Higher algebra of A_∞-algebras and the n-multiplihedra

Thibaut Mazuir

IMJ-PRG - Sorbonne Université

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The results presented in this talk are taken from my two recent papers : *Higher algebra of A_∞ and ΩBAs-algebras in Morse theory I* (arXiv:2102.06654) and *Higher algebra of A_∞ and ΩBAs-algebras in Morse theory II* (arxiv:2102.08996).

The talk will be divided in three parts : quick recollections on A_∞-algebras and A_∞-morphisms ; definition of *higher morphisms between A_∞-algebras*, or *$n - A_\infty$-morphisms*, and their properties ; definition of the *n-multiplihedra*, which are new families of polytopes generalizing the standard multiplihedra and which encode $n - A_\infty$-morphisms between A_∞-algebras.

1. A_∞-algebras and A_∞-morphisms
2. Higher algebra of A_∞-algebras
3. The n-multiplihedra

Suspension : Let A be a graded $\mathbb{Z}$-module. We denote sA, the *suspension of* A to be the graded $\mathbb{Z}$-module defined by $(sA)^i := A^{i-1}$. In other words, for $a \in A$, $|sa| = |a| - 1$.
For instance, a degree $2-n$ map $A^{\otimes n} \to A$ is equivalent to a degree $+1$ map $(sA)^{\otimes n} \to sA$.

Cohomological conventions : differentials will have degree $+1$.

1. A_∞-algebras and A_∞-morphisms
   - A_∞-algebras
   - A_∞-morphisms
   - Homotopy theory of A_∞-algebras
2. Higher algebra of A_∞-algebras
3. The n-multiplihedra

Definition

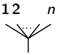
Let A be a dg-$\mathbb{Z}$-module with differential m_1. An A_∞-algebra structure on A is the data of a collection of maps of degree $2-n$

$$m_n : A^{\otimes n} \longrightarrow A \ , \ \ n \geqslant 1,$$

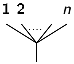
extending m_1 and which satisfy

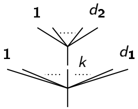
$$[m_1, m_n] = \sum_{\substack{i_1+i_2+i_3=n \\ 2\leqslant i_2 \leqslant n-1}} \pm m_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}).$$

These equations are called the A_∞-equations.

Representing m_n as 

, these equations can be written as

$[m_1,$ 

$] = \sum_{\substack{h+k=n+1 \\ 2\leqslant h\leqslant n-1 \\ 1\leqslant i\leqslant k}} \pm$ 

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In particular,

$$[m_1, m_2] = 0 \ ,$$
$$[m_1, m_3] = m_2(\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) \ ,$$

implying that m_2 descends to an associative product on $H^*(A)$. An A_∞-algebra is thus simply a correct notion of a dg-algebra whose product is associative up to homotopy.

The operations m_n are the higher coherent homotopies which keep track of the fact that the product is associative up to homotopy.

Using the universal property of the bar construction, we have the following one-to-one correspondence

$$\left\{ \begin{array}{c} \text{collections of morphisms of degree } 2-n \\ m_n : A^{\otimes n} \to A \ , \ \ n \geqslant 1, \\ \text{satisfying the } A_\infty\text{-equations} \end{array} \right\}$$

$$\longleftrightarrow \left\{ \begin{array}{c} \text{coderivations } D \text{ of degree } +1 \text{ of } \overline{T}(sA) \\ \text{such that } D^2 = 0 \end{array} \right\} .$$

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Definition

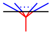
An A_∞*-morphism* between two A_∞-algebras A and B is a dg-coalgebra morphism $F : (\overline{T}(sA), D_A) \to (\overline{T}(sB), D_B)$ between their shifted bar constructions.

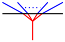
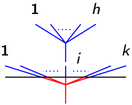
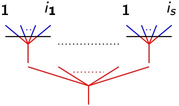
As previously, the universal property of the bar construction yields an equivalent definition in terms of operations.

Definition

An A_∞-*morphism* between two A_∞-algebras A and B is a family of maps $f_n : A^{\otimes n} \to B$ of degree $1-n$ satisfying

$$[m_1, f_n] = \sum_{\substack{i_1+i_2+i_3=n \\ i_2 \geqslant 2}} \pm f_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) + \sum_{\substack{i_1+\cdots+i_s=n \\ s \geqslant 2}} \pm m_s(f_{i_1} \otimes \cdots \otimes f_{i_s}) \ .$$

Representing the operations f_n as , the operations m_n^A in red and the operations m_n^B in blue, these equations read as

$\left[\partial,\right.$  $\left.\right] = \sum_{\substack{h+k=n+1 \\ 1\leqslant i\leqslant k \\ h\geqslant 2}} \pm$  $+ \sum_{\substack{i_1+\cdots+i_s=n \\ s\geqslant 2}} \pm$  .

We check that $[\partial, f_2] = f_1 m_2^A - m_2^B(f_1 \otimes f_1)$.

An A_∞-morphism between A_∞-algebras induces a morphism of associative algebras on the level of cohomology, and is a correct notion of morphism which preserves the product up to homotopy.

Given two coalgebra morphisms $F : \overline{T}V \to \overline{T}W$ and $G : \overline{T}W \to \overline{T}Z$, the family of morphisms associated to $G \circ F$ is given by

$$(G \circ F)_n := \sum_{i_1 + \cdots + i_s = n} \pm g_s(f_{i_1} \otimes \cdots \otimes f_{i_s}) \ .$$

This formula defines the composition of A_∞-morphisms. Hence, A_∞-algebras together with A_∞-morphisms form a category, denoted $\mathtt{A}_\infty - \mathtt{alg}$. This category can be seen as a full subcategory of $\mathtt{dg} - \mathtt{Cogc}$ of cocomplete dg-coalgebras, using the shifted bar construction viewpoint.

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The category $\mathtt{A}_\infty - \mathtt{alg}$ provides a framework that behaves well with respect to homotopy-theoretic constructions, when studying homotopy theory of associative algebras. See for instance [LH02] and [Val20].

It is because this category is encoded by the two-colored operad

$$A_\infty^2 := \mathcal{F}($$  $$) \ .$$

It is a quasi-free object in the model category of two-colored operads in dg-$\mathbb{Z}$-modules and a fibrant-cofibrant replacement of the two-colored operad As^2, which encodes associative algebras with morphisms of algebras,

$$A_\infty^2 \xrightarrow{\sim} As^2 \ .$$

Theorem (Homotopy transfer theorem)

Let (A, ∂_A) and (H, ∂_H) be two cochain complexes. Suppose that H is a deformation retract of A, that is that they fit into a diagram

$$h \circlearrowright (A, \partial_A) \underset{i}{\overset{p}{\rightleftarrows}} (H, \partial_H) \ ,$$

where $\mathrm{id}_A - ip = [\partial, h]$ and $pi = \mathrm{id}_H$. Then if (A, ∂_A) is endowed with an associative algebra structure, H can be made into an A_∞-algebra such that i and p extend to A_∞-morphisms, that are then A_∞-quasi-isomorphisms.

1 A_∞-algebras and A_∞-morphisms

2 Higher algebra of A_∞-algebras

3 The n-multiplihedra

Our goal now : study the *higher algebra of A_∞-algebras*.

Considering two A_∞-morphisms F, G, we would like first to determine a notion giving a satisfactory meaning to the sentence "F and G are homotopic". Then, A_∞-homotopies being defined, what is now a good notion of a homotopy between homotopies ? And of a homotopy between two homotopies between homotopies ? And so on.

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   - A simplicial enrichment of the category $\mathtt{A}_\infty - \mathtt{alg}$?

3. The n-multiplihedra

Start with a notion of homotopy. Drawn from [LH02].

Take C and C' two dg-coalgebras, F and G morphisms $C \to C'$ of dg-coalgebras. A (F,G)-*coderivation* is a map $H : C \to C'$ such that

$$\Delta_{C'} H = (F \otimes H + H \otimes G)\Delta_C \ .$$

The morphisms F and G are then said to be *homotopic* if there exists a (F,G)-coderivation H of degree -1 such that

$$[\partial, H] = G - F \ .$$

Define

$$\mathbf{\Delta}^1 := \mathbb{Z}[0] \oplus \mathbb{Z}[1] \oplus \mathbb{Z}[0<1] \ ,$$

with differential ∂^{sing}

$$\partial^{sing}([0<1]) = [1]-[0] \qquad \partial^{sing}([0]) = 0 \qquad \partial^{sing}([1]) = 0 \ ,$$

and coproduct the Alexander-Whitney coproduct

$$\begin{aligned} \Delta_{\mathbf{\Delta}^1}([0<1]) &= [0]\otimes[0<1] + [0<1]\otimes[1] \\ \Delta_{\mathbf{\Delta}^1}([0]) &= [0]\otimes[0] \\ \Delta_{\mathbf{\Delta}^1}([1]) &= [1]\otimes[1] \ . \end{aligned}$$

The elements [0] and [1] have degree 0, and the element $[0<1]$ has degree -1.

We check that there is a one-to-one correspondence between (F, G)-coderivations and morphisms of dg-coalgebras $\mathbf{\Delta}^1 \otimes C \longrightarrow C'$.

Definition

For two A_∞-algebras $(\overline{T}(sA), D_A)$ and $(\overline{T}(sB), D_B)$ and two A_∞-morphisms $F, G : (\overline{T}(sA), D_A) \to (\overline{T}(sB), D_B)$, an *$A_\infty$-homotopy from F to G* is defined to be a morphism of dg-coalgebras

$$H : \mathbf{\Delta}^1 \otimes \overline{T}(sA) \longrightarrow \overline{T}(sB) \ ,$$

whose restriction to the [0] summand is F and whose restriction to the [1] summand is G.

Using the universal property of the bar construction, this definition can be rephrased in terms of operations.

Definition

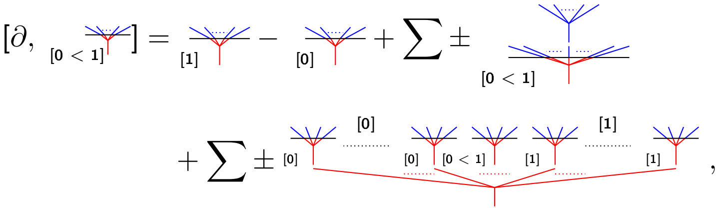
An A_∞-*homotopy* between two A_∞-morphisms $(f_n)_{n\geqslant 1}$ and $(g_n)_{n\geqslant 1}$ is a collection of maps

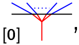
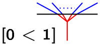
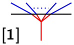
$$h_n : A^{\otimes n} \longrightarrow B \ ,$$

of degree $-n$, satisfying

$$[\partial, h_n] = g_n - f_n + \sum_{\substack{i_1+i_2+i_3=m \\ i_2\geqslant 2}} \pm h_{i_1+1+i_3}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) + \sum_{\substack{i_1+\cdots+i_s+l \\ +j_1+\cdots+j_t=n \\ s+1+t\geqslant 2}} \pm m_{s+1+t}(f_{i_1} \otimes \cdots \otimes f_{i_s} \otimes h_l \otimes g_{j_1} \otimes \cdots \otimes g_{j_t}) \ .$$

In symbolic formalism,



where we denote ,  and  respectively for the f_n, the h_n and the g_n.

The relation *being A_∞-homotopic* on the class of A_∞-morphisms is an equivalence relation. It is moreover stable under composition.

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   - A simplicial enrichment of the category $\mathtt{A}_\infty - \mathtt{alg}$?
3. The n-multiplihedra

Move on to n-morphisms between A_∞-algebras.

Define $\mathbf{\Delta}^n$ the graded $\mathbb{Z}$-module generated by the faces of the standard n-simplex Δ^n,

$$\mathbf{\Delta}^n = \bigoplus_{0 \leqslant i_1 < \cdots < i_k \leqslant n} \mathbb{Z}[i_1 < \cdots < i_k] \ .$$

The grading is $|I| := -\dim(I)$ for $I \subset \Delta^n$.

It has a dg-coalgebra structure, with differential

$$\partial_{\mathbf{\Delta}^n}([i_1 < \cdots < i_k]) := \sum_{j=1}^{k} (-1)^j [i_1 < \cdots < \widehat{i_j} < \cdots < i_k] \ ,$$

and coproduct the Alexander-Whitney coproduct

$$\Delta_{\mathbf{\Delta}^n}([i_1 < \cdots < i_k]) := \sum_{j=1}^{k} [i_1 < \cdots < i_j] \otimes [i_j < \cdots < i_k] \ .$$

Definition ([MS03])

Let I be a face of Δ^n. An *overlapping partition* of I to be a sequence of faces $(I_l)_{1 \leqslant \ell \leqslant s}$ of I such that

(i) the union of this sequence of faces is I, i.e. $\cup_{1 \leqslant \ell \leqslant s} I_l = I$;

(ii) for all $1 \leqslant \ell < s$, $\max(I_\ell) = \min(I_{\ell+1})$.

An overlapping 6-partition for $[0 < 1 < 2]$ is for instance

$$[0 < 1 < 2] = [0] \cup [0] \cup [0 < 1] \cup [1] \cup [1 < 2] \cup [2] \ .$$

Overlapping partitions are the collection of faces which naturally arise in the Alexander-Whitney coproduct.

The element $\Delta_{\mathbf{\Delta}^n}(I)$ corresponds to the sum of all overlapping 2-partitions of I. Iterating s times $\Delta_{\mathbf{\Delta}^n}$ yields the sum of all overlapping $(s+1)$-partitions of I.

We have seen that A_∞-morphisms correspond to the set

$$\mathrm{Hom}_{\mathrm{dg-Cogc}}(\overline{T}(sA), \overline{T}(sB))$$

and A_∞-homotopies correspond to the set

$$\mathrm{Hom}_{\mathrm{dg-Cogc}}(\mathbf{\Delta}^1 \otimes \overline{T}(sA), \overline{T}(sB)) \ ,$$

Definition ([Maz21b])

We define the set of n-morphisms between A and B as

$$\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A,B)_n := \mathrm{Hom}_{\mathrm{dg-Cogc}}(\mathbf{\Delta}^n \otimes \overline{T}(sA), \overline{T}(sB)) \ .$$

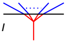
Using the universal property of the bar construction, n-morphisms admit a nice combinatorial description in terms of operations.

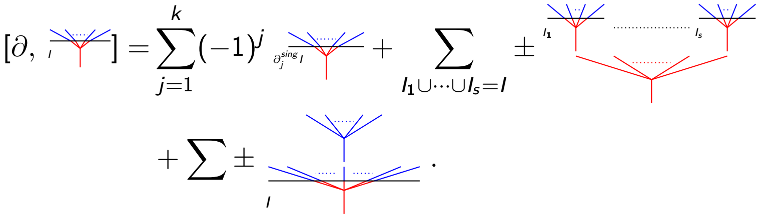
Definition ([Maz21b])

A *n-morphism* from A to B is defined to be a collection of maps $f_I^{(m)} : A^{\otimes m} \longrightarrow B$ of degree $1 - m + |I|$ for $I \subset \Delta^n$ and $m \geqslant 1$, that satisfy

$$\left[\partial, f_I^{(m)}\right] = \sum_{j=0}^{\dim(I)} (-1)^j f_{\partial_j I}^{(m)} + \sum_{\substack{i_1+\cdots+i_s=m \\ I_1\cup\cdots\cup I_s=I \\ s\geqslant 2}} \pm m_s(f_{I_1}^{(i_1)} \otimes \cdots \otimes f_{I_s}^{(i_s)})$$

$$+ (-1)^{|I|} \sum_{\substack{i_1+i_2+i_3=m \\ i_2\geqslant 2}} \pm f_I^{(i_1+1+i_3)}(\mathrm{id}^{\otimes i_1} \otimes m_{i_2} \otimes \mathrm{id}^{\otimes i_3}) \; .$$

Equivalently and more visually, a collection of maps  satisfying



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3. The n-multiplihedra

The dg-coalgebras $\mathbf{\Delta}^\bullet := \{\mathbf{\Delta}^n\}_{n \geqslant 0}$ naturally form a cosimplicial dg-coalgebra.

The sets $\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A, B)_n$ then fit into a HOM-simplicial set $\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A, B)_\bullet$. This HOM-simplicial set provides a satisfactory framework to study the higher algebra of A_∞-algebras.

Theorem ([Maz21b])

For A and B two A_∞-algebras, the simplicial set $\mathrm{HOM}_{A_\infty}(A, B)_\bullet$ is a Kan complex.

Proof : Let $\mathcal{C}$ be a model category and $C \in \mathcal{C}$. A *cosimplicial resolution* of C is defined to be a cofibrant approximation $\boldsymbol{C}$ of $const^* C$ in the Reedy model category $\mathcal{C}^\Delta$.

In other words, it is the data of a cosimplicial object $\boldsymbol{C} := \{C^n\}_{n \geqslant 0}$ of $\mathcal{C}$ together with a cosimplicial morphism $\boldsymbol{C} \to const^* C$, such that the maps $C^n \to C$ are weak equivalences in $\mathcal{C}$ and the latching maps $L_n \boldsymbol{C} \to C^n$ are cofibrations in $\mathcal{C}$.

Lemma (Lemma 16.5.3 of [Hir03])

If $\mathbf{C} \to const^* C$ is a cosimplicial resolution in $\mathcal{C}$ and D is a fibrant object of $\mathcal{C}$, then the simplicial set $\mathcal{C}(\mathbf{C}, D)$ is a Kan complex.

It is thus enough to prove that the cosimplicial cocomplete dg-coalgebra $\boldsymbol{C} := \{\boldsymbol{\Delta}^n \otimes \overline{T}(sA)\}_{n\geqslant 0}$ is a cosimplicial replacement of $\overline{T}(sA)$ and then apply the previous lemma in the model category $dg - Cogc$ defined in [LH02].

In other words, we have to prove that :

(i) the latching maps $L_n\boldsymbol{C} \to C^n = \boldsymbol{\Delta}^n \otimes \overline{T}(sA)$ are cofibrations, i.e. they are injective ;

(ii) the maps $p \otimes \mathrm{Id}_{\overline{T}(sA)} : \boldsymbol{\Delta}^n \otimes \overline{T}(sA) \to \boldsymbol{\Delta}^0 \otimes \overline{T}(sA) = \overline{T}(sA)$ are weak equivalences in the model category $dg - Cogc$, where $p : \boldsymbol{\Delta}^n \to \boldsymbol{\Delta}^0$ is the map collapsing the simplex Δ^n on one of its vertices.

Proposition

For every inner horn $\Lambda_n^k \subset \Delta^n$, there is a one-to-one correspondence

$$\left\{ \textit{fillers} \quad \begin{array}{ccc} \Lambda_n^k & \longrightarrow & \mathrm{HOM}_{A_\infty}(A,B)_\bullet \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array} \right\}$$

$$\longleftrightarrow \left\{ \begin{array}{c} \textit{families of maps of degree } 1-m-n \\ f_{\Delta^n}^{(m)} : A^{\otimes m} \to B, \ m \geqslant 1 \end{array} \right\} .$$

An inner horn $\Lambda_n^k \to \mathrm{HOM}_{A_\infty}(A,B)_\bullet$ corresponds to a collection of degree $1-m-\dim(I)$ morphisms $f_I^{(m)} : A^{\otimes m} \longrightarrow B$ for $I \subset \Lambda_n^k$ which satisfy the A_∞-equations for higher morphisms.

The previous proposition then states that filling the horn $\Lambda_n^k \subset \Delta^n$ amounts to choosing an arbitrary collection of degree $1-m-n$ morphisms $f_{\Delta^n}^{(m)} : A^{\otimes m} \to B$ and that they completely determine the collection of morphisms for the missing face $f^{(m)}_{[0<\cdots<\hat{k}<\cdots<n]}$.

The simplicial homotopy groups of the Kan complex $\mathrm{HOM}_{A_\infty}(A,B)_\bullet$ can moreover be explicitly computed. We let $F = (F^{(m)} : (sA)^{\otimes m} \to sB)_{m \geqslant 1}$ be an A_∞-morphism from A to B, i.e. a point of $\mathrm{HOM}_{A_\infty}(A,B)_\bullet$.

The set of path components $\pi_0\left(\mathrm{HOM}_{A_\infty}(A,B)_\bullet\right)$ corresponds to the set of equivalence classes of A_∞-morphisms from A to B under the equivalence relation "being A_∞-homotopic".

For $n \geqslant 1$, the set $\pi_n\left(\mathrm{HOM}_{A_\infty}(A,B)_\bullet, F\right)$ corresponds to the equivalence classes of collections of degree $-n$ maps $F_{\Delta^n}^{(m)} : (sA)^{\otimes m} \to sB$ satisfying equations

$$(-1)^n \sum_{i_1+i_2+i_3=m} F_{\Delta^n}^{(i_1+1+i_3)}\left(\mathrm{id}^{\otimes i_1} \otimes b_{i_2} \otimes \mathrm{id}^{\otimes i_3}\right)$$
$$= \sum_{\substack{i_1+\cdots+i_s+l \\ +j_1+\cdots+j_t=m}} b_{s+1+t}\left(F^{(i_1)} \otimes \cdots \otimes F^{(i_s)} \otimes F_{\Delta^n}^{(l)} \otimes F^{(j_1)} \otimes \cdots \otimes F^{(j_t)}\right) .$$

Two such collections of maps $(F^{(m)}_{\Delta^n})^{m\geqslant 1}$ and $(G^{(m)}_{\Delta^n})^{m\geqslant 1}$ are equivalent if and only if there exists a collection of degree $-(n+1)$ maps $H^{(m)} : (sA)^{\otimes m} \to sB$ such that

$$G^{(m)}_{\Delta^n} - F^{(m)}_{\Delta^n} + (-1)^{n+1} \sum_{i_1+i_2+i_3=m} H^{(i_1+1+i_3)}(\mathrm{id}^{\otimes i_1} \otimes b_{i_2} \otimes \mathrm{id}^{\otimes i_3})$$

$$= \sum_{\substack{i_1+\cdots+i_s+l \\ +j_1+\cdots+j_t=m}} b_{s+1+t}(F^{(i_1)} \otimes \cdots \otimes F^{(i_s)} \otimes H^{(l)} \otimes F^{(j_1)} \otimes \cdots \otimes F^{(j_t)}) \ .$$

(i) The composition law on $\pi_1(\mathrm{HOM}_{A_\infty}(A, B)_\bullet, F)$ is given by the formula

$$G_{\Delta^1}^{(m)} + F_{\Delta^1}^{(m)} - \sum_{\substack{i_1+\cdots+i_s+l_1 \\ +j_1+\cdots+j_t+l_2 \\ +k_1+\cdots+k_u=m}} b_{s+t+u+2}(F^{(i_1)} \otimes \cdots \otimes F^{(i_s)} \otimes F_{\Delta^1}^{(l_1)} \otimes F^{(j_1)} \otimes \cdots \otimes F^{(j_t)} \otimes G_{\Delta^1}^{(l_2)} \otimes F^{(k_1)} \otimes \cdots \otimes F^{(k_u)})$$

(ii) If $n \geqslant 2$, the composition law on $\pi_n(\mathrm{HOM}_{A_\infty}(A, B)_\bullet, F)$ is given by the formula

$$G_{\Delta^n}^{(m)} + F_{\Delta^n}^{(m)} \ .$$

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3 The n-multiplihedra

We would like to see the simplicial sets $\mathrm{HOM}_{\mathtt{A}_\infty-\mathtt{alg}}(A,B)_\bullet$ as part of a simplicial enrichment of the category $\mathtt{A}_\infty - \mathtt{alg}$. In other words, we would like to define simplicial maps

$$\mathrm{HOM}_{\mathtt{A}_\infty-\mathtt{alg}}(A,B)_n \times \mathrm{HOM}_{\mathtt{A}_\infty-\mathtt{alg}}(B,C)_n \longrightarrow \mathrm{HOM}_{\mathtt{A}_\infty-\mathtt{alg}}(A,C)_n \ ,$$

lifting the composition on the $\mathrm{HOM}_0 = \mathrm{Hom}$.

This would then endow $\mathtt{A}_\infty - \mathtt{alg}$ with a structure of ∞-category.

All the natural approaches to lift the composition in $\mathtt{A}_\infty - \mathtt{alg}$ to $\mathrm{HOM}_{\mathrm{A}_\infty-\mathrm{alg}}(A,B)_\bullet$ fail to work. Hence, it is still an open question to know whether these HOM-simplicial sets could fit into a simplicial enrichment of the category $\mathtt{A}_\infty - \mathtt{alg}$. In fact, it is unclear to the author why such a statement should be true.

1 A_∞-algebras and A_∞-morphisms

2 Higher algebra of A_∞-algebras

3 The n-multiplihedra

1 A_∞-algebras and A_∞-morphisms

2 Higher algebra of A_∞-algebras

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We define Poly to be the category of polytopes defined in [MTTV19]. Its objects are standard polytopes but beware that the morphisms of this category are not the usual affine maps. It forms a monoidal category with product the usual cartesian product, and a monoidal subcategory of CW.

The cellular chain functor $C_*^{cell} : \mathtt{Poly} \to \mathtt{dg} - \mathbb{Z} - \mathtt{mod}$ then satisfies

$$C_*^{cell}(P \times Q) = C_*^{cell}(P) \otimes C_*^{cell}(Q) \ .$$

We will in fact work with the functor

$$C_{-*}^{cell} : \mathtt{CW} \longrightarrow \mathtt{dg} - \mathbb{Z} - \mathtt{mod} \ ,$$

where $C_{-*}^{cell}(P)$ is simply the $\mathbb{Z}$-module $C_*^{cell}(P)$ taken with its opposite grading.

Definition

Given P and Q two operads seen as their Schur functors S_P and S_Q, let $R = \{R_n\}$ be a $\mathbb{N}$-module of $\mathcal{C}$ seen as its Schur functor S_R. A (P, Q)-*operadic bimodule structure* on R is a (S_P, S_Q)-bimodule structure $\lambda : S_P \circ S_R \to S_R$ and $\mu : S_R \circ S_Q \to S_R$ on S_R in $(\mathrm{End}(\mathcal{C}), \circ, Id_{\mathcal{C}})$.

Equivalently, a (P, Q)-*operadic bimodule structure* on R is the data of action-composition maps

$$R_k \otimes Q_{i_1} \otimes \cdots \otimes Q_{i_k} \underset{\mu_{i_1,\dots,i_k}}{\longrightarrow} R_{i_1+\cdots+i_k} ,$$

$$P_h \otimes R_{j_1} \otimes \cdots \otimes R_{j_h} \underset{\lambda_{j_1,\dots,j_h}}{\longrightarrow} R_{j_1+\cdots+j_h} ,$$

which are compatible with one another, with identities, and with compositions in P and Q.

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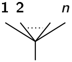
The $\mathtt{dg}-\mathbb{Z}-\mathtt{mod}$-operad A_∞ encoding A_∞-algebras stems from a Poly-operad. This was fully proven in [MTTV19].

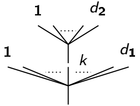
There exists a collection of polytopes, called the *associahedra* and denoted $\{K_n\}$, endowed with a structure of operad in the category Poly and whose image under the functor C_{-*}^{cell} yields the operad A_∞.

In particular K_n has a unique cell $[K_n]$ of dimension $n-2$ whose image under ∂_{cell} is the A_∞-equation, that is such that

$$\partial_{cell}[K_n] = \sum \pm \circ_i ([K_k] \otimes [K_h]) \ .$$

Recall that the A_∞-equations read as

$\partial($ 

$) = \sum_{\substack{h+k=n+1 \\ 2 \leqslant h \leqslant n-1 \\ 1 \leqslant i \leqslant k}} \pm$ 

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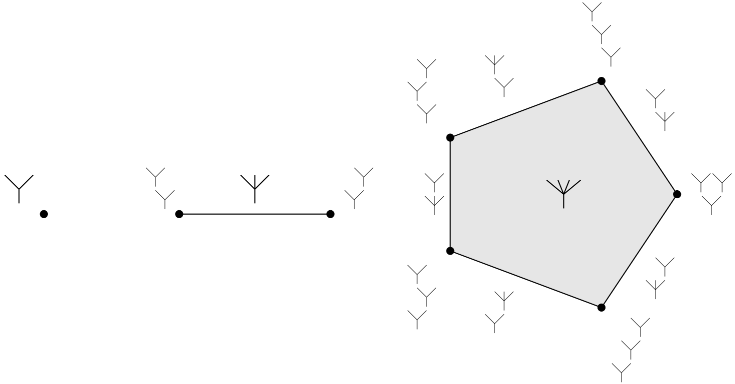

Figure: The associahedra K_2, K_3 and K_4, with cells labeled by the operations they define in A_∞

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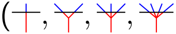
Define $A_\infty - \mathrm{Morph}$ to be quasi-free (A_∞, A_∞)-operadic bimodule encoding A_∞-morphisms between A_∞-algebras

$$A_\infty - \mathrm{Morph} = \mathcal{F}^{A_\infty, A_\infty}($$

$$, \cdots) \ .$$

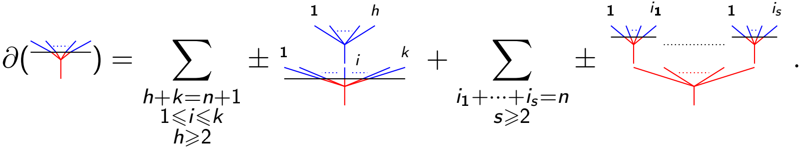
This operadic bimodule also stems from a Poly-operadic bimodule. Work in progress : [MMV].

There exists a collection of polytopes, called the *multiplihedra* and denoted $\{J_n\}$, endowed with a structure of $(\{K_n\}, \{K_n\})$-operadic bimodule, whose image under the functor C_{-*}^{cell} yields the (A_∞, A_∞)-operadic bimodule $A_\infty - \mathrm{Morph}$.

Again, J_n has a unique $n-1$-dimensional cell $[J_n]$ whose image under ∂_{cell} is the A_∞-equation for A_∞-morphisms, that is such that

$$\partial_{cell}[J_n] = \sum \pm \circ_i ([J_k] \otimes [K_h]) + \sum \pm \mu([K_s] \otimes [J_{i_1}] \otimes \cdots \otimes [J_{i_s}]) \ .$$

Recall that the A_∞-equations read as



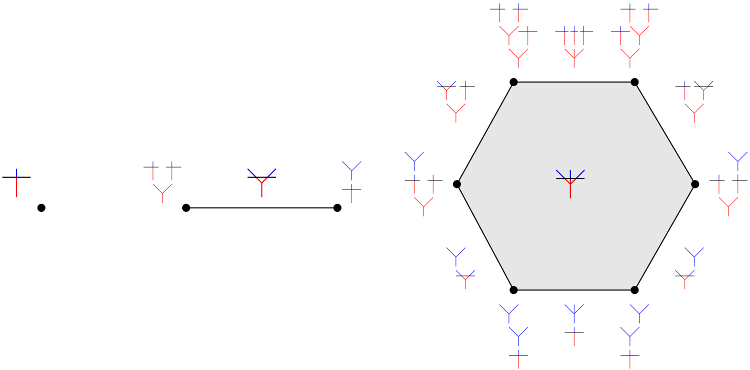


Figure: The multiplihedra J_1, J_2 and J_3 with cells labeled by the operations they define in $A_\infty - \mathrm{Morph}$

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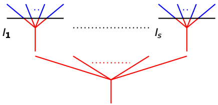
We would like to define a family of polytopes encoding n-morphisms between A_∞-algebras. These polytopes will then be called *n-multiplihedra*.

We have seen that A_∞-morphisms $\overline{T}(sA) \to \overline{T}(sB)$ are encoded by the multiplihedra. n-morphisms being defined as the set of morphisms $\mathbf{\Delta}^n \otimes \overline{T}(sA) \to \overline{T}(sB)$, a natural candidate would thus be $\{\Delta^n \times J_m\}_{m \geqslant 1}$.

However, $\Delta^n \times J_m$ does not fulfill that property as it is. Faces correspond to the data of a face of $I \subset \Delta^n$, and of a broken two-colored tree labeling a face of J_m. This labeling is too coarse, as it does not contain the trees



,

that appear in the A_∞-equations for n-morphisms.

We thus want to lift the combinatorics of overlapping partitions to the level of the n-simplices Δ^n.

Proposition ([Maz21b])

For each $s \geqslant 1$, there exists a polytopal subdivision of the standard n-simplex Δ^n whose top-dimensional cells are in one-to-one correspondence with all s-overlapping partitions of Δ^n.

Taking the realizations

$$\begin{aligned}\Delta^n &:= \mathrm{conv}\{(1,\dots,1,0,\dots,0) \in \mathbb{R}^n\} \\ &= \{(z_1,\dots,z_n) \in \mathbb{R}^n | 1 \geqslant z_1 \geqslant \cdots \geqslant z_n \geqslant 0\} \ ,\end{aligned}$$

this polytopal subdivision can be realized as the subdivision obtained after dividing Δ^n by all hyperplanes $z_i = (1/2)^k$, for $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant s$.

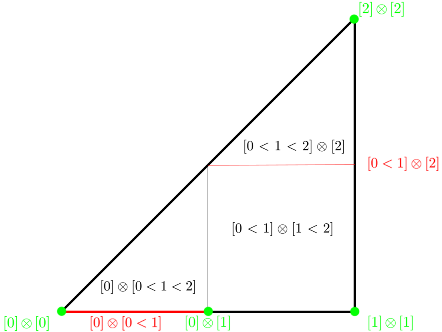


Figure: The subdivision of Δ^2 by overlapping 2-partitions

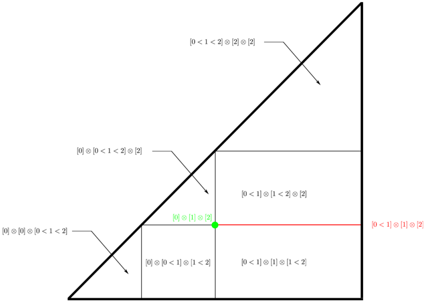


Figure: The subdivision of Δ^2 by overlapping 3-partitions

The previous issue can then be solved by constructing a thinner polytopal subdivision of $\Delta^n \times J_m$.

Consider a face F of J_m, with exactly s unbroken two-colored trees appearing in the two-colored broken tree labeling it. We refine the polytopal subdivision of $\Delta^n \times F$ into $\Delta^n_s \times F$, where Δ^n_s denotes Δ^n endowed with the subdivision encoding s-overlapping partitions.

This refinement process can be done consistently for each face F of J_m, in order to obtain a new polytopal subdivision of $\Delta^n \times J_m$.

Definition ([Maz21b])

The *n-multiplihedra* are defined to be the polytopes $\Delta^n \times J_m$ endowed with the previous polytopal subdivision. We denote them $n - J_m$.

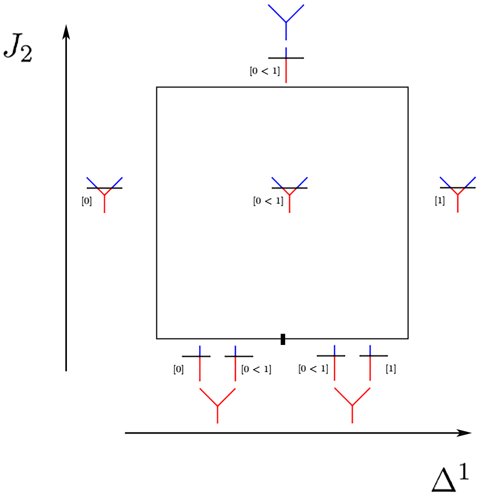


Figure: The 1-multiplihedron $\Delta^1 \times J_2$

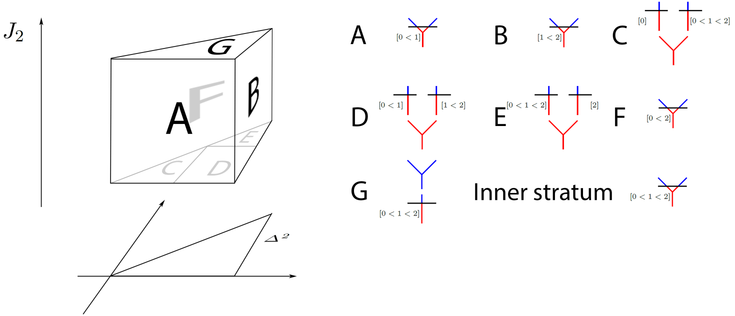


Figure: The 2-multiplihedron $\Delta^2 \times J_2$

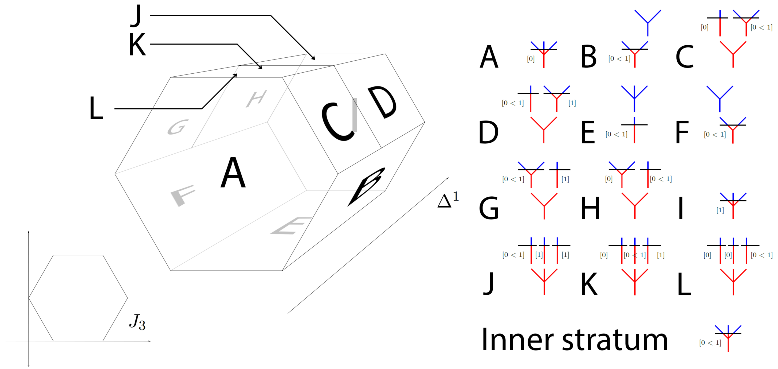
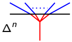


Figure: The 1-multiplihedron $\Delta^1 \times J_3$

The polytope $n - J_m$ has a unique $(n + m - 1)$-dimensional cell $[n - J_m]$, is labeled by 

. By construction :

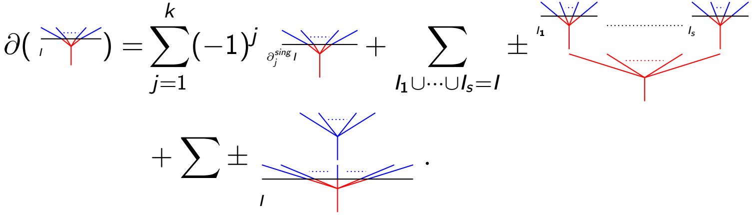
Proposition ([Maz21b])

The boundary of the cell $[n - J_m]$ is given by

$$\partial^{sing}[n - J_m] \cup \bigcup_{\substack{h+k=m+1 \\ 1 \leqslant i \leqslant k \\ h \geqslant 2}} [n - J_k] \times_i [K_h] \cup \bigcup_{\substack{i_1+\cdots+i_s=m \\ I_1 \cup \cdots \cup I_s = \Delta^n \\ s \geqslant 2}} [K_s] \times [\mathrm{dim}(I_1) - J_{i_1}] \times \cdots \times [\mathrm{dim}(I_s) - J_{i_s}] \ ,$$

where $I_1 \cup \cdots \cup I_s = \Delta^n$ is an overlapping partition of Δ^n.

Recall that the $n - A_\infty$-equations read as



In other words, the n-multiplihedra encode n-morphisms between A_∞-algebras.

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Let M be an oriented closed Riemannian manifold endowed with a Morse function f together with a Morse-Smale metric. The Morse cochains $C^*(f)$ form a deformation retract of the singular cochains $C^*_{sing}(M)$ as shown in [Hut08].

$$h \circlearrowright (C^*_{sing}, \partial_{sing}) \underset{i}{\overset{p}{\rightleftarrows}} (C^*(f), \partial_{Morse}) .$$

The cup product naturally endows the singular cochains $C^*_{sing}(M)$ with a dg-algebra structure. The homotopy transfer theorem ensures that it can be transferred to an A_∞-algebra structure on the Morse cochains $C^*(f)$.

The differential on the Morse cochains is defined by a count of moduli spaces of gradient trajectories. Is it then possible to define higher multiplications m_n on $C^*(f)$ by a count of moduli spaces such that they fit in a structure of A_∞-algebra ?

Question solved for the first time by Abouzaid in [Abo11], drawing from earlier works by Fukaya ([Fuk97] for instance). See also [Mes18] and [AL18]. In [Maz21a] I prove that this A_∞-algebra structure actually stems from an ΩBAs-algebra structure, but I will not dwell on that notion today.

We prove in [Maz21a] and [Maz21b] that given two Morse functions f and g, one can in fact construct n-morphisms between their Morse cochain complexes $C^*(f)$ and $C^*(g)$ through a count of geometric moduli spaces of perturbed Morse gradient trees. This gives a realization of this higher algebra of A_∞-algebras in Morse theory.

These constructions stem from the fact that the associahedra can be realized as the compactified moduli spaces of stable metric ribbon trees and the multiplihedra can be realized as the compactified moduli spaces of stable two-colored metric ribbon trees.

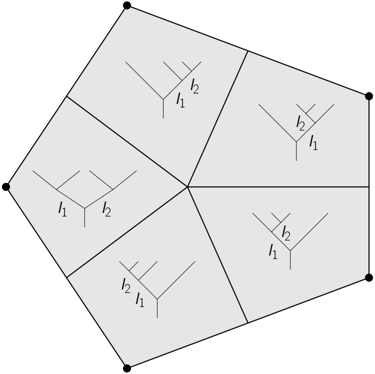
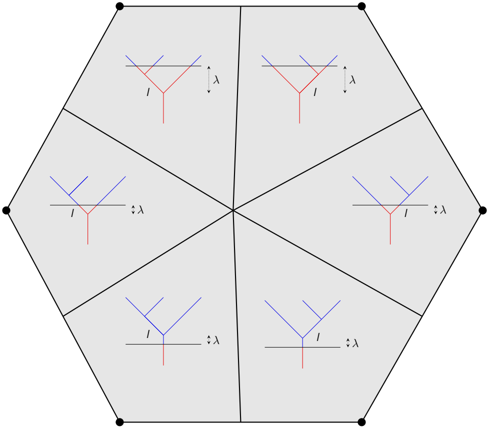


Figure: The compactified moduli space $\overline{\mathcal{T}}_4$



The compactified moduli space $\overline{\mathcal{CT}}_3$

It is also quite clear that given two compact symplectic manifolds M and N, one should be able to construct n-morphisms between their Fukaya categories $\mathrm{Fuk}(M)$ and $\mathrm{Fuk}(N)$ through counts of moduli spaces of quilted disks (under the correct technical assumptions).

Links between the n-multiplihedra and the 2-associahedra of Bottman (see [Bot19a] and [Bot19b] for instance) ? We are currently inspecting this matter with Nate Bottman.

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