K_3 and K_4, with cells labeled by the operations they define in A_∞

Corriger les dessins en mettant des jauges intersectant les sommets

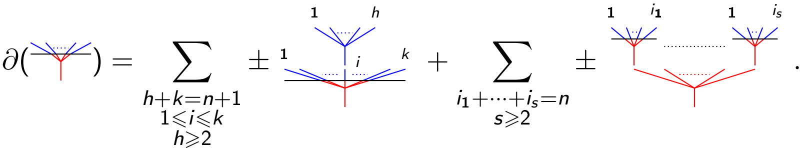
The $\mathtt{dg} - \mathbb{Z} - \mathtt{mod}$-operadic bimodule $A_\infty - \mathrm{Morph}$ also stems from a Poly-operadic bimodule. Work in progress : [MMV].

There exists a collection of polytopes, called the *multiplihedra* and denoted $\{J_n\}$, endowed with a structure of $(\{K_n\}, \{K_n\})$-operadic bimodule, whose image under the functor C_{-*}^{cell} yields the (A_∞, A_∞)-operadic bimodule $A_\infty - \mathrm{Morph}$.

Again, J_n has a unique $n-1$-dimensional cell $[J_n]$ whose image under ∂_{cell} is the A_∞-equation for A_∞-morphisms, that is such that

$$\partial_{cell}[J_n] = \sum \pm \circ_i ([J_k] \otimes [K_h]) + \sum \pm \mu([K_s] \otimes [J_{i_1}] \otimes \cdots \otimes [J_{i_s}]) .$$

Recall that the A_∞-equations read as



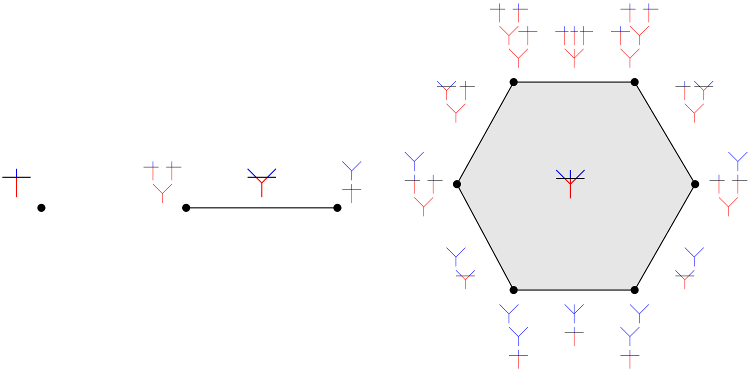
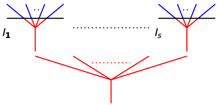


Figure: The multiplihedra J_1, J_2 and J_3 with cells labeled by the operations they define in $A_\infty - \mathrm{Morph}$

We would like to define a family of polytopes encoding n-morphisms between A_∞-algebras. These polytopes will then be called *n-multiplihedra*.

We have seen that A_∞-morphisms $\overline{T}(sA) \to \overline{T}(sB)$ are encoded by the multiplihedra. n-morphisms being defined as the set of morphisms $\mathbf{\Delta}^n \otimes \overline{T}(sA) \to \overline{T}(sB)$, a natural candidate would thus be $\{\Delta^n \times J_m\}_{m \geqslant 1}$.

However, $\Delta^n \times J_m$ does not fulfill that property as it is. Faces correspond to the data of a face of $I \subset \Delta^n$, and of a broken two-colored tree labeling a face of J_m. This labeling is too coarse, as it does not contain the trees



,

that appear in the A_∞-equations for n-morphisms.

We thus want to lift the combinatorics of overlapping partitions to the level of the n-simplices Δ^n.

Proposition ([Maz21b])

For each $s \geqslant 1$, there exists a polytopal subdivision of the standard n-simplex Δ^n whose top-dimensional cells are in one-to-one correspondence with all s-overlapping partitions of Δ^n.

Taking the realizations

$$\begin{aligned}\Delta^n &:= \mathrm{conv}\{(1,\dots,1,0,\dots,0) \in \mathbb{R}^n\} \\ &= \{(z_1,\dots,z_n) \in \mathbb{R}^n | 1 \geqslant z_1 \geqslant \cdots \geqslant z_n \geqslant 0\} ,\end{aligned}$$

this polytopal subdivision can be realized as the subdivision obtained after dividing Δ^n by all hyperplanes $z_i = (1/2)^k$, for $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant s$.

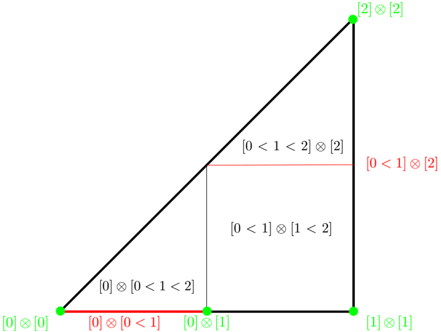


Figure: The subdivision of Δ^2 by overlapping 2-partitions

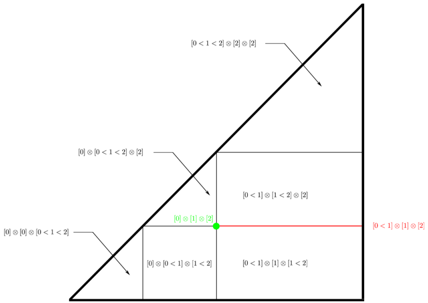


Figure: The subdivision of Δ^2 by overlapping 3-partitions

The previous issue can then be solved by constructing a thinner polytopal subdivision of $\Delta^n \times J_m$.

Consider a face F of J_m, with exactly s unbroken two-colored trees appearing in the two-colored broken tree labeling it. We refine the polytopal subdivision of $\Delta^n \times F$ into $\Delta^n_s \times F$, where Δ^n_s denotes Δ^n endowed with the subdivision encoding s-overlapping partitions.

This refinement process can be done consistently for each face F of J_m, in order to obtain a new polytopal subdivision of $\Delta^n \times J_m$.

Definition ([Maz21b])

The *n-multiplihedra* are defined to be the polytopes $\Delta^n \times J_m$ endowed with the previous polytopal subdivision. We denote them $n - J_m$.

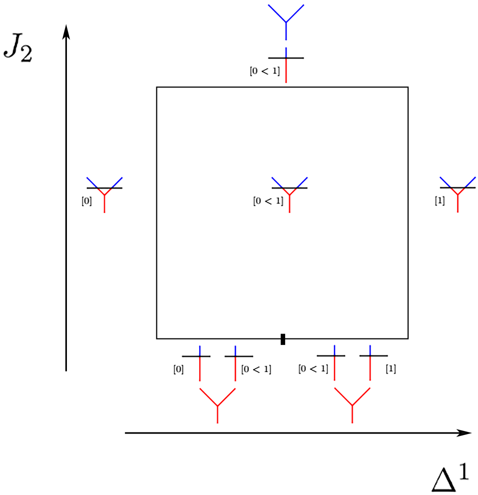


Figure: The 1-multiplihedron $\Delta^1 \times J_2$

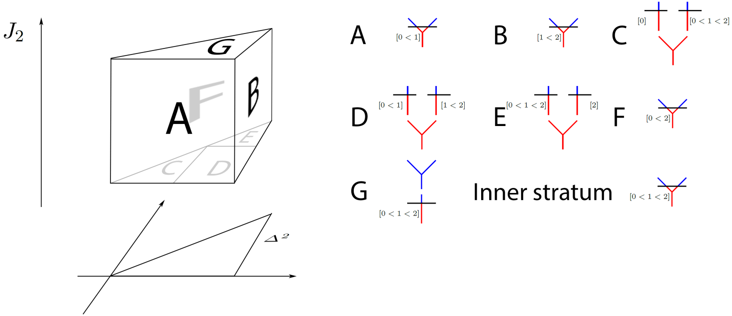


Figure: The 2-multiplihedron $\Delta^2 \times J_2$

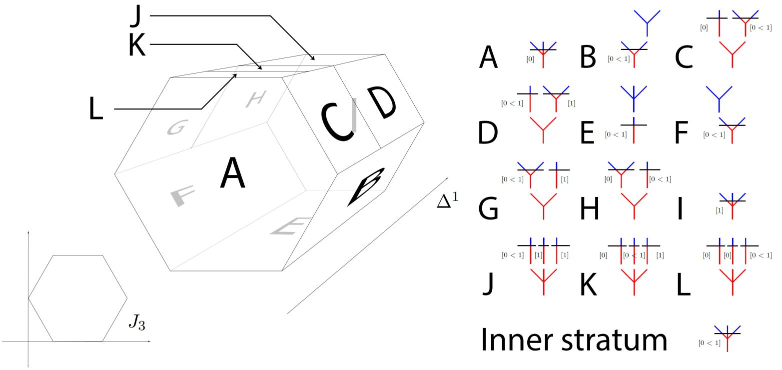
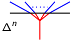


Figure: The 1-multiplihedron $\Delta^1 \times J_3$

The polytope $n - J_m$ has a unique $(n + m - 1)$-dimensional cell $[n - J_m]$, is labeled by 

. By construction :

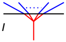
Proposition ([Maz21b])

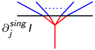
The boundary of the cell $[n - J_m]$ is given by

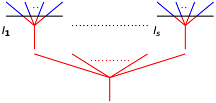
$$\partial^{sing}[n - J_m] \cup \bigcup_{\substack{h+k=m+1 \\ 1 \leqslant i \leqslant k \\ h \geqslant 2}} [n - J_k] \times_i [K_h] \cup \bigcup_{\substack{i_1+\cdots+i_s=m \\ I_1 \cup \cdots \cup I_s = \Delta^n \\ s \geqslant 2}} [K_s] \times [\dim(I_1) - J_{i_1}] \times \cdots \times [\dim(I_s) - J_{i_s}] \ ,$$

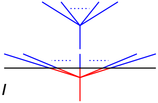
where $I_1 \cup \cdots \cup I_s = \Delta^n$ is an overlapping partition of Δ^n.

Recall that the $n - A_\infty$-equations read as

$\partial($ 

$) = \sum_{j=1}^{k} (-1)^j$ 

$+ \sum_{I_1 \cup \cdots \cup I_s = I} \pm$ 

$+ \sum \pm$ 

.

In other words, the n-multiplihedra encode n-morphisms between A_∞-algebras.

En parler plus - Parler également de ΩBAs

We prove in [Maz21a] and [Maz21b] that given two Morse functions f and g, one can construct n-morphisms between their Morse cochain complexes $C^*(f)$ and $C^*(g)$ through a count of moduli spaces of perturbed Morse gradient trees. This gives a realization of this higher algebra of A_∞-algebras in Morse theory.

It is also quite clear that given two compact symplectic manifolds M and N, one should be able to construct n-morphisms between their Fukaya categories $\mathrm{Fuk}(M)$ and $\mathrm{Fuk}(N)$ through counts of moduli spaces of quilted disks (under the correct technical assumptions).

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Thanks for your attention !

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