Introduction to algebraic operads

Exercise sheet 3: Operads

Exercise 1 (Closed symmetric monoidal categories). *Let V and W be two graded vector spaces. We denote*

$$\mathrm{Hom}(V,W)_r := \prod_{n\in\mathbb{Z}} \mathrm{Hom}(V_n, W_{n+r})$$

the vector space of linear maps $V \to W$ of degree r and define

$$\underline{\mathrm{Hom}}(V,W) := \bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}(V,W)_r \ .$$

1. Prove that the symmetric monoidal category $(\mathtt{grVect}, \otimes)$ is closed with internal hom the graded vector space $\underline{\mathrm{Hom}}(V,W)$.

Assume now that V and W are dg vector spaces. For a linear map $f : V \to W$ of degree $|f|$ we denote

$$[\partial, f] := \partial_W f - (-1)^{|f|} f \partial_V \ .$$

2. Prove that the symmetric monoidal category $(\mathtt{dgVect}, \otimes)$ is closed with internal hom the dg vector space $\big(\underline{\mathrm{Hom}}(V,W), [\partial, \cdot]\big)$.

Exercise 2 (Invariants and coinvariants).

Let V be a vector space together with a left $\mathfrak{S}_n$-action on V. Prove that if $\mathrm{char}(\mathbb{K}) = 0$, the vector spaces of invariants and coinvariants are isomorphic $V^{\mathfrak{S}_n} \simeq V_{\mathfrak{S}_n}$.

Exercise 3 (Operads and ns operads).

Prove that there exists an adjunction $\mathtt{nsOp} \leftrightarrows \mathtt{Op}$.

Exercise 4 (Diagonal of an operad). *We define a* diagonal *of an operad $\mathscr{P}$ in $\mathtt{Vect}$ to be a morphism of operads $\mathscr{P} \to \mathscr{P} \otimes \mathscr{P}$.*

Prove that a diagonal on $\mathscr{P}$ defines a bifunctor $\mathscr{P}\text{-}\mathtt{alg} \times \mathscr{P}\text{-}\mathtt{alg} \to \mathscr{P}\text{-}\mathtt{alg}$.

Exercise 5 (Enveloping algebra). *Let $\alpha : \mathscr{P} \to \mathbb{Q}$ be a morphism of operads in $\mathtt{Vect}$. Recall that it defines a functor $\alpha^* : \mathbb{Q}\text{-}\mathtt{alg} \to \mathscr{P}\text{-}\mathtt{alg}$.*

1. Prove that the functor α^* has a left adjoint $\alpha_!$.

2. Prove that if α is the morphism of operads $\mathscr{Lie} \to \mathscr{Ass}$, the functor $\alpha_!$ is exactly the enveloping algebra construction.

Exercise 6 (Operad as monoids and Schur functors). *1.* Check that an operad in $\mathtt{Vect}$ can be equivalently defined as a monoid in $(\mathfrak{S}\text{-}\mathtt{mod}, \circ)$ and that a cooperad in $\mathtt{Vect}$ can be equivalently defined as a comonoid in $(\mathfrak{S}\text{-}\mathtt{mod}, \bar{\circ})$.

2. Check that the Schur functor $S_{-} : (\mathfrak{S}\text{-}\mathtt{mod}, \circ) \to \mathrm{EndoFun}(\mathtt{Vect})$ and the coSchur functor $S^{-} : (\mathfrak{S}\text{-}\mathtt{mod}, \bar{\circ}) \to \mathrm{EndoFun}(\mathtt{Vect})$ are strong monoidal.

Exercise 7 (The free ns operad as a colimit). *For two $\mathbb{N}$-modules $\mathcal{M}$ and $\mathcal{N}$ in* $\mathtt{Vect}$ *we define their* direct sum *as* $\mathcal{M} \oplus \mathcal{N}(n) = \mathcal{M}(n) \oplus \mathcal{N}(n)$. *A bifunctor* $\mathcal{F} : \mathbb{N}\text{-}\mathtt{mod} \times \mathbb{N}\text{-}\mathtt{mod} \to \mathbb{N}\text{-}\mathtt{mod}$ *is then defined to be* linear on the left *if*

$$\mathcal{F}(\mathcal{M}_1 \oplus \mathcal{M}_2, \mathcal{N}) = \mathcal{F}(\mathcal{M}_1, \mathcal{N}) \oplus \mathcal{F}(\mathcal{M}_2, \mathcal{N}) \ .$$

Linearity on the right *is defined in a similar fashion.*

1. Prove that the composite bifunctor $- \circ - : \mathbb{N}\text{-}\mathtt{mod} \times \mathbb{N}\text{-}\mathtt{mod} \to \mathbb{N}\text{-}\mathtt{mod}$ is linear on the left but not on the right.

We denote $\mathbb{K} := (0, \mathbb{K}, 0, \ldots, 0, \ldots)$. *We define a sequence of functors* $\mathcal{F}_n : \mathbb{N}\text{-}\mathtt{mod} \to \mathbb{N}\text{-}\mathtt{mod}$ *by induction as*

$$\mathcal{F}_0(\mathcal{M}) = \mathbb{K} \qquad \mathcal{F}_1(\mathcal{M}) = \mathbb{K} \oplus \mathcal{M} \qquad \mathcal{F}_n(\mathcal{M}) = \mathbb{K} \oplus (\mathcal{M} \circ \mathcal{F}_{n-1}(\mathcal{M})) \ .$$

2. Prove by induction on n that there exists natural maps $\gamma_{n,m} : \mathcal{F}_n(\mathcal{M}) \circ \mathcal{F}_m(\mathcal{M}) \to \mathcal{F}_{n+m}(\mathcal{M})$.

3. Prove that the maps $\gamma_{n,m}$ are associative, i.e. that for $\mathcal{M} \in \mathbb{N}\text{-}\mathtt{mod}$ and $p, q, r \geqslant 0$,

$$\gamma_{p+q,r}(\gamma_{p,q} \circ \mathrm{id}_{\mathcal{F}_r(\mathcal{M})}) = \gamma_{p,q+r}(\mathrm{id}_{\mathcal{F}_p(\mathcal{M})} \circ \gamma_{q,r})$$

where $\circ$ denotes the composite product on $\mathbb{N}\text{-}\mathtt{mod}$ and not the composition of morphisms.

We define natural transformations $\iota_n : \mathcal{F}_n \Rightarrow \mathcal{F}_{n+1}$ *by induction as*

$$\iota_0^{\mathcal{M}} : \mathbb{K} = \mathcal{F}_0(\mathcal{M}) \hookrightarrow \mathcal{F}_1(\mathcal{M}) = \mathbb{K} \oplus \mathcal{M} \qquad \iota_n^{\mathcal{M}} := \mathrm{id}_{\mathbb{K}} \oplus (\mathrm{id}_{\mathcal{M}} \circ \iota_{n-1}) : \mathcal{F}_n(\mathcal{M}) \to \mathcal{F}_{n+1}(\mathcal{M})$$

where $\circ$ denotes again the composite product on $\mathbb{N}\text{-}\mathtt{mod}$ *and not the composition of morphisms. Notice that each ι_n is a monomorphism. We then define*

$$\mathcal{F}(\mathcal{M}) := \operatorname*{colim}_{n \geqslant 0} \mathcal{F}_n(\mathcal{M}) \ .$$

4. Prove that the natural transformations $\gamma_{n,m}$ induce a natural transformation

$$\gamma : \mathcal{F}(\mathcal{M}) \circ \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{M}) \ .$$

5. Prove that the $\mathbb{N}$-module $\mathcal{F}(\mathcal{M})$ endowed with the composition γ and the unit $\mathbb{K} = \mathcal{F}_0(\mathcal{M}) \hookrightarrow \mathcal{F}(\mathcal{M})$ is a ns operad.

6. Prove that the free ns operad $\mathcal{T}_{ns}(\mathcal{M})$ is isomorphic to the ns operad $\mathcal{F}(\mathcal{M})$.

The same constructions and results also hold in the symmetric case.

Exercise 8 (Dual).

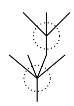
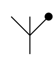
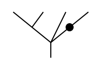
1. Given a subgroup $H \subset G$ and a vector space V with a left H-action, prove that there are natural isomorphisms

$$\mathrm{Hom}(\mathrm{Ind}_H^G(V), \mathbb{K}) \simeq \mathrm{Coind}_H^G(V^\vee) \qquad \left(\mathrm{Coind}_H^G(V)\right)^G = V^H \ .$$

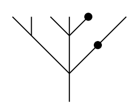
2. Prove that the arity-wise linear dual of a May cooperad $\mathscr{C}$ in `Vect` is an operad $\mathscr{C}^\vee$, and that the arity-wise linear dual of an arity-wise finite-dimensional operad $\mathscr{P}$ in `Vect` is a May cooperad $\mathscr{P}^\vee$.

3. Prove that the dual of a $\mathscr{A}ss^\vee$-coalgebra C is a standard conilpotent noncounital coassociative coalgebra $C^\vee$.

Exercise 9 (The monad $\mathbb{PT}$). *Given a planar tree t and a planar tree $t_v \in \mathrm{PT}_{|\mathrm{inc}(v)|}$ for every $v \in \mathrm{Vert}(t)$, we can define a new tree $\mathrm{sub}(t; \{t_v\}_{v\in\mathrm{Vert}(t)})$ by* substituting *every vertex v in t by the tree t_v. For instance, for*

$t =$  $\qquad t_1 =$  $\qquad t_2 =$ 

the substitution reads as

$\mathrm{sub}(t; t_1, t_2) =$  .

The free ns algebra construction moreover defines a functor

$$\mathbb{PT} : \mathbb{N}\text{-}\mathtt{mod} \longrightarrow \mathbb{N}\text{-}\mathtt{mod}$$

with $\mathbb{PT}(\mathscr{M}) := \mathscr{T}_{ns}(\mathscr{M})$.

1. Prove that substitution of trees defines a monad structure on the endofunctor $\mathbb{PT}$.

2. Prove that a $\mathbb{PT}$-algebra structure on a $\mathbb{N}$-module $\mathscr{P}$ is then equivalent to a structure of ns operad on $\mathscr{P}$.

3. Prove that the free $\mathbb{PT}$-algebra and the free ns operad structures on $\mathscr{T}_{ns}(\mathscr{M})$ coincide.