Introduction to algebraic operads

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Contents

CHAPTER 1

Homological algebra

We fix a field $\mathbb{K}$ for the rest of these notes.

1. Categories

1.1. Categories and functors.

DEFINITION 1.1. *A* category $\mathtt{C}$ *corresponds to the following data:*

(1) A class of objects $\mathrm{Ob}(\mathtt{C})$.
(2) For each pair of objects $X, Y \in \mathrm{Ob}(\mathtt{C})$ *a set* $\mathtt{C}(X, Y)$*, called the set of* morphisms *from* X *to* Y*. It is also denoted* $\mathrm{Hom}_{\mathtt{C}}(X, Y)$.
(3) For each object X *of* $\mathtt{C}$*, an element* $\mathrm{id}_X \in \mathtt{C}(X, X)$ *called the* identity *of* X.
(4) For each triple of objects X*,* Y *and* Z *of* $\mathtt{C}$ *of a map* $\circ : \mathtt{C}(Y, Z) \times \mathtt{C}(X, Y) \to \mathtt{C}(X, Z)$ *called the* composition.

These data are such that for every $f \in \mathtt{C}(X, Y)$*,* $g \in \mathtt{C}(Y, Z)$ *and* $h \in \mathtt{C}(Z, T)$ *we have that* $(h \circ g) \circ f = h \circ (g \circ f)$ *and for very* $f \in \mathtt{C}(X, Y)$ *we have that* $\mathrm{id}_Y \circ f = f = f \circ \mathrm{id}_X$.

A category is said to be *small* if $\mathrm{Ob}(\mathtt{C})$ is a set. A morphism $f \in \mathtt{C}(X, Y)$ is said to be an *isomorphism* if there exists a morphism $g \in \mathtt{C}(Y, X)$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. It will be denoted as $f : X \xrightarrow{\sim} Y$.

EXAMPLE 1.2. (i) The category $\mathtt{Set}$ whose objects are sets and morphisms the maps between them.
(ii) The category $\mathtt{Vect}$ whose objects are $\mathbb{K}$-vector spaces and morphisms the linear maps between them.

Both of these categories are not small.

DEFINITION 1.3. *(i) Given a category* $\mathtt{C}$*, the* opposite category $\mathtt{C}^{op}$ *is the category whose objects are the objects of* $\mathtt{C}$*, whose morphisms sets are* $\mathtt{C}^{op}(X, Y) := \mathtt{C}(Y, X)$ *and whose composition is defined as* $g \circ_{\mathtt{C}^{op}} f := f \circ_{\mathtt{C}} g$.
(ii) Given $\mathtt{C}$ *and* $\mathtt{D}$ *two categories, the* product category $\mathtt{C} \times \mathtt{D}$ *is defined to be the category whose objects are pairs of objects* $X \times Y := (X, Y)$ *for* $X \in \mathrm{Ob}(\mathtt{C})$ *and* $Y \in \mathrm{Ob}(\mathtt{D})$*, whose sets of morphisms are defined to be*

$$\mathrm{Hom}_{\mathtt{C}\times\mathtt{D}}(X_1 \times Y_1, X_2 \times Y_2) := \mathrm{Hom}_{\mathtt{C}}(X_1, X_2) \times \mathrm{Hom}_{\mathtt{D}}(Y_1, Y_2)$$

and whose composition is defined as $(f_2 \times g_2) \circ (f_1 \times g_1) := (f_2 \circ_{\mathtt{C}} f_1) \times (g_2 \circ_{\mathtt{D}} g_1)$ *with identities* $\mathrm{id}_{X \times Y} := \mathrm{id}_X \times \mathrm{id}_Y$.

Definition 1.4. *Given* $\mathtt{C}$ *and* $\mathtt{D}$ *two categories, a* functor $\mathscr{F} : \mathtt{C} \to \mathtt{D}$ *associates to every object* $X \in \mathtt{C}$ *an object* $\mathscr{F}(X) \in \mathtt{D}$ *and to every morphism* $f \in \mathtt{C}(X,Y)$ *a morphism* $\mathscr{F}(f) \in \mathtt{D}(\mathscr{F}(X), \mathscr{F}(Y))$ *such that* $\mathscr{F}(\mathrm{id}_X) = \mathrm{id}_{\mathscr{F}(X)}$ *and such that* $\mathscr{F}(g \circ f) = \mathscr{F}(g) \circ \mathscr{F}(f)$.

The class of functors between two categories $\mathtt{C}$ and $\mathtt{D}$ will be denoted $\mathrm{Fun}(\mathtt{C},\mathtt{D})$. If the categories $\mathtt{C}$ and $\mathtt{D}$ are small then $\mathrm{Fun}(\mathtt{C},\mathtt{D})$ is a set.

Example 1.5.
(i) The *forgetful functor* $\mathtt{Vect} \to \mathtt{Set}$ mapping a vector space to its underlying set by forgetting its vector space structure.
(ii) For a category $\mathtt{C}$ and an object $C \in \mathtt{C}$, the functor $\mathtt{C}(C, \cdot) : \mathtt{C} \to \mathtt{Set}$ associating to each object $D \in \mathtt{C}$ the set $\mathtt{C}(C,D)$.
(iii) The tensor product functor $- \otimes - : \mathtt{Vect} \times \mathtt{Vect} \to \mathtt{Vect}$.

A functor from a product category $\mathtt{C}_1 \times \mathtt{C}_2 \to \mathtt{D}$ will be called a *bifunctor.*

Definition 1.6. *The category* $\mathtt{Cat}$ *is defined to be the category whose objects are small categories and morphisms the functors between them, where two functors* $\mathscr{F} : \mathtt{C}_1 \to \mathtt{C}_2$ *and* $\mathscr{G} : \mathtt{C}_2 \to \mathtt{C}_3$ *are composed as* $\mathscr{G} \circ \mathscr{F}(X) := \mathscr{G}(\mathscr{F}(X))$ *and* $\mathscr{G} \circ \mathscr{F}(f) := \mathscr{G}(\mathscr{F}(f))$.

Definition 1.7. *Given* $\mathtt{C}$ *and* $\mathtt{D}$ *two categories and* $\mathscr{F}, \mathscr{G} : \mathtt{C} \to \mathtt{D}$ *two functors, a* natural transformation $\eta : \mathscr{F} \Rightarrow \mathscr{G}$ *is defined to be the datum of a morphism* $\eta_X \in \mathtt{D}(\mathscr{F}(X), \mathscr{G}(X))$ *for every* $X \in \mathtt{C}$ *such that for every* $f \in \mathtt{C}(X,Y)$*, the following diagram commutes*

$$\begin{array}{ccc} \mathscr{F}(X) & \xrightarrow{\mathscr{F}(f)} & \mathscr{F}(Y) \\ \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} \\ \mathscr{G}(X) & \xrightarrow{\mathscr{G}(f)} & \mathscr{G}(Y) \end{array} .$$

We will denote $\mathrm{Nat}(\mathscr{F}, \mathscr{G})$ the class of natural transformations between two functors $\mathscr{F}, \mathscr{G} : \mathtt{C} \to \mathtt{D}$. It is a set if the categories $\mathtt{C}$ and $\mathtt{D}$ are small.

Example 1.8. The abelianization $G^{ab} := G/[G,G]$ of a group G defines a functor $-^{ab} : \mathtt{Gr} \to \mathtt{Gr}$ where $\mathtt{Gr}$ denotes the category whose objects are groups and morphisms group morphisms. The collection of maps $\pi_G : G \to G^{ab}$ then defines a natural transformation $\pi : \mathrm{id}_{\mathtt{Gr}} \Rightarrow -^{ab}$.

The *identity natural transformation* $\mathrm{id}_{\mathscr{F}} : \mathscr{F} \Rightarrow \mathscr{F}$ of a functor $\mathscr{F}$ is defined for every $X \in \mathtt{C}$ as $(\mathrm{id}_{\mathscr{F}})_X = \mathrm{id}_{\mathscr{F}(X)}$. Two natural transformations $\eta : \mathscr{F}_1 \Rightarrow \mathscr{F}_2$ and $\gamma : \mathscr{F}_2 \Rightarrow \mathscr{F}_3$ can moreover be composed as $\gamma \circ \eta : \mathscr{F}_1 \Rightarrow \mathscr{F}_3$ by setting $(\gamma \circ \eta)_X := \gamma_X \circ \eta_X$.

1.2. Equivalences and adjoints.

Definition 1.9. *A natural transformation* $\eta : \mathscr{F} \Rightarrow \mathscr{G}$ *is said to be a* natural equivalence *if each morphism* η_X *is an isomorphism. In that case, we say that* $\mathscr{F}$ *and* $\mathscr{G}$ *are* naturally equivalent *and denote* $\mathscr{F} \simeq \mathscr{G}$.

For a category $\mathtt{C}$, a functor $\mathscr{F} : \mathtt{C} \to \mathtt{Set}$ is said to be *representable* if there exists an object $C \in \mathtt{C}$ such that $\mathscr{F} \simeq \mathtt{C}(C, \cdot)$.

Exercise 1.10. Prove that the morphisms $\tau_X := \eta_X^{-1}$ define a natural equivalence $\tau : \mathcal{G} \Rightarrow \mathcal{F}$ such that $\tau \circ \eta = \mathrm{id}_{\mathcal{F}}$ and that $\eta \circ \tau = \mathrm{id}_{\mathcal{G}}$.

Definition 1.11. *An* equivalence of categories *is a pair of functors* $\mathcal{F} : \mathtt{C} \rightleftarrows \mathtt{D} : \mathcal{G}$ *such that* $\mathcal{G} \circ \mathcal{F} \simeq \mathrm{id}_{\mathtt{C}}$ *and* $\mathcal{F} \circ \mathcal{G} \simeq \mathrm{id}_{\mathtt{D}}$.

Exercise 1.12. Prove that a functor $\mathcal{F}$ defines an equivalence of categories if and only if for every objects $X, Y \in \mathtt{C}$, the map $\mathcal{F} : \mathtt{C}(X, Y) \to \mathtt{D}(\mathcal{F}(X), \mathcal{F}(Y))$ is bijective (the functor $\mathcal{F}$ is *fully faithful*) and for every object $D \in \mathtt{D}$ there exists an object $C \in \mathtt{C}$ and an isomorphism $\mathcal{F}(C) \to D$ (the functor $\mathcal{F}$ is *essentially surjective*).

Definition 1.13. *Two functors* $\mathcal{F} : \mathtt{C} \leftrightarrows \mathtt{D} : \mathcal{G}$ *are said to be* adjoints *if there exists natural bijections*

$$\phi_{X,Y} : \mathtt{D}(\mathcal{F}(X), Y) \xrightarrow{\sim} \mathtt{C}(X, \mathcal{G}(Y))$$

where $X \in \mathtt{C}$ *and* $Y \in \mathtt{D}$. *The functors* $\mathcal{F}$ *and* $\mathcal{G}$ *are respectively the* left and right adjoints *and are denoted* $\mathcal{F} \dashv \mathcal{G}$.

In other words, two functors $\mathcal{F} : \mathtt{C} \leftrightarrows \mathtt{D} : \mathcal{G}$ are adjoints if the two functors

$$\mathtt{D}(\mathcal{F}(\cdot), \cdot), \mathtt{C}(\cdot, \mathcal{G}(\cdot)) : \mathtt{C}^{op} \times \mathtt{D} \to \mathtt{Set}$$

are naturally equivalent.

Example 1.14. Let W be a vector space. The functors $-\otimes W : \mathtt{Vect} \to \mathtt{Vect}$ and $\mathrm{Hom}_{\mathtt{Vect}}(W, \cdot) : \mathtt{Vect} \to \mathtt{Vect}$ are adjoints: for every vector spaces V_1 and V_2 we have that

$$\mathrm{Hom}_{\mathtt{Vect}}(V_1 \otimes W, V_2) = \mathrm{Hom}_{\mathtt{Vect}}(V_1, \mathrm{Hom}_{\mathtt{Vect}}(W, V_2)) \ .$$

1.3. Monoidal categories.

1.3.1. *Symmetric monoidal categories.*

Definition 1.15. *A* monoidal category *is a category* $\mathtt{C}$ *endowed with the following data:*

(1) A bifunctor $\boxtimes : \mathtt{C} \times \mathtt{C} \to \mathtt{C}$ *and an object* I *of* $\mathtt{C}$.
(2) A natural equivalence $\alpha : \boxtimes \circ (\boxtimes \times \mathrm{id}_{\mathtt{C}}) \simeq \boxtimes \circ (\mathrm{id}_{\mathtt{C}} \times \boxtimes)$ *called* the associator, *such that for every* $A, B, C, D \in \mathtt{C}$ *the following diagram commutes:*

$$\begin{array}{ccccc} ((A \boxtimes B) \boxtimes C) \boxtimes D & \xrightarrow{\alpha_{A,B,C} \boxtimes \mathrm{id}_D} & (A \boxtimes (B \boxtimes C)) \boxtimes D & \xrightarrow{\alpha_{A,B \boxtimes C,D}} & A \boxtimes ((B \boxtimes C) \boxtimes D) \\ \downarrow{\scriptstyle \alpha_{A \boxtimes B,C,D}} & & & & \downarrow{\scriptstyle \mathrm{id}_A \boxtimes \alpha_{B,C,D}} \\ (A \boxtimes B) \boxtimes (C \boxtimes D) & & \xrightarrow{\qquad \alpha_{A,B,C \boxtimes D} \qquad} & & A \boxtimes (B \boxtimes (C \boxtimes D)) \end{array} .$$

(3) A natural equivalence $\lambda : I \boxtimes \mathrm{id}_{\mathtt{C}} \simeq \mathrm{id}_{\mathtt{C}}$ *and a natural equivalence* $\rho : \mathrm{id}_{\mathtt{C}} \boxtimes I \simeq \mathrm{id}_{\mathtt{C}}$ *such that*

$$\rho_I = \lambda_I : I \boxtimes I \xrightarrow{\sim} I$$

and such that the following diagram commutes for every $A, B \in \mathtt{C}$

$$\begin{array}{ccc} (A \boxtimes I) \boxtimes B & \xrightarrow{\alpha_{A,I,B}} & A \boxtimes (I \boxtimes B) \\ & {\scriptstyle \rho_A \boxtimes \mathrm{id}_B} \searrow & \downarrow{\scriptstyle \mathrm{id}_A \boxtimes \lambda_B} \\ & & A \boxtimes B \end{array} .$$

A monoidal category is said to be *strict* if the natural transformations α, λ and ρ are identities.

MacLane's coherence theorem states that the commutativity of the diagram for the associator implies that given two bracketings of $A_1 \boxtimes \cdots \boxtimes A_n$, two sequences of morphisms made of iterations of the associator from one bracketing to the other have equal composition. In other words, two bracketings are naturally equivalent through a unique natural equivalence made of iterations of the associator.

DEFINITION 1.16. *A monoidal category* $\mathtt{C}$ *is said to be* symmetric *if there exists a natural equivalence*

$$\sigma_{A,B} : A \boxtimes B \xrightarrow{\sim} B \boxtimes A$$

called the braiding, *such that* $\sigma_{B,A}\sigma_{A,B} = \mathrm{id}_{A\boxtimes B}$ *and such that the following diagram commutes*

$$\begin{array}{ccccc} (A \boxtimes B) \boxtimes C & \xrightarrow{\alpha_{A,B,C}} & A \boxtimes (B \boxtimes C) & \xrightarrow{\sigma_{A,B\boxtimes C}} & (B \boxtimes C) \boxtimes A \\ \downarrow{\scriptstyle \sigma_{A,B}\boxtimes \mathrm{id}_C} & & & & \downarrow{\scriptstyle \alpha_{B,C,A}} \\ (B \boxtimes A) \boxtimes C & \xrightarrow{\alpha_{B,A,C}} & B \boxtimes (A \boxtimes C) & \xrightarrow{\mathrm{id}_B\boxtimes\sigma_{A,C}} & B \boxtimes (C \boxtimes A) \end{array} .$$

A counterpart of MacLane's coherence theorem taking the natural transformation σ into account also holds in the case of symmetric monoidal categories.

EXAMPLE 1.17. The category `Set` endowed with the cartesian product and the category `Vect` endowed with the tensor product $- \otimes_{\mathbb{K}} -$ are symmetric monoidal categories.

1.3.2. *Closed symmetric monoidal categories.*

DEFINITION 1.18. *A symmetric monoidal category* $\mathtt{C}$ *is said to be* closed *if for every* $Y \in \mathtt{C}$ *the functor* $- \boxtimes Y : \mathtt{C} \to \mathtt{C}$ *admits a right adjoint denoted* $\underline{\mathrm{Hom}}_{\mathtt{C}}(Y, \cdot) : C \to C$ *such that the bijections*

$$\mathtt{C}(X \otimes Y, Z) \simeq \mathtt{C}(X, \underline{\mathrm{Hom}}_{\mathtt{C}}(Y, Z))$$

are natural in X, Y *and* Z.

In a closed symmetric monoidal category, the set (X, Y) is called *the external hom* while the object $\underline{\mathrm{Hom}}_{\mathtt{C}}(X, Y)$ is called the *internal hom.*

EXAMPLE 1.19. (i) The symmetric monoidal categories `Set` and `Vect` are closed with internal homs their external homs.
(ii) The symmetric monoidal category `grVect` and `Vect` are closed, as proven in Exercise sheet 3.

1.3.3. *Monoids.*

DEFINITION 1.20. *A* monoid M *in a monoidal category* $\mathtt{C}$ *is an object* $M \in \mathtt{C}$ *together with:*

(1) A morphism $\mu : M \boxtimes M \to M$ *called the* multiplication, *which is* associative *i.e. makes the following diagram commute*

$$\begin{array}{ccccc} (M \boxtimes M) \boxtimes M & \xrightarrow{\alpha_{M,M,M}} & M \boxtimes (M \boxtimes M) & \xrightarrow{\mathrm{id}_M\boxtimes\mu} & M \boxtimes M \\ \downarrow{\scriptstyle \mu\boxtimes\mathrm{id}_M} & & & & \downarrow{\scriptstyle \mu} \\ M \boxtimes M & & \xrightarrow{\quad\quad\mu\quad\quad} & & M \end{array} .$$

(2) A morphism $\eta : I \to M$ called the unit, *which makes the following diagram commute*

$$\begin{array}{ccccc} I \boxtimes M & \xrightarrow{\eta \boxtimes \mathrm{id}_M} & M \boxtimes M & \xleftarrow{\mathrm{id}_M \boxtimes \eta} & M \boxtimes I \\ & {\scriptstyle \lambda_M} \searrow & \downarrow {\scriptstyle \mu} & \swarrow {\scriptstyle \rho_M} & \\ & & M & & \end{array} .$$

If $\mathtt{C}$ *is symmetric, a monoid is moreover said to be* commutative *if* $\mu = \mu\sigma_{A,A}$.

Example 1.21. A monoid in $\mathtt{Set}$ is a standard monoid while a monoid in $\mathtt{Vect}$ is a standard unital associative algebra (see Section 1.1.1 of Chapter 2).

Exercise 1.22. Prove that the unit of a monoid is unique.

1.3.4. *Lax monoidal functors.*

Definition 1.23. *A* lax monoidal functor $\mathscr{F}$ *between two monoidal categories* $\mathtt{C}$ *and* $\mathtt{D}$ *is a functor* $\mathscr{F} : \mathtt{C} \to \mathtt{D}$ *together with*

(1) a natural transformation $\phi_{A,B} : \mathscr{F}(A) \boxtimes_{\mathtt{D}} \mathscr{F}(B) \to \mathscr{F}(A \boxtimes_{\mathtt{C}} B)$,
(2) and a morphism $\psi : I_{\mathtt{D}} \to \mathscr{F}(I_{\mathtt{C}})$

which are such that for every $A, B, C \in \mathtt{C}$ *the following three diagrams commute*

$$\begin{array}{ccccc} (\mathscr{F}(A) \boxtimes_{\mathtt{D}} \mathscr{F}(B)) \boxtimes_{\mathtt{D}} \mathscr{F}(C) & \xrightarrow{\phi_{A,B} \boxtimes_{\mathtt{D}} \mathrm{id}_{\mathscr{F}(C)}} & \mathscr{F}(A \boxtimes_{\mathtt{C}} B) \boxtimes_{\mathtt{D}} \mathscr{F}(C) & \xrightarrow{\phi_{A \boxtimes_{\mathtt{C}} B, C}} & \mathscr{F}((A \boxtimes_{\mathtt{C}} B) \boxtimes_{\mathtt{C}} C) \\ \downarrow {\scriptstyle \alpha^{\mathtt{D}}_{\mathscr{F}(A),\mathscr{F}(B),\mathscr{F}(C)}} & & & & \downarrow {\scriptstyle \mathscr{F}\left(\alpha^{\mathtt{C}}_{A,B,C}\right)} \\ \mathscr{F}(A) \boxtimes_{\mathtt{D}} (\mathscr{F}(B) \boxtimes_{\mathtt{D}} \mathscr{F}(C)) & \xrightarrow[\mathrm{id}_{\mathscr{F}(A)} \boxtimes_{\mathtt{D}} \phi_{B,C}]{} & \mathscr{F}(A) \boxtimes_{\mathtt{D}} \mathscr{F}(B \boxtimes_{\mathtt{C}} C) & \xrightarrow[\phi_{A, B \boxtimes_{\mathtt{C}} C}]{} & \mathscr{F}(A \boxtimes_{\mathtt{C}} (B \boxtimes_{\mathtt{C}} C)) \end{array} ,$$

$$\begin{array}{ccc} I_{\mathtt{D}} \boxtimes_{\mathtt{D}} \mathscr{F}(A) & \xrightarrow{\psi \boxtimes_{\mathtt{D}} \mathrm{id}_{\mathscr{F}(A)}} & \mathscr{F}(I_{\mathtt{C}}) \boxtimes_{\mathtt{D}} \mathscr{F}(A) \\ \downarrow {\scriptstyle \lambda^{\mathtt{D}}_{\mathscr{F}(A)}} & & \downarrow {\scriptstyle \phi_{I_{\mathtt{C}},A}} \\ \mathscr{F}(A) & \xleftarrow[\mathscr{F}(\lambda^{\mathtt{C}}_A)]{} & \mathscr{F}(I \boxtimes_{\mathtt{C}} A) \end{array} ,$$

$$\begin{array}{ccc} \mathscr{F}(A) \boxtimes_{\mathtt{D}} I_{\mathtt{D}} & \xrightarrow{\mathrm{id}_{\mathscr{F}(A)} \boxtimes_{\mathtt{D}} \psi} & \mathscr{F}(A) \boxtimes_{\mathtt{D}} \mathscr{F}(I_{\mathtt{C}}) \\ \downarrow {\scriptstyle \rho^{\mathtt{D}}_{\mathscr{F}(A)}} & & \downarrow {\scriptstyle \phi_{A,I_{\mathtt{C}}}} \\ \mathscr{F}(A) & \xleftarrow[\mathscr{F}(\rho^{\mathtt{C}}_A)]{} & \mathscr{F}(A \boxtimes_{\mathtt{C}} I) \end{array} .$$

A lax monoidal functor is said to be *strong* if ϕ and ψ are isomorphisms and is *strict* if they are identities.

Example 1.24. The forgetful functor $\mathtt{Vect} \to \mathtt{Set}$ is lax monoidal but not strong monoidal.

Exercise 1.25. Prove that the image of a monoid under a lax monoidal functor is again a monoid.

DEFINITION 1.26. *Let* $\mathtt{C}$ *and* $\mathtt{D}$ *be two symmetric monoidal categories. A functor* $\mathcal{F} : \mathtt{C} \to \mathtt{D}$ *is said to be* lax symmetric monoidal *if it is lax monoidal and the following diagram commutes for every* $A, B \in \mathtt{C}$

$$\begin{array}{ccc} \mathcal{F}(A) \boxtimes_{\mathtt{D}} \mathcal{F}(B) & \xrightarrow{s^{\mathtt{D}}_{\mathcal{F}(A),\mathcal{F}(B)}} & \mathcal{F}(B) \boxtimes_{\mathtt{D}} \mathcal{F}(A) \\ \downarrow \phi_{A,B} & & \downarrow \phi_{B,A} \\ \mathcal{F}(A \boxtimes_{\mathtt{C}} B) & \xrightarrow{\mathcal{F}(s^{\mathtt{C}}_{A,B})} & \mathcal{F}(B \boxtimes_{\mathtt{C}} A) \end{array} .$$

EXAMPLE 1.27.
(i) The singular chains functor $C_*(-) : \mathtt{Top} \to \mathtt{dgVect}$ is lax symmetric monoidal.
(ii) The homology functor $H_*(-) : \mathtt{dgVect} \to \mathtt{grVect}$ is strong symmetric monoidal.
(iii) The free vector space functor $\mathbb{K}[-] : \mathtt{Set} \to \mathtt{Vect}$ is strong symmetric monoidal.

2. Homological algebra

2.1. Chain and cochain complexes.

2.1.1. *(Co)chain complexes and (co)homology.*

DEFINITION 2.1. *A* chain complex *corresponds to the data of a vector space* C_n *for every* $n \in \mathbb{Z}$ *together with linear maps*

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots ,$$

which are such that $\partial_n \circ \partial_{n+1} = 0$. *The collection of these maps is called a* differential.

DEFINITION 2.2. *A* cochain complex *corresponds to the data of a vector space* C^n *for every* $n \in \mathbb{Z}$ *together with linear maps*

$$\cdots \to C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \to \cdots ,$$

which are such that $\partial^n \circ \partial^{n-1} = 0$.

Chain complexes are usually denoted as C_* while cochain complexes are denoted as C^*. Dualizing a chain complex (C_n, ∂_n) moreover gives in particular a cochain complex $(C_n^\vee, \partial_n^\vee)$.

DEFINITION 2.3.
(i) The homology *of a chain complex* C_* *is defined to be the collection of vector spaces* $H_n(C) := \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n+1})$.
(ii) The cohomology *of a cochain complex* C^* *is defined to be the collection of vector spaces* $H^n(C) := \mathrm{Ker}(\partial^n)/\mathrm{Im}(\partial^{n-1})$.

A (co)chain complex is said to be *acyclic* if its (co)homology is null.

2.1.2. *Chain maps and homotopies.*

DEFINITION 2.4. *A* chain map *between two chain complexes* $f_* : C_* \to D_*$ *is defined to be a collection of maps* $f_n : C_n \to D_n$ *such that for all* $n \in \mathbb{Z}$, $f_n \partial_n^C = \partial_n^D f_n$. *A chain map between cochain complexes is defined similarly.*

A chain map $f_* : C_* \to D_*$ is usually represented as

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\ \cdots & \longrightarrow & D_{n+1} & \longrightarrow & D_n & \longrightarrow & D_{n-1} & \longrightarrow & \cdots \ . \end{array}$$

Definition 2.5. *The* composition *of two chain maps $f_* : A_* \to B_*$ and $g_* : B_* \to C_*$ is the chain map $(g \circ f)_* : A_* \to C_*$ defined as $(g \circ f)_n := g_n \circ f_n$.*

Proposition 2.6. *A chain map $f_* : C_* \to D_*$ induces well-defined maps $H_n(C) \to H_n(D)$.*

Definition 2.7. *A chain map f_* for which all maps induced in homology are isomorphisms is called a* quasi-isomorphism.

Definition 2.8. *Two chain maps $f_*, g_* : C_* \to D_*$ are said to be* homotopic *and denoted $f_* \simeq g_*$, if there exists a collection of linear maps $h_n : C_n \to D_{n+1}$ such that for all $n \in \mathbb{Z}$*

$$\partial^D_{n+1} h_n + h_{n-1} \partial^C_n = g_n - f_n \ .$$

The collection of maps h_n is then called a (chain) homotopy *between f_* and g_*.*

Proposition 2.9. *Two homotopic chain maps induce the same map in homology.*

Definition 2.10. *A chain map $f_* : C_* \to D_*$ is said to be a* homotopy equivalence *if there exists a chain map $g_* : D_* \to C_*$ such that $g_* \circ f_* \simeq \mathrm{id}_{C_*}$ and $f_* \circ g_* \simeq \mathrm{id}_{D_*}$.*

Following Proposition 2.9, a chain equivalence is in particular a quasi-isomorphism.

2.2. The differential graded viewpoint.

Definition 2.11. *(i) A* graded vector space *is a vector space V together with a direct sum decomposition $V = \oplus_{n \in \mathbb{Z}} V_n$.*

(ii) A linear map of degree r *between two graded vector spaces V and W is a linear map $f : V \to W$ such that $f(V_n) \subset W_{n+r}$ for all $n \in \mathbb{Z}$.*

An element x of a dg module C_* is said to have *degree n* if $x \in C_n$. Its degree will then be written $|x| := n$.

Lemma 2.12. *(i) The datum of a chain complex is equivalent to the data of a graded vector space C together with a map $\partial : C \to C$ of degree -1 such that $\partial \circ \partial = 0$.*

(ii) The datum of cochain complex is equivalent to the data of a graded vector space C together with a map $\partial : C \to C$ of degree $+1$ such that $\partial \circ \partial = 0$.

A (co)chain complex seen from the viewpoint of Lemma 2.12 will be refered to as a *differential graded vector space*, or *dg vector space.* It will be said to be *homologically graded* if the differential has degree -1 and *cohomologically graded* if the differential has degree $+1$.

Lemma 2.13. *The datum of a chain map $f_* : C_* \to D_*$ is equivalent to the datum of a linear map $f : C \to D$ of degree 0 such that $\partial_D f = f \partial_C$.*

Definition 2.14. *Given C_* and D_* two dg vector spaces, their* tensor product *is defined to be the dg vector space whose degree n part is* $(C \otimes D)_n := \oplus_{p+q=n} C_p \otimes D_q$ *and whose differential is defined as*

$$\partial_{C\otimes D}(c \otimes d) = \partial_C c \otimes d + (-1)^{|c|} c \otimes \partial_D d \ .$$

Exercise 2.15. Prove that the formula of Definition 2.14 defines indeed a differential.

Definition 2.16. *The* suspension sC *of a dg vector space C is the dg vector space whose degree n part is* $(sC)_n := C_{n-1}$ *and whose differential is* $\partial_{sC} := -\partial_C$.

2.3. Exact sequences.

Definition 2.17. *A sequence of linear maps $f_n : A_n \to A_{n-1}$ is said to be* exact *if for every n,* $f_n \circ f_{n-1} = 0$ *and* $\mathrm{Im}(f_{n-1}) = \mathrm{Ker}(f_n)$.

In particular, a (co)chain complex is exact if and only if its (co)homology is null.

Definition 2.18. *A* short sequence *of chain maps between chain complexes*

$$0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$$

is said to be exact *if for every $n \in \mathbb{Z}$ the sequence of linear maps*

$$0 \to A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \to 0$$

is exact, i.e. if f_n is injective, g_n is surjective and $\mathrm{Ker}(g_n) = \mathrm{Im}(f_n)$.

Lemma 2.19. *A commutative diagram of the form*

$$\begin{array}{ccccccc} & & A_1 & \xrightarrow{s_A} & A_2 & \xrightarrow{t_A} & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow{s_B} & B_2 & \xrightarrow{t_B} & B_3 & & \end{array}$$

whose rows are exact induces a short exact sequence

$$\mathrm{Ker}(f_1) \to \mathrm{Ker}(f_2) \to \mathrm{Ker}(f_3) \to B_1/\mathrm{Im}(f_1) \to B_2/\mathrm{Im}(f_2) \to B_3/\mathrm{Im}(f_3) \ .$$

This result is called the *snake lemma.* The map $\mathrm{Ker}(f_3) \to B_1/\mathrm{Im}(f_1)$ is moreover called the *connecting morphism.*

Theorem 1. *A* short exact sequence *of chain complexes*

$$0 \to A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \to 0$$

induces a long exact sequence *of linear maps*

$$\cdots \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to \cdots \ .$$

This long exact sequence is moreover natural in the short exact sequence.

2.4. Koszul conventions. Consider two maps $f : A_* \to B_*$ and $g : C_* \to D_*$ of respective degree $|f|$ and $|g|$. In the rest of this course, we will denote $f \otimes g : A_* \otimes C_* \to B_* \otimes D_*$ the map of degree $|f| + |g|$ defined as

$$(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b) \ .$$

This convention is called the ***Koszul sign convention.*** The motto is that "moving a in front of g adds the sign $(-1)^{|g||a|}$".

EXAMPLE 2.20. Under the Koszul sign convention, we have in particular that

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (-1)^{|f_1||g_2|} (f_2 \otimes f_1) \circ (g_2 \otimes g_1)$$

and that $\partial_{C \otimes D} = \partial_C \otimes \mathrm{id}_D + \mathrm{id}_C \otimes \partial_D$.

CHAPTER 2

Standard algebraic structures

We work with homologically graded dg vector spaces in this chapter. We will denote `dgVect` the category of dg vector spaces with chain maps between them, and `grVect` the category of graded vector spaces with linear maps of degree 0 between them.

1. Associative algebras and coalgebras

1.1. Associative algebras.

1.1.1. *Definitions.*

DEFINITION 1.1. *(i) Let A be a vector space. An* associative algebra *structure on A corresponds to the datum of a map $\mu : A \otimes A \to A$ such that $\mu(\mu \otimes \mathrm{id}) = \mu(\mathrm{id} \otimes \mu)$.*

(ii) A morphism of algebras *is defined to be a linear map $f : A \to B$ such that $f\mu_A = \mu_B(f \otimes f)$.*

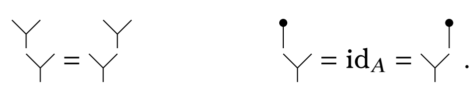
This definition is equivalent to the standard axiomatic definition of an algebra, whose multiplication is defined to be $a \cdot b := \mu(a, b)$. The field $\mathbb{K}$ is moreover naturally a $\mathbb{K}$-algebra.

DEFINITION 1.2. *(i) An algebra A is said to be* unital *if there exists a morphism of algebras $u : \mathbb{K} \to A$ such that $\mu(u, \cdot) = \mathrm{id} = \mu(\cdot, u)$.*

(ii) A morphism of unital algebras *is a morphism $f : A \to B$ between unital algebras such that $fu_A = u_B$.*

Setting $u(1_{\mathbb{K}}) = 1_A$, we recover the usual axiom for a unital algebra. A morphism of unital algebras is then simply a morphism of algebras $f : A \to B$ such that $f(1_A) = 1_B$.

Representing the multiplication μ and the unit map u respectively as  and , the axioms for a unital algebra read as



This viewpoint will be used systematically in the rest of this course.

We denote `As-alg` the category of associative algebras with morphisms of algebras between them, and `uAs-alg` the category of unital associative algebras with morphisms of unital algebras between them.

DEFINITION 1.3. *A unital algebra A is said to be* augmented *if there exists a morphism of unital algebras $\varepsilon : A \to \mathbb{K}$. This morphism is then called an* augmentation *of A.*

1.1.2. *Free associative algebra.* Let V be a vector space. Given $v_1, \ldots, v_n \in V$, we will denote $v_1 \cdots v_n := v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$ in the rest of this section.

Definition 1.4. *The* free tensor (unital) algebra *on V is defined to be the vector space*

$$T(V) := \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

endowed with the concatenation multiplication

$$v_1 \cdots v_n \otimes v_{n+1} \cdots v_{n+m} \longmapsto v_1 \cdots v_n v_{n+1} \cdots v_{n+m}$$

and unit the inclusion in the first summand $\mathbb{K} \hookrightarrow T(V)$.

The *free reduced tensor algebra* on V is the vector space

$$\overline{T}(V) := V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

endowed with the concatenation multiplication. It is not unital.

Proposition 1.5. *The free tensor algebra construction defines a functor $T(-) : \mathtt{Vect} \to \mathtt{uAs\text{-}alg}$ which is a left adjoint to the forgetful functor $\mathtt{uAs\text{-}alg} \to \mathtt{Vect}$. In other words, for every vector space V and unital algebra A, there is a natural bijection*

$$\mathrm{Hom}_{\mathtt{uAs\text{-}alg}}(T(V), A) = \mathrm{Hom}_{\mathtt{Vect}}(V, A) \ .$$

Similarly, we have that $\mathrm{Hom}_{\mathtt{As\text{-}alg}}(\overline{T}(V), A) = \mathrm{Hom}_{\mathtt{Vect}}(V, A)$.

1.2. Coassociative coalgebras.

1.2.1. *Definitions.*

Definition 1.6. (i) *Let C be a vector space. A* coassociative coalgebra *structure on A corresponds to the datum of a map $\Delta : C \to C \otimes C$ such that* $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.

(ii) *A morphism of coalgebras is defined to be a linear map $f : C_1 \to C_2$ such that $(f \otimes f)\Delta_{C_1} = \Delta_{C_2} f$.*

We will denote $\mathtt{As\text{-}cog}$ the category of coassociative coalgebras with morphisms of coalgebras. The field $\mathbb{K}$ is in particular a coassociative coalgebra.

Proposition 1.7. *(i) The dual of a coassociative coalgebra is an associative algebra.*
(ii) The dual of a finite-dimensional associative algebra is a coassociative coalgebra.

The image of an element $c \in C$ under the comultiplication Δ has the form

$$\Delta(c) := \sum_{i=1}^{n} c_i^{(1)} \otimes c_i^{(2)} \ .$$

This is often written for short as $\Delta(c) = c^{(1)} \otimes c^{(2)}$. Beware that it is a mere notation which does not mean that $\Delta(c)$ is a pure element of $C \otimes C$. It is called *Sweedler's notation.* Under this notation, a morphism of coalgebras is then simply a linear map $f : C_1 \to C_2$ such that $f(c)^{(1)} \otimes f(c)^{(2)} = f(c^{(1)}) \otimes f(c^{(2)})$.

If we write

$$(\Delta \otimes \mathrm{id})\Delta(c) = \Delta(c^{(1)}) \otimes c^{(2)} = c^{(1)(1)} \otimes c^{(2)(1)} \otimes c^{(2)}$$

and

$$(\mathrm{id} \otimes \Delta)\Delta(c) = c^{(1)} \otimes \Delta(c^{(2)}) = c^{(1)} \otimes c^{(1)(2)} \otimes c^{(2)(2)} ,$$

the coassociativity relation can be rephrased as

$$c^{(1)(1)} \otimes c^{(2)(1)} \otimes c^{(2)} = c^{(1)} \otimes c^{(1)(2)} \otimes c^{(2)(2)} .$$

This is often written as $(\mathrm{id} \otimes \Delta)\Delta(c) = (\Delta \otimes \mathrm{id})\Delta(c) = c^{(1)} \otimes c^{(2)} \otimes c^{(3)}$.

More generally, we denote $\Delta^n : C \to C^{\otimes n+1}$ the *iterated coproduct*

$$\Delta^n := (\Delta \otimes \mathrm{id}^{\otimes n-1}) \circ \cdots \circ \Delta ,$$

and write $\Delta^n(c) = c^{(1)} \otimes \cdots \otimes c^{(n+1)}$.

DEFINITION 1.8. *(i) A coassociative coalgebra C endowed with a map $\varepsilon : C \to \mathbb{K}$ such that $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}_C = (\mathrm{id} \otimes \varepsilon) \circ \Delta$ is said to be* counital. *The map ε is then called the* counit *of C.*

(ii) A morphism of coalgebras $f : C_1 \to C_2$ between counital coalgebras C_1 and C_2 is a morphism of counital coalgebras *if it preserves the counits, i.e. $\varepsilon_{C_1} = \varepsilon_{C_2} f$.*

(iii) A counital coalgebra C endowed with a morphism of counital coalgebras $u : \mathbb{K} \to C$ is said to be coaugmented. *The map u is then called its* coaugmentation.

(iv) A morphism of counital coalgebras $f : C_1 \to C_2$ between coaugmented coalgebras C_1 and C_2 is a morphism of coaugmented coalgebras *if it preserves the coaugmentations, i.e. $u_{C_1} = f u_{C_2}$.*

Writing the coproduct and counit respectively as ⅄ and ⫯, the axioms of a counital coalgebra can be represented as

Given a coaugmented coalgebra C, write $\overline{C} := \mathrm{Ker}(\varepsilon)$ and $1_C := u(1_{\mathbb{K}})$. Then $\Delta(1_C) = 1_C \otimes 1_C$ and the vector space C decomposes as $C := \overline{C} \oplus \mathbb{K}1_C$. The vector space $\overline{C}$ endowed with the coproduct $\overline{\Delta} : \overline{C} \to \overline{C} \otimes \overline{C}$,

$$\overline{\Delta}(x) := \Delta(x) - x \otimes 1_C - 1_C \otimes x ,$$

is then a coassociative coalgebra.

We denote $\overline{\Delta}^n : \overline{C} \to \overline{C}^{\otimes n+1}$ the iterated coproduct of $\overline{C}$.

DEFINITION 1.9. *The coradical filtration of C is defined for $r \geqslant 0$ as*

$$F_r C := \mathbb{K}1_C \oplus \{x \in \overline{C},\ \overline{\Delta}^r(x) = 0\} ,$$

where $F_0 C := \mathbb{K}1_C$.

We point out that it is indeed a *filtration*, as $F_r C \subset F_{r+1} C$ for all $r \geqslant 0$.

DEFINITION 1.10. *A coaugmented coalgebra C is said to be* conilpotent, *if $C = \cup_{r \geqslant 0} F_r C$.*

We denote conil-As-cog the category of conilpotent coalgebras with morphisms of coaugmented coalgebras between them. The coalgebra $\mathbb{K}$ is in particular a conilpotent coalgebra, with counit and coaugmentation the identity map.

1.2.2. *Cofree coassociative coalgebra.*

Definition 1.11. *The* cofree tensor (conilpotent) coalgebra *on a vector space V is defined to be the vector space*

$$T^c(V) := \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \oplus \cdots$$

endowed with the deconcatenation comultiplication

$$\Delta : v_1 \cdots v_n \longmapsto \sum_{i=0}^{n} v_1 \cdots v_i \otimes v_{i+1} \cdots v_n$$

where $\Delta(1) = 1 \otimes 1$, with counit the natural projection $T^c(V) \twoheadrightarrow \mathbb{K}$ and with coaugmentation the natural inclusion $\mathbb{K} \hookrightarrow T^c(V)$.

The cofree tensor coalgebra $T^c(V)$ is indeed conilpotent: its coradical filtration is in particular given by $F_r T^c(V) = \bigoplus_{n \leqslant r} V^{\otimes n}$.

Proposition 1.12. *The free tensor coalgebra construction defines a functor $T^c(-)$: $\mathtt{Vect} \to \mathtt{conil\text{-}As\text{-}cog}$ which is a right adjoint to the functor $\overline{(-)}$: $\mathtt{conil\text{-}As\text{-}cog} \to \mathtt{Vect}$ mapping C to $\overline{C}$. In other words, for every vector space V and conilpotent coalgebra C, there is a natural bijection*

$$\mathrm{Hom}_{\mathtt{Vect}}(\overline{C}, V) = \mathrm{Hom}_{\mathtt{conil\text{-}As\text{-}cog}}(C, T^c(V)) \ .$$

Proof. Let $f : C \to T^c(V)$ be a morphism of conilpotent coalgebras. We denote for every $n \geqslant 0$, $f_n := \pi_{V^{\otimes n}} f$. The fact that f preserves the counit and coaugmentation implies the following:

(1) $f_0(c) = \varepsilon_C(c)$ for every $c \in C$,
(2) $f(1_C) = 1_{\mathbb{K}}$,
(3) f maps $\overline{C}$ to $\overline{T}^c(V) = \bigoplus_{n \geqslant 1} V^{\otimes n}$.

We will denote $\overline{f} : \overline{C} \to \overline{T}^c(V)$ this induced morphism and $\overline{f}_n := \pi_{V^{\otimes n}} \overline{f}$ for $n \geqslant 1$. Every $c \in C$ moreover decomposes as $c = \varepsilon(c)1_C + \overline{c}$ where $\overline{c} := c - \varepsilon(c)1_c \in \overline{C}$. For every $c \in C$ we then have that

$$f(c) = \varepsilon(c) + \overline{f}(\overline{c}) \ .$$

The morphism $\overline{f}$ is in fact a morphism of coassociative coalgebras $(\overline{C}, \overline{\Delta}_C) \to (\overline{T}^c V, \overline{\Delta}_{T^c V})$. This implies that for all $n \geqslant 1$, the following diagram commutes

$$\begin{array}{ccc} \overline{C} & \xrightarrow{\overline{f}} & \overline{T}^c(V) \\ {\scriptstyle \overline{\Delta}_C^{n-1}}\downarrow & & \downarrow{\scriptstyle \overline{\Delta}_{T^c V}^{n-1}} \\ \overline{C}^{\otimes n} & \xrightarrow{\overline{f}^{\otimes n}} & \overline{T}^c(V)^{\otimes n} \end{array} .$$

Projecting to the factor $V^{\otimes n}$ of $\overline{T}^c(V)^{\otimes n}$ yields for every $\overline{c} \in \overline{C}$ the equality

$$\overline{f}_n(\overline{c}) = \overline{f}_1(\overline{c}^{(1)}) \otimes \cdots \otimes \overline{f}_1(\overline{c}^{(n)}) \ ,$$

where the Sweedler's notation are w.r.t. $\overline{\Delta}_C$. Hence the morphism $\overline{f}_1 : \overline{C} \longrightarrow V$ completely determines $f : C \longrightarrow T^c(V)$.

Given conversely a morphism $f : \overline{C} \longrightarrow V$, one can define a morphism of conilpotent coalgebras $F : C \longrightarrow T^c(V)$ by the formula

$$F(c) := \varepsilon(c) + \sum_{n=1}^{+\infty} f(\overline{c}^{(1)}) \otimes \cdots \otimes f(\overline{c}^{(n)}) \ .$$

This morphism is well-defined: C is conilpotent hence the above sum is always finite. □

1.3. Differential graded (co)associative (co)algebras.

1.3.1. *Definitions.*

Definition 1.13. *(i) Let A be a graded vector space. A* graded associative algebra *structure on A corresponds to the datum of a map $\mu : A \otimes A \longrightarrow A$ such that (A, μ) is an associative algebra and the map μ has degree 0, i.e. for all $p, q \in \mathbb{Z}$, $\mu(A_p \otimes A_q) \subset A_{p+q}$.*

(ii) Let (A, ∂) be a dg vector space. A differential graded associative algebra *or* dg algebra *structure on A corresponds to the datum of a map $\mu : A \otimes A \longrightarrow A$ such that (A, μ) is a graded associative algebra and μ is a chain map, i.e. satisfies*

$$\partial\mu = \mu(\partial \otimes \mathrm{id}) + \mu(\mathrm{id} \otimes \partial) \ .$$

The equality of Item (ii) reads on two elements $a_1, a_2 \in A$ as

$$\partial\mu(a_1, a_2) = \mu(\partial a_1, a_2) + (-1)^{|a_1|}\mu(a_1, \partial a_2) \ .$$

The notions of *graded coassociative coalgebra* and of *dg coalgebra* can be defined in a similar fashion. We moreover point out that a standard (co)associative (co)algebra can then simply be seen as a graded associative coalgebra concentrated in degree 0.

Example 1.14. Let X be a topological space. The singular cochains $C^*(X)$ form a dg algebra for the *cup product* $\cup$ and the singular chains $C_*(X)$ form a dg coalgebra for the *Alexander-Whitney coproduct.*

Definition 1.15. *(i) Given A_1 and A_2 two graded associative algebras, a* morphism of graded associative algebras $A_1 \longrightarrow A_2$ *is defined to be a morphism of algebras of degree 0.*

(ii) Given A_1 and A_2 two dg algebras, a morphism of dg algebras $A_1 \longrightarrow A_2$ *is defined to be a morphism of algebras which is a chain map.*

The notion of *morphism of graded coassociative coalgebras* and of *morphism of dg coalgebras* is defined in a similar fashion. We will respectively denote `As-alg` and `As-cog` the category of dg algebras and the category of dg coalgebras. These notations are in conflict with the notation of the category of associative algebras and the category of coassociative coalgebras, but it will always be clear from the context which categories they refer to.

Definition 1.16. *(i) A graded associative algebra (A, μ) is* unital *if there exists an element $e \in A_0$ which makes (A, μ) into an associative algebra.*

(ii) A dg algebra is unital if it is unital *as a graded associative algebra and its unit satisfies $\partial(e) = 0$.*

1.3.2. *Free graded associative algebra.* Let V be a graded vector space. The grading on V induces a grading on the free tensor algebra $T(V)$ defined as $|v_1 \cdots v_n| = \sum_{i=1}^{n} |v_i|$. This algebra is then a graded associative algebra with respect to this grading.

Proposition 1.17. *The free tensor algebra construction defines a functor $T(-) : \mathtt{grVect} \to \mathtt{gr\text{-}uAs\text{-}alg}$ which is a left adjoint to the forgetful functor $\mathtt{gr\text{-}uAs\text{-}alg} \to \mathtt{grVect}$. In other words, for every graded vector space V and unital graded associative algebra A, there is a natural bijection*

$$\mathrm{Hom}_{\mathtt{gr\text{-}uAs\text{-}alg}}(T(V), A) = \mathrm{Hom}_{\mathtt{grVect}}(V, A) .$$

An analogous result holds for the free tensor coalgebra $T^c(V)$ seen as a conilpotent graded coassociative coalgebra with respect to the same grading.

We also point out that an alternative grading can be defined on $T(V)$, by setting the degree of an element of $V^{\otimes n}$ to be n. This grading is usually called the *weight* and the elements of weight greater that n will be denoted $T(V)^{\geqslant n}$. The algebra $T(V)$ is then a *weight-graded associative algebra*, meaning that the product preserves both the grading defined in the previous paragraph and the weight. The same holds for the coalgebra $T^c(V)$.

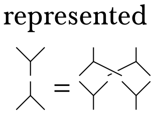
1.4. Bialgebras and Hopf algebras. Given two algebras A_1 and A_2, the tensor product $A_1 \otimes A_2$ can naturally be endowed with an algebra structure by setting $(x_1 \otimes x_2) \cdot (y_1 \otimes y_2) = x_1 \cdot y_1 \otimes x_2 \cdot y_2$. Using the switching map $\tau : A_1 \otimes A_2 \to A_2 \otimes A_1$ of Section 2.1, this can be rewritten as $\mu_{A_1 \otimes A_2} := (\mu_{A_1} \otimes \mu_{A_2})(\mathrm{id} \otimes \tau \otimes \mathrm{id})$. This last equality can be used to define an algebra structure on $A_1 \otimes A_2$ in the dg setting. We also point out that if A_1 and A_2 are unital, then $A_1 \otimes A_2$ is unital with unit $1_{A_1} \otimes 1_{A_2}$.

Definition 1.18. *Given a vector space H, a* bialgebra *structure on H is defined to be the data of a unital algebra structure (H, μ, u) on H and of a counital coalgebra structure (H, Δ, ε) on H such that Δ and ε are morphisms of unital algebras, or equivalently such that μ and u are morphisms of counital coalgebras.*

The fact that Δ and ε are morphisms of unital algebras can be rephrased using the four following axioms

(1) $\Delta\mu = (\mu \otimes \mu)(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta \otimes \Delta)$, the *Hopf relation.*
(2) $\Delta u = u \otimes u$.
(3) $\varepsilon\mu = \varepsilon \otimes \varepsilon$.
(4) $\varepsilon u = \mathrm{id}_{\mathbb{K}}$.

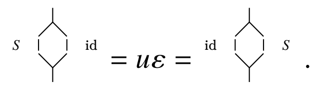
The Hopf relation can in particular be represented as

 .

DEFINITION 1.19. *A* Hopf algebra *is defined to be a bialgebra* $(H, \mu, u, \Delta, \varepsilon)$ *for which there exists a map* $S : H \to H$*, called the* antipode, *satisfying the relation*

$$\mu(S \otimes \mathrm{id})\Delta = u\varepsilon = \mu(\mathrm{id} \otimes S)\Delta \ .$$

The antipode relation can be represented as



2. Commutative algebras

2.1. Symmetric groups actions on tensor products. Given a graded vector space V, the symmetric group $\mathfrak{S}_n$ acts on the left of $V^{\otimes n}$ as

$$\tau_\sigma(v_1 \otimes \cdots \otimes v_n) = (-1)^\varepsilon v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)} \ ,$$

where $\sigma \in \mathfrak{S}_n$ and $(-1)^\varepsilon$ is the sign obtained by rearranging $v_1 \otimes \cdots \otimes v_n$ into $v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$ under the Koszul sign convention. The map $\tau := \tau_{(12)} : V \otimes V \to V \otimes V$ acts for instance on elements $v, w \in V$ as

$$\tau(v \otimes w) = (-1)^{|v|\cdot|w|} w \otimes v \ .$$

The symmetric group $\mathfrak{S}_n$ acts also on the right of $V^{\otimes n}$ as

$$(v_1 \otimes \cdots \otimes v_n) \cdot \sigma = (-1)^\varepsilon v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \ .$$

2.2. Commutative algebras.

2.2.1. *Definitions.*

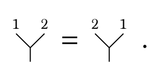
DEFINITION 2.1. *(i) A (graded) associative algebra is said to be* (graded) commutative *if its multiplication* μ *satisfies* $\mu = \mu\tau$.

(ii) A dg algebra is said to be graded commutative *if it is graded commutative as a graded associative algebra. It is then called a* cdg algebra *for short.*

In the non-graded case with recover the usual equality $x \cdot y = y \cdot y$. In the graded case, the axioms of a graded commutative algebra read as

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \qquad\qquad x \cdot y = (-1)^{|x||y|} y \cdot x \tag{2.2.1}$$

The second condition is usually represented as



REMARK 2.2. A graded commutative algebra is sometimes referred to as a *commutative superalgebra* or *supercommutative algebra*. This terminology, coming from the theory of supersymmetry in theoretical physics, will not be used in this course.

EXAMPLE 2.3. The singular cohomology $(H^*(X), \cup)$ of a topological space X is a graded commutative algebra. The singular cochains $(C^*(X), \cup)$ however do not form a cdg algebra, as the cup product is not graded commutative on the chain level. It is however graded commutative *up to homotopy*, i.e. there exists a degree -1 map $h : C^*(X) \otimes C^*(X) \to C^*(X)$ such that $\partial h + h\partial = \cup - \cup\tau$.

2.2.2. *Free graded commutative algebra.*

DEFINITION 2.4. *The free graded commutative algebra ΛV on a graded vector space V is defined to be the unital graded commutative algebra whose underlying vector space is*

$$T(V)/\langle x \otimes y - (-1)^{|x||y|} y \otimes x\rangle \ ,$$

whose grading is defined as in Section 1.3.2 and whose multiplication is given by the concatenation product.

The free graded commutative algebra ΛV admits the same weight grading as in Section 1.3.2, which determines a decomposition $\Lambda V := \bigoplus_{i=0}^{+\infty} \Lambda^i V$.

Proposition 2.5. *The functor* $\Lambda(-) : \mathtt{grVect} \to \mathtt{gr\text{-}uCom\text{-}alg}$ *is left adjoint to the forgetful functor* $\mathtt{gr\text{-}uCom\text{-}alg} \to \mathtt{grVect}$. *In other words, for every graded vector space V and graded commutative algebra C, there is a natural bijection*

$$\mathrm{Hom}_{\mathtt{gr\text{-}uCom\text{-}alg}}(\Lambda V, C) = \mathrm{Hom}_{\mathtt{grVect}}(V, C) \ .$$

2.3. Sullivan models and rational homotopy theory. In this section, we work with cohomological conventions and set $\mathbb{K} = \mathbb{Q}$. We moreover assume that all cdg algebras are unital and that all morphisms are morphisms of unital cdg algebras.

2.3.1. *Sullivan models.*

DEFINITION 2.6. *A* Sullivan algebra *is a cdg algebra of the form* $(\Lambda V, \partial)$ *such that*

(1) $V = V^{\geqslant 1}$ *is a graded vector space concentrated in degree* $\geqslant 1$,
(2) $V = \cup_{k\geqslant 0} V(k)$ *where* $V(k)$ *is an increasing sequence of graded vector spaces* $V(0) \subset V(1) \subset \cdots$,
(3) $\partial(V(0)) = 0$ *and* $\partial(V(k)) \subset \Lambda V(k-1)$ *for* $k \geqslant 1$.

Item 3 is called the *nilpotence condition.*

DEFINITION 2.7. *(i) A* Sullivan model *for a cdg algebra C is a quasi-isomorphism* $(\Lambda V, \partial) \xrightarrow{\sim} C$.
(ii) A Sullivan model is said to be minimal *if* $\partial(V) \subset \Lambda^{\geqslant 2} V$.

In Definition 2.7, *quasi-isomorphism* means that the map is a morphism of cdg algebras which induces an isomorphism in cohomology.

Proposition 2.8. *Every cdg algebra A satisfying $H^0(A) = \mathbb{K}$ admits a minimal Sullivan model. It is moreover unique up to isomorphism.*

PROOF. The unicity up to isomorphism of a minimal Sullivan model stems from the fact that if two Sullivan models $(\Lambda V_1, \partial_1)$ and $(\Lambda V_2, \partial_2)$ are quasi-isomorphic, then (V_1, ∂_1) and

(V_2, ∂_2) are quasi-isomorphic. The minimality assumption then ensures that $\partial_i = 0$ on V_i, hence that V_1 and V_2 are isomorphic. □

2.3.2. *Sullivan models of topological spaces.* Recall from Example 2.3 that the singular cochains $C^*(X, \mathbb{Q})$ form a dg algebra which is not graded commutative.

THEOREM 2. *For any topological space X, there exists a dg algebra $D(X)$ and a cdg algebra $A_{PL}(X)$ that fit into the following diagram of quasi-isomorphisms of dg algebras*

$$C^*(X, \mathbb{Q}) \xrightarrow{\sim} D(X) \xleftarrow{\sim} A_{PL}(X) \ .$$

This diagram is moreover natural in X.

The cdg algebra $A_{PL}(X)$ is called the *algebra of polynomial differential forms* on X.

DEFINITION 2.9. *A* (minimal) Sullivan model *for a path connected topological space X is defined to be a (minimal) Sullivan model for the cdg algebra $A_{PL}(X)$.*

DEFINITION 2.10. *(i) Two cdg algebras A and B are said to be* weakly equivalent *if there exists a zig-zag of quasi-isomorphisms of cdg algebras*

$$A = C_0 \leftarrow C_1 \rightarrow \cdots \leftarrow C_{n-1} \rightarrow C_n = B \ .$$

(ii) A cdg algebra C satisfying $H^0(C) = \mathbb{K}$ is said to be formal *if it is weakly equivalent to the cdg algebra $H^*(C)$ with trivial differential.*

(iii) A path connected topological space X is said to be formal *if the cdg algebra $A_{PL}(X)$ is formal.*

Proposition 2.11. *Two cdg algebras A and B satisfying $H^0(A) = H^0(B) = \mathbb{K}$ are weakly equivalent if and only if their minimal Sullivan models are isomorphic.*

The minimal Sullivan model of a formal topological space X can thereby be directly computed from its cohomology $H^*(X, \mathbb{Q})$.

2.3.3. *Rational homotopy theory.*

DEFINITION 2.12. *(i) A topological space is of* finite rational type *if for every $n \geqslant 0$, the vector space $H^n(X, \mathbb{Q})$ is finite-dimensional.*

(ii) Two topological spaces are said to have the same rational homotopy type *if their exists a zig-zag of continuous maps*

$$X = X_0 \leftarrow X_1 \rightarrow \cdots \leftarrow X_{n-1} \rightarrow X_n = Y$$

inducing a weak equivalence zig-zag between $H^*(X, \mathbb{Q})$ and $H^*(Y, \mathbb{Q})$ in rational cohomology.

THEOREM 3. *Two simply connected spaces of finite rational type have same rational homotopy type if and only if their minimal Sullivan models are isomorphic as cdg algebras.*

In other words, the rational homotopy type of a simply connected topological spaces of finite rational type is completely determined by its minimal model.

DEFINITION 2.13. *Let X be a topological space. We define the* free loop space *LX to be the space of continuous maps $\mathbb{S}^1 \rightarrow X$.*

Theorem 4. *Let X be a simply connected topological space and $(\Lambda V, \partial)$ a minimal model of X. Then its free loop space LX admits a minimal model of the form $(\Lambda V \otimes \Lambda sV, \delta)$ where sV denotes the suspension of V.*

This last theorem is an important tool for the computation of singular cohomologies of free loop spaces. See also Section 3.4.2.

3. Lie algebras

3.1. Lie algebras.

3.1.1. *Lie algebras.*

Definition 3.1. *A* Lie algebra *is defined to be a vector space* $\mathfrak{g}$ *endowed with a map* $[\cdot,\cdot] : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$, *called the* Lie bracket, *such that*

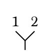
(1) $[x, y] = -[y, x]$ *: the bracket is* antisymmetric.
(2) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ *: the bracket satifies the* Jacobi identity.

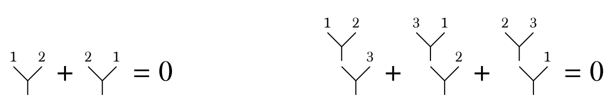
If we denote $c := [\cdot,\cdot]$ these two conditions can be rephrased as

$$c \circ (\mathrm{id} + \tau_{(12)}) = 0 \qquad c(c \otimes \mathrm{id}) \circ (\mathrm{id} + \tau_{(123)} + \tau_{(321)}) = 0 ,$$

where we point out that $\tau^2_{(12)} = \mathrm{id}_{\mathfrak{g}^{\otimes 2}}$, $\tau^2_{(123)} = \tau_{(321)}$ and $\tau^3_{(123)} = \mathrm{id}$. We will denote these last two equalities as

$$c^{\mathrm{id}+(12)} = 0 \qquad c(c \otimes \mathrm{id})^{\mathrm{id}+(123)+(321)} = 0 , \tag{3.1.1}$$

Writing the bracket $[\cdot,\cdot]$ as  , they can also be represented as

(3.1.2) 

Remark 3.2. The Jacobi relation is also sometimes replaced by the *Leibniz relation*

$$[[x, y], z] = [[x, z], y] + [x, [y, z]] ,$$

which amounts to say that for all z the map $[\cdot, z] : \mathfrak{g} \to \mathfrak{g}$ is a derivation for the bracket $[\cdot,\cdot]$, i.e. satisfies the Leibniz relation w.r.t. this bracket.

Definition 3.3. *Given two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, a morphism of Lie algebras is defined to be a linear map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that $fc_1 = c_2(f \otimes f)$, i.e. such that $[f(x), f(y)]_{\mathfrak{g}_2} = f([x, y]_{\mathfrak{g}_1})$.*

We denote `Lie-alg` the category of Lie algebras with morphisms of Lie algebras.

Any associative algebra A can be endowed with a Lie algebra structure, by setting

$$[x, y] := xy - yx .$$

This defines a forgetful functor `As-alg` $\to$ `Lie-alg`.

Definition 3.4. *Let* $\mathfrak{g}$ *be a Lie algebra and denote* $T(\mathfrak{g})$ *the free tensor algebra over the vector space* $\mathfrak{g}$*. The* universal enveloping algebra *of* $\mathfrak{g}$ *is defined to be the quotient of* $T(\mathfrak{g})$ *by the two-sided ideal generated by the elements*

$$x \otimes y - y \otimes x - [x, y] \ ,$$

and is denoted $U(\mathfrak{g})$.

Proposition 3.5. *The universal enveloping algebra construction defines a functor* $U(-) : \mathtt{Lie\text{-}alg} \to \mathtt{uAs\text{-}alg}$ *which is left adjoint to the forgetful functor* $\mathtt{uAs\text{-}alg} \to \mathtt{Lie\text{-}alg}$. *In other words, for every unital associative algebra* A *and Lie algebra* $\mathfrak{g}$*, there is a natural bijection*

$$\mathrm{Hom}_{\mathtt{uAs\text{-}alg}}(U(\mathfrak{g}), A) = \mathrm{Hom}_{\mathtt{Lie\text{-}alg}}(\mathfrak{g}, A) \ .$$

3.1.2. *Lie algebra of a Lie group.* Let M be a smooth manifold. We denote $\Gamma(TM)$ the vector space of smooth vector fields on M, i.e. the vector space of sections of the tangent bundle $TM \to M$. Recall that $\Gamma(TM)$ is isomorphic to the vector space $\mathrm{Der}(\mathscr{C}^\infty(M, \mathbb{R}))$ of derivations of the algebra $\mathscr{C}^\infty(M, \mathbb{R})$. For $X \in \Gamma(TM)$ and f a smooth function, we will thereby denote Xf the image of f under the derivation X. The *Lie bracket* of two vector fields X and Y is then defined to be the vector field $[X, Y]$ acting on smooth functions as

$$[X, Y]f = X(Yf) - Y(Xf) \ .$$

Proposition 3.6. *The vector space of smooth vector fields* $\Gamma(TM)$ *endowed with the Lie bracket is a Lie algebra.*

Let now G be a Lie group, whose unit we denote $e \in G$. For $g \in G$ we denote $L_g : G \to G$ the left multiplication, i.e. $L_g(h) = g \cdot h$. A vector field X on G is then said to be *left-invariant* if for every $g \in G$ it is invariant under L_g, i.e. $(L_g)_*(X) = X$ or equivalently $d(L_g)_h(X_h) = X_{gh}$ for all $h \in G$. We denote $\mathrm{Lie}(G)$ the vector space of left-invariant vector fields on G.

Proposition 3.7.
(i) The vector space $\mathrm{Lie}(G)$ *is a finite-dimensional vector space of dimension* $\dim(G)$*. More precisely, the linear map* $\mathrm{Lie}(G) \to T_eG$ *mapping* X *to* X_e *is an isomorphism.*
(ii) The vector space $\mathrm{Lie}(G)$ *is stable under the Lie bracket, i.e. is a finite-dimensional Lie subalgebra of* $\Gamma(TG)$.

The functor Lie from the category of real finite-dimensional Lie algebras to the category of simply-connected Lie groups is in fact an equivalence of category:

Theorem 5.
(i) Let G *and* H *be Lie groups. If* G *is simply-connected, then every morphism of Lie algebras* $f : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ *admits a unique lift to a morphism of Lie groups* $F : G \to H$ *such that* $\mathrm{Lie}(F) = f$.
(ii) For every real finite-dimensional Lie algebra $\mathfrak{g}$ *there exists a unique simply-connected Lie group* G *such that* $\mathrm{Lie}(G) = \mathfrak{g}$.

Item (i) and Item (ii) are respectively called *Lie's second theorem* and the *Cartan-Lie theorem.*

3.1.3. *dg Lie algebras and Maurer-Cartan elements.*

DEFINITION 3.8. *A* dg Lie algebra *is defined to be a dg vector space* $\mathfrak{g}$ *endowed with a linear map* $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ *such that*

(1) The map $[\cdot,\cdot]$ *is a chain map.*
(2) $[x, y] = (-1)^{|x||y|+1}[y, x]$.
(3) $(-1)^{|x||z|}[[x, y], z] + (-1)^{|x||y|}[[y, z], x] + (-1)^{|y||z|}[[z, x], y] = 0$.

We point out that Items 2 and 3 can be reformulated using the Koszul sign rules as in Equation (3.1.2). The notion of a *graded Lie algebra* can be defined similarly. A *morphism of dg Lie algebras* is moreover defined as a chain map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ which preserves the Lie bracket.

DEFINITION 3.9. *A* Maurer-Cartan element *of a dg Lie algebra* $\mathfrak{g}$ *over a field of characteristic* $\neq 2$ *is defined to be an element* $\alpha \in \mathfrak{g}_{-1}$ *of degree* -1 *such that*

$$\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0 \ .$$

The *set of Maurer-Cartan elements* of a dg Lie algebra $\mathfrak{g}$ will be denoted as $\mathrm{MC}(\mathfrak{g})$. We point out that a morphism of dg Lie algebras $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ preserves the Maurer-Cartan equation hence induces a map $\mathrm{MC}(\mathfrak{g}_1) \to \mathrm{MC}(\mathfrak{g}_2)$.

Proposition 3.10. *Given a dg Lie algebra* $\mathfrak{g}$ *and* $\alpha \in \mathrm{MC}(\mathfrak{g})$*, the map*

$$\partial^\alpha := \partial + [\alpha, \cdot]$$

is a derivation w.r.t. the bracket $[\cdot,\cdot]$ *and satisfies* $\partial^\alpha \circ \partial^\alpha = 0$.

The differential ∂^α is called the *twisted differential.* The dg Lie algebra $(\mathfrak{g}, \partial^\alpha, [\cdot,\cdot])$ will then be denoted $\mathfrak{g}^\alpha$ and called the *twisted dg Lie algebra.*

Proposition 3.11. *For a dg Lie algebra* $\mathfrak{g}$ *and* $\alpha \in \mathrm{MC}(\mathfrak{g})$*, we have that* $\beta \in \mathrm{MC}(\mathfrak{g}^\alpha)$ *if and only if* $\beta + \alpha \in \mathrm{MC}(\mathfrak{g})$.

We will see in **??** that the problem of the *deformation* of an algebraic structure is usually encoded by a dg Lie algebra whose Maurer-Cartan elements correspond to the deformations of the considered structure.

3.2. pre-Lie algebras.

DEFINITION 3.12. *A* pre-Lie algebra *is defined to be a vector space* A *endowed with a map* $\{\cdot,\cdot\} : A \otimes A \to A$ *satisfying the relation*

$$\{\{x, y\}, z\} - \{x, \{y, z\}\} = \{\{x, z\}, y\} - \{x, \{z, y\}\} \ .$$

An associative algebra is in particular a pre-Lie algebra with bracket its multiplication.

Proposition 3.13. *Let* $(A, \{\cdot,\cdot\})$ *be a pre-Lie algebra. The map* $[\cdot,\cdot] : A \otimes A \to A$ *defined as* $[x, y] := \{x, y\} - \{y, x\}$ *endows* A *with a Lie algebra structure.*

A dg pre-Lie algebra $(A, \partial, \{\cdot,\cdot\})$ is defined similarly. The bracket of the induced dg Lie algebra then reads as $[x, y] := \{x, y\} - (-1)^{|x||y|}\{y, x\}$. The Maurer-Cartan equation for a dg pre-Lie algebra hence reads as $\partial\alpha + \{\alpha, \alpha\} = 0$.

3.3. Poisson algebras.

Definition 3.14. *A* Poisson algebra *structure on a vector space P corresponds to the data of a multiplication $\mu : P \otimes P \to P$ and of a bracket $[\cdot,\cdot] : P \otimes P \to P$ such that*

(1) (P, μ) is an associative algebra,
(2) $(P, [\cdot,\cdot])$ is a Lie algebra,
(3) for all $x \in P$ the map $[x, \cdot] : P \to P$ is a derivation w.r.t. the multiplication, i.e.

$$[x, y \cdot z] = [x, y] \cdot z + y \cdot [x, z] \ .$$

The relation of Item 3 is called the *Poisson identity.*

A *morphism of Poisson algebras* is defined to be a linear map $f : P_1 \to P_2$ that commutes with both the brackets and the multiplications of P_1 and P_2. We denote `Pois` the category of Poisson algebras.

Example 3.15. See Exercise sheet 2.

3.4. Gerstenhaber and Batalin-Vilkovisky algebras.

3.4.1. *Definitions.*

Definition 3.16. *A* Gerstenhaber algebra *structure on a graded vector space G is defined to be the data of a multiplication $\mu : G \otimes G \to G$ and of a bracket $[\cdot,\cdot] : sG \otimes sG \to sG$ such that*

(1) (G, μ) is a graded commutative algebra,
(2) $(sG, [\cdot,\cdot])$ is a graded Lie algebra,
(3) for all $x \in G$ the map $[x, \cdot] : G \to G$ is a derivation w.r.t. the multiplication, i.e. satisfies

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{(|y|+1)(|x|+1)} y \cdot [x, z] \ .$$

Let us make explicit the relations that a Gerstenhaber algebra has to satisfy:

(1) the relations of Equation (2.2.1) for the graded commutative algebra (G, μ),
(2) the degree 0 map $[\cdot,\cdot] : sG \otimes sG \to sG$ induces a degree 1 map $[\cdot,\cdot] : G \otimes G \to G$ which satisfies the relations for the graded Lie algebra $(sG, [\cdot,\cdot])$,
$$[x, y] = (-1)^{(|x|+1)(|y|+1)+1}[y, x]$$
$$(-1)^{(|x|+1)(|z|+1)}[[x, y], z] + (-1)^{(|x|+1)(|y|+1)}[[y, z], x] + (-1)^{(|y|+1)(|z|+1)}[[z, x], y] = 0 \ ,$$
where we use the fact that $|x|_{sG} = |x|_G + 1$,
(3) the *Gerstenhaber relation* of Item 3.

Beware that a Gerstenhaber algebra is not the same thing as a graded Poisson algebra !

Example 3.17. See Exercise sheet 2.

DEFINITION 3.18. *A* Batalin-Vilkovisky algebra *structure or* BV algebra *structure on a graded vector space A is the data of a multiplication $\mu : A \otimes A \to A$ and of a linear map $\Delta : A \to A$, called the* BV operator, *such that*

(1) (A, μ) *is a graded commutative algebra,*
(2) Δ *has degree 1 and* $\Delta^2 = 0$,
(3) $\Delta(- \cdot - \cdot -) = (\Delta(- \cdot -) \cdot -)^{\mathrm{id}+(123)+(321)} - (\Delta(-) \cdot - \cdot -)^{\mathrm{id}+(123)+(321)}$

The notations of Item 3 are as in Equation (3.1.1). The BV relation moreover reads on elements as

$$\begin{aligned}\Delta(abc) =& \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{(|a|+1)|b|}b\Delta(ac)\\ &- \Delta(a)bc - (-1)^{|a|}a\Delta(b)c - (-1)^{|a|+|b|}ab\Delta(c)\end{aligned}$$

Proposition 3.19. *A BV algebra* (A, μ, Δ) *is in particular a Gerstenhaber algebra* $(A, \mu, [\cdot, \cdot])$ *whose bracket is defined as*

$$[a, b] = (-1)^{|a|} \Big(\Delta(a \cdot b) - (-1)^{|a|} a \cdot \Delta(b) - \Delta(a) \cdot b\Big) \ .$$

The bracket can be interpreted as the obstruction to Δ being a derivation. It can also be checked that Δ is then a derivation for $[\cdot, \cdot]$.

3.4.2. *Loop homology.* Let M be a smooth orientable closed manifold of dimension m. The free loop space LM defined in Definition 2.13 comes with an evaluation map $\mathrm{ev} : \gamma \in LM \mapsto \gamma(0) \in M$, where we define $\mathbb{S}^1$ as the quotient of $[0, 1]$ by the relation $0 = 1$.

DEFINITION 3.20. *We denote* $\mathbb{H}_*(LM) := H_{*+m}(LM)$ *and call it the* loop homology *of* M.

Consider two singular chains $\sigma_i \in C_i(LM)$ and $\sigma_j \in C_j(LM)$. We introduce the map

$$\phi := (\mathrm{ev} \circ \sigma_i, \mathrm{ev} \circ \sigma_j) : \Delta^i \times \Delta^j \to M \times M \ ,$$

and denote $D := \{(x, x), x \in M\} \subset M \times M$. The *Chas-Sullivan product* $\sigma_i \bullet \sigma_j$ is then the chain in $C_{i+j-m}(LM)$ defined as

$$\begin{aligned}\phi^{-1}(D) \times \mathbb{S}^1 &\to M\\ (\delta_1, \delta_2, t) &\mapsto \begin{cases}\sigma_1(\delta_1, 2t) \text{ if } t \in [0, 1/2]\\ \sigma_2(\delta_2, 2t - 1) \text{ if } t \in [1/2, 1]\end{cases}\end{aligned} \ .$$

REMARK 3.21. For a smooth map $\phi : M \to N$ transverse to a smooth submanifold $S \subset N$, the space $\phi^{-1}(S) \subset M$ is a submanifold of M of dimension

$$\dim(\phi^{-1}(S)) = \dim(M) + \dim(S) - \dim(N) \ .$$

Applying this idea to the map $\phi : \Delta^i \times \Delta^j \to M \times M$ and the submanifold $D \subset M \times M$, we find

$$\dim(\phi^{-1}(D)) = i + j + m - 2m = i + j - m \ .$$

For a singular chain $\sigma \in C_i(LM)$ we moreover define the chain $\Delta(\sigma) \in C_{i+1}(M)$ as

$$\begin{aligned}(\mathbb{S}^1 \times \Delta^i) \times \mathbb{S}^1 &\to M \\ (s, \delta, t) &\mapsto \sigma(\delta, s+t) \ .\end{aligned}$$

The Chas-Sullivan product and the operator Δ induce operations on loop homology

$$\bullet : \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LM) \to \mathbb{H}_*(LM) \qquad \Delta : \mathbb{H}_*(LM) \to \mathbb{H}_{*+1}(LM) \ .$$

We point out that the degree shift $H_{*+m}(LM)$ was necessary in order for $\bullet$ to have degree 0.

We refer to **[CS99]**, **[LO15]** and **[CHV06]** for more details on these two operations.

Theorem 6. *The loop homology $\mathbb{H}_*(LM)$ endowed with the Chas-Sullivan product $\bullet$ and the operator Δ is a BV algebra.*

We have proven in Exercise sheet 2 that for an associative algebra A, the Hochschild cohomology $HH^*(A, A)$ is a Gerstenhaber algebra. The same result holds for Hochschild homology of a dg algebra (see Exercise sheet 4). We also recall from Example 1.14 that the singular cochains $C^*(M)$ form a dg algebra.

Theorem 7. *We assume that M is simply connected and that* $\mathrm{char}(\mathbb{K}) = 0$. *Then there exists a BV algebra structure on $HH^*(C^*(M), C^*(M))$ such that*

(1) the BV algebras $HH^*(C^*(M), C^*(M))$ and $\mathbb{H}_*(LM)$ are isomorphic as BV algebras,
(2) the induced Gerstenhaber algebra structure on $HH^*(C^*(M), C^*(M))$ is its standard Gerstenhaber algebra structure.

CHAPTER 3

Operads

1. Operads and $\mathcal{P}$-algebras

We fix a closed symmetric monoidal category $(\mathtt{C}, \boxtimes, I)$ in this chapter. Recall from Section 1.3.1 that given n objects $C_1, \ldots, C_n$ in the monoidal category $\mathtt{C}$, two bracketings of $C_1 \boxtimes \cdots \boxtimes C_n$ are always equivalent through a unique natural equivalence made of associators. We will hence write $C_1 \boxtimes \cdots \boxtimes C_n$ for any representative of this equivalence class of bracketings.

1.1. Group actions. Let $\mathtt{C}$ be a category and $X \in \mathtt{C}$. Then the composition map $(g, f) \mapsto g \circ f$ of $\mathtt{C}$ and the identity id_X naturally endow the set $\mathtt{C}(X, X)$ with a structure of monoid.

Definition 1.1. *Let G be a group.*

(i) A left group action *of G on X is defined to be a morphism of monoids $G \to \mathtt{C}(X, X)$.*
(ii) A right group action *of G on X is defined to be a morphism of monoids $G^{op} \to \mathtt{C}(X, X)$, where G^{op} denotes the set G endowed with the multiplication $(g_1, g_2) \mapsto g_2 g_1$.*

This definition recovers the usual notions of left and right group actions when X is a set. We will write the image of $g \in G$ in $\mathtt{C}(X, X)$ as τ_g.

Definition 1.2. *(i) A morphism $f : X_1 \to X_2$ between two objects $X_1, X_2 \in \mathtt{C}$ with a left/right G-action is said to be* G-equivariant *if $f\tau_g^{X_1} = \tau_g^{X_2} f$ for every $g \in G$.*
(ii) A morphism $f : X \to Y$ between two objects $X, Y \in \mathtt{C}$ where G acts on the left/right of X is said to be G-invariant *if $f\tau_g = f$ for every $g \in G$.*

Definition 1.3. *Let V be a vector space together with a left action of a group G on V.*

(i) The vector space of coinvariants *V_G is defined as $V_G := V/\langle v - gv, g \in G\rangle$.*
(ii) The vector space of invariants *V^G is defined as $V^G := \{v \in V,\ gv = v\ \forall g \in G\}$.*

Remark 1.4. A left group action of G on V is in fact equivalent to a left $\mathbb{K}[G]$-module structure on V. Endowing $\mathbb{K}$ with its trivial right $\mathbb{K}[G]$-algebra structure, we then have that

$$V_G = \mathbb{K} \otimes_{\mathbb{K}[G]} V \qquad\qquad V^G = \mathrm{Hom}_{\mathbb{K}[G]\text{-}\mathtt{mod}}(\mathbb{K}, V)\ .$$

1.2. May's original definition. We set $\mathfrak{S}_0 := \{*\}$ to be the trivial group with one element.

Definition 1.5. *A $\mathfrak{S}$-* module *is defined to be a sequence of objects $\{\mathcal{M}(n)\}_{n \geqslant 0}$ in $\mathtt{C}$ together with a right action of $\mathfrak{S}_n$ on $\mathcal{M}(n)$ for $n \geqslant 0$.*

Let $\{\mathcal{M}(n)\}_{n\geqslant 0}$ be a $\mathfrak{S}$-module, $\boldsymbol{l} = l_1, \dots, l_k$ be a sequence of integers $\geqslant 0$ and $\sigma \in \mathfrak{S}_k$. We will denote $\mathcal{M}(\boldsymbol{l}) := \mathcal{M}(l_1) \boxtimes \cdots \boxtimes \mathcal{M}(l_s)$, $|\boldsymbol{l}| := \sum_{i=1}^{k} l_i$ and $\boldsymbol{l}\sigma := l_{\sigma(1)}, \dots, l_{\sigma(k)}$.

Definition 1.6. *For $k \geqslant 1$, $\boldsymbol{i} = i_1, \dots, i_k$ a sequence of integers $\geqslant 0$ and $\sigma \in \mathfrak{S}_k$, we define the* block permutation $\sigma_{\boldsymbol{i}} \in \mathfrak{S}_{i_1+\cdots+i_k}$ *as the permutation which permutes the k intervals*

$$[[i_1 + \cdots + i_{s-1} + 1, i_1 + \cdots + i_s]] \ , \ 1 \leqslant s \leqslant k$$

as prescribed by the permutation σ.

Definition 1.7. *An* operad *structure on a $\mathfrak{S}$-module $\{\mathcal{P}(n)\}_{n\geqslant 0}$ is defined to be the following data:*

(1) A morphism

$$\gamma_{i_1,\cdots,i_k} : \mathcal{P}(k) \boxtimes \mathcal{P}(i_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k) \to \mathcal{P}(i_1 + \cdots + i_k)$$

for all $k \geqslant 1$ and $i_1, \dots, i_k \geqslant 0$, called a composition morphism.

(2) A morphism $\eta : I \to \mathcal{P}(1)$ called the unit.

These data have to satisfy the following properties:

(1) The maps $\gamma_{i_1,\cdots,i_k}$ are equivariant under the right action of $\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_k}$, where $\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_k}$ is seen as a subgroup of $\mathfrak{S}_{i_1+\cdots+i_k}$.

(2) Given an integer $k \geqslant 1$, a sequence $\boldsymbol{i} = i_1, \dots, i_k$ of integers $\geqslant 1$ and k sequences $\boldsymbol{j}_h$ of integers $\geqslant 0$ of length i_h for $1 \leqslant h \leqslant k$, the following diagram commutes

$$\begin{array}{ccc}
\mathcal{P}(k) \boxtimes \mathcal{P}(\boldsymbol{i}) \boxtimes \mathcal{P}(\boldsymbol{j}) & \longrightarrow & \mathcal{P}(k) \boxtimes \mathcal{P}(i_1) \boxtimes \mathcal{P}(\boldsymbol{j}_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k) \boxtimes \mathcal{P}(\boldsymbol{j}_k) \\
 & & \Big\downarrow {\scriptstyle \mathrm{id}_{\mathcal{P}(k)} \boxtimes \gamma_{\boldsymbol{j}_1} \boxtimes \cdots \boxtimes \gamma_{\boldsymbol{j}_k}} \\
\Big\downarrow {\scriptstyle \gamma_{i_1,\dots,i_k} \boxtimes \mathrm{id}_{\mathcal{P}(\boldsymbol{j})}} & & \mathcal{P}(k) \boxtimes \mathcal{P}(|\boldsymbol{j}_1|) \boxtimes \cdots \boxtimes \mathcal{P}(|\boldsymbol{j}_k|) \\
 & & \Big\downarrow {\scriptstyle \gamma_{|\boldsymbol{j}_1|,\dots,|\boldsymbol{j}_k|}} \\
\mathcal{P}(i_1 + \cdots + i_k) \boxtimes \mathcal{P}(\boldsymbol{j}) & \xrightarrow{\quad \gamma_{\boldsymbol{j}} \quad} & \mathcal{P}(|\boldsymbol{j}_1| + \cdots + |\boldsymbol{j}_k|)
\end{array}$$

where $\boldsymbol{j}$ denotes the concatenated sequence $\boldsymbol{j} = \boldsymbol{j}_1 \dots \boldsymbol{j}_k$ and the top arrow is the composition of braidings rearranging $\mathcal{P}(\boldsymbol{i}) \boxtimes \mathcal{P}(\boldsymbol{j})$ into $\mathcal{P}(i_1) \boxtimes \mathcal{P}(\boldsymbol{j}_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k) \boxtimes \mathcal{P}(\boldsymbol{j}_k)$.

(3) For $k \geqslant 1$, $i_1, \dots, i_k \geqslant 0$ and $\sigma \in \mathfrak{S}_k$, the following diagram commutes

$$\begin{array}{ccc}
\mathcal{P}(k) \boxtimes \mathcal{P}(i_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k) & \longrightarrow & \mathcal{P}(k) \boxtimes \mathcal{P}(i_{\sigma^{-1}(1)}) \boxtimes \cdots \boxtimes \mathcal{P}(i_{\sigma^{-1}(k)}) \\
 & & \Big\downarrow {\scriptstyle \gamma_{\boldsymbol{i}\sigma^{-1}}} \\
\Big\downarrow {\scriptstyle \sigma \boxtimes \mathrm{id}_{\mathcal{P}(i_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k)}} & & \mathcal{P}(i_1 + \cdots + i_k) \\
 & & \Big\downarrow {\scriptstyle \sigma_{\boldsymbol{i}}} \\
\mathcal{P}(k) \boxtimes \mathcal{P}(i_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k) & \xrightarrow{\quad \gamma_{i_1,\dots,i_k} \quad} & \mathcal{P}(i_1 + \cdots + i_k) \ ,
\end{array}$$

where the top arrow is the composition of braidings rearranging $\mathcal{P}(i_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k)$ into $\mathcal{P}(i_{\sigma^{-1}(1)}) \boxtimes \cdots \boxtimes \mathcal{P}(i_{\sigma^{-1}(k)})$.

(4) For $n \geqslant 0$, the composite morphisms

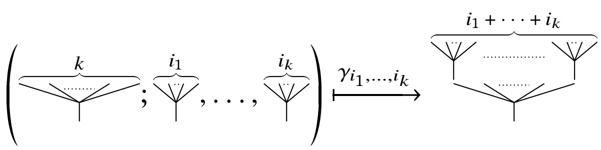
$$\mathcal{P}(n) \boxtimes I^{\boxtimes n} \xrightarrow{\mathrm{id}_{\mathcal{P}(n)} \boxtimes \eta^{\boxtimes n}} \mathcal{P}(n) \boxtimes \mathcal{P}(1) \boxtimes \cdots \boxtimes \mathcal{P}(1) \xrightarrow{\gamma_{1,\dots,1}} \mathcal{P}(n),$$

$$I \boxtimes \mathcal{P}(n) \xrightarrow{\eta \boxtimes \mathrm{id}_{\mathcal{P}(n)}} \mathcal{P}(1) \boxtimes \mathcal{P}(n) \xrightarrow{\gamma_n} \mathcal{P}(n)$$

are respectively equal to the iterations of the right unit of C *and to the left unit of* C.

The object $\mathcal{P}(n)$ *is then called the* space of operations of arity n *of the operad* $\mathcal{P}$.

Assuming that the objects of C are sets and representing the elements of $\mathcal{P}(n)$ as corollae of arity n, the composition morphisms $\gamma_{i_1,\cdots,i_k} : \mathcal{P}(k) \boxtimes \mathcal{P}(i_1) \boxtimes \cdots \boxtimes \mathcal{P}(i_k) \to \mathcal{P}(i_1 + \cdots + i_k)$ can naturally be represented as



The elements of $\mathcal{P}(0)$ will moreover be represented as ⫯ .

EXAMPLE 1.8. (i) The operad $uAss$ is the operad in Set whose arity n set of operations is $\mathfrak{S}_n$ for $n \geqslant 0$. The group $\mathfrak{S}_n$ acts on the right of $uAss(n)$ by multiplication on the right. The composition maps are given by mapping a permutation $\sigma \in uAss(k)$ and permutations $\sigma_j \in uAss(i_j)$ for $1 \leqslant j \leqslant k$ to the composite permutation $\sigma_{\boldsymbol{i}}(\sigma_1 \times \cdots \times \sigma_k)$ of $uAss(i_1 + \cdots + i_k)$, where $\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_k}$ is seen as a subgroup of $\mathfrak{S}_{i_1+\cdots+i_k}$.

(ii) The operad Com is the operad in Set whose arity n set of operations is a singleton $\{*\}$ for $n \geqslant 1$ and the empty set for $n = 0$. The action of the symmetric groups as well as the composition maps and unit are then all trivial.

(iii) A monoid structure on an object $C \in \mathsf{C}$ is equivalent to a structure of operad on the $\mathfrak{S}$-module $(0, M, 0, \dots, 0, \dots)$, where we assume that the closed symmetric monoidal category C has an initial object $0 \in \mathsf{C}$.

We will denote $I := (0, I, 0, \dots)$ the $\mathfrak{S}$-module concentrated in arity 1. The object I being a monoid in C, the previous example implies that the $\mathfrak{S}$-module I is an operad.

DEFINITION 1.9. *A* morphism of operads $\mathcal{P} \to \mathcal{Q}$ *is defined to be a sequence of* $\mathfrak{S}_n$*-equivariant maps* $\mathcal{P}(n) \to \mathcal{Q}(n)$ *for* $n \geqslant 0$ *which commute with the composition maps and preserve the units. It is an* isomorphism of operads *if each map* $\mathcal{P}(n) \to \mathcal{Q}(n)$ *is an isomorphism.*

DEFINITION 1.10. *An* augmented *operad is defined to be the data of an operad* $\mathcal{P}$ *together with a morphism of operads* $\varepsilon : \mathcal{P} \to I$ *called the* augmentation.

DEFINITION 1.11. *Let* $\mathcal{P}$ *and* $\mathcal{Q}$ *be two operads. We define their* Hadamard product $\mathcal{P} \boxtimes \mathcal{Q}$ *as the operad whose underlying* $\mathfrak{S}$*-module is* $(\mathcal{P} \boxtimes \mathcal{Q})(n) := \mathcal{P}(n) \boxtimes \mathcal{Q}(n)$*, whose unit is*

$$\eta_{\mathcal{P}\boxtimes\mathcal{Q}} : I \xrightarrow{\lambda_I^{-1}=\rho_I^{-1}} I \boxtimes I \xrightarrow{\eta_{\mathcal{P}} \boxtimes \eta_{\mathcal{Q}}} \mathcal{P}(1) \boxtimes \mathcal{Q}(1) = (\mathcal{P} \boxtimes \mathcal{Q})(1)$$

and whose composition maps $\gamma^{\mathcal{P}\boxtimes\mathbb{Q}}_{i_1,\ldots,i_k}$ are defined as

$$\begin{array}{c}\mathcal{P}(k)\boxtimes\mathbb{Q}(k)\boxtimes\mathcal{P}(i_1)\boxtimes\mathbb{Q}(i_1)\boxtimes\cdots\boxtimes\mathcal{P}(i_k)\boxtimes\mathbb{Q}(i_k)\\ \downarrow\\ \mathcal{P}(k)\boxtimes\mathcal{P}(i_1)\boxtimes\cdots\boxtimes\mathcal{P}(i_k)\boxtimes\mathbb{Q}(k)\boxtimes\mathbb{Q}(i_1)\boxtimes\cdots\boxtimes\mathbb{Q}(i_k)\\ \downarrow{\scriptstyle \gamma^{\mathcal{P}}_{i_1,\ldots,i_k}\boxtimes\gamma^{\mathbb{Q}}_{i_1,\ldots,i_k}}\\ (\mathcal{P}\boxtimes\mathbb{Q})(i_1+\cdots+i_k)\end{array}$$

where the top arrow is the composition of braidings rearranging the factors of the top expression into those of the bottom expression.

Proposition 1.12. *Let* C *and* D *be two closed symmetric monoidal categories and $\mathcal{F}:$ C $\to$ D *be a strong symmetric monoidal functor. Then $\mathcal{F}$ maps operads in the category* C *to operads in the category* D.

1.3. Algebras over an operad.

DEFINITION 1.13. *Let C be an object of* C.

(i) The endormorphism operad *of C is defined to be the $\mathfrak{S}$-module* $\mathrm{End}_C(n) := \underline{\mathrm{Hom}}(C^{\boxtimes n}, C)$ *where the right action of $\mathfrak{S}_n$ is defined by permuting the inputs using composites of braidings, whose unit is* id_C *and whose compositions are defined for $g : C^{\boxtimes k} \to C$ and $f_j : C^{\boxtimes i_j} \to C$ as*

$$\gamma_{i_1,\ldots,i_k}(g; f_1,\ldots,f_k) = g\circ(f_1\boxtimes\cdots\boxtimes f_k)\ .$$

(ii) The coendormorphism operad *of C is defined to be the $\mathfrak{S}$-module* $\mathrm{coEnd}_C(n) := \underline{\mathrm{Hom}}(C, C^{\boxtimes n})$ *where $\mathfrak{S}_n$ acts on the right by permuting the outputs, whose unit is* id_C *and whose compositions $\gamma_{i_1,\ldots,i_k}$ are defined for $g : C \to C^{\boxtimes k}$ and $f_j : C \to C^{\boxtimes i_j}$, $1 \leqslant j \leqslant k$ as*

$$\gamma_{i_1,\ldots,i_k}(g; f_1,\ldots,f_k) = (f_1\boxtimes\cdots\boxtimes f_k)\circ g\ .$$

We point out that for the sake of clarity, we have written the definition as if the internal homs are sets. It is however possible to spell out Definition 1.13 without this assumption, using diagrams in C involving internal homs.

DEFINITION 1.14. *Let $\mathcal{P}$ be an operad.*

(i) A structure of $\mathcal{P}$-algebra *on $A \in$ C is defined to be a morphisms of operads $\mathcal{P} \to$* End_A.
(ii) A structure of $\mathcal{P}$-coalgebra *on $C \in$ C is defined to be a morphism of operads $\mathcal{P} \to$* coEnd_C.

In other words, a $\mathcal{P}$-algebra structure on A is a way to translate each operation of arity n in $\mathcal{P}(n)$ as an arity n morphism $A^{\boxtimes n} \to A$, such that a composition of operations in $\mathcal{P}$ translates into a composition of morphisms in C.

Using the adjunction $-\boxtimes A^{\boxtimes n} \dashv \underline{\mathrm{Hom}}(A^{\boxtimes n}, \cdot)$, a $\mathcal{P}$-algebra structure on A induces in particular a morphism $\mathcal{P}(n)\boxtimes A^{\boxtimes n} \to A$ that is $\mathfrak{S}_n$-invariant for every $n \geqslant 0$. We refer to Section 1.4.2 for more details on that viewpoint.

Definition 1.15. *Given A_1 and A_2 two $\mathcal{P}$-algebras, a morphism $f \in \mathtt{C}(A_1, A_2)$ is said to be a* morphism of $\mathcal{P}$-algebras *if for every $n \geqslant 0$ the following diagram commutes*

$$\begin{array}{ccc} \mathcal{P}(n) \boxtimes A_1^{\boxtimes n} & \longrightarrow & A_1 \\ \downarrow{\scriptstyle \mathrm{id}_{\mathcal{P}(n)} \boxtimes f^{\boxtimes n}} & & \downarrow{\scriptstyle f} \\ \mathcal{P}(n) \boxtimes A_2^{\boxtimes n} & \longrightarrow & A_2 \end{array} .$$

We will denote $\mathcal{P}$-$\mathtt{alg}$ the *category of $\mathcal{P}$-algebras* with morphisms of $\mathcal{P}$-algebras between them. The category $\mathcal{P}$-$\mathtt{cog}$ of $\mathcal{P}$-coalgebras can be defined in a similar fashion.

Example 1.16. (i) A $uAss$-algebra structure on a set X is exactly a monoid structure on X.
(ii) Denote $u\mathcal{A}ss := \mathbb{K}[uAss]$ and $\mathcal{C}om := \mathbb{K}[Com]$. Then a vector space A with a $u\mathcal{A}ss$-algebra/$\mathcal{C}om$-algebra structure is exactly a unital associative algebra/a commutative algebra.
(iii) Seeing the operads $u\mathcal{A}ss$ and $\mathcal{C}om$ as operads in $\mathtt{dgVect}$ concentrated in degree 0 with null differential, a $u\mathcal{A}ss$-algebra/$\mathcal{C}om$-algebra structure is exactly a unital dg algebra/a cdg algebra.

Proposition 1.17. *A morphism of operads $\mathbb{Q} \to \mathcal{P}$ induces a functor $\mathcal{P}$-$\mathtt{alg} \to \mathbb{Q}$-$\mathtt{alg}$.*

1.4. Operads as monoids. In this section, we let $\mathtt{C}$ be one the following three categories: $\mathtt{Vect}$, $\mathtt{grVect}$ and $\mathtt{dgVect}$. An operad in one of these categories is usually called an *algebraic operad.*

1.4.1. *Monoidal category structure on $\mathfrak{S}$-$\mathtt{mod}$.* Recall that for a group G, a vector space with a left/right G-action is equivalent to a left/right $\mathbb{K}[G]$-module.

Definition 1.18. *Let G be a group.*

(1) Given two vector spaces V and W respectively with a right and left G-action, we define

$$V \otimes_G K := V \otimes_{\mathbb{K}[G]} W .$$

(2) Given a subgroup $H \subset G$ and a vector space V with a right H-action, we define

$$\mathrm{Ind}_H^G V := V \otimes_H \mathbb{K}[G] .$$

(3) Given a subgroup $H \subset G$ and a vector space V with a left H-action, we define

$$\mathrm{Coind}_H^G V := \mathrm{Hom}_{\mathbb{K}[H]}(\mathbb{K}[G], V) .$$

Definition 1.19. *Given two $\mathfrak{S}$-modules $\mathcal{M}$ and $\mathcal{N}$, we define their* composite *as the $\mathfrak{S}$-module $\mathcal{M} \circ \mathcal{N}$ given in arity $n \geqslant 0$ by*

$$\mathcal{M} \circ \mathcal{N}(n) = \bigoplus_{k \geqslant 0} \mathcal{M}(k) \otimes_{\mathfrak{S}_k} \bigoplus_{i_1 + \cdots + i_k = n} \mathrm{Ind}_{\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_k}}^{\mathfrak{S}_n} (\mathcal{N}(i_1) \otimes \cdots \otimes \mathcal{N}(i_k)) ,$$

where the summand is equal to $\mathcal{M}(0)$ when $k = 0$.

The composite defines in fact a bifunctor $- \circ - : \mathfrak{S}$-$\mathtt{mod} \times \mathfrak{S}$-$\mathtt{mod} \to \mathfrak{S}$-$\mathtt{mod}$. We moreover denote $\mathbb{K} := (0, \mathbb{K}, 0, \dots, 0, \dots)$.

Proposition 1.20. *The category* $(\mathfrak{S}\text{-mod}, \circ, \mathbb{K})$ *is a monoidal category.*

Proposition 1.21. *Given a* $\mathfrak{S}$*-module* $\mathcal{P}$*, a structure of operad on* $\mathcal{P}$ *corresponds exactly to a structure of monoid on* $\mathcal{P}$ *in* $(\mathfrak{S}\text{-mod}, \circ, \mathbb{K})$.

1.4.2. *Schur functor.*

Definition 1.22. *Let* $\mathcal{M}$ *be a* $\mathfrak{S}$*-module. The* Schur functor *associated to* $\mathcal{M}$ *is the endofunctor* $S_{\mathcal{M}} : \mathtt{C} \to \mathtt{C}$ *defined as*

$$S_{\mathcal{M}}(V) := \bigoplus_{n\geqslant 0} (\mathcal{M}(n) \otimes V^{\otimes n})_{\mathfrak{S}_n} \ ,$$

where $\mathcal{M}(n) \otimes V^{\otimes n}$ *is endowed with the diagonal right* $\mathfrak{S}_n$*-action.*

This construction defines in fact a functor $S_{-} : \mathfrak{S}\text{-mod} \to \mathrm{EndoFun}(\mathtt{C})$. Recall moreover from Exercice sheet 1 that the category $(\mathrm{EndoFun}(\mathtt{C}), \circ, \mathrm{id}_{\mathtt{C}})$ is a strict monoidal category.

Proposition 1.23. *The functor* S_{-} *is strong monoidal. In particular, an operad structure on* $\mathcal{P}$ *induces a monoid structure on its Schur functor* $S_{\mathcal{P}}$.

In other words, an operad $\mathcal{P}$ defines a *monad* $S_{\mathcal{P}}$.

1.4.3. *Free algebra over an operad.*

Proposition 1.24. *(i) A* $\mathcal{P}$*-algebra structure on an object* A *of* $\mathtt{C}$ *is equivalent to a* $S_{\mathcal{P}}$*-algebra structure on* A*, i.e. to a morphism* $\mu : S_{\mathcal{P}}(A) \to A$ *such that the following diagrams commute*

$$\begin{array}{ccc} (S_{\mathcal{P}} \circ S_{\mathcal{P}})(A) = S_{\mathcal{P}}(S_{\mathcal{P}}(A)) & \xrightarrow{S_{\mathcal{P}}(\mu)} & S_{\mathcal{P}}(A) \\ \downarrow{\scriptstyle \gamma_A} & & \downarrow{\scriptstyle \mu} \\ S_{\mathcal{P}}(A) & \xrightarrow{\quad \mu \quad} & S_{\mathcal{P}}(A) \end{array} \qquad \begin{array}{ccc} A = \mathrm{id}_{\mathtt{C}}(A) & \xrightarrow{\eta_A} & S_{\mathcal{P}}(A) \\ & \searrow^{\mathrm{id}_A} & \downarrow{\scriptstyle \mu} \\ & & A \end{array} .$$

(ii) A morphism $f : A_1 \to A_2$ *is then a morphism of* $\mathcal{P}$*-algebras if and only if the following diagram commutes*

$$\begin{array}{ccc} S_{\mathcal{P}}(A_1) & \xrightarrow{\gamma_{A_1}} & A_1 \\ \downarrow{\scriptstyle S_{\mathcal{P}}(f)} & & \downarrow{\scriptstyle f} \\ S_{\mathcal{P}}(A_2) & \xrightarrow{\gamma_{A_2}} & A_2 \end{array} .$$

Definition 1.25. *For an operad* $\mathcal{P}$ *and a vector space* V *of* $\mathtt{C}$*, the* free $\mathcal{P}$-algebra *on* V *is defined to be the object* $S_{\mathcal{P}}(V)$ *whose* $\mathcal{P}$*-algebra structure is given by the morphism*

$$S_{\mathcal{P}}(S_{\mathcal{P}}(V)) = (S_{\mathcal{P}} \circ S_{\mathcal{P}})(V) \xrightarrow{\gamma_V} S_{\mathcal{P}}(V) \ .$$

Proposition 1.26. *The free* $\mathcal{P}$*-algebra construction defines a functor* $S_{\mathcal{P}} : \mathtt{C} \to \mathcal{P}\text{-}\mathtt{alg}$ *which is left adjoint to the forgetful functor* $\mathcal{P}\text{-}\mathtt{alg} \to \mathtt{C}$*. In other words, for every* $V \in \mathtt{C}$ *and* $\mathcal{P}$*-algebra* A*, there is a natural bijection*

$$\mathrm{Hom}_{\mathcal{P}\text{-}\mathtt{alg}}(S_{\mathcal{P}}(V), A) = \mathrm{Hom}_{\mathtt{C}}(V, A) \ .$$

Example 1.27. If $\mathtt{C} = \mathtt{grVect}$, the free $u\mathcal{A}ss$-algebra on a graded vector space V is the free graded tensor algebra $T(V)$ and the free $\mathcal{C}om$-algebra on V is the reduced free graded commutative algebra $\overline{\Lambda}V = \bigoplus_{n\geqslant 1} \Lambda^n V$.

1.5. Nonsymmetric operads. Let $\mathtt{C}$ be a closed symmetric monoidal category.

DEFINITION 1.28. *A* $\mathbb{N}$- module *is defined to be a sequence of objects* $\{\mathscr{M}(n)\}_{n\geqslant 0}$ *in* $\mathtt{C}$.

DEFINITION 1.29. *A* nonsymmetric operad *or* ns operad *structure on a* $\mathbb{N}$*-module* $\{\mathscr{P}(n)\}_{n\geqslant 0}$ *is defined to be the data of composition morphisms*

$$\gamma_{i_1,\cdots,i_k} : \mathscr{P}(k) \boxtimes \mathscr{P}(i_1) \boxtimes \cdots \boxtimes \mathscr{P}(i_k) \to \mathscr{P}(i_1 + \cdots + i_k)$$

and of a unit morphism $\eta : I \to \mathscr{P}(1)$ *which satisfy Item 2 and Item 4 of Definition 1.7.*

Every symmetric operad yields a ns operad by forgetting about the symmetric groups actions: this defines a forgetful functor $\mathtt{Op} \to \mathtt{nsOp}$. The endomorphism and coendormophism operads End_C and coEnd_C of an object $C \in \mathtt{C}$ are thereby in particular ns operads. Given a ns operad $\mathscr{P}$, a $\mathscr{P}$*-algebra* structure on A is then defined as a morphism of ns operads $\mathscr{P} \to \mathrm{End}_A$, and a $\mathscr{P}$*-coalgebra* structure on C is defined as a morphism of ns operads $\mathscr{P} \to \mathrm{coEnd}_C$.

EXAMPLE 1.30. (i) The ns operad uAs is the ns operad in $\mathtt{Set}$ whose arity n set of operations is a singleton $\{*\}$ for every $n \geqslant 0$. A uAs-algebra X is then a monoid X in $\mathtt{Set}$.
(ii) The ns operad $u\mathscr{A}s := \mathbb{K}[uAs]$ is the ns operad in vector spaces which encodes unital associative/unital graded associative/unital dg algebras. Beware that $u\mathscr{A}s$ is not the image of the operad $u\mathscr{A}ss$ under the forgetful functor $\mathtt{Op} \to \mathtt{nsOp}$, but the image of the operad $u\mathscr{C}om$!

Assume now that the category $\mathtt{C}$ is either $\mathtt{Vect}$, $\mathtt{grVect}$ or $\mathtt{dgVect}$. The *Schur functor* associated to a $\mathbb{N}$-module $\mathscr{M}$ is the endofunctor $S_{\mathscr{M}} : \mathtt{C} \to \mathtt{C}$ defined as

$$S_{\mathscr{M}}(V) := \bigoplus_{n\geqslant 0} \mathscr{M}(n) \otimes V^{\otimes n} \ .$$

All constructions and propositions of Section 1.4 then still hold in the ns case. For instance, the free $u\mathscr{A}s$-algebra on a vector space V is again the free tensor algebra $S_{u\mathscr{A}s}(V) = T(V)$.

2. Free operad

In this section, $\mathtt{C}$ either denotes $\mathtt{Vect}$, $\mathtt{grVect}$ or $\mathtt{dgVect}$.

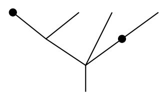
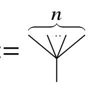
2.1. Trees.

2.1.1. *Planar and nonplanar trees.*

DEFINITION 2.1. *A* planar tree t *is defined to be a tree which satisfies the following conditions:*

(1) it has a distinguished outgoing edge called the root,
(2) each vertex v *of* t *is endowed with a linear order on the set of its incoming edges* $\mathrm{inc}(v)$ *(a way to embed the tree in the plane),*
(3) its input edges can be capped by stumps *i.e. vertices with no incoming edge,*
(4) vertices with only one incoming edge are allowed.

These conditions imply in particular that the set of non-capped input edges of t is linearly ordered. We will moreover denote $\mathrm{Vert}(t)$ the set of vertices of t, and PT_n the *set of planar trees* with n non-capped input edges. The *trivial tree* | is then an element of PT_1. The corolla of arity $n \geqslant 1$ will moreover be denoted $c_n \in \mathrm{PT}_n$.

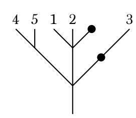
 $\in \mathrm{PT}_3$ $\in \mathrm{PT}_0$ $c_n :=$ 

$\in \mathrm{PT}_n$

Unary vertices and capping vertices are represented with bullets in the above planar trees.

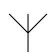
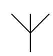
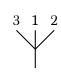
Definition 2.2. *A* nonplanar tree *is defined to be a planar tree t together with a permutation of its non-capped input edges.*

We will denote $\mathrm{T}_n \simeq \mathrm{PT}_n \times \mathfrak{S}_n$ the *set of nonplanar trees* with n non-capped input edges.



$\in \mathrm{T}_5$

We will use three different representation for an element of T_n depending on the context, as illustrated below.

 $\cdot\, (123) =$  $^{(123)} =$ 

$\in \mathrm{T}_3$

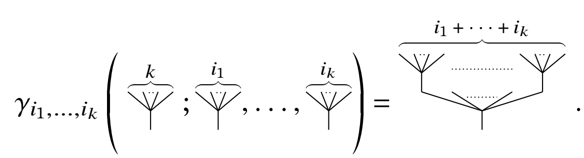
An element of T_n will moreover either be denoted as $t \cdot \sigma$ where t denotes its underlying planar tree and σ the associated permutation, or simply as t depending on the context.

2.1.2. *The ns operad $\mathscr{PT}$.*

Definition 2.3. *The ns operad $\mathscr{PT}$ is the $\mathbb{N}$-module $\mathscr{PT} := \{\mathrm{PT}_n\}_{n\geqslant 0}$ with composition maps*

$$\gamma_{i_1,\ldots,i_k} : \mathrm{PT}_k \times \mathrm{PT}_{i_1} \times \cdots \times \mathrm{PT}_{i_k} \longrightarrow \mathrm{PT}_{i_1+\cdots+i_k}$$

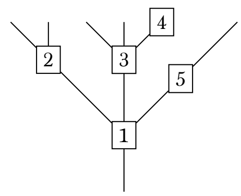
given by grafting the root of a tree in PT_{i_j} to the j-th non-capped input edge of a tree in PT_k.



This ns operad structure permits us to define an order on the vertices of a planar tree as follows. Every planar tree $\neq$ ⫯, | can be written as a composition $t := \gamma(c_k; t_1, \ldots, t_k)$, where c_k is the corolla of arity $k \geqslant 1$. If $t_1, \ldots, t_k$ are all equal to the trivial tree |, $t = c_k$ and there is only one vertex. Otherwise, we put the vertex v_{c_k} of c_k in first position and then proceed by induction to concatenate the orders on the vertices of the t_i, i.e.

$$\mathrm{Vert}(t) = v_{c_k} < \mathrm{Vert}(t_1) < \cdots < \mathrm{Vert}(t_k)\ .$$

We represent an example of an ordering of the vertices of a planar tree below.



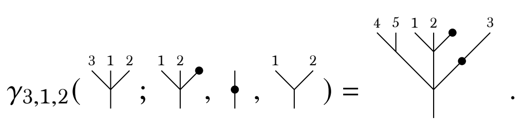
2.1.3. *The operad $\mathcal{T}$.*

Definition 2.4. *The sets T_n for $n \geqslant 0$ define a $\mathfrak{S}$-module $\mathcal{T}$ in* Set*, where $\mathfrak{S}_n$ acts on the right of T_n by permuting the n non-capped input edges of the nonplanar tree. This $\mathfrak{S}$-module is an operad with composition maps*

$$\gamma_{i_1,\ldots,i_k} : \mathrm{T}_k \times \mathrm{T}_{i_1} \times \cdots \times \mathrm{T}_{i_k} \to \mathrm{T}_{i_1+\cdots+i_k}$$

given by grafting the root of a tree $t_j \cdot \sigma_j \in \mathrm{T}_{i_j}$ to the j-th non-capped input edge of a tree $t \cdot \sigma \in \mathrm{T}_k$. The choice of permutation on the planar tree obtained under this composition is then defined to be $\sigma_{\boldsymbol{i}}(\sigma_1 \times \cdots \times \sigma_k)$.

We have for instance that



We moreover linearly order the vertices of a nonplanar tree by ordering the vertices of its underlying planar tree as in the previous section.

2.2. Free operad.

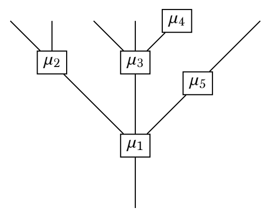
2.2.1. *Free ns operad.*

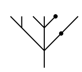
Definition 2.5. *Let $\mathcal{M}$ be a $\mathbb{N}$-module and t a planar tree. We define*

$$\mathcal{M}(t) := \bigotimes_{v \in \mathrm{Vert}(t)} \mathcal{M}(|\mathrm{inc}(v)|)\ ,$$

where the set of vertices is ordered as explained in Section 2.1.2 and we set $\mathcal{M}(\,|\,) := \mathbb{K}$.

An element of $\mathcal{M}(t)$ can be represented as a linear combination of labelings of the vertices of t by operations in $\mathcal{M}$. For instance



$\in \mathcal{M}($  $)$

where $\mu_1 \in \mathcal{M}(3)$, $\mu_2 \in \mathcal{M}(2)$, $\mu_3 \in \mathcal{M}(3)$, $\mu_4 \in \mathcal{M}(0)$ and $\mu_5 \in \mathcal{M}(1)$.

Definition 2.6. *The* free ns operad $\mathcal{T}_{ns}(\mathcal{M})$ *on a* $\mathbb{N}$*-module* $\mathcal{M}$ *is defined to be the* $\mathbb{N}$*-module*

$$\mathcal{T}_{ns}(\mathcal{M})(n) := \bigoplus_{t \in \mathrm{PT}_n} \mathcal{M}(t) \ ,$$

endowed with the composition maps

$$\mathcal{T}_{ns}(\mathcal{M})(k) \otimes \mathcal{T}_{ns}(\mathcal{M})(i_1) \otimes \cdots \otimes \mathcal{T}_{ns}(\mathcal{M})(i_k) \to \mathcal{T}_{ns}(\mathcal{M})(i_1 + \cdots + i_k)$$

defined on each summand as

$$\mathcal{M}(t') \otimes \mathcal{M}(t_1) \otimes \cdots \otimes \mathcal{M}(t_k) \to \mathcal{M}(\gamma(t; t_1, \ldots, t_k)) \hookrightarrow \mathcal{T}_{ns}(\mathcal{M})(i_1 + \cdots + i_k)$$

where $t' \in \mathrm{PT}_k$, $t_j \in \mathrm{PT}_{i_j}$ *for* $1 \leqslant j \leqslant k$ *and the left arrow corresponds to reordering the factors of* $\mathcal{M}(t') \otimes \mathcal{M}(t_1) \otimes \cdots \otimes \mathcal{M}(t_k)$ *into the factors of* $\mathcal{M}(\gamma(t; t_1, \ldots, t_k))$.

In other words, the composition of the operad $\mathcal{T}_{ns}(\mathcal{M})$ is given by the grafting of trees whose vertices are labeled by operations of $\mathcal{M}$.

Example 2.7. (i) The free ns operad on the $\mathbb{N}$-module $V = (0, V, 0, \ldots, 0, \ldots)$ is

$$\mathcal{T}_{ns}(V) = (0, T(V), 0, \ldots, 0)$$

where $T(V)$ is the free tensor algebra of Definition 1.4.

(ii) The free ns operad on the $\mathbb{N}$-module $(\mathbb{K}, \mathbb{K}, \mathbb{K}, \ldots)$ is the operad $\mathbb{K}[\mathcal{PT}]$.

A free ns operad $\mathcal{T}_{ns}(\mathcal{M})$ is in particular weight graded, where for $m \geqslant 0$ the $\mathbb{N}$-module $\mathcal{T}_{ns}(\mathcal{M})^{(m)}$ is defined as

$$\mathcal{T}_{ns}(\mathcal{M})^{(m)}(n) = \bigoplus_{\substack{t \in \mathrm{PT}_n \\ |\mathrm{Vert}(t)| = m}} \mathcal{M}(t) \ .$$

Proposition 2.8. *The functor* $\mathcal{T}_{ns}(-) : \mathbb{N}\text{-}\mathtt{mod} \to \mathtt{nsOp}$ *is left adjoint to the forgetful functor* $\mathtt{nsOp} \to \mathbb{N}\text{-}\mathtt{mod}$. *In other words, for every* $\mathbb{N}$*-module* $\mathcal{M}$ *and ns operad* $\mathcal{P}$ *there is a natural bijection*

$$\mathrm{Hom}_{\mathtt{nsOp}}(\mathcal{T}_{ns}(\mathcal{M}), \mathcal{P}) \simeq \mathrm{Hom}_{\mathbb{N}\text{-}\mathtt{mod}}(\mathcal{M}, \mathcal{P}) \ .$$

In particular, a structure of $\mathcal{T}_{ns}(\mathcal{M})$-algebra on A simply corresponds to a morphism of $\mathbb{N}$-modules $\mathcal{M}(n) \to \underline{\mathrm{Hom}}(A^{\otimes n}, A)$.

2.2.2. *Free operad.* For a nonplanar tree $t\cdot\sigma$ and a $\mathfrak{S}$-module $\mathcal{M}$, we define $\mathcal{M}(t\cdot\sigma) := \mathcal{M}(t)$ as in Definition 2.5, where the set of vertices is ordered as in Section 2.1.3 and we set $\mathcal{M}(\,|\,) := \mathbb{K}$. We will now define the underlying $\mathfrak{S}$-module $\mathcal{T}(\mathcal{M})$ of the free operad on $\mathcal{M}$. For every $n \geqslant 0$, the vector space $\bigoplus_{t\in \mathrm{T}_n} \mathcal{M}(t)$ admits a right $\mathfrak{S}_n$-action defined on a summand as

$$\tau_\sigma : \mathcal{M}(t') \xrightarrow{\mathrm{id}} \mathcal{M}(t'\cdot\sigma) \hookrightarrow \bigoplus_{t\in \mathrm{T}_n} \mathcal{M}(t) \ .$$

Definition 2.9. *We define the $\mathfrak{S}$-module $\mathcal{T}(\mathcal{M})$ as*

$$\mathcal{T}(\mathcal{M})(n) := (\bigoplus_{t\in \mathrm{T}_n} \mathcal{M}(t))/\sim \ .$$

where the quotient is defined by induction as explained below.

Any nonplanar tree $\neq$ ⫯, $|$ can be written as a composition $t := \gamma(c_k\cdot\sigma; t_1,\ldots,t_k)\cdot\sigma'$ where:

(1) c_k is the corolla of arity $k \geqslant 1$ and $\sigma \in \mathfrak{S}_k$ is a permutation of its incoming edges,
(2) $t_j \in \mathrm{PT}_{i_j}$ for $1 \leqslant j \leqslant k$ and $i_1 + \cdots + i_k = n$,
(3) σ' is a permutation of the non-capped input edges of $\gamma(c_k\cdot\sigma; t_1,\ldots,t_k)$.

Write $\mu\otimes\boldsymbol{\nu}$ an element of $\mathcal{M}(t) = \mathcal{M}(k)\otimes\cdots$ where μ is the operation labeling the unique vertex of the corolla c_k. The quotient is then defined by induction on the number of vertices of the tree t by the identification

$$\mathcal{M}(t) \ni \mu\otimes\boldsymbol{\nu} = \mu\cdot\sigma\otimes\boldsymbol{\nu}' \in \mathcal{M}(t')$$

where

(1) $t' = \gamma(c_k; t_{\sigma^{-1}(1)},\ldots,t_{\sigma^{-1}(k)})\sigma_{\boldsymbol{i}}\sigma'$ where $\sigma_{\boldsymbol{i}}$ is defined in Definition 1.6,
(2) $\boldsymbol{\nu}'$ is obtained by reordering the factors of $\boldsymbol{\nu}$ following the ordering on the vertices of t'.

We moreover check that the action of $\mathfrak{S}_n$ is still well-defined after quotienting.

The operations of $\mathcal{T}(\mathcal{M})$ are thereby to be understood as linear combinations of labelings of the vertices of nonplanar trees by operations of $\mathcal{M}$, such that permuting the incoming edges at a vertex is equal to applying this permutation to the label of this vertex.

Definition 2.10. *The* free operad *on the $\mathfrak{S}$-module $\mathcal{M}$ is defined to be the $\mathfrak{S}$-module $\mathcal{T}(\mathcal{M})$ endowed with the composition given by the grafting of nonplanar trees as in Definition 2.6.*

The operad $\mathcal{T}(\mathcal{M})$ is again weight graded by the number of vertices of a nonplanar tree $\mathcal{T}_{ns}(\mathcal{M}) = \bigoplus_{m\geqslant 0}\mathcal{T}_{ns}(\mathcal{M})^{(m)}$.

Example 2.11. (i) The free operad on the $\mathfrak{S}$-module $V = (0, V, 0, \ldots, 0, \ldots)$ is

$$\mathcal{T}(V) = (0, T(V), 0, \ldots, 0) \ .$$

(ii) The free operad on the $\mathfrak{S}$-module $\{K[\mathfrak{S}_n]\}_{n\geqslant 0}$ is the operad $\mathbb{K}[\mathcal{T}]$.

Proposition 2.12. *The functor $\mathcal{T}(-) : \mathfrak{S}\text{-}\mathtt{mod} \to \mathtt{Op}$ is left adjoint to the forgetful functor $\mathtt{Op} \to \mathfrak{S}\text{-}\mathtt{mod}$. In other words, for every $\mathfrak{S}$-module $\mathcal{M}$ and operad $\mathcal{P}$ there is a natural bijection*

$$\mathrm{Hom}_{\mathtt{Op}}(\mathcal{T}_{ns}(\mathcal{M}), \mathcal{P}) \simeq \mathrm{Hom}_{\mathfrak{S}\text{-}\mathtt{mod}}(\mathcal{M}, \mathcal{P}) \ .$$

2.3. Presentation of an operad.

DEFINITION 2.13. *Let $\mathcal{P}$ be an operad.*

(i) An ideal *of $\mathcal{P}$ is defined to a $\mathfrak{S}$-module $\mathcal{M}$ such that $\mathcal{M}(n) \subset \mathcal{P}(n)$ for every $n \geqslant 0$ and such that for any family of operations $\nu; \mu_1, \ldots, \mu_k$ of $\mathcal{P}$, if one of them is in $\mathcal{M}$ then theire composition $\gamma(\nu; \mu_1, \ldots, \mu_k)$ is in $\mathcal{M}$.*

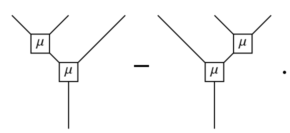
(ii) Given a collection of operations $r_j \in \mathcal{P}$, the ideal generated *by the r_j is defined to be the smallest ideal of $\mathcal{P}$ which contains the r_j.*

We will say that an operad $\mathcal{P}$ admits a *presentation*, if there exists a $\mathfrak{S}$-module $\mathcal{M}$ and an ideal $\mathcal{I} \subset \mathcal{T}(\mathcal{M})$ such that

$$\mathcal{P} = \mathcal{T}(\mathcal{M})/\mathcal{I}\ .$$

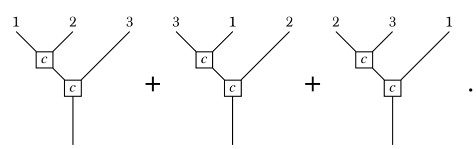
In most cases, we will assume that the ideal $\mathcal{I}$ is generated by some r_j in $\mathcal{T}(\mathcal{M})$. Choosing for every $n \geqslant 0$ a basis μ_i^n for the vector space $\mathcal{M}(n)$, we will call the elements $\mu_i^n \in \mathcal{M}(n)$ the *generating operations* of the operad $\mathcal{P}$ and the elements $r_j \in \mathcal{T}(\mathcal{M})$ its *relators.*

EXAMPLE 2.14. (i) The ns operad $\mathcal{A}s$ encoding associative algebras admits a presentation with generating $\mathbb{N}$-module $(0, 0, \mathbb{K}\mu, 0, \ldots)$ and with ideal generated by the relator



(ii) The operad $\mathcal{A}ss$ encoding associative algebras admits a presentation with generating $\mathfrak{S}$-module $(0, 0, \mathbb{K}\mu \oplus \mathbb{K}\lambda, 0, \ldots)$ where $\mu \cdot (12) = \lambda$ and with ideal generated by the same relator as the ns operad $\mathcal{A}s$.

(iii) The operad $\mathcal{L}ie$ encoding Lie algebra admits a presentation with generating $\mathfrak{S}$-module $(0, 0, \mathbb{K}c, 0, \ldots)$ where $c \cdot (12) = -c$ and with ideal generated by the relator



3. Cooperads

In this section, we let `C` be one the following three closed symmetric monoidal categories: `Vect`, `grVect` and `dgVect`.

3.1. Definitions.

3.1.1. *May cooperad.*

DEFINITION 3.1. *A* May cooperad *structure on a $\mathfrak{S}$-module $\{\mathcal{C}(n)\}_{n \geqslant 0}$ is defined to be data of:*

(1) A morphism

$$\delta^{i_1, \ldots, i_k} : \mathcal{C}(n) \to \mathcal{C}(k) \otimes \mathcal{C}(i_1) \otimes \cdots \otimes \mathcal{C}(i_k)$$

for all $k \geqslant 1$ and $i_1, \ldots, i_k \geqslant 0$ such that $i_1 + \cdots + i_k = n$, called a decomposition morphism.

(2) A morphism $\varepsilon : \mathscr{C}(1) \to \mathbb{K}$ *called the* counit.

These data have to satisfy the following properties:

(1) The maps $\delta^{i_1,\ldots,i_k}$ *are equivariant under the right action of* $\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_k}$.

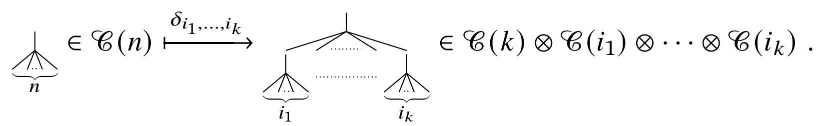
(2) The maps

$$\prod_{i_1+\cdots+i_k=n} \delta^{i_1,\ldots,i_k} : \mathscr{C}(n) \to \prod_{i_1+\cdots+i_k=n} \mathscr{C}(k) \otimes \mathscr{C}(i_1) \otimes \cdots \otimes \mathscr{C}(i_k)$$

factor through the invariants of $\mathfrak{S}_k$ *on the target.*

(3) The decomposition morphisms and the counit satisfy coassociativity and counit axioms dual to the ones of an operad.

The vector space $\mathscr{C}(n)$ is then called the *space of cooperations of arity* n of the May cooperad $\mathscr{C}$. The decomposition morphisms can be represented as



Example 3.2. A May cooperad structure on the $\mathfrak{S}$-module $(0, C, 0, \ldots, 0, \ldots)$ is exactly a counital coassociative coalgebra structure on C

We will moreover denote Δ^n the arity n *total decomposition map*

$$\Delta^n = \prod_{k\geqslant 1} \prod_{i_1+\cdots+i_k=n} \delta i_1, \ldots, i_k : \mathscr{C}(n) \to \prod_{k\geqslant 1} \prod_{i_1+\cdots+i_k=n} \mathscr{C}(k) \otimes \mathscr{C}(i_1) \otimes \cdots \mathscr{C}(i_k) \ .$$

3.1.2. *Cooperads and* $\mathscr{C}$*-coalgebras.*

Definition 3.3. *(i) The* cocomposite product *on* $\mathfrak{S}$-mod *is defined as*

$$\mathscr{M} \bar{\circ} \mathscr{N}(n) = \bigoplus_{k\geqslant 0} \left(\mathscr{M}(k) \otimes \bigoplus_{i_1+\cdots+i_k=n} \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{i_1}\times\cdots\times\mathfrak{S}_{i_k}} \left(\mathscr{N}(i_1) \otimes \cdots \otimes \mathscr{N}(i_k)\right) \right)^{\mathfrak{S}_k} \ .$$

(ii) The coSchur functor *associated to* $\mathscr{M}$ *is the endofunctor* $S^{\mathscr{M}} : \mathtt{C} \to \mathtt{C}$ *defined as*

$$S^{\mathscr{M}}(V) := \bigoplus_{n\geqslant 0} (\mathscr{M}(n) \otimes V^{\otimes n})^{\mathfrak{S}_n} \ .$$

Proposition 3.4. *(i) The cocomposite product* $\bar{\circ}$ *defines a monoidal structure on the category* $\mathfrak{S}$-mod.

(ii) The functor $S^{-} : (\mathfrak{S}\text{-mod}, \bar{\circ}) \to (\mathrm{EndoFun}(\mathtt{C}), \circ)$ *is strong monoidal i.e.* $S^{\mathscr{M}\bar{\circ}\mathscr{N}} = S^{\mathscr{M}} \circ S^{\mathscr{N}}$

Definition 3.5. *A* cooperad $\mathscr{C}$ *is defined to be a May cooperad such that the total decomposition maps* Δ^n *factor through*

$$\Delta^n : \mathscr{C}(n) \to \bigoplus_{k\geqslant 1} \bigoplus_{i_1+\cdots+i_k=n} \mathscr{C}(k) \otimes \mathscr{C}(i_1) \otimes \cdots \mathscr{C}(i_k) \ .$$

In other words, a cooperad is a May cooperad such that for each $n \geqslant 0$ and $\mu \in \mathscr{C}(n)$ there exists only a finite numbers of $i_1, \dots, i_k \geqslant 0$ with $i_1 + \cdots + i_k = n$ such that $\delta^{i_1,\dots,i_k}(\mu) \neq 0$.

Generalizing Sweedler's notation to cooperads, we will denote the image of a cooperation $\mu \in \mathscr{C}(n)$ under Δ^n as the finite sum

$$\Delta^n(\mu) = \sum (\nu; \nu_1, \dots, \nu_k) \in \bigoplus_{k \geqslant 1} \bigoplus_{\substack{i_1+\cdots+i_k=n \\ i_1,\dots,i_k \geqslant 0}} \mathscr{C}(k) \otimes \mathscr{C}(i_1) \otimes \cdots \otimes \mathscr{C}(i_k) \ .$$

Proposition 3.6. *A cooperad structure on a $\mathfrak{S}$-module $\mathscr{C}$ is equivalent to a comonoid structure on $\mathscr{C}$ in $(\mathfrak{S}\text{-}\mathtt{mod}, \bar{\circ})$.*

A cooperad structure on a $\mathfrak{S}$-module $\mathscr{C}$ then induces in particular a comonad structure on its coSchur functor $S^{\mathscr{C}}$.

Definition 3.7. *Let $\mathscr{C}$ be a cooperad. A $\mathscr{C}$-*coalgebra *structure on a vector space C is defined to be a $S^{\mathscr{C}}$-coalgebra structure on C, i.e. the datum of a linear map $\Delta_C : C \to \bigoplus_{n \geqslant 0} (\mathscr{C}(n) \otimes C^{\otimes n})^{\mathfrak{S}_n} = S^{\mathscr{C}}(C)$ such that the following diagrams commute*

$$\begin{array}{ccc} C & \xrightarrow{\Delta_C} & S^{\mathscr{C}}(C) \\ \downarrow{\scriptstyle \Delta_C} & & \downarrow{\scriptstyle S^{\mathscr{C}}(\Delta_C)} \\ S^{\mathscr{C}} & \xrightarrow{\Delta_C^{\mathscr{C}}} & S^{\mathscr{C}}(S^{\mathscr{C}}(C)) \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\Delta_C} & S^{\mathscr{C}}(C) \\ & \searrow^{\mathrm{id}_C} & \downarrow{\scriptstyle \varepsilon_C} \\ & & C \end{array}$$

In other words, a $\mathscr{C}$-coalgebra C corresponds to the data of $\mathfrak{S}_n$-invariant linear maps

$$\Delta^n : C \to \mathscr{C}(n) \otimes C^{\otimes n}$$

for every $n \geqslant 0$ that are compatible with the decomposition of the cooperad $\mathscr{C}$ and such that for every $c \in C$, only a finite number of $\Delta^n(c)$ are $\neq 0$.

Remark 3.8. Hence, a $\mathscr{C}$-coalgebra is in some sense always conilpotent. See Exercice 8 in Exercice sheet 3 for more details.

3.2. Conilpotent cooperads. Recall that the $\mathfrak{S}$-module $\mathbb{K}$ is defined as the $\mathfrak{S}$-module concentrated in arity 1 $(0, \mathbb{K}, 0, \dots)$. It carries an obvious structure of cooperad.

Definition 3.9. *A* coaugmented cooperad *is defined to be the data of a cooperad $\mathscr{C}$ together with a morphism of cooperads $\eta : \mathbb{K} \to \mathscr{C}$ called the* coaugmentation.

Let $\mathscr{C}$ be a coaugmented cooperad. We define $\mathrm{id} := \eta(1_{\mathbb{K}}) \in \mathscr{C}(1)$ and call it the *identity.* We also define $\bar{\mathscr{C}}(n) := \mathscr{C}(n)$ for $n \neq 1$ and $\bar{\mathscr{C}}(1) := \mathrm{Ker}(\varepsilon)$, where ε is the counit of $\mathscr{C}$.

We moreover set $\bar{T}_n := T_n$ for $n \neq 1$ and $\bar{T}_1 := T_1 - \{\, | \,\}$. Given $\mathscr{M}$ a $\mathfrak{S}$-module we then denote $\bar{\mathscr{T}}^{\wedge}(\mathscr{M})$ the $\mathfrak{S}$-module given in arity n by

$$\bar{\mathscr{T}}^{\wedge}(\mathscr{M}) := \prod_{t \in \bar{T}_n} \mathscr{M}(t)/\sim \ ,$$

where $\sim$ is defined as in Definition 2.9. For $m \geqslant 0$, we also denote

$$\bar{\mathcal{T}}^{\wedge}(\mathcal{M})^{(m)}(n) := \prod_{\substack{t \in \bar{T}_n \\ |\mathrm{Vert}(t)|=m}} \mathcal{M}(t)/\sim \ .$$

Proposition 3.10. *A coaugmented cooperad* $\mathscr{C}$ *determines a morphism of* $\mathfrak{S}$*-modules*

$$\bar{\Delta} : \bar{\mathscr{C}} \to \bar{\mathcal{T}}^{\wedge}(\bar{\mathscr{C}}) \ .$$

Proof. The projection $\bar{\mathscr{C}} \to \bar{\mathcal{T}}^{\wedge}(\bar{\mathscr{C}})^{(1)} = \bar{\mathscr{C}}$ is equal to the identity. The projection $\bar{\mathscr{C}} \to \bar{\mathcal{T}}^{\wedge}(\bar{\mathscr{C}})^{(2)}$ is defined for $\mu \in \bar{\mathscr{C}}(n)$ as the sum $\sum \nu_1 \otimes \nu_2$ for all decompositions of the form $(\nu_1; \mathrm{id}, \ldots, \mathrm{id}, \nu_2, \mathrm{id}, \ldots, \mathrm{id})$ appearing in $\Delta^n(\mu)$. The sum $\Delta^n(\mu) - (\mu; \mathrm{id}, \ldots, \mathrm{id}) - (\mathrm{id}; \mu)$ determines in fact the projection of $\bar{\Delta}$ to the factors associated to 2-leveled trees in $\bar{\mathcal{T}}^{\wedge}(\bar{\mathscr{C}})$. More generally, the factors of $\bar{\Delta}(\mu)$ are obtained by iterating the total decomposition maps Δ^n and discarding the summands featuring an intermediate cooperation equal to id or a final level of cooperations equal to id. □

In other words, the map $\bar{\Delta}$ is defined by iterating the total decomposition maps Δ^n and ensuring that the equivalent decompositions of a cooperation contribute exactly once to the associated factor in $\bar{\mathcal{T}}^{\wedge}(\bar{\mathscr{C}})$.

Definition 3.11. *Let* $\mathscr{C}$ *be a coaugmented cooperad. The* coradical filtration *on* $\bar{\mathscr{C}}$ *is defined as*

$$F_r\bar{\mathscr{C}} := \mathrm{Ker}\left(\bar{\Delta}^{\geqslant r} : \bar{\mathscr{C}} \to \bar{\mathcal{T}}^{\wedge}(\bar{\mathscr{C}})^{(\geqslant r)}\right) \ ,$$

where $\bar{\Delta}^{\geqslant r}$ *denotes the projection of* $\bar{\Delta}$ *to* $\bar{\mathcal{T}}^{\wedge}(\bar{\mathscr{C}})^{(\geqslant r)}$.

It is indeed a filtration as

$$0 = F_0\bar{\mathscr{C}} \subset F_1\bar{\mathscr{C}} \subset \cdots \subset F_r\bar{\mathscr{C}} \subset \cdots \ .$$

Definition 3.12. *A coaugmented cooperad is said to be* conilpotent *if its coradical filtration is complete, i.e. if* $\bar{\mathscr{C}} = \bigcup_{r \geqslant 0} F_r\bar{\mathscr{C}}$.

In other words, a coaugmented cooperad $\mathscr{C}$ is *conilpotent* if a sequence of nontrivial decompositions of any cooperation in $\bar{\mathscr{C}}$ always terminates.

Remark 3.13. There are two main obstructions for a cooperad to be conilpotent.

(1) The *vertical obstruction* coming from the coassociative coalgebra structure on $\mathscr{C}(1)$ with coproduct δ^1. This is in particular one of the reasons why we define conilpotency using $\bar{\mathscr{C}}$ and not $\mathscr{C}$, as the identity can be decomposed ad libitum.
(2) The *horizontal obstruction* which comes from the existence of arity 0 elements.

Proposition 3.14. *(i) A May cooperad* $\mathscr{C}$ *with* $\mathscr{C}(0) = 0$ *is a cooperad.*
(ii) A coaugmented cooperad $\mathscr{C}$ *with* $\mathscr{C}(0) = 0$ *and* $\mathscr{C}(1) = \mathbb{K}\mathrm{id}$ *is conilpotent.*

3.3. Cofree cooperad. Let t be a nonplanar tree. A *degrafting* of t is defined to be a collection of trees $t', t_1, \ldots, t_k$ such that $t = \gamma(t'; t_1, \ldots, t_k)$. Degrafting of trees defines a cooperad structure on the $\mathfrak{S}$-module $\{\mathrm{T}_n\}_{n \geqslant 0}$ in $\mathtt{Set}$. We will denote it as $\mathcal{T}^c$.

DEFINITION 3.15. *The* cofree cooperad *on a $\mathfrak{S}$-module $\mathcal{M}$ is defined to be the $\mathfrak{S}$-module $\mathcal{T}(\mathcal{M})$ endowed with the decomposition given by the degrafting of nonplanar trees. Its is denoted $\mathcal{T}^c(\mathcal{M})$.*

The cofree cooperad $\mathcal{T}^c(\mathcal{M})$ on $\mathcal{M}$ is moreover conilpotent: a sequence of nontrivial degraftings of a nonplanar tree whose vertices are labeled by operations of $\mathcal{M}$ always terminates when all the pieces of the degrafted tree are equal to corollae labeled by an operation of $\mathcal{M}$.

Proposition 3.16. *The functor $\mathcal{T}^c(-) : \mathfrak{S}\text{-}\mathtt{mod} \to \mathtt{conil\text{-}Coop}$ is right adjoint to the functor $\mathcal{C} \in \mathtt{conil\text{-}Coop} \to \bar{\mathcal{C}} \in \mathfrak{S}\text{-}\mathtt{mod}$. In other words, for every $\mathfrak{S}$-module $\mathcal{M}$ and conilpotent cooperad $\mathcal{C}$ there is a natural bijection*

$$\mathrm{Hom}_{\mathfrak{S}\text{-}\mathtt{mod}}(\bar{\mathcal{C}}, \mathcal{M}) \simeq \mathrm{Hom}_{\mathtt{conil\text{-}Coop}}(\mathcal{C}, \mathcal{T}^c(\mathcal{M})) .$$

EXAMPLE 3.17. (i) The cofree cooperad on the $\mathfrak{S}$-module $V = (0, V, 0, \ldots, 0, \ldots)$ is the cofree coalgebra

$$\mathcal{T}^c(V) = (0, T^c(V), 0, \ldots, 0) .$$

(ii) The cofree cooperad on the $\mathfrak{S}$-module $\{K[\mathfrak{S}_n]\}_{n \geqslant 0}$ is the cooperad $\mathbb{K}[\mathcal{T}^c]$.

4. Applications in algebraic topology

4.1. Recognition principle for k-fold loop spaces.

4.1.1. *k-fold loop spaces.*

DEFINITION 4.1. *Let X be a topological space and $x \in X$. The* (based) loop space $\Omega_x X$ *is defined to be the topological space of pointed continuous maps $(\mathbb{S}^1, 0) \to (X, x)$.*

The topology taken on $\Omega_x X$ is the compact-open topology and it is naturally pointed by the constant loop at x. We will write $\Omega X := \Omega_x X$ in the rest of this section for the sake of readability. The space ΩX being pointed, we can define $\Omega^2 X := \Omega(\Omega X)$ and $\Omega^k X$ for any $k \geqslant 0$ inductively. We then call a space of the form $\Omega^k X$ a *k-fold loop space.*

4.1.2. *Little k-cubes operad.*

DEFINITION 4.2. *A* linear embedding with parallel axes $I^n \to I^n$ *is a map of the form*

$$(t_1, \ldots, t_n) \mapsto (c_1(t_1), \ldots, c_n(t_n))$$

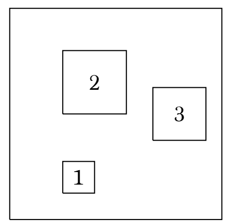
where $c_i(t_i) = (1 - t_i)x_i + t_i y_i$ for fixed $0 < x_i < y_i \leqslant 1$.

DEFINITION 4.3. *The* little k-cubes operad $\mathcal{C}_k := \{\mathcal{C}_k(n)\}_{n \geqslant 1}$ *is the operad in topological spaces defined as follows:*

(1) The arity n space $\mathcal{C}_k(n)$ is the space of ordered collections of n linear embeddings with parallel axes $I^k \to I^k$ with disjoint interiors. A collection of n linear embeddings will be written $\bigsqcup_{i=1}^n I^k \to I^k$.
(2) The action of $\mathfrak{S}_n$ on $\mathcal{C}_k(n)$ is given by permuting the linear embeddings.

(3) *Given $l \geqslant 1$, collections of linear embeddings $\bigsqcup_{i=1}^{n_j} I^k \to I^k$ in $\mathscr{C}_k(n_j)$ for $1 \leqslant l \leqslant k$ and a collection of linear embeddings $\bigsqcup_{i=1}^{l} I^k \to I^k$ in $\mathscr{C}_k(l)$, their composition in $\mathscr{C}_k(n_1 + \cdots + n_l)$ is given by inserting each collection of linear embeddings of $\mathscr{C}_k(n_j)$ in the j-th slot of $\bigsqcup_{i=1}^{l} I^k \to I^k$.*

An element of $\mathscr{C}_2(3)$ can for instance be represented as

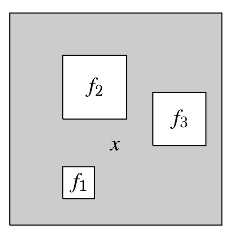


.

4.1.3. *Recognition principle for k-fold loop spaces.*

Proposition 4.4. *Every k-fold loop space $\Omega^k X$ is an algebra over the little k-cubes operad.*

Proof. An element of the k-fold loop space $\Omega^k X$ can be equivalently defined as a map $I^n \to X$ which maps the boundary of I^n to the base point x. The structure maps $\mathscr{C}_k(n) \times (\Omega^k X)^{\times n} \to \Omega^k X$ endowing $\Omega^k X$ with a $\mathscr{C}_k$-algebra structure are then defined as follows. Consider an element $\bigsqcup_{i=1}^{n} I^k \to I^k$ in $\mathscr{C}_k(n)$ and n elements $f_i : I^k \to X$ of $\Omega^k X$. Their image under the $\mathscr{C}_k$-algebra structure map is defined to be the map $I^k \to X$ whose restriction to the i-th embedded cube is f_i and whose restriction to the complement of the interiors of embedded cubes is the constant map x. An element of $\mathscr{C}_2(3)$ acts for instance on f_1, f_2 and f_3 as



.

□

Theorem 8. *If a connected space X is an algebra over the little k-cubes operad then there exists a pointed topological space X_k such that X is homotopy equivalent to $\Omega^k X_k$.*

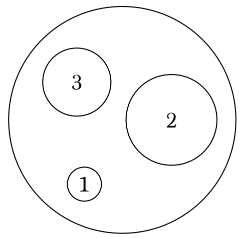
4.2. Framed little disks operad.

4.2.1. *Little k-disks operad.*

Definition 4.5. *The* little k-disks operad $\mathscr{D}_k := \{\mathscr{D}_k(n)\}_{n \geqslant 1}$ *is the operad in topological spaces defined as follows:*

(1) *The arity n space $\mathscr{D}_k(n)$ is the space of ordered collections of n embeddings $\mathbb{D}^k \to \mathbb{D}^k$ by translation and dilation with disjoint interiors.*
(2) *The action of $\mathfrak{S}_n$ on $\mathscr{D}_k(n)$ and the composition maps are defined as in Definition 4.3.*

An element of $\mathcal{D}_2(3)$ can for instance be represented as



.

The functor $H_*(\cdot) : \mathtt{Top} \to \mathtt{grVect}$ mapping a topological space to its singular homology with coefficients in $\mathbb{K}$ is strong monoidal, i.e. $H_*(X \times Y) \simeq H_*(X) \otimes H_*(Y)$. Following Proposition 1.12, the functor $H_*(\cdot)$ thereby maps operads in topological spaces to operads in graded vector spaces.

Proposition 4.6. *For every $k \geqslant 1$ the little k-disks operad $\mathcal{D}_k$ and the little k-cubes operad $\mathcal{C}_k$ are weakly equivalent, i.e. there exists a zig-zag of morphisms of operads in topological spaces*

$$\mathcal{D}_k =: \mathcal{P}_0 \leftarrow \mathcal{P}_1 \rightarrow \cdots \leftarrow \mathcal{P}_{n-1} \rightarrow \mathcal{P}_n := \mathcal{C}_k$$

that induces a zig-zag of isomorphisms of operads in graded vector spaces

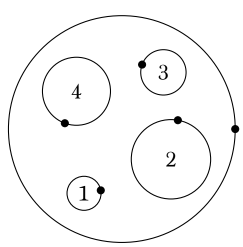
$$H_*(\mathcal{D}_k) = H_*(\mathcal{P}_0) \xleftarrow{\sim} H_*(\mathcal{P}_1) \xrightarrow{\sim} \cdots \xleftarrow{\sim} H_*(\mathcal{P}_{n-1}) \xrightarrow{\sim} H_*(\mathcal{P}_n) = H_*(\mathcal{C}_k) \ .$$

4.2.2. *Framed little disks operad.*

Definition 4.7. *The framed little disks operad $f\mathcal{D}_2 := \{f\mathcal{D}_2(n)\}_{n\geqslant 1}$ is the operad in topological spaces defined as follows:*

(1) The arity n space $f\mathcal{D}_2(n)$ is the space of ordered collections of n embeddings $\mathbb{D}^2 \to \mathbb{D}^2$ by translation, dilation AND rotation with disjoint interiors.
(2) The action of $\mathfrak{S}_n$ on $f\mathcal{D}_2(n)$ and the composition maps are defined as in Definition 4.3 .

An element of $f\mathcal{D}_2(4)$ can for instance be represented as



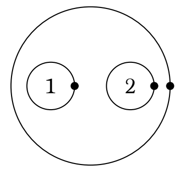
.

where the marked points on the boundaries of the disks represents the rotations used in the embeddings.

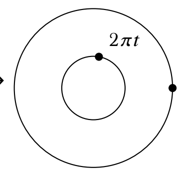
Theorem 9. *The singular homology of the framed little disks operad is isomorphic to the BV operad as an operad in graded vector spaces*

$$H_*(f\mathcal{D}_2) \simeq BV \ .$$

Proof. The following element of $f\mathcal{D}_2(2)$ seen as a 0-chain



is mapped in homology to the multiplication operation of the operad BV, while the following 1-cycle in $f\mathcal{D}_2(1)$

$t \in [0,1] \mapsto$ 

is mapped in homology to the Δ operation of the operad BV. □

In particular if X is a topological space with a $f\mathcal{D}_2$-algebra structure, its singular homology $H_*(X)$ inherits a BV algebra structure.

CHAPTER 4

Twisting morphisms

1. Twisting morphisms

1.1. Convolution algebra and twisting morphisms. Let A be a dg algebra and C be a dg coalgebra.

DEFINITION 1.1. *We define the* convolution algebra *of A and C to be the dg vector space*

$$\underline{\mathrm{Hom}}(C,A) := \underline{\mathrm{Hom}}_{\mathtt{dgVect}}(C,A) = \bigoplus_{r\in\mathbb{Z}} \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathtt{Vect}}(C_n, A_{n+r})$$

endowed with the differential $\partial := [\partial, \cdot]$ and the convolution product

$$f \star g := \mu_A(f \otimes g)\Delta_C \ .$$

The convolution algebra $(\underline{\mathrm{Hom}}(C,A), \star, \partial)$ is a dg algebra. If A is unital with unit u_A and C is counital with counit ε_C, it is moreover a unital dg algebra with unit $u_A\varepsilon_C$.

DEFINITION 1.2. *A* twisting morphism *$\alpha : C \longrightarrow A$ is defined to be a linear map of degree -1 such that*

$$\partial(\alpha) + \alpha \star \alpha = 0 \ .$$

If C is coaugmented with coaugmentation u_C and A is augmented with augmentation ε_A, the linear map α also has to satisfy

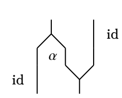
$$\varepsilon_A \alpha = 0 \qquad\qquad \alpha u_B = 0 \ .$$

We denote $\mathrm{Tw}(C,A)$ the *set of twisting morphisms $C \longrightarrow A$*. We point out that a twisting morphism is exactly a Maurer-Cartan element in the dg algebra $\underline{\mathrm{Hom}}(C,A)$.

We now define the map $\partial^r_\alpha : C \otimes A \longrightarrow C \otimes A$ as

$$\partial^r_\alpha := (\mathrm{id}_C \otimes \mu_A)(\mathrm{id}_C \otimes \alpha \otimes \mathrm{id}_A)(\Delta_C \otimes \mathrm{id}_A) \ ,$$

which can be represented as

$\partial^r_\alpha =$  .

In other words $\partial^r_\alpha(c \otimes a) = c^{(1)} \otimes \mu_A(c^{(2)}, a)$.

Proposition 1.3. *Let α be a twisting morphism. Then the degree -1 map*

$$\partial_\alpha := \partial^r_\alpha + \partial_{C\otimes A}$$

defines a differential on $C \otimes A$.

Proof. $\partial_\alpha^2 = (\partial_\alpha^r)^2 + \partial_{C\otimes A}\partial_\alpha^r + \partial_\alpha^r\partial_{C\otimes A} + \partial_{C\otimes A}^2 = \partial_{\alpha\star\alpha}^r + \partial_{\partial(\alpha)}^r + 0 = \partial_{\partial\alpha+\alpha\star\alpha}^r = 0.$ □

We will denote $C \otimes_\alpha A := (C \otimes A, \partial_\alpha)$ and call it the *(right) twisted tensor product.* The *(left) twisted tensor product* $A_\alpha \otimes C$ can be defined in a similar fashion.

1.2. Homology of fiber spaces.

Twisted differentials are used in the computation of the singular homology of fiber spaces. We sketch the results of [**Bro59**] in this section.

1.2.1. *Space of Moore loops.*

Definition 1.4. *Let* (B, b_0) *be a pointed topological space. We define the* space of Moore loops $\Omega_{b_0}^M X$ *as*

$$\Omega_{b_0}^M B := \{\gamma : [0, r] \to B, \gamma(0) = \gamma(r) = b_0, r \geqslant 0\} \ .$$

We then have a natural inclusion $\Omega B \subset \Omega^M B$ where ΩB is the standard based loop space (we drop the subscript $\cdot_{b_0}$ for the sake of readability). This inclusion is in fact a strong deformation retract, which implies in particular that these two spaces are homotopy equivalent.

The space of Moore loops carries a topological monoid structure, given by the concatenation of loops: for $\gamma_1 : [0, r] \to B$ and $\gamma_2 : [0, s] \to B$, we define

$$\gamma_1 * \gamma_2 : t \in [0, r+s] \mapsto \begin{cases} \gamma_1(t) \text{ if } t \in [0, r] \\ \gamma_2(t-r) \text{ if } t \in [r, r+s] \end{cases} \ .$$

Its unit is moreover given by e_{b_0} the *constant path* at b_0 of length 0. This implies that the singular chains $C_*(\Omega^M B)$ form a dg algebra.

Remark 1.5. The based loop space ΩB endowed with the concatenation of based loops is not a topological monoid, as concatenation of loops is not associative. We will however see in ?? that it is an A_∞-space, i.e. a monoid whose multiplication is associative up to homotopy and higher coherent homotopies.

Definition 1.6. (i) *The* space of Moore paths *of a topological space* B *is defined as*

$$P(B) := \{\gamma : [0, r] \to B, r \geqslant 0\} \ .$$

(ii) *The* space of based Moore paths *of a pointed topological space* (B, b_0) *is the subspace* $E(B) \subset P(B)$ *of Moore paths that end at* b_0.

We point out that two paths $\alpha, \beta \in P(B)$ such that $\alpha(r) = \beta(0)$ can still be concatenated to a path $\alpha * \beta$.

1.2.2. *Weakly transitive Hurewicz fibrations.*

Definition 1.7. *A continuous map* $p : E \to B$ *is said to be a* Hurewicz fibration *if it admits the* homotopy-lifting property, *i.e. if for every topological space* X *the following diagram can be filled*

$$\begin{array}{ccc} X \times \{0\} & \longrightarrow & E \\ \big\downarrow & \nearrow & \big\downarrow p \\ X \times [0,1] & \longrightarrow & B \end{array} \ .$$

We define the *fiber* over a point $b \in B$ to be the space $p^{-1}(b)$. For a Hurewicz fibration, all fibers over a path component of B can be proven to be homotopy equivalent: one can thereby speak of the fiber $F := p^{-1}(b)$ of a Hurewicz fibration when B is path connected. This is usually denoted as $F \hookrightarrow E \to B$.

Example 1.8. (i) The *path space fibration* is defined as $\mathrm{ev}_0 : \gamma \in E(B) \mapsto \gamma(0) \in B$. It is an Hurewicz fibration with fiber $ev_0^{-1}(b_0) = \Omega^M B$, i.e.

$$\Omega^M B \hookrightarrow E(B) \to B \ .$$

(ii) Covering spaces and vector bundles are Hurewicz fibrations

Let $p : E \to B$ be a Hurewicz fibration and assume that we have chosen a point $b_0 \in B$. We consider the fiber product

$$U_p := P(B)_{\mathrm{ev}} \times_p E = \{(\gamma, e) \in P(B) \times E, \gamma(r) = p(e)\} \ .$$

We define a *lifting function* of a Hurewicz fibration $p : E \to B$ to be a map $\lambda : U_p \to E$ such that $p\lambda = \mathrm{ev}_0$. The homotopy lifting property ensures that a Hurewicz fibration always admits a lifting function.

Definition 1.9. *A lifting function for p is said to be* weakly transitive *if*

(1) $\lambda(e_{b_0}, x) = x$ *for every* $x \in p^{-1}(b_0)$,
(2) $\lambda(\alpha * \beta, x) = \lambda(\alpha, \lambda(\beta, x))$ *for every* $\alpha, \beta \in P(B)$ *such that* $\alpha(r) = \beta(0) = b_0$.

A Hurewicz fibration is said to be *weakly transitive* if it admits a weakly transitive lifting function. A weakly transitive lifting function defines in particular a left action of $\Omega^M B$ on the fiber $F = p^{-1}(b_0)$ by setting

$$\alpha \cdot x = \lambda(\alpha, x) \ .$$

This action induces a left $C_*(\Omega^M B)$-module structure on $C_*(F)$.

Example 1.10. The path space fibration is weakly transitive. The action of $\Omega^M B$ on the fiber $\Omega^M B \hookrightarrow E(B)$ is then simply given by the concatenation of Moore loops.

Remark 1.11. The weakly transitive property for a Hurewicz fibration should be compared to the the fact the the fundamental group of a connected topological space B acts on the fiber of its universal cover $\tilde{B} \to B$.

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