MORIWAKI HEIGHTS

1. Introduction

This is some lecture notes from the mini-course I gave on Moriwaki heights in Dijon in September 2025. The purpose of these lectures was to introduce Moriwaki heights and to explain their usefulness for complex dynamics.

Why is arithmetic dynamics interesting for complex dynamics ? Well I'll motivate that by the following introducting example.

Consider the transformation $f : z \mapsto z^2 + c$ over $\mathbf{C}$ where $c \in \mathbf{Q}$ is a rational number. There are several dynamical objects associated to f. For example its *Green function*.

$$G_f(p) = \frac{1}{2^n} \log^+ \| f^n(p) \| . \tag{1}$$

We have that $G_f(p) = 0$ if and only if the orbit of p under f is bounded. Furthermore the measure

$$\mu_f = dd^c G_f \tag{2}$$

is the unique measure of maximal entropy of f. Its support is the Julia set of f. Now since c is in $\mathbf{Q}$, f induces a map also over $\mathbf{Q}_p$ the space of p-adic rational numbers and also on $\mathbf{C}_p$ which is the completion of the algebraic closure of $\mathbf{Q}_p$. The Green function $G + f, p$ of f can also be defined over $\mathbf{C}_p$, the measure $mu_{f,p} = dd^C G_{f,p}$ can also be defined. Notice that for almost every prime $p, G_{f,p} = \log^+ |z|_p$. This is an *adelic* object. More precisely it is an *adelic divisor*. Now, for a point $q \in \mathbf{Q}$ we can look at

$$h_f(p) := G_f(p) + \sum_p G_{f,q}(q). \tag{3}$$

And we have the following result $h_f(p) = 0$ if and only if p is preperiodic. This is because h_f is a *height* that satisfies the *Northcott* property. Thus studying the dynamics of f for every prime p allows one to detect preperiodic points. Furthermore Yuan's arithmetic equidistribution results shows that for any generic sequence (p_n) of preperiodic points. The normalised sum of the Dirac measures over the Galois orbits of p_n converges towards the measure μ_f over $\mathbf{C}$ and also over $\mathbf{C}_p$.

Now all of this works the same if $c \in \overline{\mathbf{Q}}$. However if c is transcendental, then there needs to be another theory of heights. Namely the issue is that there infinitely many embeddings $\mathbf{Q}(c) \hookrightarrow \mathbf{C}$ and there is no reason to pick one more than the others. One idea could be to use *specialisation arguments*. We view c as a variable and then for every algebraic value we cover a dynamical system over $\overline{\mathbf{Q}}$ and can apply the theory of heights. But this approach is quite involved and is not well behaved in higher dimension. One other way to tackle this is to use *geometric heights*. The field $\mathbf{Q}(c)$ is the function field of $\mathbf{P}^1_{\mathbf{Q}}$. We can consider the absolute value given by $|\cdot|_w = e^{-\operatorname{ord}_w}$ where $w \in \mathbf{P}^1_{\mathbf{Q}}$ is a closed point and ord_w is the order of vanishing at w. We can define the Green function

$$G_{d,w}(q) = \lim_n \frac{1}{2^n} \log^+ |f^n(q)|_w . \tag{4}$$

The problem is that in general geometric height do not satisfy the Northcott property. Here it will hold because of result on *isotriviality* of families on transformation for rational maps over $\mathbf{P}^1$ but these results are also hard to obtain.

The idea I want to present here is to use *Moriwaki heights*. They are a generalisation of heights for finitely generated fields over $\mathbf{Q}$ that were introduced by Moriwaki in [Mor00]. We still have a Northcott property for thoses heights and an arithmetic equidistribution theorem. The purpose of these notes is to explain how they work and how they are constructed. We will illustrate their use by proving the following result of Baker and DeMarco. If $f : X \to X$ is an endomorphism of a projective variety defined over a field K, we write $\mathrm{Preper}(f)$ for the set of preperiodic points of f. That is

$$\mathrm{Preper}(f) := \left\{p \in X(\overline{K}) : \exists n \neq m, f^n(p) = f^m(p)\right\}. \tag{5}$$

Theorem 1.1 ([BD11]). *Let K be any field of characteristic zero. Let $f, g \in K(x)$ be two endomorphisms of $\mathbf{P}^1_K$ of degree $\geqslant 2$. Then the following are equivalent.*

(1) $\mathrm{Preper}(f) \cap \mathrm{Preper}(g)$ *is infinite.*
(2) $\mathrm{Preper}(f) = \mathrm{Preper}(g)$.

The initial proof of Baker and DeMarco goes like this. If K is a number field, one can construct the canonical heights of f and g. By arithmetic equidistribution f and g have the same Julia set and equilibrium measure for any absolute value over K and by the Northcott property this implies that $\mathrm{Preper}(f) = \mathrm{Preper}(g)$. If K is not a number field, then the proof uses geometric canonical heights and results about isotriviality to conclude. We will give a new proof here using Moriwaki heights.

2. Absolute values

2.1. **Definitions.** A general reference for this subsection is [Rob00, §2.1 and §2.2]. Let K be a field. An *absolute value* over K is a map $|\cdot| : K \to \mathbf{R}_{\geqslant 0}$ such that

(1) $|x| = 0 \Leftrightarrow x = 0$.
(2) $|xy| = |x| \cdot |y|$.
(3) $|x + y| \leqslant \max(|x|, |y|)$.

We say that $|\cdot|$ is *non-archimedean* if it satisfies the ultrametric inequality

$$|x + y| \leqslant \max(|x|, |y|). \tag{6}$$

It is *archimedean* otherwise. An absolute value defines a distance by setting

$$d(x, y) := |x - y|. \tag{7}$$

And we say that two absolute values $|\cdot|_1, |\cdot|_2$ are *equivalent* if they induce the same topology on K. This is equivalent to the existence of $\alpha > 0$ such that

$$|\cdot|_1 = |\cdot|_2^{\alpha}. \tag{8}$$

We give some examples.

Example 2.1. The trivial absolute value over K is defined at $|0|_{\mathrm{triv}} = 0$ and $\forall x \in K \setminus \{0\}, |x|_{\mathrm{triv}} = 1$. It is non-archimedean.

Example 2.2. Let $K = \mathbf{Q}$ be the field of rational numbers. Then we have the classical absolute value $|\cdot|_\infty$ which is the restriction of the complex absolute value. We also have arithmetic ones which are defined as follows. Let p be a prime number. Then we define

$$\forall x \in \mathbf{Q}, \quad |x|_p := p^{-v_p(x)} \tag{9}$$

where $v_p(x)$ is the p-adic valuation of x. It is a result of Ostrowski that every absolute value over $\mathbf{Q}$ is equivalent to $|\cdot|_\infty$. If p is a prime number. We define $\mathbf{Q}_p$ as the completion of $\mathbf{Q}$ with respect to the absolute value $|\cdot|_p$. It is a field and the p-adic absolute value extends uniquely to $\mathbf{Q}_p$ making it a non-archimedean complete field.

Proposition 2.3 ([Rob00, Theorem in §2.3.4]). *Let $K/\mathbf{Q}_p$ be a finite Galois extension. Then, there exists a unique extension of $|\cdot|_p$ to K. It is defined by*

$$\forall x \in K, \quad |x|_p = |N(x)|_p^{1/d} \tag{10}$$

where $d = [K : \mathbf{Q}_p]$ *and* $N(x)$ *is the* norm *of* x *defined by*

$$N(x) := \prod_{\sigma \in \mathrm{Gal}(K/\mathbf{Q}_p)} \sigma(x). \tag{11}$$

Furthermore, K *is complete with respect to this metric.*

Definition 2.4. Let K be a field. A *place* of K is an equivalence class of absolute values. We write $\mathcal{M}(K)$ for the set of places of K. We say that $v \in \mathcal{M}(K)$ is *non-archimedean* if every absolute value in v is non-archimedean.

If K_v is a complete field with respect to a non-archimedean place v, then we write O_v for the subring of elements of absolute value $\leqslant 1$. It is a ring and its unique maximal ideal $\mathfrak{m}_v$ is the set of elements of absolute value < 1. The *residue field* of O_v is defined as

$$\kappa_v := O_v/\mathfrak{m}_v. \tag{12}$$

If v is *discretely valued* i.e $\log|K|$ is a discrete subgroup of $\mathbf{R}$, then $\mathfrak{m}_v$ is a principal maximal ideal generated by one element $\omega \in O_v$. This is the case for example for finite extensions of $\mathbf{Q}_p$.

Example 2.5. Let F be a function field over $\mathbf{Q}$, i.e the function field of some normal projective variety B over $\mathbf{Q}$. If $\Gamma \subset B$ is an irreducible codimension 1 subvariety, then it induces an absolute value over F defined by

$$\forall f \in F, \quad |f|_\Gamma := e^{-\mathrm{ord}_\Gamma(f)} \tag{13}$$

where ord_Γ is the order of vanishing along Γ. Notice that it extends the trivial absolute value over $\mathbf{Q}$ and it is non-archimedean.

2.2. **Product formula.** If K is a number field, we make the following normalisation. If v is an archimedean place of K, then $|\cdot|_v$ is the complex modulus associated to the embedding $K \hookrightarrow \mathbf{C}$. If v is non-archimedean, then we set

$$\forall \lambda \in K^\times, \quad |\lambda|_v = (\#\kappa(v))^{-\mathrm{ord}_{\mathfrak{m}_v}(\lambda)}. \tag{14}$$

With this normalisation we have the following relation which is called the *product formula* (see [Neu99, Proposition II.1.3].)

$$\forall \lambda \in K^\times, \qquad \sum_{v \in \mathcal{M}(K)} \log|\lambda|_v = 0. \tag{15}$$

Notice that $|\cdot|_v$ does not extend the p-adic absolute value but

$$|\cdot|_v^{\frac{1}{[K_v:\mathbf{Q}_p]}} \tag{16}$$

does.

Suppose L/K is a finite extension of number fields. If v is a place of K, then there are finitely many places w over L such that w extends v. Recall the normalisation of $|\cdot|_w$ and $|\cdot|_v$. We study the relation between the two. By Equation (16) and Proposition 2.3 we have that

$$|\cdot|_w^{\frac{1}{[L_w:\mathbf{Q}_p]}} = |\cdot|_v^{\frac{1}{[K_v:\mathbf{Q}_p]}}. \tag{17}$$

Thus, we infer

$$|\cdot|_w = |\cdot|_v^{\frac{[L_w:\mathbf{Q}_p]}{[K_v:\mathbf{Q}_p]}} = |\cdot|_v^{[L_w:K_v]}. \tag{18}$$

And we also have the following formula

$$\sum_{w|v} [L_w : K_v] = [L : K]. \tag{19}$$

3. BERKOVICH SPACES

A general reference for Berkovich spaces is [Ber12]. Let A be an integral ring. A *seminorm* over A is a function $|\cdot| : A \to \mathbf{R}_{\geqslant 0}$ satisfying the same axioms as an absolute value except that we can have nonzero elements $\phi \in A$ such that $|\phi| = 0$. The *kernel* of $|\cdot|$ is the set

$$\ker(|\cdot|) := \{\phi \in A : |\phi| = 0\}. \tag{20}$$

It is a prime ideal.

Let K_v be a complete metrised field. Let X be a projective variety over K_v. The *Berkovich analytification* of X is denoted by X^{an}. It is defined as follows. If $U \subset X$ is an open affine subset with ring A, then U^{an} consists of the set of seminorms over A extending the absolute value over K_v. Then, for any $x \in U^{\mathrm{an}}$ and any $\phi \in A$ we define $|\phi(x)|$ as

$$\phi(x) := |\phi|_x \tag{21}$$

where $|\cdot|_x$ is the associated seminorm of x. It is equipped with the finest topology such that the evaluation maps

$$\mathrm{ev}_x : \phi \in A \mapsto |\phi(x)| \tag{22}$$

is continuous. If $X = \bigcup U_i$ is an open affine cover of X, then the U_i^{an} glue together to define X^{an}. We have that X^{an} is compact Hausdorff. There is the *contraction map*

$$c : X^{\mathrm{an}} \to X \tag{23}$$

defined on open affine subset as $c(x) = \ker(|\cdot|_x)$. In particular, if $Y \subset X$ is a closed subvariety, then

$$Y^{\mathrm{an}} = c^{-1}(Y). \tag{24}$$

There is also a canonical map $X(\overline{K}_v) \to X^{\mathrm{an}}$ defined as follows. By Proposition 2.3, there is a canonical extension of the absolute value of K_v to $\overline{K}_v$. We still denote it by $|\cdot|$. It is Galois invariant by construction. Let $q \in X(\overline{K}_v)$, it is a closed point of X. Let $U = \operatorname{Spec} A$ be an open affine neighbourhood of q, then q defines a seminorm on A via

$$\phi \in A \mapsto |\phi(q)|. \tag{25}$$

Notice that the images of q and of any of its Galois conjugates are the same. We thus have an embedding

$$X(\overline{K}_v)/\mathrm{Gal}(\overline{K}_v/K_v) \hookrightarrow X^{\mathrm{an}}. \tag{26}$$

And the image is dense. We will still write $X(\overline{K}_v)$ for the image.

Example 3.1. If $K_v = \mathbf{C}$ equipped with its usual archimedean absolute value, then

$$X^{\mathrm{an}} = X(\mathbf{C}). \tag{27}$$

If $K_v = \mathbf{C}$ equipped with the trivial absolute value, then X^{an} is related to the Riemann-Zariski space of X.

3.1. **The reduction map.** We give here an important construction for the notion of arithmetic divisors and metrised line bundles. Let K_v be a non-archimedean non-trivially valued complete field with valuation ring O_v. Let X be a projective variety over K_v. A *model* of X over O_v is a normal flat projective scheme $\mathscr{X}$ over O_v such that the generic fiber of $\mathscr{X}$ is X. We write $\mathscr{X}_s$ for the special fiber of $\mathscr{X}$. For any model $\mathscr{X}$ of X, we define the *reduction map*

$$r_{\mathscr{X}} : X^{\mathrm{an}} \to \mathscr{X}_s \tag{28}$$

as follows.

We study the reduction map for points in X^{an} coming from $X(\overline{K}_v)$. If $q \in X(\overline{K}_v)$, then its image is a closed point of X. The closure of q in $\mathscr{X}$ is a curve over O_v which intersects the special fiber at a unique closed point which is $r_{\mathscr{X}}(q)$.

We give an example. Let $X = \mathbf{P}^1_{\mathbf{Q}_p}$, a model of X over $\mathbf{Z}_p$ is $\mathscr{X} = \mathbf{P}^1_{\mathbf{Z}_p} = \operatorname{Proj} \mathbf{Z}_p[u,v]$. We study $r_{\mathscr{X}}(q)$ for $q \in X(\mathbf{Q}_p)$. A point $q \in X(\mathbf{Q}_p)$ is of the form

$$q = [x : y] \tag{29}$$

with $x,y\in\mathbf{Q}_p$. We can assume that $x,y\in\mathbf{Z}_p$ such that x or y is not divisible by p. Then

$$r_{\mathscr{X}}(q)=[\bar{x}:\bar{y}]\in\mathbf{P}^1(\mathbf{F}_p)\subset\mathscr{X}_s=\mathbf{P}^1_{\mathbf{F}_p}. \tag{30}$$

Proposition 3.2. *Let K_v be a complete non-archimedean field with a discretely valued valuation. Let X be a projective variety over K_v and $\mathscr{X}$ a model of X over O_v. For every codimension 1 irreducible component E of the special fiber $\mathscr{X}_s$, there is a unique point $x_E\in X^{\mathrm{an}}$ such that $r_{\mathscr{X}}(x_E)=\eta_E$ the generic point of E. Furthermore, x_E is equivalent to*

$$e^{-\operatorname{ord}_E}. \tag{31}$$

We call such points *divisorial points*.

3.2. **For varieties over number field.** Let K be a number field and let X be a projective variety over K. For every $v\in\mathcal{M}(K)$ we write K_v for the completion of K with respect to $|\cdot|_v$. We define $X_v=X\times_{\operatorname{Spec}K}\operatorname{Spec}K_v$ and

$$X^{\mathrm{an}}:=\bigsqcup_{v\in\mathcal{M}(K)}X_v^{\mathrm{an}}. \tag{32}$$

Furthermore, we write

$$X_\Sigma(\mathbf{C}):=\bigsqcup_{v\in\mathcal{M}_\infty(K)}X_v=\bigsqcup_{K\hookrightarrow\mathbf{C}}X_{\mathbf{C}}(\mathbf{C}). \tag{33}$$

4. Arithmetic divisors and metrised line bundles

4.1. **Over C.** Let X be a complex projective variety and D be a Cartier divisor over X. A *Green function* of D is a function

$$g:X(\mathbf{C})\setminus\operatorname{Supp}D\to\mathbf{R} \tag{34}$$

such that for any $q\in\operatorname{Supp}D$, if ξ is a local equation of D at q, then

$$g+\log|\xi| \tag{35}$$

extends to a continuous function at q.

An *arithmetic divisor* over X is the data of a Cartier divisor D over X and a Green function of D. A *model arithmetic divisor* is an arithmetic divisor $\overline{D}=(D,g)$ where g is smooth. If $D=\operatorname{div}(P)$ is a principal divisor, then it induces an arithmetic divisor

$$\widehat{\operatorname{div}}(P):=(\operatorname{div}(P),-\log|P|). \tag{36}$$

We call such arithmetic divisors *principal*. The space of arithmetic divisor is a group with respect to the sum.

We say that an arithmetic divisor $\overline{D}$ is *effective* if its Green function is $\geqslant 0$. This implies in particular that the underlying divisor is effective.

Let L be a line bundle over X, a *metric* over L is a family of metric $(|\cdot|_x)_{x\in X(\mathbf{C})}$ over the space of local section of L at x such that for every local section s of L, the map

$$x\mapsto|s|_x \tag{37}$$

is continuous. A *metrised line bundle* $\overline{L}$ over X is the data of a line bundle L over X and a metric on it. The space of metrised line bundle is a group where the group law is given by tensor product.

If $\overline{L}$ is a metrised line bundle and s is a rational section of L with divisor $\operatorname{div}(s)$, then this yields an arithmetic divisor $\widehat{\operatorname{div}}(s)$ given by

$$\widehat{\operatorname{div}}(s):=(\operatorname{div}(s),\log|s|). \tag{38}$$

Conversely, if $\overline{D}=(D,g)$ is an arithmetic divisor, then it yields a metrised line bundle $O_X(\overline{D})$ such that

$$|s_D|=g \tag{39}$$

where s_D is the canonical rational section of $O_X(D)$ such that $\mathrm{div}(s_D) = D$. If D_1, D_2 are linearly equivalent then $O_X(\overline{D}_1), O_X(\overline{D}_2)$ are isometric.

The space of metrics over the trivial line bundle O_X is given by the space of continuous function over $X(\mathbf{C})$. The bijection is given by

$$\overline{M} \mapsto \log|1|_{\overline{M}}. \tag{40}$$

We put a distance on the space of metrics of a line bundle L as follows. If $\overline{L}_1, \overline{L}_2$ are two metrisations of L, then $\overline{L}_1 - \overline{L}_2$ is a metrisation of the trivial line bundle O_X and it is therefore given by a continuous function ϕ_{12}. We define

$$d_\infty(\overline{L}_1, \overline{L}_2) := \max_{X(\mathbf{C})} |\phi_{12}|. \tag{41}$$

Definition 4.1. A model metrised line bundle $\overline{L}$ is *semipositive* if L is nef and the metric of L is plurisubharmonic. A metrised line bundle $\overline{L}$ is *semipositive* if there exists a sequence of semipositive model metrised line bundles L_n such that $d_\infty(L_n, L) \to 0$.

If $\overline{L}$ is a semipositive metrised line bundle, then we can define its associated $(1,1)$-current $c_1(\overline{L})$ defined locally by

$$c_1(\overline{L}) = \frac{1}{2\pi} dd^c \log|s| \tag{42}$$

where s is a local non-vanishing section. It is a closed positive current. By the theory of Bedford and Taylor if $\overline{L}_1, \ldots, \overline{L}_n$ are semipositive metrised line bundles, then the measure

$$c_1(\overline{L}_1) \cdots c_1(\overline{L}_n) \tag{43}$$

is well defined. It is a positive measure of total mass $L_1 \cdots L_n$.

Example 4.2. If $X = \mathbf{P}^n_{\mathbf{C}}$ and $L = O(1)$ then we can consider the Fubini-Study metric of L. We have that

$$c_1(\overline{L}) = \omega_{FS}. \tag{44}$$

We can also consider the *Weil* metric given by

$$\| a_0X_0 + \cdots + a_nX_n \| = \frac{|a_0X_0 + \cdots + a_nX_n|}{\max(|X_0|, \ldots, |X_n|)}. \tag{45}$$

It is also semipositive and

$$c_1(\overline{L})^n \tag{46}$$

is the normalised Lebesgue measure on the n-dimensional torus $|X_1| = |X_2| = \cdots = |X_n|$.

4.2. **Over a non-archimedean complete field.** Over a non-archimedean complete field, the definitions of arithmetic divisors and metrised line bundles are analogous. The space $X(\mathbf{C})$ will be replaced by the Berkovich analytification X^{an}. We just have to define what model metrics and model Green functions are.

Let K_v be a non-archimedean complete field and let X be a projective variety over K_v. A *model* of X is a projective variety $\mathscr{X}$ over $\operatorname{Spec} O_v$. A *model arithmetic divisor* is the data of $(\mathscr{X}, \mathscr{D})$ where $\mathscr{X}$ is a model of X and $\mathscr{D}$ is a $\mathbf{Q}$-Cartier divisor over $\mathscr{X}$. Every model arithmetic divisor induces a Green function of $D = \mathscr{D}_{|X}$ over X^{an} as follows. If $x \in X^{\mathrm{an}}$, let ξ be a local equation of $\mathscr{D}$ at $r_{\mathscr{X}}(x)$, then

$$g_{\mathscr{D}}(x) := -\log|\xi(x)|_v. \tag{47}$$

It does not depend on the choice of the local equation ξ because the quotient of two such local equations is a unit in O_v and thus has absolute value 1. We will write $\overline{\mathscr{D}}$ for model arithmetic divisor to have coherent notations. We say that $\overline{\mathscr{D}}$ is semipositive if $O_{\mathscr{X}}(\mathscr{D})$ is nef.

We have a similar definition for model metrised line bundles.

Definition 4.3. An *arithmetic divisor* over X is the data of $\overline{D}=(D,g)$ where D is a $\mathbf{Q}$-Cartier divisor and g is a continuous Green function of D over X^{an}. It is *semipositive* if it is the uniform limit of semipositive model arithmetic divisors. It is *effective* if its Green function is $\geqslant 0$.

A *metrised line bundle* over X is the data $\overline{L}$ of a $\mathbf{Q}$-line bundle L over X and a continuous family of metrics of the local sections of L over X^{an}. It is *semipositive* if it the uniform limit of semipositive model metrics of L. It is *effective* if there exists $\overline{D}$ effective such that $\overline{L}=O_X(\overline{D})$.

4.3. **Chambert-Loir measures.** To define a local intersection number in the same fashion as in the complex case. We need to define the measure

$$c_1(\overline{L}_1)\cdots c_1(\overline{L}_n) \tag{48}$$

over X^{an}. There has been recent work on developping a pluripotential theory and a theory of differential forms and currents on Berkovich spaces (see [CD12, GK16]). This allows to define the $(1,1)$-current $c_1(\overline{L})$ for a semipositive metrised line bundle. However, I will not use this theory here. Before these works, Chambert-Loir proposed a definition for the measure in (48) for model metrised line bundles which then can be extended to integrable metrised line bundles by a limit argument, see [Cha03, Cha11]. Let $\overline{\mathscr{L}}_1,\dots\overline{\mathscr{L}}_n$ be model metrised line bundles. We can assume that they are all defined over the same model $\mathscr{X}$. Then, we define

$$c_1(\overline{\mathscr{L}}_1)\cdots c_1(\overline{\mathscr{L}}_n):=\sum_{E\subset\mathscr{X}_s}\left(\mathscr{L}_{1|E}\cdots\mathscr{L}_{n|E}\right)\delta_{x_E}. \tag{49}$$

where the sum is over the codimension 1 irreducible components of the special fiber and δ_E is the Dirac measure at the divisorial point x_E associated to E.

Proposition 4.4. *If $\overline{L}_1,\dots,\overline{L}_n$ are semipositive metrised line bundles, then for any sequence $(\mathscr{L}_{i,k})_{k\geqslant 0}$ of semipositive model metrised line bundles converging towards $\overline{L}_i$ the sequence of measures*

$$c_1(\overline{\mathscr{L}}_{1,k})\cdots c_1(\overline{\mathscr{L}}_{n,k}) \tag{50}$$

converges over X^{an} to a measure $c_1(\overline{L}_1)\cdots c_1(\overline{L}_n)$ which does not depend on the choice of the sequences of semipositive model metrised line bundles.

Thus the measure $c_1(\overline{L}_1)\cdots c_1(\overline{L}_n)$ is well defined for integrable metrised line bundles. Furthermore,

$$c_1(\overline{L}_1)\cdots c_1(\overline{L}_n)(X^{\mathrm{an}})=L_1\cdots L_n. \tag{51}$$

4.4. **Local intersection number.** Let X be a projective variety over K_v. Let $D_0,\dots,D_n$ be Cartier divisors over X. We say that they *intersect properly* if for every $J\subset\{0,\dots,n\}$,

$$\dim\bigcap_{j\in J}\operatorname{Supp}D_j=n-\#J+1. \tag{52}$$

If $\overline{D}_0,\dots,\overline{D}_n$ are integrable arithmetic divisors over X that intersect properly, then following [Cha03] we define their *local intersection number* by induction as

$$\overline{D}_0\cdots\overline{D}_n=\left(\overline{D}_0\cdots\overline{D}_{n-1}\right)_{|\operatorname{Supp}D_n}+\int_{X^{\mathrm{an}}}g_n c_1(\overline{D}_0)\cdots c_1(\overline{D}_{n-1}). \tag{53}$$

And if $n=0$, then $\overline{D}_0=g_0\in\mathbf{R}$.

Proposition 4.5 ([CM21, Proposition 3.5.5])**.** *The intersection number is well defined. It is multilinear and symmetric. Furthermore, if $P\in K_v(X)$, then*

$$\widehat{\operatorname{div}}(P)\cdot\overline{D}_1\cdots\overline{D}_n=\sum_i -a_i\log|P(q_i)| \tag{54}$$

where $D_1\cdots D_n=\sum a_iq_i$ as zero cycles.

Proof. We prove the result by induction on n. If $n=1$, then $D=\sum_i a_i p_i$ and

$$\widehat{\operatorname{div}}(P)\cdot\overline{D}=(\widehat{\operatorname{div}}(P))_{|D}+\int_{X^{\mathrm{an},v}} g_{\overline{D}}c_1(\widehat{\operatorname{div}}(P)). \tag{55}$$

The first term is the result we want and the integral vanishes because the measure $c_1(\widehat{\operatorname{div}}(P))=0$. In the complex case it is because $\log|P|$ is harmonic outside the support of $\operatorname{div}(P)$ and in the non-archimedean case, $O_{\mathscr{X}}(\operatorname{div}(P))$ is the trivial line bundle so it has degree 0 over every irreducible component of the special fiber on any model.

Now, by induction we have

$$\widehat{\operatorname{div}}(P)\cdot\overline{D}_1\cdot\overline{D}_n=\left(\widehat{\operatorname{div}}(P)\cdot\overline{D}_1\cdots\overline{D}_{n-1}\right)_{|D_n}+\int_{X^{\mathrm{an},v}} g_{\overline{D}_n}c_1(\widehat{\operatorname{div}}(f))\cdot c_1(\overline{D}_1)\cdots c_1(\overline{D}_{n-1}). \tag{56}$$

By induction the first term is the result we want to prove and the integral vanishes by the same argument as in the case $n=1$ because $O_{\mathscr{X}}(\operatorname{div}(P))$ is the trivial line bundle. □

In particular, this proposition shows that the local intersection product is not well defined over metrised line bundles. However, this issue disappears when we use global intersection number thanks to the product formula.

If K_v' is a complete extension of K_v and $\overline{D}_0,\dots,\overline{D}_n$ are arithmetic integrable divisor over X intersecting properly then this remains true when doing the base change over $X_{K_v'}$. If $\overline{D}_i'$ is the pull-back by the base change, then

$$\overline{D}_0\cdots\overline{D}_n=\overline{D}_0'\cdots\overline{D}_n'. \tag{57}$$

5. Adelic divisors and line bundles

5.1. **Model adelic divisors and line bundles.** These are objects defined from Arakelov geometry. Let K be a number field and X a projective variety over K.

A *model adelic divisor* over X is the data of a model $\mathscr{X}$ of X over O_K and a $\mathbf{Q}$-Cartier divisor $\mathscr{D}$ over $\mathscr{X}$ together with a Green function of $D=\mathscr{D}_{|X}$ over $X_\Sigma(\mathbf{C})$. We denote such an object by $\overline{\mathscr{D}}$.

A *model adelic line bundle* over X is the data of a model $\mathscr{X}$ of X over O_K and a $\mathbf{Q}$-line bundle $\mathscr{L}$ over $\mathscr{X}$ together with a metric of $L=\mathscr{L}_{|X}$ over $X_\Sigma(\mathbf{C})$. We denote such an object by $\overline{\mathscr{L}}$.

5.2. **Definitions.** Let K be a number field and let X be a projective variety over K. An *adelic divisor* $\overline{D}=\left(D,(g_v)_{v\in\mathcal{M}(K)}\right)$ over X is the data of a $\mathbf{Q}$-Cartier divisor D over X and a Green function g_v of D_v over X_v such that there exists a sequence $\overline{\mathscr{D}}_n$ of model adelic divisors such that $\mathscr{D}_{n|X}=D$ and an open subset $U\subset\operatorname{Spec}O_K$ such that for every $n\geqslant 1$

$$(\mathscr{X}_{n|U},\mathscr{D}_{n|U})=(\mathscr{X}_{1|U},\mathscr{D}_{1|U}). \tag{58}$$

For the finitely many reminaining places v which are not in U (this includes the archimedean ones), the Green functions g_v are uniform limits of the Green functions $g_{\overline{\mathscr{D}}_n,v}$. It is *semipositive* if for every v the induced arithmetic divisor is semipositive. We still write g for the data of $(g_v)_v$. We can view g as a function on the disjoint union of the Berkovich analytifications of X with respect to all the places in $\mathcal{M}(K)$. We say that $\overline{D}$ is *effective* if $g\geqslant 0$, this implies in particular that the Cartier divisor D over X is effective.

The definition for adelic line bundles is similar. If $\overline{L}$ is an adelic line bundle, then the *height* of a point $p\in X(\overline{K})$ is given by

$$h_{\overline{L}}(p)=\frac{1}{\left|\operatorname{Gal}(\overline{K}/K)\cdot p\right|}\sum_{q\in\operatorname{Gal}(\overline{K}/K)\cdot p)}\sum_v -\log|s(q)|_v \tag{59}$$

where s is a local regular section of L at p not vanishing at p. By the product formula it does not depend on the choice of the local section s.

We say that a model adelic line bundle $\overline{\mathscr{L}}$ is *nef* if it is semipositive and for any point $p \in X(\overline{K}), h_{\overline{\mathscr{L}}}(p) \geqslant 0$. An adelic line bundle is *nef* if it is the uniform limit of nef model adelic line bundles. An adelic line bundle is *integrable* if it is the difference of two nef adelic line bundles.

5.3. **Examples.** Let $K = \mathbf{Q}$ and $X = \mathbf{P}^n_{\mathbf{Q}}$. Consider the line bundle $H = O(1)$ with the following metrics. For $v \in \mathcal{M}_{\mathbf{Q}}$ we set

$$\| a_0X_0 + \cdots a_nX_n \|_v = \frac{|a_0X_0 + \cdots + a_nX_n|_v}{\max\left(|X_0|_v \cdots |X_n|_v\right)}. \tag{60}$$

We call this family of metrics the *Weil metric* on $O(1)$. We write $\overline{H}_W$ for this adelic line bundle.

Proposition 5.1. *The adelic line bundle $\overline{H}_W$ is a nef model adelic line bundle. We have that $\overline{H}_W$ is $O(1)_{|\mathbf{P}^n_{O_K}}$ with the Weil metric over the archimedean places. Furthermore, for v archimdean, $c_1(\overline{H}_W)^n_v$ is the Lebesgue measure on the n-dimensional torus $|X_0|_\infty = |X_1|_\infty = \cdots = |X_n|_\infty$. And for every $v \in \mathcal{M}_f(K)$, the measure $c_1(\overline{H}_W)^n$ is the Dirac measure at the Gauss point of $\mathbf{P}^{n,\mathrm{an}}_{K_v}$.*

Proof. Indeed, consider the model $\mathbf{P}^n_{O_K}$ with the line bundle $\mathscr{L} = O(1)_{\mathbf{P}^n_{O_K}}$, then $\overline{H}_W = \overline{\mathscr{L}}$ where the metric at the archimedean place is given by the Weil metric. It is equivalent to show that the adelic divisors $\widehat{\mathrm{div}}(X_0)$ induced by the global section X_0 are equal for both adelic line bundles. For simplicity we assume $K = \mathbf{Q}$ and we show the computation for $\mathbf{Q}_p$ points. Over $\{X_0 \neq 0\} = \mathbf{A}^n_{\mathbf{Q}}$, we have that the Green function of $\widehat{\mathrm{div}}(s_0)$ for $\overline{H}_w$ is

$$g_v = \log^+ \max(|x_1|_v, \ldots, |x_n|_v) \tag{61}$$

where $x_i = \frac{X_i}{X_0}$. The global section X_0 is also a global section of $O(1)_{\mathbf{P}^n_{\mathbf{Z}}}$ and its zero divisor is $\mathscr{D} = \{X_0 = 0\} \subset \mathbf{P}^n_{\mathbf{Z}}$. Fix a prime p and make the base change to $\mathbf{Z}_p$. Take a point $q = [q_0 : \cdots : q_n] \in \mathbf{P}^n(\mathbf{Q})$ with $q_0 \neq 0$. We know that its reduction is

$$r(q) = [\overline{q}_0 : \cdots : \overline{q}_n] \in \mathbf{P}^n(\mathbf{F}_p). \tag{62}$$

where $q_0, \ldots, q_n \in \mathbf{Z}_p$ are not all divisible by p and $\overline{q}_i$ is the reduction mod p. If $\overline{q}_0 = 1$, then $r(q) \notin \operatorname{Supp} \mathscr{D}$ so that $g_{\overline{\mathscr{D}}}(q) = 0 = \log^+ \max \left|\frac{q_i}{q_0}\right|$. Notice that in that case $q_0 \in \mathbf{Z}_p^\times$. Otherwise, let i be an index such that $\overline{q}_i$ is not zero. This means that $|q_i|_p = 1 = \max(|q_j|_p)$. Then, $r(q) \in \operatorname{Supp} \mathscr{D}$ and $\frac{X_0}{X_i}$ is a local equation of $\mathscr{D}$ at $r(q)$. Then we have

$$g_{\overline{\mathscr{D}}}(q) = -\log\left|\frac{q_0}{q_i}\right|_p = \log\left|\frac{q_i}{q_0}\right|_p = \log^+ \max\left(\left|\frac{q_j}{q_0}\right|\right). \tag{63}$$

Notice in particular that the Green function of $\overline{\mathscr{D}}$ is $\geqslant 0$ so that the height function of $\overline{H}_W$ is $\geqslant 0$ on $\mathbf{A}^n(\overline{\mathbf{Q}})$. Taking another global section X_{i_0} we get that the height function is $\geqslant 0$ over $\mathbf{P}^n_{\mathbf{Q}}$. The line bundle $\mathscr{L}$ is obviously nef, the Weil metric at the archimdean place is semipositive so that $\overline{H}_w$ is nef. In particular, the height function of $\overline{H}_W$ is the classical Weil height over $\mathbf{P}^n$.

For the non-archimedean places. Notice that for every prime p, the fiber over p in $\mathbf{P}^1_{\mathbf{Z}}$ is irreducible and it is actually $\mathbf{P}^1_{\mathbf{F}_p}$. So if $\Gamma = \mathbf{P}^1_{\mathbf{F}_p}$, then Γ is a principal Cartier divisor equal to $\mathrm{div}(p)$. So what is $\mathrm{ord}_\Gamma(P)$ when $P \in \mathbf{Q}[x_1, \ldots, x_n]$? Well it is equal to

$$\mathrm{ord}_\Gamma(P) = \min_{a \in coeff(P)} v_p(a) = -(\max_a -v_p(a)). \tag{64}$$

So using the right normalisation we get that $e^{-(\log p)\,\mathrm{ord}_\Gamma}$ is exactly the Gauss norm over $\mathbf{Q}_p(x_1, \ldots, x_n)$. The result follows from $O(1)^n_{\mathbf{P}^n_{\mathbf{F}_p}} = 1$. □

Exercice 5.2. Let $\overline{\mathscr{D}}_n$ be the model adelic divisor $\{X_0 = 0\} \subset \mathbf{P}^1_{\mathbf{Z}}$ but with complex Green function given by

$$g_n(z) = \log^+ \left|\frac{z}{n}\right|. \tag{65}$$

Show that

$$h_{\mathcal{O}^1_{\mathbf{P}}(\overline{\mathscr{D}}_n)}(\infty) = -\log n. \tag{66}$$

In particular, it is not nef when $n \geqslant 2$ but it is semipositive.

We define another model adelic line bundle over $\mathbf{P}^n_{\mathbf{Q}}$ that we denote by $\overline{H}_{FS}$. It is the model line bundle $\mathcal{O}_{\mathbf{P}^n_{\mathbf{Z}}}(1)$ equipped with the Fubini-Study metric at the archimedean place

$$\| X_i \| = \frac{|X_i|_{\mathbf{C}}}{\left(|X_0|^2 + \cdots |X_n|^2\right)^{1/2}}. \tag{67}$$

It is also a nef model adelic metric.

5.4. **Global intersection number and heights.** When $\overline{\mathscr{L}}_0, \overline{\mathscr{L}}_n$ are model adelic line bundles, the arithmetic intersection number

$$\overline{\mathscr{L}}_0 \cdots \overline{\mathscr{L}}_n \tag{68}$$

was first introduced by Gillet-Soulé in [GS90]. Then, Chambert-Loir in [Cha03] showed that the intersection number can be given by the sum of the local intersection numbers introduced in §4.4. We explain this now. Let $\overline{L}_0, \ldots, \overline{L}_n$ be model semipositive adelic line bundles over X. There exist rational sections $s_i \in L_i$ such that $\operatorname{div}(s_0), \ldots, \operatorname{div}(s_n)$ intersect properly. Then we define

$$\overline{L}_0 \cdots \overline{L}_n = \sum_{v \in \mathcal{M}(K)} \left(\widehat{\operatorname{div}}(s_0) \cdots \widehat{\operatorname{div}}(s_n)\right)_v. \tag{69}$$

This does not depend on the choice of the rational sections by Proposition 4.5 and (15). Notice that if $\overline{\mathscr{L}}_0, \ldots, \overline{\mathscr{L}}_n$ are model adelic line bundles defined over the same model $\mathscr{X}$, then by the definition of the local intersection number, if s_0 is a rational section of L_0 we have

$$\begin{aligned} \overline{\mathscr{L}}_0 \cdots \overline{\mathscr{L}}_n &= \left(\overline{\mathscr{L}}_1 \cdots \overline{\mathscr{L}}_n\right)_{\operatorname{div}(s_0)} + \sum_{v \in \mathcal{M}(K)} \int_{X_v^{\mathrm{an}}} -\log |s_0|_{\overline{\mathscr{L}}_0, v} c_1(\overline{\mathscr{L}}_1)_v \cdots c_1(\overline{\mathscr{L}}_n)_v \\ &= \sum_{\Gamma \subset X^{(1)}} \operatorname{ord}_\Gamma(s_0) \left(\overline{\mathscr{L}}_{1|\Gamma} \cdots \overline{\mathscr{L}}_{n|\Gamma}\right) \\ &\quad + \sum_{E \subset \mathscr{X}_s} C_E \operatorname{ord}_E(s_0) \left(\mathscr{L}_{1|E} \cdots \mathscr{L}_{n|E}\right) \\ &\quad + \int_{X_\Sigma(\mathbf{C})} -\log |s_0|_{\overline{\mathscr{L}}_0, \mathbf{C}} c_1(\overline{\mathscr{L}}_1)_{\mathbf{C}} \cdots c_1(\overline{\mathscr{L}}_n)_{\mathbf{C}}. \end{aligned} \tag{70}$$

We see that the global intersection number can be decomposed into three separated contributions. The first sum is over the codimension 1 subvarieties of X this is a contribution from the geometric absolute values. The second one is over the codimension 1 subvarieties of $\mathscr{X}$ contained in a special fiber, this is a contribution from the arithmetic absolute values.

This is important for the notions of Moriwaki heights and more generally for the notions of *adelic curves* developped by Chen and Moriwaki. We are going to make the following definition. Let $\Gamma \subset \mathscr{B}$ be a codimension 1 subvariety. If $\Gamma = \overline{\Gamma}_{hor}$ where $\Gamma_{hor} \subset X^{(1)}$ then we set

$$(\overline{\mathscr{L}}_{1|\Gamma} \cdots \overline{\mathscr{L}}_{n|\Gamma}) = (\overline{\mathscr{L}}_{1|\Gamma_{hor}} \cdots \overline{\mathscr{L}}_{n|\Gamma_{hor}}). \tag{71}$$

Furthermore we have that s_0 is also a rational section of $\mathscr{L}_0$ over $\mathscr{X}$ and we have that

$$\operatorname{ord}_\Gamma(s_0) = \operatorname{ord}_{\Gamma_{hor}}(s_0). \tag{72}$$

If $\Gamma = E \subset \mathscr{X}_s$ is vertical, then we define

$$\overline{\mathscr{L}}_{1|E} \cdots \overline{\mathscr{L}}_{n|E} := \mathscr{L}_{1|E} \cdots \mathscr{L}_{n|E}. \tag{73}$$

Using this convention, which actually comes from the arithmetic intersection theory of Gillet-Soulé, we have

$$\overline{L}_0\cdots\overline{L}_n = \sum_{\Gamma\subset\mathscr{B}} -\log \| s_0 \|_\Gamma \,(\overline{L}_{1|\Gamma}\cdots\overline{L}_{n|\Gamma}) + \int_{X(\mathbf{C})} -\log|s_0|_{\overline{L}_0,\mathbf{C}}\,. \tag{74}$$

Proposition 5.3. *The global intersection number for model adelic line bundles is well defined. It is symmetric and multilinear. Furthermore if $\overline{L}_0,\dots,\overline{L}_n$ are nef, then*

$$\overline{\mathscr{L}}_0\cdots\overline{\mathscr{L}}_n \geqslant 0. \tag{75}$$

Theorem 5.4. *If $\overline{L}_0,\dots,\overline{L}_n$ are nef adelic line bundles then for any sequence of nef model adelic line bundles $(\overline{\mathscr{L}}_{i,k})$ converging towards $\overline{L}_i$ we have*

$$\overline{L}_0\cdots\overline{L}_n = \lim_k \overline{\mathscr{L}}_{0,k}\cdots\overline{\mathscr{L}}_{n,k} \tag{76}$$

and the formulas (70) *and* (74) *hold.*

Example 5.5. Let $\overline{H}_W$ be the nef model adelic divisor defined in the previous section. Then,

$$\overline{H}_W^{n+1} = 0. \tag{77}$$

Indeed, we show the result by induction over n. If $n=1$, then

$$\overline{H}_W^2 = h_{\overline{H}_w}(\infty) + \sum_v \int_{\mathbf{P}_v^{1,\mathrm{an}}} \log^+|z|_v\, c_1(\overline{H}_W)_v. \tag{78}$$

Now the Weil height of $\infty=[1:0]$ is zero. We analyse the sum of integrals. If $v=\infty$, then $c_1(\overline{H}_W)$ is the Lebesgue measure on the unit circle and the function $\log^+|z|$ is zero there, so this integral vanishes. If $v=p$ is a prime number, then the integral is the evaluation of z at the Gauss point and therefore $|z|_{p,Gauss}=0$ so the integral vanishes as well and we get $\overline{H}_W^2=0$. Now, over $\mathbf{P}^n$ with $n\geqslant 2$. We use the integration by parts formula with global section X_n of $O(1)$. Notice that $\overline{H}_{W|X_0=0}$ is exactly the Weil metric over $\mathbf{P}^{n-1}$ after the identification $\mathbf{P}^{n-1}=\{X_0=0\}$. So that we get by induction

$$\overline{H}_W^{n+1} = \overline{H}_W^n + \sum_v \int_{\mathbf{P}_v^{n-1,\mathrm{an}}} \log^+\max\left(|z_1|_v,\cdots,|z_n|_v\right) c_1(\overline{H}_W)_v^n. \tag{79}$$

The arithmetic intersection number vanishes by induction and every integral vanishes because the function integrated vanishes on the support of the measures (either the Lebesgue measure on the $(n-1)$-dimensional torus or the Dirac at the Gauss point).

Exercice 5.6. Show that $h_{\overline{H}_{FS}}(\infty)=0$ and that $\overline{H}_{FS}^2>0$.

If $Z\subset X$ is a closed subvariety and $\overline{L}$ is an integrable adelic line bundle, then the *height* of Z with respect to $\overline{L}$ is given by

$$h_{\overline{L}}(Z) := \frac{\overline{L}_{|Z}^{\dim Z+1}}{(\dim Z+1)\deg_L(Z)} \tag{80}$$

whenever $\deg_L(Z)>0$. This extends the formula for points. In particular, if $q\in X(K)$, then

$$h_{\overline{L}}(q) = \sum_{v\in\mathcal{M}(K)} -\log|s(q)|_v\,. \tag{81}$$

Furthermore, if Z is a closed subvariety of $X_{\overline{K}}$, then there exists a finite extension L of K such that Z is defined over L. Let $X_L = X\times_{\operatorname{Spec}K}\operatorname{Spec}L$ be the base change with the projection $\pi: X_L\to X$. Then, if $\overline{L}$ is a metrised line bundle over X, its pullback $\pi^*\overline{L}$ is a metrised line bundle over X_L and

$$h_{\overline{L}}(Z) := h_{\pi^*\overline{L}}(Z). \tag{82}$$

It does not depend on the choice of the completion L thanks to our choice of renormalisation from (16). In particular if $q \in X(\overline{K})$ we recover the classical formula

$$h_{\overline{L}}(q) = \frac{1}{\left|\mathrm{Gal}(\overline{K}/K)\cdot q\right|} \sum_{v\in\mathcal{M}(K)} \sum_{p\in\mathrm{Gal}(\overline{K}/K)} -\log|s(p)|_v\,. \tag{83}$$

5.5. **Zhang's fundamental inequality.** Let X be a projective variety over a number field K. Let $\overline{L}$ be a nef adelic line bundle over X with L big. Then the fundamental inequality of Zhang states that

$$e_1(X,\overline{L}) \geqslant h_{\overline{L}}(X) \tag{84}$$

where

$$e_1(\overline{X},\overline{L}) = \sup_{U\subset X} inf_{q\in U} h_{\overline{L}}(q). \tag{85}$$

This motivates the following definition. We say that $(q_n) \subset X(\overline{K})$ is a sequence of *small points* if $h_{\overline{L}} \to h_{\overline{X}}$.

5.6. **Positivity.** Let X be a projective variety of dimension d over a number field K and let L be a line bundle over X. If s is a global section of X and $\overline{L}$ is an adelic metrisation of L then we define

$$\| s \|_{\sup} := \sup_{v\in\mathcal{M}(K)} \sup_{X(K_v)} |s(x)|_{\overline{L},v}\,. \tag{86}$$

We say that s is *small* if $\| s \|_{\sup} \leqslant 1$. This corresponds to integral global sections of s. They are in bijection with effective adelic divisors such that $\overline{L} = O_X(\overline{D})$. We write $H^0(X,\overline{L})$ for the set of small global sections of $\overline{L}$. We define

$$\widehat{h}^0(X,\overline{L}) = \log\left|H^0(X,\overline{L})\right|. \tag{87}$$

We define the *arithmetic volume* of $\overline{L}$ by

$$\widehat{\mathrm{vol}}(\overline{L}) = \lim_{m\to+\infty} \frac{d!}{m^d}\widehat{h}^0(X,m\overline{L}). \tag{88}$$

We say that $\overline{L}$ is *big* if $\widehat{\mathrm{vol}}(\overline{L}) > 0$.

Theorem 5.7 (Arithmetic Hilbert-Samuel formula)**.** *Let $\overline{L}$ be a nef adelic line bundle, then*

$$\widehat{\mathrm{vol}}(\overline{L}) = \overline{L}^{d+1}. \tag{89}$$

Furthermore, if $\overline{M}$ is another nef adelic line bundle, then

$$\widehat{\mathrm{vol}}(X,\overline{L}-\overline{M}) \geqslant \overline{L}^{d+1} - \overline{L}^d \cdot \overline{M}. \tag{90}$$

5.7. **Functoriality.** Let X,Y be projective varieties over a number field K. Let $f : X \to Y$ be dominant morphism, then we have a pullback operator

$$f^*\widehat{\mathrm{Pic}}(Y) \to \widehat{\mathrm{Pic}}(X), \quad f^*\widehat{\mathrm{Div}}(Y) \to \widehat{\mathrm{Div}}(X). \tag{91}$$

This operator sends model adelic line bundles to model ones, preserves semipositivity and nefness and also effectiveness. Furthermore if X,Y have the same dimension then we have

$$f^*\overline{L}_0 \cdots f^*\overline{L}_n = (\deg f)\overline{L}_0 \cdots \overline{L}_n. \tag{92}$$

5.8. **Arithmetic equidistribution theorem.** Let $\overline{L}$ be a nef adelic line bundle such that L is big. For any $v \in \mathcal{M}(K)$, the *equilibrium measure* of $\overline{L}$ over X_v^{an} is the probability measure defined as

$$\mu_{\overline{L},v} = \frac{c_1(\overline{L})_v^n}{L^n}. \tag{93}$$

It is in particular a probability measure.

Theorem 5.8. *Let X be a projective variety over a number field K and let $\overline{L}$ be a semipositive adelic line bundle such that L is big. Then, for any generic sequence $(p_n) \subset X(\overline{K})$ such that $h_{\overline{L}}(p_n) \to h_{\overline{L}}(X) = \overline{L}^{n+1}$, the sequence of probability measures*

$$\delta_n := \frac{1}{\mathrm{Gal}(\overline{K}/K) \cdot p_n} \sum_{q \in \mathrm{Gal}(\overline{K}/K) \cdot p_n} \delta_q \tag{94}$$

weakly converges towards $\mu_{\overline{L}}$.

5.9. **Canonical height associated to a polarised endomorphism.** Let X be a projective variety over a number field K and let $f : X \to X$ be a dominant endomorphism. We say that f is *polarised* if there exists an ample line bundle L such that $f^*L = dL$ with $d \geqslant 2$.

Theorem 5.9. *There exists a unique semipositive adelic line bundle $\overline{L}$ with underlying line bundle L such that*

$$f^*\overline{L} = d\overline{L}. \tag{95}$$

In particular, we have that $\overline{L}^{n+1} = 0$ and $p \in \mathrm{Preper}(f) \Leftrightarrow h_{\overline{L}}(p) = 0$.

Proof. Start with any semipositive adelic model metrisation $\overline{L}_0 = (\mathscr{X}, \overline{\mathscr{L}})$ of L. For any $n \geqslant 0$, let ϕ_n be an isomorphism between $\frac{1}{d^n}(f^n)^*L$ and L. The sequence $\overline{L}_n = \frac{1}{d^n}(\phi_n^*)(f^n)^*\overline{L}$ is a sequence of model semipositive adelic line bundles with underlying line bundle L. We show that it converges towards an adelic line bundle $\overline{L}$. There exists an open subset $U \subset \operatorname{Spec} O_K$ such that f induces a dominant endomorphism $f_U : \mathscr{X}_U \to \mathscr{X}_U$ and

$$f_U^*\mathscr{L}_U = d\mathscr{L}_U. \tag{96}$$

In particular, for any finite place v above U the metric of $\overline{L}_n$ is induced by $\mathscr{L}_U$. Now pick v to be a place not above U. The metrised line bundle $\overline{M}_v := \overline{L}_{1,v} - \overline{L}_{0,v}$ is vertical and we have

$$\overline{L}_{n+1,v} - \overline{L}_{n,v} = \frac{1}{d^n}(f^n)^*\overline{M}. \tag{97}$$

Let $C > 0$ be such that $\max_{X_v^{\mathrm{an}}} g_{\overline{M}} \leqslant C$, then $d_\infty(\overline{L}_{n+1,v}, L_{n,v}) \leqslant \frac{C}{d^n}$ and this shows that the metrics converge uniformly over $X^{\mathrm{an},v}$. It is clear that the limit $\overline{L}$ is semipositive and satisfies $f^*\overline{L} = d\overline{L}$.

Now we have

$$\left(f^*\overline{L}\right)^{n+1} = d_{\mathrm{top}}(f)\overline{L}^{n+1} = d^{n+1}\overline{L}^{n+1}. \tag{98}$$

Since $d_{\mathrm{top}}(f) = d^n$ we get that $\overline{L}^{n+1} = 0$.

Finally, we have that $h_{\overline{L}} \circ f = dh_{\overline{L}}$. This implies that $p \in \mathrm{Preper}(f) \Rightarrow h_{\overline{L}}(p) = 0$. Conversely, if $h_{\overline{L}}(p) = 0$, then we have that for every $k \geqslant 0$

$$h_{\overline{L}}(f^k(p)) = 0 \tag{99}$$

and by the Northcott property, this implies that $p \in \mathrm{Preper}(f)$. □

5.10. **Proof of Theorem 1.1 in the number field case.** Suppose $f, g \in K(x)$ are endomorphisms of $\mathbf{P}^1$ defined over a number field K that share infinitely many preperiodic points. We can extract a subsequence of common preperiodic points which is generic. We call (p_n) such a sequence. Both endomorphisms are polarised with $L = O(1)$. Thus we can contruct the canonical heights of f and g. The sequence (p_n) is a sequence of small points for both $\overline{L}_f$ and $\overline{L}_g$ so that we can apply Theorem 5.8 to get that for every $v \in \mathcal{M}(K)$,

$$\mu_{f,v} = \mu_{g,v}. \tag{100}$$

We show that this implies that $h_f = h_g$. Fix a place v. The metrised line bundles $L_{f,v}$ and $L_{g,v}$ are both metrisation of the line bundle $O(1)$. Thus, the difference $\overline{M}_v$ is an integrable metrisation of the trivial line bundle. Furthermore we have by construction that $\mu_{f,v} = c_1(\overline{L}_{f,v}) = c_1(\overline{L}_{g,v}) = \mu_{g,v}$. In particular, $c_1(\overline{M}_v) = 0$ and $g_{\overline{M}_v}$ is harmonic. But since $\mathbf{P}^{1,\mathrm{an}}_v$ is compact we have by the maximum principle that $g_{\overline{M}_v}$ is constant equal to λ_v. In particular, if $\overline{M} = \overline{L}_f - \overline{L}_g$, then the height function $h_{\overline{M}}$ is constant since

$$\forall q \in \mathbf{P}^1(\overline{K}), \quad h_{\overline{M}}(q) = \frac{1}{|\mathrm{Gal}(q)|} \sum_v \sum_{p \in \mathrm{Gal}(q)} g_{\overline{M},v}(p) = \sum_v \lambda_v. \tag{101}$$

But $h_{\overline{M}}(q) = 0$ for any $q \in \mathrm{Preper}(f) \cap \mathrm{Preper}(g)$. So we get the result.

6. Moriwaki heights

6.1. **Absolute values over finitely generated fields.** Let $F = \mathbf{Q}(t)$. How can we describe say all the archimedean absolute values of F ? They correspond exactly to field embedding $\mathbf{Q}(t) \hookrightarrow \mathbf{C}$ which is characterised by the image of t in $\mathbf{C} \backslash \overline{\mathbf{Q}}$. Now $\mathbf{Q}(t)$ is also the function field of $\mathbf{P}^1_{\mathbf{Q}}$ and we have

$$F^{\mathrm{an},\mathbf{C}} = \mathbf{C} \backslash \overline{\mathbf{Q}} = \mathbf{P}^1(\mathbf{C}) \backslash \mathbf{P}^1(\overline{\mathbf{Q}}) \hookrightarrow \mathbf{P}^{1,\mathrm{an}}_{\mathbf{Q}} = \mathbf{P}^1(\mathbf{C}). \tag{102}$$

Notice that for any semipositive metrised line bundle $\overline{H}$ over $\mathbf{P}^1_{\mathbf{C}}$ the set $\mathbf{P}^1(\overline{\mathbf{Q}})$ is of $c_1(\overline{H})$-measure zero. More generally, let K be the algebraic closure of $\mathbf{Q}$ in F. It is a number field. Since $K \subset F$ every absolute value over F induces an absolute value over K so we have a decomposition.

$$\mathcal{M}(F) = \bigsqcup_{v \in \mathcal{M}(K)} \mathcal{M}_v(F) \bigsqcup \mathcal{M}_0(F). \tag{103}$$

where $\mathcal{M}_F(F)$ is the set of places of F extending v over K and $\mathcal{M}_(F)$ is the set of places extending the trivial absolute value over K.

Let $\mathcal{B}$ be a *projective model* of F over O_K. That is a projective variety over $\mathrm{Spec}\, O_K$ with function field F. We explain how $\mathcal{B}$ induces elements of $\mathcal{M}(F)$. Write $B = \mathcal{B}_K$. First, every archimedean place v over K corresponds to an embedding $K \hookrightarrow \mathbf{C}$. Fixing this embedding we have that

$$\mathcal{M}_v(F) = B(\mathbf{C}) \backslash \bigcup_{Y \subset B} Y(\mathbf{C}). \tag{104}$$

We write

$$B_\Sigma(\mathbf{C}) := \bigsqcup_{K \hookrightarrow \mathbf{C}} B(\mathbf{C}). \tag{105}$$

Now, let $\Gamma \subset \mathcal{B}$ be an irreducible codimension 1 subvariety. There are two possibilities. Either Γ is *horizontal*, i.e it surjects onto $\mathrm{Spec}\, O_K$. In that case Γ is the closure of a prime divisor $C \subset B$ and the absolute value $e^{-\mathrm{ord}_C} = e^{-\mathrm{ord}_\Gamma}$ induces an element of $\mathcal{M}_0(F)$. If Γ is *vertical*, that is it lies on the fiber over a maximal ideal $\mathfrak{m}_v$ of $\mathrm{Spec}\, O_K$. Then, the absolute value $e^{-\mathrm{ord}_\Gamma}$ is an arithmetic absolute value of F. And there exists a unique $C_\Gamma > 0$ such that

$$|\cdot|_\Gamma := e^{-C_\Gamma \mathrm{ord}_\Gamma} \tag{106}$$

extends the absolute value $|\cdot|_v$ of K.

6.2. **First example.** Let F be finitely generated field over $\mathbf{Q}$ of transcendence degree $d \geqslant 1$. Let K be the algebraic closure of $\mathbf{Q}$ in F. It is a number field. An *arithmetic polarisation* of F is the data of a projective variety $\mathscr{B}$ over $\operatorname{Spec} O_{\mathbf{K}}$ with function field F and the data of nef model arithmetic line bundles $\overline{H}_1, \ldots, \overline{H}_n$ over $\mathscr{B}$. We write $\overline{H}$ for the data of $\overline{H}_1, \ldots, \overline{H}_n$. Notice that an arithmetic polarisation $\overline{H}$ induces a measure $\mu_{\overline{H}} := c_1(\overline{H}_1) \cdots c_1(\overline{H}_n)$ over $B(\mathbf{C})$. In particular, the measure $\mu_{\overline{H}}$ gives zero measure to every algebraic subvariety of B.

The *naive height* over $\mathbf{P}^n_F$ with respect to $\overline{H}$ is given by

$$\forall [x_0 : \cdots : x_n] \in \mathbf{P}^n(F), \quad h^{\overline{H}}_{naive}([x_0 : \cdots : x_n]) = \sum_{\Gamma \subset B} \log \max \left(|x_0|_\Gamma, \cdots, |x_n|_\Gamma \right) \left(\overline{H_1}_{|\Gamma} \cdots \overline{H_n}_{|\Gamma} \right)$$
$$+ \int_{B_\Sigma(\mathbf{C})} \log \left(\max |x_i(b)| \right) d\mu_{\overline{H}}(b)$$

There are several comments to be made about this formula. First, every point in $p \in P^n(F)$ yields a rational section $p : B \dashrightarrow \mathbf{P}^n_B = \mathbf{P}^n \times B$. The indeterminacy locus of this rational section is a closed subvariety of B which has zero $\mu_{\overline{H}}$-measure which makes the integration formula well defined.

Secondly, this is well defined as for any $f \in F$, using (74) we have

$$0 = \widehat{\operatorname{div}}(1/f) \cdot \overline{H}_1 \cdots \overline{H}_n = \sum_{\Gamma \subset \mathscr{B}} \log |f|_\Gamma \left(\overline{H_1}_{|\Gamma} \cdots \overline{H_n}_{|\Gamma} \right) + \int_{B_\Sigma(\mathbf{C})} \log |f| \, d\mu_{\overline{H}}. \tag{107}$$

So that $h^{\overline{H}}_{naive}([fx_0 : \cdots : fx_n]) = h^{\overline{H}}_{naive}([x_0 : \cdots : x_n])$.

6.3. **Definition.** Let F be a finitely generated field over $\mathbf{Q}$ and let X be a projective variety over F. We need to define the notion of adelic line bundles over X. We will first define model adelic line bundles and then define a suitable topology for them. Adelic line bundles will then be limits of model adelic line bundles. First we set $\mathscr{B}$ a projective variety over $\operatorname{Spec} \mathbf{Z}$ such that the function field of $\mathscr{B}$ is F.

Definition 6.1. A *projective model* of X over $\mathscr{B}$ is the data of a projective variety $\mathscr{X}$ over $\mathscr{B}$ such that the generic fiber is isomorphic to $X \to \operatorname{Spec} F$.

Lemma 6.2. *Two projective models $\mathscr{X}$ and $\mathscr{X}'$ of X over $\mathscr{B}$ satisfy the following property. There exists an open subset $\mathscr{V} \subset \mathscr{B}$ such that $\mathscr{X}_V$ and $\mathscr{X}'_V$ are isomorphic.*

A *model adelic line bundle* over X is the data of a model adelic line bundle $\overline{\mathscr{L}}$ over some projective model of $\mathscr{X}$ over $\mathscr{B}$.

The boundary topology. Let $\mathscr{X}$ be a projective model of X over $\mathscr{B}$ and fix an open subset $\mathscr{V} \subset \mathscr{B}$. A *boundary divisor* $\overline{\mathscr{E}}_0$ of $\mathscr{V}$ is a model adelic divisor $\overline{\mathscr{D}}$ such that $\operatorname{Supp} \mathscr{D} = \mathscr{B} \backslash \mathscr{V}$ and for any archimdean place $v, g_{\overline{\mathscr{D}}_v} > 0$.

If $(\mathscr{X}, \overline{\mathscr{L}})$ and $(\mathscr{X}', \overline{\mathscr{L}}')$ are two model adelic line bundles such that there exists $\mathscr{V} \subset \mathscr{B}$ satisfying $\mathscr{X}_V \simeq \mathscr{X}'_V$ and $\mathscr{L}_V \simeq \mathscr{L}'_V$ using the same isomorphism, then the difference of the two adelic line bundles yields an adelic divisor which supports is above $\mathscr{B} \backslash \mathscr{V}$. Thus we define

$$d_{\overline{\mathscr{E}}_0}(\overline{\mathscr{L}}, \overline{\mathscr{L}}') = \inf \left\{ \varepsilon > 0, \quad -\varepsilon \overline{\mathscr{E}}_0 \leqslant \overline{\mathscr{L}} - \overline{\mathscr{L}}' \leqslant \varepsilon \overline{\mathscr{E}}_0 \right\}. \tag{108}$$

This defines a topology over the space of model adelic line bundles $\overline{\mathscr{L}}$ such that $\mathscr{L}_V$ is isomorphic to a fixed line bundle $\mathscr{M}_V$ over $\mathscr{X}_V$. An *adelic line bundle* over X is a Cauchy sequence of such model adelic line bundles. The notions of semipositivity, nefness and integrability follow.

Now how do we define the height of a point ? Fix a nef model adelic line bundle $\overline{H}$ over $\mathscr{B}$. We call this an *arithmetic polarisation* of F. Let $\overline{\mathscr{L}}$ be a model adelic line bundle defined over a model $\mathscr{X}$. Then, for any point $p \in X(\overline{F})$, the closure Δ_p of p in $\mathscr{X}$ is a projective variety over $\operatorname{Spec} \mathbf{Z}$ of same dimension as $\mathscr{B}$. Then the Moriwaki height with respect to $\overline{H}$ is defined as

$$\forall p \in X(\overline{F}), \quad h^{\overline{H}}_{\overline{\mathscr{L}}} := \overline{\mathscr{L}}_{|\Delta_p} \cdot \left(\pi^* \overline{H}_{|\Delta_p} \right)^d. \tag{109}$$

We explicit the formula for $p \in X(F)$. Let s be a rational section of L over X such that $p \notin \operatorname{Supp} \operatorname{div}(s)$. Then, s to a rational section $s(p) := s_{|\Delta_p}$ of $\mathscr{L}_{|\Delta_p}$. Using Formula (74), we get

$$h^{\overline{H}}_{\overline{\mathscr{L}}}(p) = \sum_{\Gamma \subset \mathscr{B}} -\log |s(p)|_\Gamma \left(\overline{H}_{1|\Gamma} \cdots \overline{H}_{n|\Gamma}\right) + \int_{B(\mathbf{C})} -\log \|s(p(b))\|_{\overline{\mathscr{L}},\mathbf{C}} d\mu_{\overline{H}}(b). \tag{110}$$

Proposition 6.3. *Let X be a projective variety over a finitely generated field F. Let $\overline{L}$ be an integrable adelic line bundle over X and $(\mathscr{B}, \overline{H})$ an arithmetic polarisation of F, then for any sequence of model adelic line bundles $\overline{\mathscr{L}}_n$ converging towards $\overline{L}$ and for any $p \in X(\overline{F})$, the sequence $h^{\overline{H}}_{\overline{\mathscr{L}}_n}(p)$ converges towards a number $h^{\overline{H}}_{\overline{L}}(p)$ which does not depend on the choice of the sequence $(\overline{\mathscr{L}}_n)$. Furthermore, the height formula 110 also holds for $\overline{L}$.*

Proof. It suffices to show the result when $\overline{L}$ is strongly nef. Let $\overline{\mathscr{L}}_n$ be a sequence of model adelic line bundles over $\mathscr{B}$ converging towards $\overline{L}$ and let $p \in X(F)$. Let s be a rational section of L over X having no zeroes or poles at p. Define the model adelic divisor $\overline{\mathscr{D}}_n = \widehat{\operatorname{div}}(s)_{\overline{\mathscr{L}}_n}$ and let $\overline{\mathscr{E}}_0$ be a boundary divisor over $\mathscr{B}$. By definition of the boundary topology, there exists a sequence of positive rational numbers $\varepsilon_n \to 0$ such that

$$\forall m \geqslant n > 0, \quad -\varepsilon_n \overline{\mathscr{E}}_0 \leqslant \overline{\mathscr{D}}_n - \overline{\mathscr{D}}_m \leqslant \varepsilon_n \overline{\mathscr{E}}_0. \tag{111}$$

If g_n is the Green function of $\overline{\mathscr{D}}_n$, then we have

$$u_n := h^{\overline{H}}_{\overline{\mathscr{L}}_n}(p) = \sum_{\Gamma \subset \mathscr{B}} g_{n,\Gamma}(p) \left(\overline{H}_{1|\Gamma} \cdots \overline{H}_{n|\Gamma}\right) + \int_{B(\mathbf{C})} g_n(p(b)) d\mu_{\overline{H}}(\mathbf{C}). \tag{112}$$

We can further assume that every $g_0 \geqslant 0$ by adding a suitable positive multiple of $\overline{\mathscr{E}}_0$ to $\overline{\mathscr{D}}_0$. We are going to apply Lebesgue's dominated convergence theorem. Take the following measured space $\Omega = \{\Gamma\}_{\Gamma \subset \mathscr{B}} \bigsqcup B(\mathbf{C})$ with the measure

$$\mu := \sum_{\Gamma} \left(\overline{H}_{1|\Gamma} \cdots \overline{H}_{n|\Gamma}\right) \delta_\Gamma + \mu_{\overline{H}}. \tag{113}$$

For every n we have the measurable function $g_n(p) : \Omega \to \mathbf{R}$ and u_n is the integral of $g_n(p)$ with respect to μ. Furthermore, the function $g_0(p) \geqslant 0$ is integrable and there exists a constant $C > 0$ such that for all $n \geqslant 0, |g_n(p)| \leqslant g_0(p) + C g_{\overline{\mathscr{E}}_0}(p)$ and the right hand side is integrable and $\geqslant 0$ with respect to μ. Take for example $C = \max \varepsilon_n$. Now, the functions $g_n(p)$ converges to $g(p) = -\log |s(p)|_{\overline{L}}$. So by the Lebesgue dominated convergence theorem we have the result. □

The takeaway from this is that Moriwaki heights have a similar form as heights over number fields. We have an infinite sum of local heights coming from non-archimedean places and the new feature is that the contribution from the archimedean places is now an integral and not a finite sum anymore.

Proposition 6.4. *If $\mathscr{X} = \mathbf{P}^n_{\mathscr{B}}$, $\overline{\mathscr{L}} = p_1^*(O_{\mathbf{P}^n_{\mathbf{Z}}}(1), \|\cdot\|_W)$ and $\overline{H}$ is an arithmetic polarisation of F over $\mathscr{B}$, then*

$$h^{\overline{H}}_{\overline{\mathscr{L}}} = h^{\overline{H}}_{naive}. \tag{114}$$

Proof. Let $p \in \mathbf{P}^n(F)$. We assume that $X_0(p) \neq 0$ where $X_0, \ldots, X_n$ are the canonical global section of $O(1)_{\mathbf{P}^n_F}$. By Proposition 6.3 and Formula (110) we have

$$h^{\overline{H}}_{\overline{\mathscr{L}}}(p) = \sum_{\Gamma \subset \mathscr{B}} -\log |X_0(p)|_\Gamma \left(\overline{H}_{1|\Gamma} \cdots \overline{H}_{n|\Gamma}\right) + \int_{B(\mathbf{C})} -\log \|X_0(p(b))\|_{\overline{\mathscr{L}},\mathbf{C}} d\mu_{\overline{H}}(b). \tag{115}$$

Now, the norm of the section X_0 at a point is given by

$$\| X_0(p) \| = \frac{|X_0(p)|}{\max\left(|X_0(p)|, \cdots, |X_n(p)|\right)}. \tag{116}$$

This yields by the product formula

$$h_{\overline{\mathscr{L}}(p)} = \sum_{\Gamma} \log\max(|x_i|_{\Gamma}) \left(\overline{H}_{1|\Gamma}\cdots\overline{H}_{n|\Gamma}\right) + \int \log\max\left(|x_i(p(b))|\right) d\mu_{\overline{H}}(b). \tag{117}$$

□

Proposition 6.5 ([YZ23]). *Let X be a projective variety over a finitely generated field F over $\mathbf{Q}$. Let L be a line bundle over X, then*

(1) If $\overline{L},\overline{L}'$ are two adelic metrisation of L, then

$$h_{\overline{L}}^{\overline{H}} = h_{\overline{L}'}^{\overline{H}} + O(1). \tag{118}$$

(2) If $\overline{H}'$ is another arithmetic polarisation of F such that $\overline{H}_i' - \overline{H}_i$ is effective, then

$$h_{\overline{L}}^{\overline{H}} \leqslant h_{\overline{L}}^{\overline{H}'}. \tag{119}$$

Proof. There exists an open subset $\mathscr{V} \subset \mathscr{B}$ such that $\overline{L}$ and $\overline{L}'$ are induced by Cauchy sequences of model adelic divisor $\overline{\mathscr{L}}_n, \overline{\mathscr{L}}_n'$ such that $\overline{\mathscr{L}}_{n,\mathscr{V}} = \overline{\mathscr{L}}'_{n,\mathscr{V}}$. Thus, if $\overline{\mathscr{E}}_{\mathscr{V}}$ is a boundary divisor of $\mathscr{V}$ we have that there exists $A > 0$ such that $-A\overline{\mathscr{E}}_{\mathscr{V}} \leqslant \overline{L} - \overline{L}' \leqslant A\overline{\mathscr{E}}_{\mathscr{V}}$. Thus,

$$-A\overline{\mathscr{E}}_V \cdot \overline{H}_1 \cdots \overline{H}_d \leqslant h_{\overline{L}}^{\overline{H}} - h_{\overline{L}'}^{\overline{H}} \leqslant A\overline{\mathscr{E}}_V \cdot \overline{H}_1 \cdots \overline{H}_d. \tag{120}$$

The complex measure $c_1(\overline{H}_1^W)\cdots c_1(\overline{H}_d^W)$ is the normalised product Lebesgue measure on $(\mathbb{S}^1)^d$. If $P \in \mathbf{C}[z_1,\dots,z_d]$, its logarithmic Mahler measure is

$$m(P) := \int_{[0,1]^d} \log\left|P\left(e^{2i\pi\theta_1},\dots,e^{2i\pi\theta_d}\right)\right| d\theta_1\cdots d\theta_d. \tag{121}$$

In particular, we have if $d = 1$ that

$$m(P) = \log|a| + \sum_i \log^+|\alpha_i| \tag{122}$$

where $P = a(z-\alpha_1)\cdots(z-\alpha_{\deg P})$. Its Mahler measure is $M(P) = \exp(m(P))$. We also define

$$\| P \| = \max|coeff(P)|. \tag{123}$$

□

Proposition 6.6. *For any $m \geqslant 1$, let $C_m = \max_k \binom{m}{k}$. Then, for any $P \in \mathbf{C}[z_1,\dots,z_d]$*

$$\| P \| \leqslant C_{\deg_1(P)}\cdots C_{\deg_d}(P)M(P). \tag{124}$$

In particular, the set $\left\{P \in \mathbf{Z}[z_1,\dots,z_d] : m(P) \leqslant A, \forall j, \deg_j(P) \leqslant B\right\}$ is finite.

Proof. We prove the result by induction of d. The result is clear if $d = 1$ because the coefficients of P are obtained as the elementary symmetric functions of the α_i's.

Now assume $d \geqslant 2$. Writing

$$P(z_1,\dots,z_d) = \sum_{k=0}^{\deg_d(P)} a_k(z_1,\dots,z_{d-1})z_d^k \tag{125}$$

we have

$$m(P) = \int_{(\mathbb{S}^1)^{d-1}} \left(\int_{\mathbb{S}^1} \log|P|\, d\theta_d \right) d\theta_1 \cdots d\theta_{d-1} \tag{126}$$

$$\geqslant -\log C_{\deg_d(P)} + \max_{a_k \neq 0} \int_{(\mathbb{S}^1)^{d-1}} \log(|a_k(z_1,\ldots,z_{d-1})|)\, d\theta_1 \cdots d\theta_{d-1} \tag{127}$$

$$\geqslant -\log C_{\deg_d(P)} + \max_{a_k \neq 0} -(\log\deg_1(a_k) + \cdots + \log\deg_{d-1}(a_k)) + \parallel a_k \parallel \tag{128}$$

$$\geqslant -\log C_{\deg_d(P)} - \log\deg_{d-1}(P) - \cdots - \log\deg_1(P) + \log \parallel P \parallel . \tag{129}$$

□

Now, we can write

$$p = [P_0(z_1,\ldots,z_d) : \cdots : P_d(z_1,\ldots,z_d)] \tag{130}$$

such that $P_i \in \mathbf{Q}[z_1,\ldots,z_d]$ and they are coprime. We have

$$h_{nv}^{\overline{H}_1^W,\cdots,\overline{H}_d^W}(P) = \sum_{\Gamma} \max(-\mathrm{ord}_{\Gamma}(P_i)) + \int_{(\mathbb{S}^1)^d} \log\max(|P_i|) d\theta_1 \cdots d\theta_d. \tag{131}$$

The integral term is $\geqslant \max m(P_i)$. Now we study the sum, this is a sum of nonnegative term because since the P_i's are coprime polynomials with integer coefficients for every Γ there must exists P_j such that $\mathrm{ord}_{\Gamma}(P_j) = 0$. Let's look at the term where $\Gamma = \Delta_j = \mathbf{P}^1_{\mathbf{Z}} \times \cdots \times \mathbf{P}^1_{\mathbf{Z}} \times \{\infty\} \times \mathbf{P}^1_{Z} \times \cdots \times (\mathbf{P}^1_{\mathbf{Z}})^d$ where $\{\infty\}$ is at the j-th term. Then, we have that $\overline{H}^W_{j\ |\Delta_j}$ is the vertical adelic line bundle over $(\mathbf{P}^1_{\mathbf{Z}})^{d-1}$ with constant metric equal to 1 over $\mathbf{C}$ and zero for every other places. It follows that

$$\overline{H}^W_{1\ |\Delta_i} \cdots \overline{H}^W_{d\ |\Delta_j} = 1. \tag{132}$$

Thus we get that

$$h_{nv}^{\overline{H}_1^W,\ldots,\overline{H}_d^W}(P) \geqslant \sum_j \max_i \deg_j(P_i) + \max m(P_i). \tag{133}$$

Finally, let's look at $\Gamma_p \subset \mathscr{B}$ the prime divisor induce by $p = 0$. We have that $\Gamma_p = (\mathbf{P}^1_{\mathbf{F}_p})^d$ and therefore $\overline{H}_{1|\Gamma_p} \cdots \overline{H}_{d|\Gamma_p} = 1$. Furthermore, we have that

$$\max\log\max(\parallel P_i \parallel_{\Gamma_p}) = \log \parallel q \parallel_p \tag{134}$$

where q is the point in a large enough projective space which coordinates are the coefficients of all the polynomials P_i. Thus by Proposition 6.6 we have that

$$h_{nv}^{\overline{H}_1^W,\cdots,\overline{H}_d^W}([P_0 : \cdots : P_n]) \geqslant \sum_j \max\deg_j(P_i) + Ch_{nv}(q). \tag{135}$$

Thus we have a bound on the number of coefficients appearing in $P_0,\ldots,P_n$ and a bound on a naive height of the induced point q in some projective space. By the Northcott property of the naive height over a number field we have that the set

$$\left\{ p \in \mathbf{P}^n(\mathbf{Q}(z_1,\ldots,z_d)) : h_{nv}^{\overline{H}}(p) \leqslant A \right\} \tag{136}$$

is finite.

Now, to get the result for any big and nef arithmetic polarisation $\overline{H}_1,\ldots,\overline{H}_d$ we have that there exists an integer N_i such that $N_i\overline{H}_i - \overline{H}_i^W$ is effective. So that

$$h_{nv}^{\overline{H}_1^W,\cdots,\overline{H}_d^W} \leqslant N_1 h_{nv}^{\overline{H}_1,\overline{H}_2^W,\ldots,\overline{H}_d^W} + O(1) \leqslant \cdots \leqslant N_1 \cdots N_d h_{nv}^{\overline{H}_1,\ldots,\overline{H}_d} + O(1). \tag{137}$$

6.4. **Arithmetic equidistribution.**

Theorem 6.7 ([YZ23, CM]). *Let L be an ample line bundle over a projective variety X and let $\overline{L}$ be a nef adelic metrisation of L defined over $\mathscr{X}_{\mathscr{V}} \to \mathscr{B}_{\mathscr{V}}$. Let $\overline{H}$ be a big and nef arithmetic polarisation of F defined over $\mathscr{B}$. Suppose $(p_n) \subset X(\overline{F})$ is a generic sequence such that $h_L^{\overline{H}}(p_n) \to h_L^{\overline{H}}(X)$, then*

(1) For any $\Gamma \subset \mathscr{B}$, the sequence of measures $\delta_{n,\Gamma}$ converges weakly towards $c_1(\overline{L})^n_\Gamma$.

(2) For the archimedean place of $\mathbf{Q}$ we have over $X_V(\mathbf{C})$ the convergence of currents

$$\delta_n \wedge c_1(\overline{H})^d_{\mathbf{C}} \to c_1(\overline{L})^n_{\mathbf{C}} \wedge c_1(\overline{H})^d_{\mathbf{C}}. \tag{138}$$

where δ_n is the integration current

$$\delta_n = \frac{1}{\left|\mathrm{Gal}(\overline{F}/F)\cdot p_n\right|} \sum_{q \in \mathrm{Gal}(\overline{F}/F)\cdot p_n} \delta_{\Delta_q} \tag{139}$$

where Δ_q is the Zariski closure of q in $X_V(\mathbf{C})$.

Remark 6.8. The statement here in this theorem is modeled from [CM, Theorem F]. In [YZ23] there is an extra condition on the polarisation called the *Moriwaki condition* which was introduced in the first paper of Moriwaki to prove an equidistribution theorem at the archimedean places. Chen and Moriwaki removed this condition by showing the differentiability of the arithmetic volume function with respect to the polarisation $\overline{H}$.

6.5. **Canonical Moriwaki heights for a polarised endomorphism.** We follow §6.1.1 of [YZ23]. Let X be a projective variety over a field F and let f be a polarised endomorphism with polarisation L. Let $(\mathscr{X}, \overline{\mathscr{L}})$ be a nef model adelic metrisation of L over a base $\mathscr{B}$. There exists an open subset $\mathscr{V} \subset \mathscr{B}$ such that

(1) f restricts to a surjective endomorphism $f_{\mathscr{V}} : \mathscr{X}_{\mathscr{V}} \to \mathscr{X}_{\mathscr{V}}$.

(2) $f_{\mathscr{V}}^* \mathscr{L}_{\mathscr{V}} = d\mathscr{L}_{\mathscr{V}}$.

This implies that for a boundary divisor $\overline{\mathscr{E}}_0$ of $\mathscr{V}$, there exists a constant $C > 0$ such that

$$-C\overline{\mathscr{E}}_0 \leqslant \frac{1}{d} f^*\overline{\mathscr{L}} - \overline{\mathscr{L}} \leqslant C\overline{\mathscr{E}}_0. \tag{140}$$

Furthermore, $f^*\overline{\mathscr{E}}_0 = \mathscr{E}_0$ so that we get

$$-\frac{C}{d^n}\overline{\mathscr{E}}_0 \leqslant \frac{1}{d^{n+1}}(f^{n+1})^*\overline{\mathscr{L}} - \frac{1}{d^n}(f^n)^*\overline{\mathscr{L}} \leqslant \frac{C}{d^n}\overline{\mathscr{E}}_0. \tag{141}$$

So that this defines a Cauchy sequence with respect to the boundary topology. We write $\overline{L}_f$ for the limit.

Theorem 6.9. *The adelic line bundle $\overline{L}_f$ is the unique strongly nef adelic metrisation of L such that*

$$f^*\overline{L}_f = d \cdot \overline{L}_f. \tag{142}$$

7. Proof of Theorem 1.1

The proof is now very similar as in the number field case. Let F be a finitely generated field over $\mathbf{Q}$ such that f and g are defined over F and share infinitely many preperiodic points in common. Let $\overline{H}$ be a big and nef arithmetic polarisation of F over a model $\mathscr{B}$ over $\mathbf{Z}$. Let $\overline{L}_f, \overline{L}_g$ be the canonical adelic line bundles associated to f and g respectively. By the Northcott property we have that

$$\mathrm{Preper}(f) = \left\{ h_f^{\overline{H}} = 0 \right\}. \tag{143}$$

And the same formula holds for g. By the arithmetic equidistribution theorem we have the following.

(1) If $\Gamma \subset \mathscr{B}$, then $\mu_{f,\Gamma} = \mu_{g,\Gamma}$. Which implies that the local metrics of $\overline{L}_{f,\Gamma}$ and $\overline{L}_{g,\Gamma}$ differ by a constant λ_Γ because their difference is a harmonic function over $\mathbf{P}^{1,\mathrm{an}}_\Gamma$.

(2) For the archimedean places, they are contained in $\mathscr{B}(\mathbf{C})$. We have that for $c_1(\overline{H})^d_{\mathbf{C}}$-almost every $b \in \mathscr{B}(\mathbf{C})$, the measures $\mu_{f,b}, \mu_{g,b}$ are equal so that the metrics of $\overline{L}_{f,b}$ and $\overline{L}_{g,b}$ differ by a constant $\lambda(b)$.

We show the second statement at the end of the proof.

This implies that the Moriwaki heights $h_f^{\overline{H}}$ and $h_f^{\overline{H}}$ differ by a constant. Indeed, for any $q \in \mathbf{P}^1(\overline{F})$, the set of preperiodic points.

$$h^{\overline{H}}_{\overline{L}_f}(q) - h_{\overline{L}_g}(q) = \int_{\mathscr{B}(\mathbf{C})} \lambda(b) d\mu_{\overline{H},\mathbf{C}}(b) + \sum_{\Gamma \subset \mathscr{B}} \overline{H}^d_{|\Gamma} \lambda_\Gamma. \tag{144}$$

For some nonnegative numbers $n_\Gamma \geqslant 0$. Since the heights coincide on common preperiodic points we have that this constant is zero and the height functions are equal. Thus we get the equality of the set of preperiodic points.

Proof of (2). The set $\mathscr{X}_{\mathscr{V}}(\mathbf{C}) = X_V(\mathbf{C})$ is a complex analytic space containing $X^{\mathrm{an},\mathbf{C}}$. Let ψ be a compactly supported function over $X_V(\mathbf{C})$. We show that

$$\int_{X_b} \psi \mu_{f,b} = \int_{X_b} \psi \mu_{g,b} \tag{145}$$

is of full measure with respect to $c_1(\overline{H})^d_{\mathbf{C}}$. Introduce for $\varepsilon > 0$ the set

$$U_\varepsilon := \left\{ b \in V(\mathbf{C}) : \int_{X_b} \psi \mu_{f,b} \geqslant \int_{X_b} \psi \mu_{g,b} + \varepsilon \right\}. \tag{146}$$

The U^+_ε is measurable and since $c_1(\overline{H})^d_{\mathbf{C}}$ is a Radon measure there exists a compact subset K_δ and an open subset T_δ such that

$$K_\delta \subset U_\varepsilon \subset T_\delta \subset V(\mathbf{C}) \tag{147}$$

and

$$\mu_{\overline{H}}(U_\varepsilon) - \delta \leqslant \mu_{\overline{H}}(K_\delta) \leqslant \mu_{\overline{H}}(T_\delta) \leqslant \mu_{\overline{H}}(U_\varepsilon) + \delta. \tag{148}$$

Now, there exists a compactly supported function ϕ over $V(\mathbf{C})$ such that $\phi_{|K_\delta} = 1$, $\phi_{|V(\mathbf{C}) \setminus T_\delta} = 0$ and $0 \leqslant \phi \leqslant 1$. We have that

$$\int_{V(\mathbf{C})} \left(\int_{X_b} \phi(b) \psi(x(b)) d\mu_{f,b} \right) d\mu_{\overline{H}}(b) = \int_{V(\mathbf{C})} \left(\int_{X_b} \phi(b) \psi(x(b)) d\mu_{g,b} \right) d\mu_{\overline{H}}(b). \tag{149}$$

If we call T the difference of the two sides we have

$$0 = T \geqslant \varepsilon \mu_{\overline{H}}(K_\delta) - 2\delta M \geqslant \varepsilon(\mu_{\overline{H}}(U_\varepsilon) - \delta) - 2\delta M. \tag{150}$$

where $M = \max \psi$. Letting $\delta \to 0$ we get that $\mu_{\overline{H}}(U_\varepsilon) = 0$. Reversing the role of f and g we get that the set

$$W_\varepsilon = \left\{ b \in V(\mathbf{C}) : \left| \int_{X_b} \psi \mu_{f,b} - \int_{X_b} \psi \mu_{g,b} \right| > \varepsilon \right\} \tag{151}$$

has $\mu_{\overline{H}}$ measure zero. Thus, we get

$$\mu_{\overline{H}} \left(\bigcup_{\varepsilon \in \mathbf{Q}_{>0}} W_\varepsilon \right) = 0. \tag{152}$$

Now, the space of compactly supported function over $X_V(\mathbf{C})$ is separable. So we take a dense sequence ψ_n, By what we have proven we have that for $\mu_{\overline{H}}$-almost every $b \in V(\mathbf{C})$

$$\int_{X_b} \psi_n \mu_{f,b} = \int_{X_b} \psi_n \mu_{g,b}. \tag{153}$$

By density, we get that (153) holds for any compactly supported function ψ over $X_V(\mathbf{C})$. Now, for any $b \in V(\mathbf{C})$, the restriction $\psi_{|X_b}$ is a function with compact support over X_b. By the Stone-Weierstrass theorem, the restrictions to X_b of compactly supported function over $X_V(\mathbf{C})$ is a dense subset of the set of continuous functions over X_b. So that we we get for $\mu_{\overline{H}}$-almost every $b \in V(\mathbf{C})$ that $\mu_{f,b} = \mu_{g,b}$.

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