Introduction to algebraic operads

Exercise sheet 1: Homological algebra

1. Categories

Exercise 1 (Yoneda lemma). Given a small category $\mathscr{C}$, a functor $\mathscr{F} : \mathscr{C} \to \texttt{Set}$ and an object C of $\mathscr{C}$, prove that the map

$$\begin{aligned} \mathrm{Nat}(\mathscr{C}(C,\cdot),\mathscr{F}) &\longrightarrow \mathscr{F}(C) \\ \tau &\longmapsto \tau_C(\mathrm{id}_C) \end{aligned}$$

is a bijection. This result is known as the *Yoneda lemma.*

Exercise 2 (The 2-category `Cat`**).** *We define a* 2*-category* $\mathscr{C}$ *to be the following data:*

- *(i) A class of objects* $\mathrm{Ob}(\mathscr{C})$.
- *(ii) For every object* $X, Y \in \mathscr{C}$ *a category* $\mathscr{C}(X,Y)$ *and for every* $X \in \mathscr{C}$ *an identity object* id_X *in* $\mathscr{C}(X,X)$.
- *(iii) For every objects* $X, Y, Z \in \mathscr{C}$ *a composition bifunctor* $\mathscr{C}(Y,Z) \times \mathscr{C}(X,Y) \to \mathscr{C}(X,Z)$.

These data have to satisfy the same associativity and identity axioms as a standard category. The elements $f \in \mathscr{C}(X,Y)$ *are called* 1-morphisms *and denoted* $f : X \to Y$ *and the elements* $\tau \in \mathscr{C}(X,Y)(f,g)$ *are called* 2-morphisms *and denoted* $\tau : f \Rightarrow g$.

Prove that `Cat` is a 2-category whose 1-morphisms are functors and 2-morphisms are natural transformations.

Exercise 3 (Adjunction).

1. Given two functors $\mathscr{F}, \mathscr{G} : \mathscr{C} \to \mathscr{D}$, prove that a collection of bijections

$$\phi_{X,Y} : \mathscr{C}(\mathscr{F}(X), Y) \xrightarrow{\sim} \mathscr{D}(X, \mathscr{G}(Y))$$

defines a natural equivalence if and only if for all objects $X, X' \in \mathscr{C}$ and $Y, Y' \in \mathscr{D}$, and all morphisms $f : X \to X'$, $h : \mathscr{F}(X') \to Y$ and $g : Y \to Y'$ we have that

$$\phi_{X,Y'}(g \circ h \circ \mathscr{F}(f)) = G(g) \circ \phi_{X',Y}(h) \circ f\ .$$

2. Given two adjoint functors $\mathscr{F} \dashv \mathscr{G}$ as in the previous question, prove that the diagram

$$\begin{array}{ccc} \mathscr{F}(X) & \xrightarrow{\ h\ } & Y \\ \downarrow{\scriptstyle \mathscr{F}(f)} & & \downarrow{\scriptstyle g} \\ \mathscr{F}(X') & \xrightarrow{\ h'\ } & Y' \end{array}$$

is commutative if and only if the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi_{X,Y}(h)} & \mathcal{G}(Y) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathcal{G}(g)} \\ X' & \xrightarrow{\phi_{X',Y'}(h')} & \mathcal{G}(Y') \end{array}$$

is commutative.

3. Prove that two functors $\mathcal{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathcal{G}$ form an adjunction $\mathcal{F} \dashv \mathcal{G}$ if and only if there exists two natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}$ and $\varepsilon : \mathcal{F} \circ \mathcal{G} \Rightarrow \mathrm{id}_{\mathcal{D}}$ such that $\varepsilon\mathcal{F} \circ \mathcal{F}\eta = \mathrm{id}_{\mathcal{F}}$ and $\mathcal{G}\varepsilon \circ \eta\mathcal{G} = \mathrm{id}_{\mathcal{G}}$. The natural transformations η and ε are then respectively called the *unit* and the *counit* of the adjunction.

Exercise 4 (Monad). *We define an* endofunctor *of a category* $\mathcal{C}$ *to be a functor* $\mathcal{C} \to \mathcal{C}$ *and denote* $\mathrm{EndoFun}_{\mathcal{C}}$ *the category of endofunctors of* $\mathcal{C}$ *(see* Exercise 2*).*

1. Prove that the composition $\circ$ and the identity functor $\mathrm{id}_{\mathcal{C}}$ endow the category $\mathrm{EndoFun}_{\mathcal{C}}$ with a structure of strict monoidal category.

We define a monad *to be a monoid in the monoidal category* $\mathrm{EndoFun}_{\mathcal{C}}$.

2. Prove that an adjunction $\mathcal{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathcal{G}$ with unit η and counit ε gives rise to a monad structure on the endofunctor $\mathcal{G}\mathcal{F} : \mathcal{C} \to \mathcal{C}$ with multiplication $\mathcal{G}\varepsilon\mathcal{F}$ and unit η.

An algebra over a monad $(\mathcal{M}, \mu, \eta)$ *is defined to be an object* C *of* $\mathcal{C}$ *together with a morphism* $\gamma_A : \mathcal{M}(A) \to A$ *making the following diagrams commute*

$$\begin{array}{ccc} \mathcal{M}(\mathcal{M}(A)) & \xrightarrow{\mathcal{M}(\gamma_A)} & \mathcal{M}(A) \\ \downarrow{\scriptstyle \mu_A} & & \downarrow{\scriptstyle \gamma_A} \\ \mathcal{M}(A) & \xrightarrow{\gamma_A} & A \end{array} \qquad \begin{array}{ccc} \mathrm{id}_{\mathcal{C}}(A) & \xrightarrow{\eta_A} & \mathcal{M}(A) \\ & \searrow & \downarrow{\scriptstyle \gamma_A} \\ & & A \end{array} .$$

3. Prove that the forgetful functor $\mathcal{U} : \mathtt{Vect} \to \mathtt{Set}$ has a left-adjoint $\mathcal{F}$ and that an algebra structure on a set V over the monad $\mathcal{U}\mathcal{F}$ is then exactly a vector space structure on V.

2. Homological algebra

Exercise 5 (Euler characteristic). *A chain complex is said to be* bounded *if up to reindexing it is of the form*

$$0 \longrightarrow C_n \longrightarrow \cdots \longrightarrow C_0 \longrightarrow 0 .$$

1. Prove that for a bounded chain complex C_*,

$$\sum_{i=0}^{n} (-1)^i \dim(C_i) = \sum_{i=0}^{n} (-1)^i \dim(H_i(C)) .$$

This number is then called its *Euler characteristic* and denoted $\chi(C)$.

2. Prove that every short exact sequence of bounded chain complexes

$$0 \longrightarrow A_* \longrightarrow B_* \longrightarrow C_* \longrightarrow 0,$$

satisfies the relation $\chi(B) = \chi(A) + \chi(C)$.

Exercise 6 (Diagram chasing lemmas).

1. Prove that in a short exact sequence $0 \to A_* \to B_* \to C_* \to 0$ of chain complexes, if two of the three chain complexes are exact then so is the third.

2. Consider a commutative diagram of linear maps

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & D_1 & \longrightarrow & E_1 \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d & & \downarrow e \\ A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & D_2 & \longrightarrow & E_2 \end{array}$$

whose rows are exact, and such that b and d are isomorphisms, a is surjective and e is injective. Prove that the linear map c is an isomorphism. This result is known as the *five lemma.*

3. Prove that in a commutative diagram of chain maps

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \longrightarrow & B_* & \longrightarrow & C_* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & D_* & \longrightarrow & E_* & \longrightarrow & F_* & \longrightarrow & 0 \end{array}$$

whose rows are exact, if two of the vertical morphisms are quasi-isomorphisms then so is the third.

Exercise 7 (Cone). *Let C and D be two chain complexes and $f : C \to D$ be a chain map. We define the* cone of f *to be the chain complex* $\mathrm{Cone}(f)$ *whose degree n part is $C_{n-1} \oplus D_n$ and whose differential is given by*

$$\partial(c, d) := (-\partial_C c, \partial_D d - f(c)) \ .$$

This is often written as

$$\partial_{\mathrm{Cone}(f)} := \begin{pmatrix} -\partial_C & 0 \\ -f & \partial_D \end{pmatrix} \ .$$

1. Check that the above formula indeed defines a differential on $\mathrm{Cone}(f)$.

2. Prove that the sequence

$$0 \longrightarrow D \longrightarrow \mathrm{Cone}(f) \longrightarrow sC \longrightarrow 0$$

is exact and make the connecting homomorphism explicit.

3. Prove that f is a quasi-isomorphism if and only $\mathrm{Cone}(f)$ is exact.

4. Suppose that f and f' are two homotopic chain maps $C \to D$. Show that there exists a quasi-isomorphism $\mathrm{Cone}(f) \to \mathrm{Cone}(f')$.

We say that a chain complex C_* is contractible *if the chain map* id_C *is* null-homotopic*, i.e. homotopic to the null chain map.*

5. Prove that if f is a homotopy equivalence, then $\mathrm{Cone}(f)$ is contractible.

A chain complex (C_*, ∂) *is said to be* split *if there exists a collection of maps* $s_n : C_n \to s_{n+1}$ *such that* $\partial_{n+1} = \partial_{n+1} s_n \partial_{n+1}$.

6. Prove that a split chain complex is exact if and only if its contractible.

7. Prove that $\mathrm{Cone}(C) := \mathrm{Cone}(\mathrm{id}_C)$ is split exact.