Cohomology of compact Lie groups

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1 Introduction

This short note explains how to compute the de Rham cohomology of a compact connected Lie group using only its Lie algebra. This allows one to give a nice integral formula for the Poincaré polynomial formed by the Betti numbers of such a group (theorem 3.0.1) :

Theorem 1.0.1. *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. One has the following equality :*

$$\Pi_G(t) = \int_G \det(1 + tg \mid \mathfrak{g}) dg$$

where the action is through the adjoint representation of G on $\mathfrak{g}$.

We then explain how to explicitly compute this integral in concrete cases using the Weyl integration formula.

2 Invariant forms

In this first part we show that the de Rham cohomology of a compact connected Lie group can be computed by looking only at invariant differential forms. Let G be a connected compact Lie group of dimension n with Lie algebra $\mathfrak{g}$ and normalized Haar measure dg. We denote by $\mathfrak{g}_{\mathbb{C}}$ the complex Lie algebra $\mathfrak{g} \otimes \mathbb{C}$ and by :

$$\Omega^k_G$$

the locally free sheaf of smooth complex k-forms on G. Recall that this sheaf is free for we have an isomorphism :

$$\mathcal{O}_G \otimes \Lambda^k \mathfrak{g}^\vee \longrightarrow \Omega^k_G$$

where $\mathcal{O}_G$ is the sheaf of smooth complex functions on G. This isomorphism is obtained by the following extension construction : if $\alpha \in \Lambda^k \mathfrak{g}^\vee$, we denote by $\hat{\alpha}$ the (left)-extension of α to G, defined as follows :

$$\hat{\alpha}_x = (L_x)_* \alpha$$

where L_x is the left translation automorphism of G. Note that $\hat{\alpha}$ is a left-invariant form. Let us write $\Omega^k(G)$ the space of total sections of Ω^k_G, and $\Omega^k(G)^G$ the space of left-invariant forms. Also we put $\Omega^k(G)^{G,G}$ the space of bi-invariant forms (i.e. left and right invariant). Since the wedge-product and the differential of left-invariant or bi-invariant forms is again left or bi-invariant, we have inclusions of cochain complexes :

$$\Omega^*(G)^{G,G} \subseteq \Omega^*(G)^G \subseteq \Omega^*(G) \quad (*).$$

Note that the extension construction provides canonical isomorphisms :

$$\Lambda^*\mathfrak{g}^\vee \otimes \mathbb{C} \cong \Omega^*(G)^G$$

and :

$$\left(\Lambda^*\mathfrak{g}^\vee\right)^{\mathfrak{g}} \otimes \mathbb{C} \cong \Omega^*(G)^{G,G}$$

where we define the differential morphisms by transport of structure. We shall see later that these can be defined purely in terms of the Lie-algebra structure of $\mathfrak{g}$. The notation $V^{\mathfrak{g}}$ for V a representation of $\mathfrak{g}$ stands for the space of invariant elements under the representation. Of course, we have :

$$\left(\Lambda^*\mathfrak{g}^\vee\right)^{\mathfrak{g}} = \left(\Lambda^*\mathfrak{g}^\vee\right)^G$$

under the representation induced by the adjoint representation of G.
The inclusions of cochain complexes $(*)$ are actually retracts. To see this, we define, for $\omega \in \Omega^k(G)$ its left mean :

$$\langle\omega\rangle_\ell = \int_G (L_g^*\omega)dg$$

where the integral is performed pointwise on G on each finite dimensional space $\Lambda^k T^x G$. Similarly, denoting by R_x the right translation by x, we define the right-mean of ω :

$$\langle\omega\rangle_r = \int_G (R_g^*\omega)dg.$$

Since right and left translations commute, we have :

$$\langle\langle\omega\rangle_r\rangle_\ell = \langle\langle\omega\rangle_\ell\rangle_r$$

and we define this common form as the two-sided mean of ω which we write $\langle\langle\omega\rangle\rangle$. Note that if $k = 0$ or $k = n$, left-invariant and right-invariant forms are the same (it is the same argument as for the uniqueness of the normalized Haar measure on G using the modular character) so left, right and two-sided means are the same for these forms. It is easy to check that left, right and two-sided means commute with differentials, so that we have left-inverses to the inclusions in $(*)$:

$$\Omega^*(G) \xrightarrow{\langle\bullet\rangle_\ell} \Omega^*(G)^G \xrightarrow{\langle\bullet\rangle_r} \Omega^*(G)^{G,G}.$$

Hence we may write :

$$\Omega^*(G) = \Lambda^*\mathfrak{g}_{\mathbb{C}}^\vee \oplus \Omega^*(G)_0$$

where $\Omega^*(G)_0$ is the space of forms with 0-left-mean. Similarly :

$$\Lambda^*\mathfrak{g}_{\mathbb{C}}^\vee = \left(\Lambda^*\mathfrak{g}_{\mathbb{C}}^\vee\right)^{\mathfrak{g}} \oplus K^*$$

where K^* is the space of left-invariant forms with 0-right-mean.

Lemma 2.0.1. *The complexes $\Omega^*(G)_0$ and K^* are acyclic.*

Proof. For the first part, we need to show that if ω is a closed k-form with trivial left-mean, then ω is exact and has a primitive that also has a trivial left-mean. For $k = 0$, this is easy : a constant function with 0 left-mean is the 0-function. Suppose $k \geq 1$ and $d\omega = 0$ and $\langle \omega \rangle_\ell = 0$. By de Rham's theorem, we have a perfect pairing :

$$\langle \bullet, \bullet \rangle : H^k_{\mathrm{dR}}(G, \mathbb{C}) \otimes H_k(G, \mathbb{C}) \longrightarrow \mathbb{C}$$

given by integration on smooth cycles. Hence, to prove that ω is exact, it suffices to prove that :

$$\int_Z \omega = 0$$

for every smooth k-cycle Z. Now one has :

$$0 = \int_Z \left(\int_G L_g^* \omega dg \right) = \int_G \left(\int_Z L_g^* \omega \right) dg = \int_G \left(\int_{gZ} \omega \right) dg = \int_G \langle \omega, gZ \rangle dg = \int_G \langle \omega, Z \rangle dg$$

because Z and gZ have the same homology class using the path-connectedness of G to make a homotopy between Z and gZ. Hence the constant function $\langle \omega, Z \rangle$ is zero, which is what we wanted. Now we can write $\omega = d\alpha$, and we may replace α by $\alpha - \langle \alpha \rangle_\ell$ to get a primitive with trivial left-mean since $d\langle \alpha \rangle_\ell = \langle d\alpha \rangle_\ell = \langle \omega \rangle_\ell = 0$.
The second point is similar : if $d\omega = 0$ and $\langle \omega \rangle_r = 0$ and ω is left-invariant, then by a same argument as before ω is exact with a primitive α that has trivial right-mean. Now replace α by its left-mean to conclude : we still have $d\langle \alpha \rangle_\ell = \langle \omega \rangle_\ell = \omega$ since ω is left-invariant. □

As a consequence, the inclusions in (∗) are quasi-isomorphisms of cochain complexes and we have, for G a connected compact Lie group :

$$H^*_{dR}(G, \mathbb{C}) \cong H^*(\mathfrak{g}_\mathbb{C}) \cong H^*\left(\left(\Lambda^k \mathfrak{g}_\mathbb{C}^\vee \right)^\mathfrak{g} \right)$$

where we define the cohomology $H^*(\mathfrak{g}_\mathbb{C})$ as the cohomology of the complex $\Lambda^* \mathfrak{g}_\mathbb{C}^\vee$.

We now explicitly compute the differentials on the complex $\Lambda^* \mathfrak{g}_\mathbb{C}^\vee$ and see that they depend only on the Lie algebra structure of $\mathfrak{g}$. Up to some normalization of the differentials, this complex is called the Chevalley-Eilenberg complex of $\mathfrak{g}$ and it computes the Lie algebra cohomology :

$$H^*(\mathfrak{g}_\mathbb{C}, \mathbb{C}) = \mathrm{Ext}^*_{U(\mathfrak{g}_\mathbb{C})}(\mathbb{C}, \mathbb{C})$$

where $U(\mathfrak{g}_\mathbb{C})$ denotes the universal enveloping algebra of $\mathfrak{g}$. By doing this computation, we obtain the following.

Proposition 2.0.2. *The complex* $\left(\Lambda^* \mathfrak{g}_\mathbb{C}^\vee \right)^\mathfrak{g}$ *has trivial differentials.*

Proof. Recall the Cartan's identity for ω a k-form and $X_0, \ldots, X_k$ any vector fields on G :

$$d\omega(X_0, \ldots, X_k) = \sum_i (-1)^i X_i \left(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k) \right) + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \hat{X}_j, \ldots, X_k).$$

If ω is left-invariant and the vector fields are left-invariant, then the first term vanishes since $\omega(X_0, \ldots, \hat{X}_i, \ldots, X$
is a left-invariant function, hence a constant.
In this case we then have :

$$d\omega(X_0, \ldots, X_k) = \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \hat{X}_j, \ldots, X_k).$$

Evaluating this at the identity point, we obtain that the differentials on $\Lambda^*\mathfrak{g}^\vee$ depend only on the Lie algebra structure. Now if ω is bi-invariant then the explicit calculation of the representation $\Lambda^*\mathfrak{g}^\vee$ as an exterior power of a representation shows that $d\omega = 0$. □

Hence we conclude the following.

Theorem 2.0.3. *Let G be a compact connected Lie group. One has a canonical isomorphism of graded rings :*

$$H^*_{\mathrm{dR}}(G,\mathbb{C}) \cong \left(\Lambda^*\mathfrak{g}^\vee\right)^{\mathfrak{g}} \otimes \mathbb{C}.$$

3 The poincaré polynomial of a compact connected Lie group

Recall that if X is a compact smooth manifold, then we may define its Poincaré polynomial :

$$\Pi_X(t) = \sum_{i=0}^{\infty} \beta_i(X) t^i$$

where :

$$\beta_i = \dim H^i(X,\mathbb{C})$$

is the i-th Betti number of X. Recall that one has :

$$\Pi_{X\times Y} = \Pi_X \Pi_Y$$

thanks to Künneth's formula and that the Euler-characteristic of X is given by :

$$\chi(X) = \Pi_X(-1).$$

We now give an integral formula for $\Pi_G(t)$ for G a compact connected Lie group.

Theorem 3.0.1. *Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. One has the following equality :*

$$\Pi_G(t) = \int_G \det(1 + tg \mid \mathfrak{g}) dg$$

where the action is through the adjoint representation of G on $\mathfrak{g}$.

Proof. One has the following equalities for every real number t :

$$\begin{aligned}\Pi_G(t) &= \sum_k \dim H^k(G,\mathbb{C}) t^k = \sum_k \dim\left(\Lambda^k \mathfrak{g}_{\mathbb{C}}^\vee\right)^{\mathfrak{g}} t^k \\ &= \sum_k \int_G \mathrm{Tr}(g \mid \Lambda^k \mathfrak{g}_{\mathbb{C}}^\vee) t^k dg \quad \text{using the trace of the Reynolds projector} \\ &= \int_G \sum_k \mathrm{Tr}(g \mid \Lambda^k \mathfrak{g}_{\mathbb{C}}^\vee) t^k dg\end{aligned}$$

Now let f be an endomorphism of a finite dimensional complex vector space V. A classical computation in a basis shows that :

$$\sum_k \mathrm{Tr}(\Lambda^k f) t^k = \det(1 + tf).$$

Hence we have :

$$\Pi_G(t) = \int_G \det\left(1 + tg \mid \mathfrak{g}^\vee\right) dg.$$

By definition of the dual representation, we then have for $t \in \mathbb{R}$:

$$\Pi_G(t) = \int_G \overline{\det\left(1 + tg^{-1} \mid \mathfrak{g}\right)} dg = \int_G \det\left(1 + tg^{-1} \mid \mathfrak{g}\right) dg$$

since this polynomial takes real values on $\mathbb{R}$. We may then drop the inverse on g because G is compact hence unimodular. This concludes the proof. □

4 Weyl integration formula

To make the computation of the integral formula 3.0.1 easier, we may notice that we are integrating a so-called central function : that is one which is invariant under conjugation. For such functions, the integral can be performed on a maximal torus of G, which is much easier. Let us explain this in detail.

Let G be a connected compact Lie group. Fix T a maximal torus of G endowed with its normalized Haar measure. Recall all maximal tori of G are conjugates of one another. One defines the Weyl group :

$$W = N_G(T)/T$$

where $N_G(T)$ is the normalizer of T. Note that the neutral connected component of $N_G(T)$ is a connected group that acts on T, but the group of automorphisms of T is discrete (it is some $\mathrm{GL}_k(\mathbb{Z})$) so this connected component $N_G(T)^0$ acts trivially on T, hence it is contained in $Z_G(T) = T$. Hence $T = N_G(T)^0$ which implies that W is discrete and compact thus finite. Also W is well defined up to isomorphism if we change T.
Let us now consider the following function :

$$q : G/T \times T \longrightarrow G$$

which maps (xT, t) to xtx^{-1}, this being well defined since T is commutative. It is also onto since any element of G is conjugate to an element of T. It is clear from the universal property of quotient manifolds that q is a smooth function and one can in fact show that there exist dense open subsets T' and G' of T and G such that q induces a covering map :

$$q' : G/T \times T' \longrightarrow G'$$

with fiber W. We then have, for f a continuous function on G :

$$|W| \int_G f(x)dx = \int_{G/T} \int_T f(q(xT,t)) |\det(dq)| \, dt d(xT) = \int_{G/T} \int_T f(xtx^{-1}) |\det(dq)| \, dt d(xT)$$

with normalized Haar measures on G/T and T. A computation yields :

$$|\det(dq)|_{xT,t} = \left|\det(t^{-1} - 1 \mid \mathfrak{g}/\mathfrak{t})\right| = \left|\det(t^{-1} - 1 \mid \mathfrak{g})\right|$$

since t acts trivially on $\mathfrak{t}$, the Lie algebra of T.
This gives the following result.

Theorem 4.0.1. *(Weyl integration formula)*
Let G be a compact connected Lie group and T be a maximal torus of G. For any continuous function f on G, one has :

$$\int_G f(x)dx = \frac{1}{|W|}\int_{G/T}\int_T f(xtx^{-1})u(t)dtd(xT)$$

where :

$$u(t) = \left|\det(t^{-1} - 1 \mid \mathfrak{g}/\mathrm{t})\right|.$$

Moreover, if f is a central function, then one has :

$$\int_G f(x)dx = \frac{1}{|W|}\int_T f(t)u(t)dt.$$

Note that it is possible to compute u in terms of the roots of $\mathfrak{g}$.
Now we can rephrase theorem 3.0.1 as :

Corollary 4.0.2. *For a connected compact Lie group G with maximal torus T, one has the following equality :*

$$\Pi_G(\lambda) = \frac{(1+\lambda)^r}{|W|}\int_T \det(1+\lambda x \mid \mathfrak{g}/\mathrm{t})\left|\det(x^{-1} - 1 \mid \mathfrak{g}/\mathrm{t})\right| dx$$

where r is the rank of G, that is the dimension of T, and where the representation we use is the adjoint representation on $\mathfrak{g}/\mathfrak{t}$.

References