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Representations of the Yangian in Deligne's category Rep(Gl_t)

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Ce mémoire récapitule le travail effectué lors de mon stage de master au MIT, USA, sous la direction de Pavel Etingof. Le but de ce stage était d'étudier, et si possible de classifier, les représentations du Yangian $Y(\mathfrak{g})$, d'une algèbre $\mathfrak{g}$ définie comme un objet de la catégorie de Deligne $\mathbf{Rep}(Gl_t)$.

Les représentations du groupe linéaire Gl_N, où N est un entier naturel, sont assez bien connues des mathématiciens. En particulier, les représentations complexes de dimension finie de Gl_N sont complètement connues (voir [10], par exemple). En 1982, P. Deligne utilise ces résultats pour définir une catégorie, qu'on appellera plus tard $\mathbf{Rep}(Gl_t)$, pour t un complexe quelconque, dans son article sur les catégories Tannakiennes [6] (voir exemple 1.27). En 2007, dans son article fondamental [5], il définit la catégorie $\mathbf{Rep}(S_t)$, en interpolant la catégorie des représentations du groupe symétrique S_N, quand N est un grand entier. Dans la dernière partie de ce même article, il donne une définition plus précise de la catégorie $\mathbf{Rep}(Gl_t)$.

C'est cette définition que nous allons utiliser dans ce mémoire, plus particulièrement la construction détaillée par Comes et Wilson dans [3]. Un des objets de cette catégorie, noté $\mathfrak{g}$, nous intéressera particulièrement, il s'agit de l'équivalent de $\mathfrak{gl}_N$ dans la catégorie $\mathbf{Rep}(Gl_t)$. A partir de cette "algèbre" $\mathfrak{g}$, on pourra définir son Yangian $Y(\mathfrak{g})$, en suivant la définition de P. Etingof dans [8]. Là aussi, il s'agira de mimer la définition du Yangian $Y(\mathfrak{gl}_N)$.

Les Yangians forment une famille de groupes quantiques. Le premier Yangian étudié est justement $Y(\mathfrak{gl}_N)$, considéré par L. Faddeev et l'école de St Petersburg. Ils s'en servaient à l'origine pour générer des solutions de l'équation de Yang-Baxter quantique. En 1985, V. Drinfeld introduit pour la première fois le terme "Yangian", en honneur du physicien C.N. Yang. Le Yangian $Y(\mathfrak{a})$ d'une algèbre de Lie simple de dimension finie $\mathfrak{a}$ est alors défini comme une déformation canonique de l'algèbre enveloppante universelle $\mathcal{U}(\mathfrak{a}[z])$ de l'algèbre des lacets $\mathfrak{a}[z]$. Une nouvelle présentation du Yangian est donnée par V. Drinfeld en 1988. Le Yangian de $\mathfrak{gl}_N$ bénéficie d'une troisième présentation, où les relations entre générateurs peuvent s'écrire sous la forme d'une seule équation matricielle. Cette structure particulière peut expliquer pourquoi $Y(\mathfrak{gl}_N)$ avait été étudié plus précocement.

Notre but étant d'étudier les représentations du Yangian $Y(\mathfrak{g})$, il y a deux aspects distincts à présenter, d'une coté la catégorie de Deligne $\mathbf{Rep}(Gl_t)$, et de l'autre les Yangians. Nous allons commencer par rappeler les définitions et concepts généraux de théorie des catégories dont nous aurons besoin. Ensuite, nous détaillerons la définition de $\mathbf{Rep}(Gl_t)$, et nous étudierons cette catégorie, avant d'arriver à la définition de $Y(\mathfrak{g})$. Pour cela, nous donnerons d'abord la définition classique de $Y(\mathfrak{gl}_N)$, que nous utiliserons ensuite pour définir $Y(\mathfrak{g})$. Le chapitre suivant sera consacré à l'étude des représentations du Yangian de $\mathfrak{gl}_N$, pour lesquelles il existe une classification bien connue. Et enfin le dernier chapitre sera consacré aux représentations de $Y(\mathfrak{g})$. Nous présenterons des pistes pour l'étude des ces représentations, en se basant sur l'étude faite au chapitre précédent.

Introduction

The representation theory of the general linear group Gl_N has been intensively studied by mathematicians. But in 1982, in his paper [6], P. Deligne used the concept of Tannakian categories to define a category which will be later be called $\mathbf{Rep}(Gl_t)$, where t is not necessarily an integer (see example 1.27). In his fundamental paper [5] of 2007, he introduced the category $\mathbf{Rep}(S_t)$ by interpolating the well-known categories of representations of the symmetric group $\mathbf{Rep}(S_N)$, when N is a large integer, to any complex number t, and in the last section of this paper, he also gave a more detailed definition of the category $\mathbf{Rep}(Gl_t)$.

In this report, we are going to define the category $\mathbf{Rep}(Gl_t)$, following the definition in [3], and in particular, one of its objects $\mathfrak{g}$, which interpolates the algebra $\mathfrak{gl}_N$ in this category. We are going to be able to define the Yangian $Y(\mathfrak{g})$ of $\mathfrak{g}$, using the definition introduced by Etingof in [8].

A Yangian is a type of quantum group, which was first studied by L. Faddeev, in the context of the *quantum inverse scattering method.* At the time, it was mostly seen as a tool to generate solutions to the quantum Yang-Baxter equation. Faddeev and the St Petersburg School were actually studying the Yangian of the general linear algebra $\mathfrak{gl}_N$. The term "Yangian" was introduced in 1985 by V. Drinfeld, in honor of the physicist C.N. Yang, to define a Hopf algebra associated to any simple finite-dimensional Lie algebra $\mathfrak{a}$ over $\mathbb{C}$. The Yangian $Y(\mathfrak{a})$ is defined as a deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{a}[z])$ for the polynomial current Lie algebra $\mathfrak{a}[z]$. The Yangian of the general linear algebra $Y(\mathfrak{gl}_N)$ stays a special case, as there is a special presentation of it in which the defining relations can be written in the form of a single relation on the matrix of generators.

The aim of this work is to study the representation of the Yangian $Y(\mathfrak{g})$ of the algebra $\mathfrak{g}$. Hence there are two different directions to work with : the category $\mathbf{Rep}(Gl_t)$ and the Yangian of the general linear group $Y(\mathfrak{gl}_N)$. So we are going to begin by recalling some definitions and results of category theory which we are going to use later, and then study first $\mathbf{Rep}(Gl_t)$, with the results we are interested in, then explain what is a Yangian. Then we are going to get interested more precisely on representations of the Yangian $Y(\mathfrak{gl}_N)$, in order to get some results that could be used in the case of $Y(\mathfrak{g})$. The last chapter explains the results we have and want to get on representations of $Y(\mathfrak{g})$.

Contents

Chapter 1

Preliminaries on category theory

We will assume that the reader is familiar with the basic concepts of category theory, such as functors, natural transformations, etc.

As we will use some more advanced concepts precisely, we will begin by defining them.

1.1 Symmetric tensor category

The term "tensor category" can refer to different definitions, in which the conditions under which the category is called *tensor* can vary. Here, we will use a weak definition of tensor category, because it is the one we are going to need.

Definition 1.1.1. A *symmetric tensor category* is a $\mathbb{C}$-linear rigid symmetric monoidal category, in which the tensor product is biadditive and $\mathrm{End}(\mathbb{1}) = \mathbb{C}$.

We are going to give precise definitions for these terms, but we invite the interested reader to consult other sources, see [9] for example, for a more detailed discussion, and more examples.

1.1.1 $\mathbb{C}$-linear category

A $\mathbb{C}$-linear category is a category enriched over the category of $\mathbb{C}$-vector spaces. Precisely

Definition 1.1.2. A $\mathbb{C}$-*linear category* is a category $\mathcal{C}$ in which, for every objects X, Y, the Hom-space $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has the structure of a $\mathbb{C}$-vector space.

Example 1.1.3. The category of $\mathbb{C}$-vector spaces is a $\mathbb{C}$-linear category.

1.1.2 Monoidal category

The following definition is the usual definition of a monoidal category, as found in [12] for example.

Definition 1.1.4. A *monoidal category* is a category $\mathcal{C}$ equipped with :

- A bifunctor : $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the *monoidal product* or *tensor product*.
- A object $\mathbb{1}$ called the *unit object* or the *identity object*.
- A natural isomorphism α, called *associativity constraint* such that for all objects X,Y and Z, $\alpha_{X,Y,Z}$ is an isomorphism of $\mathcal{C}$ from $X(\otimes Y \otimes Z)$ to $(X \otimes Y) \otimes Z$, which satisfy the ***pentagon axiom***:

  For all objects X, Y, Z, T, the diagram

$$
\begin{array}{ccccc}
 & & X \otimes (Y \otimes (Z \otimes T)) & & \\
 & \swarrow^{1_X \otimes \alpha_{Y,Z,T}} & & \searrow^{\alpha_{X,Y,Z\otimes T}} & \\
X \otimes ((Y \otimes Z) \otimes T) & & & & (X \otimes Y) \otimes (Z \otimes T) \\
 & \searrow^{\alpha_{X,Y\otimes Z,T}} & & \swarrow^{\alpha_{X\otimes Y,Z,T}} & \\
 & (X \otimes (Y \otimes Z)) \otimes T & \xrightarrow{\alpha_{X,Y,Z} \otimes 1_T} & ((X \otimes Y) \otimes Z) \otimes T &
\end{array}
$$

is commutative.

- Two natural isomorphisms λ and ρ respectively called *left* and *right unitor*, such that for any object X of $\mathcal{C}$, λ and ρ induce isomorphisms $\lambda_X : \mathbb{1} \otimes X \to X$ and $\rho_X : X \otimes \mathbb{1} \to X$, which satisfy the following :

  For all objects X, Y, the diagram :

$$
\begin{array}{ccc}
X \otimes (\mathbb{1} \otimes Y) & \xrightarrow{\alpha_{X,\mathbb{1},Y}} & (X \otimes \mathbb{1}) \otimes Y \\
 & {\scriptstyle 1_X \otimes \lambda_Y} \searrow \quad \swarrow {\scriptstyle \rho_X \otimes 1_Y} & \\
 & X \otimes Y &
\end{array} \tag{1.1}
$$

  is commutative.

Remark 1.1.5. Mac Lane's Coherence Theorem for monoidal categories states that every "formal" diagram built using α, λ and ρ, and the $\otimes$ product commutes. We will not go into details about what formal means (see [12] for a precise definition and the proof of the Theorem). All the futures diagrams we will consider will be formal and as such we will not write λ and ρ anymore, and α only sometimes.

Example 1.1.6. The category **Set** of sets is a monoidal category, the monoidal product being the Cartesian product, and the unit object being a set with one element.

Example 1.1.7. If R is a commutative ring, the category R-**Mod** of R-modules, with the tensor product of modules $\bigotimes_R$, is a monoidal category. The unit object is R itself.

We could ask that a functor between monoidal categories preserve the tensor product and the identity object, this would give a certain definition of *monoidal functor*. But we are going to use a more general definition, where the monoidal structure is preserved up to a specified isomorphism, which should then satisfy some compatibility conditions. This leads to the following definition.

Definition 1.1.8. Let $\mathcal{C} = (\otimes, \mathbb{1}, \alpha, \lambda, \rho)$ and $\mathcal{D} = (\otimes', \mathbb{1}', \alpha', \lambda', \rho')$ be two monoidal categories. A *monoidal functor* is a tuple (F, η, χ), where

- $F : \mathcal{C} \to \mathcal{D}$ is a functor,
- $\eta_{X,Y} : F(X) \otimes' F(Y) \to F(X \otimes Y)$ is a natural isomorphism,
- $\epsilon : F(\mathbb{1}) \to \mathbb{1}'$ is an isomorphism,

such that, for every three objects X, Y, Z of $\mathcal{C}$, the following diagrams commute

$$\begin{array}{ccc} F(X)\otimes'(F(Y)\otimes' F(Z)) & \xrightarrow{\alpha'_{F(X),F(Y),F(Z)}} & (F(X)\otimes' F(Y))\otimes' F(Z) \\ \downarrow{\scriptstyle 1_{F(X)}\otimes'\eta_{Y,Z}} & & \downarrow{\scriptstyle \eta_{X,Y}\otimes' 1_{F(Z)}} \\ F(X)\otimes' F(Y\otimes Z) & & F(X\otimes Y)\otimes' F(Z) \\ \downarrow{\scriptstyle \eta_{X,Y\otimes Z}} & & \downarrow{\scriptstyle \eta_{X\otimes Y,Z}} \\ F(X\otimes(Y\otimes Z)) & \xrightarrow{F\alpha_{X,Y,Z}} & F((X\otimes Y)\otimes Z) \end{array} \tag{1.2}$$

$$\begin{array}{ccc} F(X)\otimes' F(\mathbb{1}) & \xrightarrow{1_{F(X)}\otimes'\epsilon} & F(X)\otimes'\mathbb{1}' \\ \downarrow{\scriptstyle \eta_{X,\mathbb{1}}} & & \downarrow{\scriptstyle \rho'_{F(X)}} \\ F(X\otimes\mathbb{1}) & \xrightarrow{F\rho_X} & F(X) \end{array} \qquad \begin{array}{ccc} F(\mathbb{1})\otimes' F(Y) & \xrightarrow{\epsilon\otimes 1_{F(Y)}} & \mathbb{1}'\otimes' F(Y) \\ \downarrow{\scriptstyle \eta_{\mathbb{1},Y}} & & \downarrow{\scriptstyle \lambda_{F(Y)}} \\ F(\mathbb{1}\otimes Y) & \xrightarrow{F\lambda_Y} & F(Y) \end{array} \tag{1.3}$$

The monoidal functor is said to be *strict* if $F(X)\otimes' F(Y) = F(X\otimes Y)$, and $\eta_{X,Y} = 1_{F(X)\otimes' F(Y)}$, for all objects X, Y of $\mathcal{C}$.

Example 1.1.9. The forgetful functor from the category of representation of a group G to the category of vector spaces, $\mathbf{Rep}_G \to \mathbf{Vect}$, is a monoidal functor. Actually, all forgetful functors from one monoidal category to another are monoidal functors.

Definition 1.1.10. A *monoidal natural transformation* is a natural transformation $\Gamma : F \to G$, where (F, η, ϵ) and (G, μ, ν) are monoidal functors from the monoidal category $\mathcal{C}$ to the monoidal category $\mathcal{D}$, such that the two following diagrams commute, for X, Y any objects of $\mathcal{C}$.

$$\begin{array}{ccc} F(X)\otimes' F(Y) & \xrightarrow{\Gamma_X\otimes'\Gamma_Y} & G(X)\otimes' G(Y) \\ \downarrow{\scriptstyle \eta_{X,Y}} & & \downarrow{\scriptstyle \mu_{X,Y}} \\ F(X\otimes Y) & \xrightarrow{\Gamma_{X\otimes Y}} & G(X\otimes Y) \end{array} \qquad \begin{array}{ccc} F(\mathbb{1}) & & \\ \downarrow{\scriptstyle \epsilon} & \searrow^{\Gamma_{\mathbb{1}}} & \\ \mathbb{1}' & \xleftarrow{\nu} & G(\mathbb{1}) \end{array} \tag{1.4}$$

Now we can define two functor categories.

Let $\mathcal{H}om^{\otimes}(\mathcal{C}, \mathcal{D})$ be the category of monoidal functors between two monoidal categories $\mathcal{C}$ and $\mathcal{D}$, the morphisms being the monoidal natural transformations.

Let $\mathcal{H}om^{\otimes-str}(\mathcal{C}, \mathcal{D})$ be the category of strict monoidal functors between two monoidal categories $\mathcal{C}$ and $\mathcal{D}$.

1.1.3 Symmetric category

Here again, the following definitions are the ones found in Mac Lane's book [12].

Definition 1.1.11. A *braided monoidal category* is a monoidal category $\mathcal{C}$, which is equipped with a natural isomorphism γ, called the *commutativity constraint* such that, for all objects X, Y of $\mathcal{C}$, $\gamma_{X,Y} : X\otimes Y \to Y\otimes X$ is a isomorphism which satisfies the two following ***hexagon identities***.

For all objects X, Y, Z of $\mathcal{C}$, the diagrams

$$\begin{array}{ccccc} & (X \otimes Y) \otimes Z & \xrightarrow{\gamma_{X \otimes Y, Z}} & Z \otimes (X \otimes Y) & \\ {}^{\alpha_{X,Y,Z}} \nearrow & & & & \searrow^{\alpha_{Z,X,Y}} \\ X \otimes (Y \otimes Z) & & & & (Z \otimes X) \otimes Y \\ {}_{1_X \otimes \gamma_{Y,Z}} \searrow & & & & \swarrow_{\gamma_{Z,X} \otimes 1_Y} \\ & X \otimes (Z \otimes Y) & \xrightarrow{\alpha_{X,Z,Y}} & (X \otimes Z) \otimes Y & \end{array} \tag{1.5}$$

$$\begin{array}{ccccc} & X \otimes (Y \otimes Z) & \xrightarrow{\gamma_{X, Y \otimes Z}} & (Y \otimes Z) \otimes X & \\ {}^{\alpha^{-1}_{X,Y,Z}} \nearrow & & & & \searrow^{\alpha^{-1}_{Y,Z,X}} \\ (X \otimes Y) \otimes Z & & & & Y \otimes (Z \otimes X) \\ {}_{\gamma_{X,Y} \otimes 1_Z} \searrow & & & & \swarrow_{1_Y \otimes \gamma_{Z,X}} \\ & (Y \otimes X) \otimes Z & \xrightarrow{\alpha^{-1}_{Y,X,Z}} & Y \otimes (X \otimes Z) & \end{array} \tag{1.6}$$

are commutative.

Remark 1.1.12. Using these axioms, we can prove that the braiding is also compatible with the right and left unitors.

Example 1.1.13. The category of representations of a quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ is a braided monoidal category, where γ is obtained from the universal R-matrix see [2], chap. 10., or [11], chap. VIII.

Definition 1.1.14. A *symmetric monoidal category* is a braided monoidal category in which the commutativity constraint γ satisfies

$$\gamma_{Y,X} \gamma_{X,Y} = 1_{X \otimes Y} \tag{1.7}$$

for all objects X, Y of $\mathcal{C}$.

Remark 1.1.15. We have seen in Example 1.1.13 that the category of representations of $\mathcal{U}_q(\mathfrak{g})$ is a braided monoidal category. But it is not, in general, a symmetric monoidal category.

Example 1.1.16. The category $\mathbf{Rep}_G$ of representations of a group G is a symmetric monoidal category, with the braiding $\gamma_{V,W} : v \otimes w \mapsto w \otimes v$.

Definition 1.1.17. A *braided monoidal functor* is a monoidal functor (F, η, ϵ), between two braided categories, which respects the braiding on both sides. That can be translated as the commutativity of the following diagram, for all objects X, Y of $\mathcal{C}$.

$$\begin{array}{ccc} F(X) \otimes' F(Y) & \xrightarrow{\eta_{X,Y}} & F(X \otimes Y) \\ \downarrow{\scriptstyle \gamma_{F(X),F(Y)}} & & \downarrow{\scriptstyle F\gamma_{X,Y}} \\ F(Y) \otimes' F(X) & \xrightarrow{\eta_{Y,X}} & F(Y \otimes X) \end{array} \tag{1.8}$$

A *symmetric functor* is a braided monoidal functor between two symmetric categories, with no additional condition.

1.1.4 Rigid monoidal category

Let $\mathcal{C}$ be a monoidal category (not necessarily symmetric) and X an object of $\mathcal{C}$.

Definition 1.1.18. A *dual* of X is a triplet (X^*, ev_X, ev_X), where X^* is an object of $\mathcal{C}$, and $ev_X : X^* \otimes X \to \mathbb{1}$ and $coev_X : \mathbb{1} \to X^* \otimes X$ are morphisms, such that the compositions

$$X \xrightarrow{coev_X \otimes 1_X} X \otimes X^* \otimes X \xrightarrow{1_X \otimes ev_X} X \tag{1.9}$$

$$X^* \xrightarrow{1_{X^*} \otimes coev_X} X^* \otimes X \otimes X^* \xrightarrow{ev_X \otimes 1_{X^*}} X \tag{1.10}$$

are the identity morphisms.

Remark 1.1.19. If an object X has a dual, then it is unique up to unique isomorphism.

Now let $\mathcal{D}$ be another monoidal category and $(F, \eta, \epsilon) : \mathcal{C} \to \mathcal{D}$ be a monoidal functor. Then the functor F and the dual X^* of the object X of $\mathcal{C}$ define a dual $(F(X^*), ev_{F(X)}, coev_{F(X)})$ of $F(X)$ in $\mathcal{D}$, where

$$ev_{F(X)} := F(X^*) \otimes' F(X) \xrightarrow{\eta_{X^*,X}} F(X^* \otimes X) \xrightarrow{Fev_X} F(\mathbb{1}) \xrightarrow{\epsilon} \mathbb{1}' \tag{1.11}$$

$$coev_{F(X)} := \mathbb{1}' \xrightarrow{\epsilon^{-1}} F(\mathbb{1}) \xrightarrow{Fcoev_X} F(X \otimes X^*) \xrightarrow{\eta^{-1}_{X,X^*}} F(X) \otimes' F(X^*) \tag{1.12}$$

Definition 1.1.20. A *rigid monoidal category* is a monoidal category in which every object has a dual.

Example 1.1.21. If k is a field, the category k-Vect of finite dimensional k-vector spaces is a rigid monoidal category. The dual of a vector space V is it usual dual $V^* = \{\phi : V \to k, k\text{-linear}\}$. The evaluation and coevaluation maps are defined as

$$\begin{array}{rrcl} ev_V : & V^* \otimes V & \to & k \\ & \phi \otimes v & \mapsto & \phi(v) \end{array} \quad , \quad \begin{array}{rrcl} coev_V : & k & \to & V \otimes V^* \\ & \lambda & \mapsto & \lambda \cdot \sum\limits_{i=1}^{n} e_i \otimes e_i^*, \end{array} \tag{1.13}$$

where $(e_1, \ldots, e_n)$ and $(e_1^*, \ldots, e_n^*)$ are dual basis of V and V^*, respectively.

Then, $(1_V \otimes ev_V) \circ (coev_V \otimes 1_V)v = (1_V \otimes ev_V) \left(\sum\limits_{i=1}^{n} e_i \otimes e_i^* \otimes v \right) = \sum\limits_{i=1}^{n} e_i v_i = v$. The other condition can be verified the same way.

1.2 Categorical dimension

Here we are going to work on a rigid monoidal category $\mathcal{C}$. On such a category we can define a *categorical trace* $\operatorname{tr}\phi$, for every object X and $\phi \in \operatorname{End}_{\mathcal{C}}(X)$.

$$\operatorname{tr}\phi := ev_X \circ \gamma_{X,X^*} \circ (\phi \otimes 1_{X^*}) \circ coev_X \tag{1.14}$$

Remark 1.2.1. One has $\operatorname{tr}\phi \in \operatorname{End}_{\mathcal{C}}(\mathbb{1})$, so $\operatorname{tr}\phi \in \mathbb{C}$ if $\mathcal{C}$ is a tensor category.

Definition 1.2.2. For every object X, we define the *categorical dimension* $\dim X$ by

$$\dim X := \operatorname{tr}(1_X) \in \operatorname{End}_{\mathcal{C}}(\mathbb{1})$$

Remark 1.2.3. With the rigid monoidal category structure on k-Vect given in Example 1.1.21, we can see that, if V is a k-finite-dimensional vector space, or dimension $n \in \mathbb{N}$, the categorical dimension of V is

$$\dim V = n.$$

Indeed, $\mathrm{tr}(1_V) \in \mathrm{End}_{\mathcal{C}}(k) = k$ in this case, and $\mathrm{tr}(1_V)1 = ev_V \circ \gamma_{V,V^*} \circ coev_V 1 = ev_V \circ \gamma_{V,V^*} \sum_{i=1}^{n} e_i \otimes e_i^* = ev_V \sum_{i=1}^{n} e_i^* \otimes e_i = \sum_{i=1}^{n} e_i^*(e_i) = n$.

Hence, the categorical dimension is a generalization of the dimension, to any rigid monoidal category.

1.3 Additive envelope

Constructing the additive envelope of a category is a way of enriching a category for it to become additive.

Precisely, an *additive envelope* of $\mathcal{C}$ is a pair $(\mathcal{C}^{add}, \iota)$, where $\mathcal{C}^{add}$ is a additive category, and $\iota : \mathcal{C} \longrightarrow \mathcal{C}^{add}$ is a fully-faithful preadditive functor, such that for any additive category $\mathcal{D}$, the *restriction functor*

$$\begin{array}{rcl} \mathcal{H}om^+(\mathcal{C}^{add}, \mathcal{D}) & \to & \mathcal{H}om^+(\mathcal{C}, \mathcal{D}) \\ F & \mapsto & F \circ \iota \\ (\eta : F \Rightarrow G) & \mapsto & \eta_\iota \end{array} \tag{1.15}$$

is an equivalence of categories. Thus a additive envelope, when it exists, is unique up to equivalence of categories.

Let us define the terms we used in this paragraph.

1.3.1 Additive category

In order to be additive, a category first needs to be preadditive.

Definition 1.3.1. A category $\mathcal{C}$ is *preadditive* if every hom-set $\mathrm{Hom}(X, Y)$ in $\mathcal{C}$ has the structure of an abelian group, if the composition of morphisms is bilinear, and if it has a zero object 0.

We call a *zero object* a object 0 in $\mathcal{C}$ which is both *initial* (for every object X in $\mathcal{C}$ there is a unique morphism $0 \to X$) and *terminal* (for every object X in $\mathcal{C}$ there is a unique morphism $X \to 0$).

A *preadditive functor* if a functor $F : \mathcal{C} \to \mathcal{D}$, between two preadditive categories $\mathcal{C}$ and $\mathcal{D}$, such that for any pair of morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, one has that

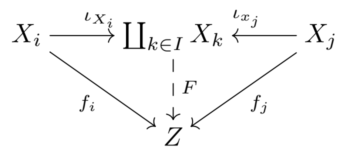
$$F(f + g) = F(f) + F(g). \tag{1.16}$$

Definition 1.3.2. A category $\mathcal{C}$ is *additive* if it is preadditive and admits finite coproducts.

The notion of *coproduct* is a categorical generalization of the notion of disjoint union in sets, it is an internal addition law.

For I a finite set and $\{X_i\}_{i\in I}$ objects in the category $\mathcal{C}$, their coproduct $\coprod_{i\in I} X_i$ is a object of $\mathcal{C}$, equipped with morphisms $\iota_{X_i} : X_i \to \coprod_{k\in I} X_k$, for every $i \in I$, satisfying a universal property.

For every object Z of $\mathcal{C}$ and set of morphisms $f_i : X_i \to Z$, there is a unique morphism $F : \coprod_{i\in I} X_i \to Z$, such that, for every $i \neq j$ in I we have a commuting diagram



(1.17)

Let us denote by $\mathcal{H}om^+(\mathcal{C},\mathcal{D})$ the category of preadditive functors between two preadditive categories $\mathcal{C}$ and $\mathcal{D}$, the morphisms being any natural transformation.

Definition 1.3.3. A object X of the category $\mathcal{C}$ is said to be *indecomposable* if when writing a coproduct decomposition of $X = X_1 \coprod X_2$, with maps $\iota_i : X_i \to X$ and $\pi_i : X \to X_i$, there is an $i \in \{1,2\}$ such that $\iota_i \circ \pi_i = 0 \in \mathrm{End}_{\mathcal{C}}(X)$.

1.3.2 Construction of the additive envelope

From an existing preadditive category $\mathcal{C}$, we can build $\mathcal{C}^{add}$, which is going to be an additive category. It is defined as follow

Objects $X = (X_j)_{j\in J}$ finite-length tuples of objects of $\mathcal{C}$.

Morphisms $\phi : (X_j)_{j\in J} \to (Y_i)_{i\in I}$, where

$$\phi = (\phi_{i,j})_{i\in I, j\in J} \qquad \text{and} \qquad \phi_{i,j} : X_j \longrightarrow Y_i$$

Composition $\phi = (\phi_{i,j})_{i\in I,j\in J} : (X_j)_{j\in J} \to (Y_i)_{i\in I}$ and $\psi = (\psi_{k,i})_{k\in K,i\in I} : (Y_j)_{j\in J} \to (Z_k)_{k\in K}$ are composed via matrix multiplication

$$(\psi \circ \phi)_{k,j} = \sum_{i\in I} \psi_{k,i} \circ \phi_{i,j} : X_j \to Z_k. \tag{1.18}$$

$\mathcal{C}^{add}$ is indeed an additive category, the coproduct being the concatenation of tuples.

$\{(X_i^j)_{i\in I_j}\}_{j\in J}$ has as coproduct $\coprod_{j\in J}(X_i^j)_{i\in I_j} := (X_i^j)_{j\in J, i\in I_j}$. The injection maps are built using the zero object 0.

Let j_0 be fixed in J, $\phi^{j_0} : (X_i^{j_0})_{i\in I_{j_0}} \to (X_i^j)_{j\in J,i\in I_j}$ is made of $\phi^{j_0}_{i,j,k} : X_i^{j_0} \to X_k^j$, which is equal to $1_{X_i^{j_0}}$ if $(i, j_0) = (k, j)$ and $X_i^{j_0} \to 0 \to X_k^j$ if not.

It can be seen easily that the universal property of the finite coproducts is satisfied.

It is easy to see why $\mathcal{C}^{add}$ is indeed an additive envelope of $\mathcal{C}$. The functor $\iota : \mathcal{C} \to \mathcal{C}^{add}$ is defined by sending any object X of $\mathcal{C}$ to the length-1 tuple (X) and any morphism ϕ of $\mathcal{C}$ to the 1x1-matrix (ϕ). It is fully-faithful.

We have to understand why the universal property of the additive envelope holds. The restriction functor $\mathcal{H}om^+(\mathcal{C}^{add},\mathcal{D}) \to \mathcal{H}om^+(\mathcal{C},\mathcal{D})$ is essentially surjective (and even surjective) because if we have a given preadditive functor $F : \mathcal{C} \to \mathcal{D}$, we can easily build an antecedent F' under the restriction functor. It is enough to see that functors on $\mathcal{C}^{add}$ are entirely defined by the images of the length-1 tuple, so by the images of the objects of $\mathcal{C}$. Furthermore, we can map each natural transformation of functors from $\mathcal{C}$ to $\mathcal{D}$ to one and only one natural transformation from $\mathcal{C}^{add}$ to $\mathcal{D}$. Thus the restriction functor is also fully-faithful and is an equivalence of categories.

Hence, the additive envelope, as we built it, in the only additive envelope, up to equivalence of categories.

1.4 Karoubi envelope

As with the additive envelope, the Karoubi envelope of an existing category $\mathcal{C}$ is a Karoubi category built from the category $\mathcal{C}$. The definition is similar.

A *Karoubi envelope* of $\mathcal{C}$ is a pair $(\mathcal{C}^{kar}, \iota)$, where $\mathcal{C}^{kar}$ is a Karoubi category, and $\iota : \mathcal{C} \longrightarrow \mathcal{C}^{kar}$ is a fully-faithful preadditive functor, such that for any Karoubi category $\mathcal{D}$, the *restriction functor*

$$\begin{array}{rcl} \mathcal{H}om^+(\mathcal{C}^{kar}, \mathcal{D}) & \to & \mathcal{H}om^+(\mathcal{C}, \mathcal{D}) \\ F & \mapsto & F \circ \iota \\ (\eta : F \Rightarrow G) & \mapsto & \eta_\iota \end{array} \tag{1.19}$$

is a equivalence of categories. Thus a Karoubi envelope, when it exists, is unique up to equivalence of categories.

Let us explain what a Karoubi category is and how to build the Karoubi envelope.

1.4.1 Splittings of idempotents and Karoubi category

Let $\mathcal{C}$ be a category and X an object in $\mathcal{C}$.

Definition 1.4.1. $e \in \mathrm{End}_{\mathcal{C}}(X)$ is called an *idempotent* of $\mathcal{C}$ if $e^2 = e$. A *splitting* of an idempotent e is a tuple $(\mathrm{im}\, e, \iota_e, \pi_e)$ where $\mathrm{im}\, e$ is an object of $\mathcal{C}$,

$$\iota_e : \mathrm{im}\, e \longrightarrow X, \qquad \pi_e : X \longrightarrow \mathrm{im}\, e,$$

are morphisms of $\mathcal{C}$, such that

1. $e = \iota_e \circ \pi_e$,
2. $\mathrm{id}_{\mathrm{im}\, e} = \pi_e \circ \iota_e$,

If there exists a splitting $(\mathrm{im}\, e, \iota_e, \pi_e)$ for an idempotent e, we will say that e *splits*, and call $\mathrm{im}\, e$ the *image* of e.

Definition 1.4.2. A category is said to be *Karoubi* if every idempotent in that category splits.

We also introduce a few more notions, for which the category $\mathcal{C}$ is supposed to be $\mathbb{C}$-linear.

Definition 1.4.3. Two idempotents e, e' of $\mathrm{End}_{\mathcal{C}}(X)$ are said to be *orthogonal* if $ee' = e'e = 0$.

An idempotent e is said to be *primitive* if it is nonzero and it cannot be written as the sum of two nonzero orthogonal idempotents.

Lemma 1.4.4. *If $\mathcal{C}$ is a $\mathbb{C}$-linear Karoubi category and X an object of $\mathcal{C}$, then X is indecomposable if and only if id_X is a primitive idempotent.*

Proof. Suppose X is indecomposable and write $\mathrm{id}_X = e + e'$, where e, e' are orthogonal idempotents. As the category is Karoubi, we have splittings : $(\mathrm{im}\, e, \iota e, \pi_e)$ and $(\mathrm{im}\, e', \iota e', \pi_{e'})$. It is enough to show that $X = \mathrm{im}\, e \oplus \mathrm{im}\, e'$, because as it is indecomposable, that would imply $e = 0$ or $e' = 0$.

Let Y be an object of $\mathcal{C}$, with morphisms $f_e : \mathrm{im}\, e \to Y$, $f_{e'} : \mathrm{im}\, e' \to Y$. The morphism $F = f_e \circ \pi_e + f_{e'} \circ \pi_{e'} : X \to Y$ satisfies the universal property of the coproduct.

The reverse can be shown in a similar way.

□

1.4.2 Construction of the Karoubi envelope

The Karoubi envelope $\mathcal{C}^{kar}$ of a category $\mathcal{C}$ is defined as follow

Objects The tuples (X, e) where X is an object of $\mathcal{C}$ and e an idempotent of $\mathrm{End}_{\mathcal{C}}(X)$.

Morphisms $\mathrm{Hom}_{\mathcal{C}^{kar}}((X,e),(Y,f)) = \{\phi \in \mathrm{Hom}_{\mathcal{C}}(X,Y) | f \circ \phi = \phi = \phi \circ e\}$

Let ϕ be a morphism $\mathcal{C}^{kar}$, and ϕ_0 denote the same morphism, considered as a morphism of $\mathcal{C}$. Then the $\mathcal{C}^{kar}$-morphisms ϕ and ψ are equal if and only if they have the same source and target in $\mathcal{C}^{kar}$ and $\phi_0 = \psi_0$.

Composition The composition of morphisms in $\mathcal{C}^{kar}$ is inherited from the composition of morphisms in $\mathcal{C}$. $\psi \circ \phi$ is defined by the source of ϕ, the target of ψ and $(\psi \circ \phi)_0 = \psi_0 \circ \phi_0$. We have $(\mathrm{id}_{(X,e)})_0 = e$.

In $\mathcal{C}^{kar}$ every idempotent has a splitting, thus it is a Karoubi category. Indeed, let ϕ be an idempotent of $\mathrm{End}_{\mathcal{C}^{kar}}(X,e)$, ϕ has a splitting $(\mathrm{im}\,\phi, \iota_\phi, \pi_\phi)$, where $\mathrm{im}\,\phi = (X,\phi)$ and $(\iota_\phi)_0 = (\pi_\phi)_0 = \phi_0$.

The functor $\iota : \mathcal{C} \to \mathcal{C}^{kar}$ defined by $\iota_X = (X, \mathrm{id}_X)$ and $(\iota_\phi)_0 = \phi$ is fully-faithful, and is such that the universal property of the Karoubi envelope holds. This is pretty straightforward to check.

Remark 1.4.5. With this construction of the Karoubi envelope, we can see that any object of the category $\mathcal{C}^{kar}$ is isomorphic to the image of an idempotent of $\mathcal{C}$.

1.5 Krull-Schmidt category

A Krull-Schmidt category is a category in which the Krull-Schmidt theorem holds, which means that every object can be uniquely written as a sum of indecomposable objects. Of course we have to work on categories that are at least additive to be able to define the sum of objects. The following definition is sometime used (in [3] for example).

Definition 1.5.1. A *Krull-Schmidt category* is an additive, Karoubi category $\mathcal{C}$ with finite-dimensional Hom-spaces.

In particular, if $\mathcal{C}$ is a preadditive category with finite-dimensional Hom-spaces, then $(\mathcal{C}^{add})^{kar}$ is a Krull-Schmidt category.

Proposition 1.5.2. *Let $\mathcal{C}$ be a $\mathbb{C}$-linear category and $\mathcal{C}^{kar}$ its Karoubi envelope, as built above. Let X be an object of $\mathcal{C}$, and e, e', e'' idempotents of* $\mathrm{End}_{\mathcal{C}}(X)$. *Then*

1. $\mathrm{im}\,e$ *is indecomposable if and only if e is a primitive idempotent, and up to isomorphism, all indecomposables are thus obtained.*
2. *if $e = e' + e''$ and e', e'' are orthogonal, then* $\mathrm{im}\,e \cong \mathrm{im}\,e' \oplus \mathrm{im}\,e''$
3. *Suppose further that* $\mathrm{End}_{\mathcal{C}}(X)$ *is finite-dimensional, then* $\mathrm{im}\,e \cong \mathrm{im}\,e'$ *if and only if e, e' are conjugate in* $\mathrm{End}_{\mathcal{C}}(X)$.

Proof. We begin by the first part. We already now that $\mathrm{im}\,e$ is indecomposable if and only if $\mathrm{id}_{\mathrm{im}\,e}$ is a primitive idempotent. So we have is to show that

$$\mathrm{id}_{\mathrm{im}\,e} \text{ is a primitive idempotent } \Leftrightarrow e \text{ is a primitive idempotent}$$

Suppose $e = e' + e''$, where e', e'' are orthogonal idempotents in $\mathcal{C}$, then

$$\mathrm{id}_{\mathrm{im}\, e} = \pi_e \circ e \circ \iota_e = \pi_e \circ e' \circ \iota_e + \pi_e \circ e'' \circ \iota_e.$$

Suppose now that $\mathrm{id}_{\mathrm{im}\, e} = f + f'$, where f, f' are primitive idempotents, then

$$e = \iota_e \circ \mathrm{id}_{\mathrm{im}\, e} \circ \pi_e = \iota_e \circ f \circ \pi_e + \iota_e \circ f'' \circ \pi_e.$$

Hence, a decomposition as a sum of orthogonal idempotents in $\mathcal{C}$ of e gives us a decomposition in sum of orthogonal idempotents of $\mathrm{id}_{\mathrm{im}\, e}$, and the other way round. Thus we have a bijection between orthogonal idempotent decompositions of e in $\mathcal{C}$ and of $\mathrm{id}_{\mathrm{im}\, e}$ in $\mathcal{C}^{kar}$. This proves the first part of the first assumption.

Moreover, as we noted in remark 1.4.5, every object of $\mathcal{C}^{kar}$ is isomorphic to the image of an idempotent of $\mathcal{C}$. the proves the second part of the first assumption.

The proof of the second assumption is really similar to the proof of Lemma 1.4.4.

For the third assumption, suppose first that $e' = \phi e \phi^{-1}$. Then the morphism $\pi_{e'} \circ \phi \circ \iota_e : \mathrm{im}\, e \to \mathrm{im}\, e'$ is an isomorphism, with inverse $\pi_e \circ \phi^{-1} \circ \iota_{e'}$. Conversely, let $\Phi : \mathrm{im}\, e \to \mathrm{im}\, e'$ be an isomorphism. Then the morphisms

$$\begin{array}{rcl} (\mathrm{End}_{\mathcal{C}})e' & \to & (\mathrm{End}_{\mathcal{C}})e \\ \alpha & \mapsto & \alpha \circ \iota_{e'} \circ \Phi \circ \pi_e \circ e \end{array} \quad , \quad \begin{array}{rcl} (\mathrm{End}_{\mathcal{C}})e & \to & (\mathrm{End}_{\mathcal{C}})e' \\ \beta & \mapsto & \beta \circ \iota_e \circ \Phi^{-1} \circ \pi_{e'} \circ e' \end{array} \tag{1.20}$$

are mutually inverse. Indeed, if we write $\alpha \in (\mathrm{End}_{\mathcal{C}})e'$ as $\alpha = \alpha' \circ e'$, the composition of the morphisms gives us $\alpha' \circ \iota_{e'} \circ \Phi \circ \pi_e \circ e \circ \iota_e \circ \Phi^{-1} \circ \pi_{e'} \circ e' = \alpha' \circ \iota_{e'} \circ \Phi \circ \Phi^{-1} \circ \pi_{e'} \circ e' = \alpha' \circ \iota_{e'} = \alpha$. The composition in the other way is symmetrical. Hence the modules $(\mathrm{End}_{\mathcal{C}})e$ and $(\mathrm{End}_{\mathcal{C}})e'$ are isomorphic. This implies that the idempotents e and e' are conjugate, as $\mathrm{End}_{\mathcal{C}}$ is finite-dimensional (see [1] for example, for a proof of this classical fact, in the case of finite-dimensional algebras). □

Using this proposition, we can now prove the following Theorem, which explains why we use the term "Krull-Schmidt category". Let us note that Proposition 1.5.2 is satisfied in a Krull-Schmidt category $\mathcal{C}$, as in this $\mathcal{C}$ is its own Karoubi envelope.

Theorem 1.5.3. *A Krull-Schmidt category $\mathcal{C}$ satisfies the Krull-Schmidt Theorem, in the sense that every object in $\mathcal{C}$ can be written as a sum of indecomposable objects.*

Proof. For every object X in $\mathcal{C}$, it is clear that $\mathrm{im}\,\mathrm{id}_X = X$, thus is it enough to shown that every object of the form $\mathrm{im}\, e$ can be written as a sum of indecomposable objects.

Let X be an object of $\mathcal{C}$ and let e be an idempotent of $\mathrm{End}_{\mathcal{C}}(X)$. If $\mathrm{im}\, e$ is not indecomposable, then by Proposition 1.5.2 e is not primitive, and if $e = e' + e''$, with e' and e'' orthogonal, then $\mathrm{im}\, e \cong \mathrm{im}\, e' \oplus \mathrm{im}\, e''$. If $\mathrm{im}\, e'$ or $\mathrm{im}\, e''$ is not indecomposable, we can decompose it the same way. We iterate this process, and we obtain $\mathrm{im}\, e \cong \bigoplus_{i=1}^{n} \mathrm{im}\, e_i$, with $e_1, \ldots, e_n \in \mathrm{End}_{\mathcal{C}}(X)$ idempotents such that $e = e_1 + \cdots + e_n$. These idempotents also satisfy various conditions of orthogonality. The conditions can be defined recursively.

$(\star)$: A sum of idempotents $e = e_1 + \cdots + e_n$ satisfies the condition $(\star)$ if

- $n = 1$, or
- $n = 2$, and e_1, e_2 are orthogonal, or
- $n > 2$, and there is a $k \in \{2, \ldots, n-1\}$ such that $e_1 + \cdots + e_k$ and $e_{k+1} + \cdots + e_n$ are orthogonal and both satisfy the condition $(\star)$.

Lemma 1.5.4. *If $e_1, \dots, e_n$ are idempotents such that $e_1 + \cdots + e_n$ satisfies the condition $(\star)$, then $(e_1, \dots, e_n)$ is free.*

We will admit this Lemma.

As $\mathrm{End}_{\mathcal{C}}(X)$ is finite-dimensional, there can not be a sum of an arbitrary number of idempotents which satisfies the condition $(\star)$. Hence, the process we detailed above stops at some point, and we can write $\operatorname{im} e \cong \bigoplus_{i=1}^{n} \operatorname{im} e_i$, where each $\operatorname{im} e_i$, for $1 \leq i \leq n$, is indecomposable. □

Chapter 2

The category Rep(Gl_t)

We are now going to define the Deligne category, $\mathbf{Rep}(Gl_t)$, which interpolates the category of finite-dimensional representations of the general linear group $\mathbf{Rep}(Gl_N)$, to complex values of $N = t$.

We are going to follow the construction of $\mathbf{Rep}(Gl_t)$ introduced by Deligne in the end of [5], and more precisely, the more detailed version of this construction, found in [3].

For that we first define a smaller category, called ***the skeleton category*** $\mathbf{Rep}_0(Gl_t)$, then we are going to take successively the additive envelope and the Karoubi envelope of this category to obtain $\mathbf{Rep}(Gl_t)$.

2.1 The category $\mathbf{Rep}_0(Gl_t)$

We start by introducing the diagrams we will use to define $\mathbf{Rep}_0(Gl_t)$.

2.1.1 Words and diagrams

Let w and w' be two finite words in the two letter alphabet $\{\circ, \bullet\}$.

Definition 2.1.1. A (w, w')-*diagram* is a graph such that

1. The vertices are the letters of w and w', placed in two rows. The upper row forms w', and the lower w.

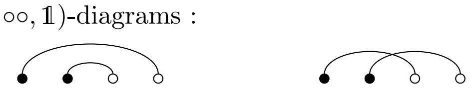
2. Each vertex is adjacent to exactly one edge.

3. An edge can be adjacent to two letters of different colors only if they are placed on the same row, it can be adjacent to two letters of the same color only if they are placed on different rows.

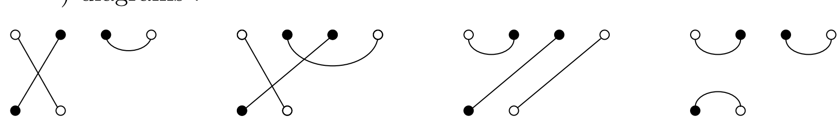
In a (w, w')-diagram, an edge between a vertex of the upper row and one of the lower row is called a *propagating edge*.

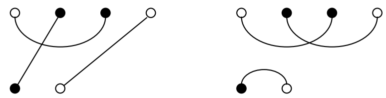
Example 2.1.2. For these examples, we list all the possible diagrams. $\mathbb{1}$ denotes the empty word.

$(\bullet\bullet\circ\circ, \mathbb{1})$-diagrams :



$(\bullet\circ, \circ\bullet\bullet\circ)$-diagrams :

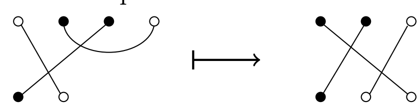




Proposition 2.1.3. *Let w and w' be two finite words in the alphabet $\{\circ, \bullet\}$. Let r (resp. r') be the number of black letters in w (resp. w') and let s (resp. s') be the number of white letters in w (resp. w'). Then*

$$\#\{(w,w')\text{-}diagrams\ \} = (r+s')! \qquad \textit{if } r+s' = r'+s$$
$$= 0 \qquad \textit{if not}$$

Proof. First of all, it is clear that the number of possible (w, w')-diagrams depends only on the number of black and white letters in each word, and not on the order in which they appear within the word.
Next, we can build a bijection from the (w, w')-diagrams to the $(r + s', r' + s)$-diagrams with only propagating edges, by putting all the blacks letters of w' and the white letters of w on the upper row, and the rest on the lower row. Example :



This is clearly a bijection, the inverse function being to exchange the white letters of the upper and lower row, while keeping the edges.
Hence we have

$$\begin{aligned}\#(w,w')\text{-diagrams }\} &= \#\{(r+s', r'+s)\text{-diagrams with only} \\ &\text{propagating edges}\} \\ &= \#\{\text{ bijections from } (r+s') \text{ to } (r'+s)\} \\ &= (r+s')! \text{ if } r+s' = r'+s \\ &= 0 \text{ if not}\end{aligned}$$

□

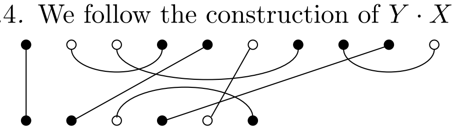
Now we are going to define a product operation on these diagrams.

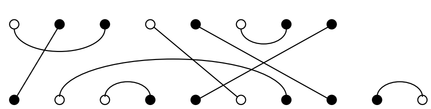
Let w, w' and w'' be three finite words on $\{\circ, \bullet\}$. Let X be a given (w, w')-diagram and Y a (w', w'')-diagram. We begin by defining the diagram $Y \star X$, which is the diagram obtained by stacking Y on top X such that the upper row of X is identified with the lower row of Y.

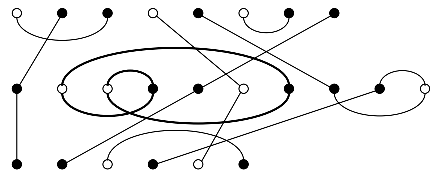
Next, we define the (w, w'')-diagram $Y \cdot X$ whose vertices are the upper row of Y and lower row of X and whose edges are obtained by composing the edges of X and Y.

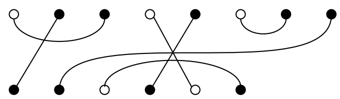
Finally, let $\ell(X, Y)$ denote the number of cycles appearing in $Y \star X$ (i.e. the number of connected components of $Y \star X$ minus the number of connected components of $Y \cdot X$).

Example 2.1.4. We follow the construction of $Y \cdot X$ on an explicit example.

Let $X =$ 

and $Y =$ 

Then $Y \star X =$ 

Thus $Y \cdot X =$  and $\ell(X,Y) = 1$

2.1.2 The skeleton category

With this setup we can now define what we are going to call the *skeleton category* $\mathbf{Rep}_0(Gl_t)$, for any $t \in \mathbb{C}$.

Definition 2.1.5. Let $t \in \mathbb{C}$, the category $\mathbf{Rep}_0(Gl_t)$ has

Objects finite words on the alphabet $\{\circ, \bullet\}$.

Morphisms for $w, w' \in \mathrm{Obj}(\mathbf{Rep}_0(Gl_t))$, $\mathrm{Hom}(w,w')$ is the $\mathbb{C}$-vector space with basis $\{(w,w')\text{-diagrams }\}$.

Composition for $w, w', w'' \in \mathrm{Obj}(\mathbf{Rep}_0(Gl_t))$,

$$\mathrm{Hom}(w',w'') \times \mathrm{Hom}(w,w') \to \mathrm{Hom}(w,w'') \tag{2.1}$$

is the $\mathbb{C}$-linear application defined by

$$(Y,X) \mapsto YX = t^{\ell(X,Y)} Y \cdot X \tag{2.2}$$

for X any (w,w')-diagram and Y any (w',w'')-diagram.

We must of course verify that this definition is correct.

Proposition 2.1.6. ***Rep**$_0(Gl_t)$ as defined above is indeed a category, for every $t \in \mathbb{C}$.*

Proof. **Associativity** As the composition is $\mathbb{C}$-linear, we only have to check the associativity on basis vectors. Let X be a (w,w')-diagram, Y a (w',w'')-diagram and Z a (w'',w''')-diagram. We have to verify that the (w,w''')-diagrams $Z(YX)$ and $(ZY)X$ are equals.

$$\begin{aligned} Z(YX) &= Z(t^{\ell(X,Y)} Y \cdot X) = t^{\ell(X,Y)} Z(Y \cdot X) \quad \text{composition is } \mathbb{C}\text{-linear} \\ &= t^{\ell(X,Y)} t^{\ell(Y \cdot X, Z)} Z \cdot (Y \cdot X) \\ &= t^{\ell(X,Y)} t^{\ell(Y \cdot X, Z)} (Z \cdot Y) \cdot X \quad \text{as the } \cdot \text{ product is clearly associative} \\ &= t^{\ell(X,Y)+\ell(Y \cdot X,Z)-\ell(X,Z \cdot Y)-\ell(Z,Y)} (ZY)X \end{aligned}$$

Thus it is enough to show that

$$\ell(X,Y) + \ell(Y \cdot X, Z) = \ell(X, Z \cdot Y) + \ell(Z,Y)$$

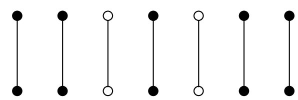
But we have

$$\begin{aligned} \ell(Y \cdot X, Z) &= \ell(Y \star X, Z) - \ell(X,Y) = \\ &= \ell(X, Z \star Y) - \ell(X,Y) = \ell(X, Z \cdot Y) + \ell(Z,Y) - \ell(X,Y) \end{aligned}$$

Identity For all w, word on $\{\circ, \bullet\}$, there is a (w, w)-diagram 1_w which satisfies the following property.

For all $w' \in \mathrm{Obj}(\mathbf{Rep}_0(Gl_t))$, $f \in Hom(w, w')$ and $g \in Hom(w', w)$, $f = f1_w$ and $g = 1_w g$.

We get 1_w by simply putting an edge between the corresponding letters on the upper and lower row. For example

$1_{\bullet\bullet\circ\bullet\circ\bullet\bullet} =$ 

$\square$

2.1.3 Tensor structure on $\mathbf{Rep}_0(Gl_t)$

Now we are going to see that the category we built can be equipped with the structure of a tensor category.

First we have to build the tensor product, which is a functor

$$\bigotimes : \mathbf{Rep}_0(Gl_t) \times \mathbf{Rep}_0(Gl_t) \longrightarrow \mathbf{Rep}_0(Gl_t).$$

Here is how this functor acts on the category $\mathbf{Rep}_0(Gl_t) \times \mathbf{Rep}_0(Gl_t)$.

On objects $(w_1, w_2) \mapsto w_1 \otimes w_2 := w_1 w_2$ concatenation of words.

On morphisms Let X_i be a (w_i, w_i')-diagram, for $i = 1, 2$. Then $(X_1, X_2) \mapsto X_1 \otimes X_2 =$ the $(w_1 w_2, w_1' w_2')$-diagram obtained by placing X_2 directly on the right of X_1.

Proposition 2.1.7. *$\mathbf{Rep}_0(Gl_t)$ equipped with*

- *$\bigotimes$ as the tensor product,*
- *the empty word $\mathbb{1}$ as the unit object,*
- *identity as the associativity constraint, and left and right unitors,*
- *the commutativity constraint γ defined by γ_{w_1,w_2} is the $(w_1 w_2, w_2 w_1)$-diagram obtained by putting an edge between the ith letter of w_1 (resp. w_2) on the upper row and the ith letter of w_1 (resp. w_2) on the lower row,*

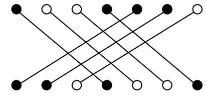
is a rigid symmetric category.

Proof. we are going to prove that statement by checking each term with regard to the definitions we gave earlier.

Monoidal $\bigotimes$ as defined is clearly a bifunctor. The associativity constraint $\alpha_{w_1 w_2 w_3}$ is equal to $1_{w_1 w_2 w_3}$, hence the pentagon axiom is satisfied. For the left and right unitors, we have $\lambda_w : \mathbb{1} \otimes w = w \to w$ is equal to 1_w, the same goes for ρ_w, so the triangle diagram is also satisfied.

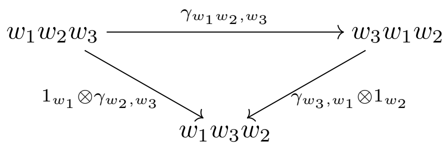
Thus $\mathbf{Rep}_0(Gl_t)$ is a monoidal category.

Symmetric First, let us take a look at what the commutativity constraint does, on an example

$\gamma_{\bullet\bullet\circ,\bullet\circ\circ\bullet} =$ 

We can see that only the number of letters in each word have an effect on the form of the diagram, so we will not draw the letters anymore in this part.

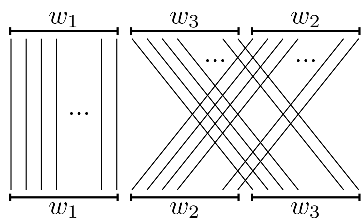
We have to prove that the hexagon identities are satisfied, but in our case there are only triangular diagrams, as the associativity constraint is the identity. We are going to prove one of these identities. For all w_1, w_2, w_3



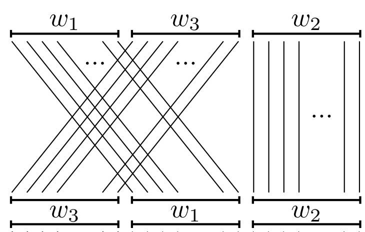
must be commutative.

Hence we have that two $(w_1w_2w_3, w_1w_3w_2)$-diagrams are equal.

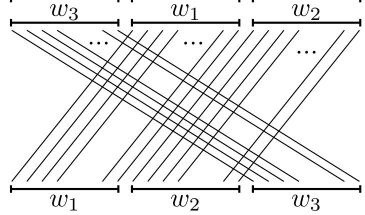
$1_{w_1} \otimes \gamma_{w_2w_3} =$



$\gamma_{w_3,w_1} \otimes 1_{w_2} =$



$\gamma_{w_1w_2,w_3} =$

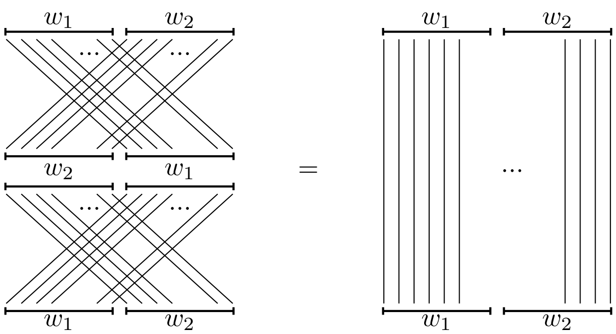


It is clear by looking at these figures that we have indeed

$$1_{w_1} \otimes \gamma_{w_2w_3} = \gamma_{w_3,w_1} \otimes 1_{w_2} + \gamma_{w_1w_2,w_3}$$

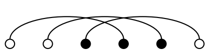
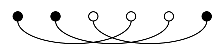
We have now shown that $\mathbf{Rep}_0(Gl_t)$ is a braided monoidal category. To show that it is a symmetric category, it is enough to show that the braiding is symmetric, i.e. that $\gamma_{w_2,w_1}\gamma_{w_1,w_2} = 1_{w_1w_2}$.

This property is illustrated in the following figure

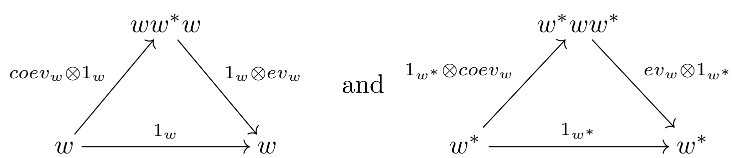


Rigid For any $w \in \mathrm{Obj}(\mathbf{Rep}_0(Gl_t))$ we can define a dual object w^* by replacing all the black letters with white letters and vice versa. Now we build the morphism ev_w, (resp. $coev_w$), which is a $(w^*w, \mathbb{1})$-diagram (resp. $(\mathbb{1}, ww^*)$-diagram in which there is an edge between the ith letter of w and the ith letter of w^* (it is correct because they are of opposite color).

For example :

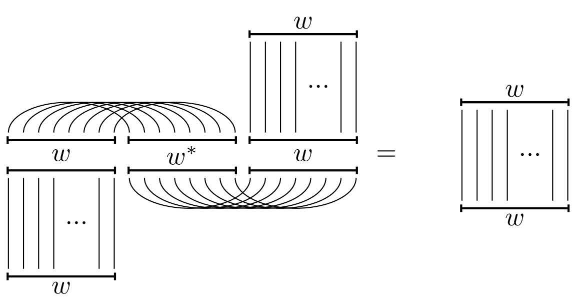
$ev_{\bullet\bullet\circ} =$  and $coev_{\bullet\bullet\circ} =$ 

We have to check that ev_w and $coev_w$ make w^* a dual to w. For that we have to check that the two following diagrams



are commutative.

Let us look at what the first diagram, for example, looks like in terms of (w, w)-diagram.



This figure clearly shows the commutativity of the first diagram, a similar figure would show the same way the commutativity of the second diagram.

Therefore we have shown that $\mathbf{Rep}_0(Gl_t)$ is a rigid symmetric category.

□

As we also have $\mathrm{End}(\mathbb{1}) = \mathbb{C}$, (there is only one $(\mathbb{1}, \mathbb{1})$-diagram, which is the empty diagram). Thus $\mathbf{Rep}_0(Gl_t)$ is a tensor category with the definition given before.

2.1.4 Universal property of $\mathbf{Rep}_0(Gl_t)$

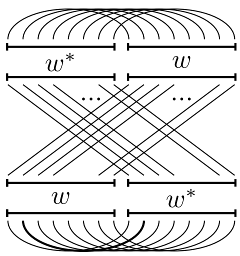
As the skeleton category $\mathbf{Rep}_0(Gl_t)$ is a tensor category, we have a concept of dimension, which is a complex number attached to each object (see 1.2).

In our situation, the dimension is quite simple.

Proposition 2.1.8. *Let w be a word in the category $\boldsymbol{Rep}_0(Gl_t)$. Then*

$$\dim w = t^{|w|} \tag{2.3}$$

Proof. Here once again the result is quite clear if we look at a diagram. Let w be a word and let us write $\mathrm{tr}(w) = ev_w \circ \gamma_{w,w^*} \circ coev_w \in \mathbb{C}$.



We have $\mathrm{tr}(w) = t^\ell$ where ℓ is the number of connected components in the diagram. Then it is obvious that the number of connected components in the diagram is equal to $|w|$, the number of letters in w.
Hence we have $\mathrm{tr}(w) = t^{|w|}$.

□

Remark 2.1.9. We can see that in this case we have

$$\dim(w_1 \otimes w_2) = \dim(w_1) \cdot \dim(w_2) = t^{|w_1|+|w_2|} \tag{2.4}$$

This result is actually true for any tensor category $\mathcal{C}$ with this definition of categorical dimension.

In particular, we have $\dim(\bullet) = t$.

In fact, $\mathbf{Rep}_0(Gl_t)$ is characterized as the universal tensor category generated by an object of dimension t and its dual. More precisely, it satisfies the following universal property

Proposition 2.1.10. *Let $\mathcal{T}$ be any tensor category and let $\mathcal{T}_t$ denote the category of t-dimensional objects in $\mathcal{T}$. Then the following functor induces an equivalence of categories*

$$\begin{array}{rrcl} \Theta : & \mathcal{H}om^{\otimes - str}(\boldsymbol{Rep}_0(Gl_t), \mathcal{T}) & \to & \mathcal{T}_t \\ & F & \mapsto & F(\bullet) \\ & (\eta : F \Rightarrow G) & \mapsto & \eta_\bullet \end{array} \tag{2.5}$$

This property was first stated by Deligne in [5],with the idea of the proof, and a more detailed proof is given in [3].

2.2 Definition of $\mathbf{Rep}(Gl_t)$

From $\mathbf{Rep}_0(Gl_t)$ we define $\mathbf{Rep}(Gl_t)$ as we announced before, which is

$$\mathbf{Rep}(Gl_t) := ((\mathbf{Rep}_0(Gl_t))^{add})^{kar} \tag{2.6}$$

As we have seen in 1.3, to be able to define an additive envelope over an existing category $\mathcal{C}$, $\mathcal{C}$ has to already be preadditive. As the hom-spaces of $\mathcal{C}$ are vector spaces, they also are abelian groups, and the composition is bilinear.

It is clear that $\mathbf{Rep}(Gl_t)$ is also a symmetric tensor category, the tensor structure being inherited from that of $\mathbf{Rep}_0(Gl_t)$.

2.2.1 Universal property of $\mathbf{Rep}(Gl_t)$

For any tensor category $\mathcal{T}$, let $\mathcal{H}om'(\mathbf{Rep}(Gl_t), \mathcal{T})$ denote the full subcategory of $\mathcal{H}om^{\otimes-str}(\mathbf{Rep}(Gl_t), \mathcal{T})$, whose objects are the functors whose restriction $\mathbf{Rep}_0(Gl_t) \to \mathcal{T}$ yields a strict tensor functor. The category $\mathbf{Rep}(Gl_t)$ satisfies the following universal property.

Proposition 2.2.1. *Let $\mathcal{T}$ be any tensor category and let $\mathcal{T}_t$ denote the category of $t-$dimensional objects in $\mathcal{T}$. Then the following functor induce an equivalence of categories.*

$$\begin{array}{rcl} \mathcal{H}om'(\boldsymbol{Rep}(Gl_t), \mathcal{T}) & \to & \mathcal{T}_t \\ F & \mapsto & F(\bullet) \\ (\eta : F \Rightarrow G) & \mapsto & \eta_\bullet \end{array} \tag{2.7}$$

As for the universal property of $\mathbf{Rep}_0(Gl_t)$, the property was first mentioned by Deligne in [5], and a more detailed statement, and proof, of it can be found in [3].

2.2.2 Interpolation of the classical case

Let us denote by $\underline{\mathrm{Rep}}(Gl_N)$ the classical category of complex finite-dimensional representations of Gl_N, where N is an integer. We are going to see the link between this category and the category $\mathbf{Rep}(Gl_t)$ which we just defined.

The category $\underline{\mathrm{Rep}}(Gl_N)$ contains the vector representation $V = \mathbb{C}^N$. It is known that all the irreducible finite-dimensional representation of Gl_N lies in some $V^{\otimes r} \otimes V^{*\otimes s}$, with r, s integers.

What we did earlier was an interpolation of this classical case, with the correspondence

$$V^{\otimes r} \otimes V^{*\otimes s} \longleftrightarrow \underbrace{\bullet\bullet\cdots\bullet}_{r}\underbrace{\circ\circ\cdots\circ}_{s}. \tag{2.8}$$

Remark 2.2.2. One can see that we have indeed

$$\dim(\underbrace{\bullet\bullet\cdots\bullet}_{r}\underbrace{\circ\circ\cdots\circ}_{s}) = \dim(V^{\otimes r} \otimes V^{*\otimes s}) = r + s$$

Hence, the definition of categorical dimension is coherent with the usual definition of dimension in this case also.

We already showed that there was some correspondence between the Deligne category we defined $\mathbf{Rep}(Gl_t)$ and the category $\underline{\mathrm{Rep}}(Gl_N)$, when $t = N$ is an integer. We can wonder if these categories are equivalent, when $t = N$ is a integer. The answer is negative, there are not equivalent.

But, Deligne showed that even if these categories are not exactly the same, the functor $\bullet \mapsto \mathbb{C}^N$, from $\mathbf{Rep}(Gl_{t=N})$ to $\underline{\mathrm{Rep}}(Gl_N)$, still induce an equivalence of categories (see [5])

$$\mathbf{Rep}(Gl_N)/\mathcal{N} \to \underline{\mathrm{Rep}}(Gl_N), \tag{2.9}$$

where $\mathcal{N}$ is the ideal of negligible morphisms of $\mathbf{Rep}(Gl_N)$.

Remark 2.2.3. Let us recall that in a $\mathbb{C}$-linear tensor category $\mathcal{C}$ a morphism $f : X \to Y$ is called *negligible* if, for all morphisms $u : Y \to X$, we have $\mathrm{tr}(f \circ u) = 0$. The trace being the categorical trace defined in 1.2.

In the rest of the paper, we will often compare the situation in $\mathbf{Rep}(Gl_t)$ to the corresponding situation in $\underline{\mathrm{Rep}}(Gl_N)$, in order to better understand the mechanics in place. As we have seen, these two categories are very similar and a lot of phenomenons are going to work the same way.

2.3 Definition of the Lie algebra $\mathfrak{g}$

With this definition of $\mathbf{Rep}(Gl_t)$ it is possible to give a meaning to Gl_t, for any complex number t, as the *fundamental group* of the category (see [6]). Then it is also possible to define $\mathrm{Lie}(Gl_t)$, as the Lie algebra of this group scheme. Details about that can be found in [8].

The result of this process is a simple object of the category $\mathbf{Rep}(Gl_t)$, and we are going to define it as such.

First, let us denote by V the *tautological object* $\bullet$, equivalent of $\mathbb{C}^N$ in the general case. It is the basis object from which all the category is obtained. As we saw with the universal property, V holds a lot of the category's structure.

Definition 2.3.1. Using the tautological object V we are going to define $\mathfrak{g} = \mathrm{Lie}(Gl_t)$ by

$$\mathfrak{g} := V \otimes V^*. \tag{2.10}$$

In other word, it is the object $\bullet\circ$ of the category $\mathbf{Rep}(Gl_t)$.

Remark 2.3.2. This is actually the interpolation in complex rank of the classical case, in which we have

$$\mathrm{Lie}(Gl_N) = \mathfrak{gl}_N = \mathbb{C}^N \otimes \mathbb{C}^{*N}. \tag{2.11}$$

$\mathfrak{g}$ as it is defined is actually an object of the category $\mathbf{Rep}_0(Gl_t)$ and thus of the category $\mathbf{Rep}(Gl_t)$. It is not necessarily a Lie algebra in a classical sense, if t is not an integer, but we are going to consider it as such.

Remark 2.3.3. If we want to see the objects of $\mathbf{Rep}(Gl_t)$ as representations, we can look at them as $\mathfrak{g}$-modules. It is clear that for every object X of $\mathbf{Rep}_0(Gl_t)$, the hom-space $\mathrm{Hom}(\mathfrak{g} \otimes X, X)$ is nonzero, with the result of Proposition 2.1.3, i.e. there is a morphism $\mathfrak{g} \otimes X \to X$. From that is it easy to see that for every object X of $\mathbf{Rep}(Gl_t)$ there is also a morphism $\mathfrak{g} \otimes X \to X$.

2.4 Indecomposable objects in $\mathbf{Rep}(Gl_t)$

The paper [3] gives all the details on the classification of indecomposable representations of $\mathbf{Rep}(Gl_t)$. Here we are only going to give the main results.

First of all, as we have seen in 1.5, indecomposables objects of $\mathbf{Rep}(Gl_t)$ correspond to primitive idempotents of $\mathbf{Rep}_0(Gl_t)$. So that is what we will have to study.

Some notations are going to be useful. First of all, for nonnegative integers r, s, let $w_{r,s}$ be the word of $\mathbf{Rep}_0(Gl_t)$

$$w_{r,s} = \underbrace{\bullet\bullet\cdots\bullet}_{r}\underbrace{\circ\circ\cdots\circ}_{s}. \tag{2.12}$$

It is clear that every object of the category $\mathbf{Rep}_0(Gl_t)$ is isomorphic to a unique $w_{r,s}$.

We write $\mathbb{C}B_{r,s}$, or simply $B_{r,s}$ for the algebra $\mathrm{End}_{\mathbf{Rep}_0(Gl_t)}(w_{r,s})$, called the *wall Brauer algebra.*

Our aim is to have a classification of conjugacy classes of primitive idempotents, for that we will use bipartitions and the action of symmetric groups.

2.4.1 Bipartitions

Definition 2.4.1. A *partition* λ is a tuple of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$, such that for all $i \geq 1$, $\lambda_i > \lambda_{i+1}$, and $\lambda_i = 0$ for all but finitely many i. We will write $|\lambda| = \sum_{i=1}^{\infty} \lambda_i$ for the *size* of λ, and $l(\lambda)$ for the *length* of λ, the smallest integer n such that $\lambda_n \neq 0$ and $\forall i > n, \lambda_i = 0$.

Let $\mathcal{P}$ be the set of all partitions.

Definition 2.4.2. A *bipartition* is an element of $\mathcal{P} \times \mathcal{P}$. For a bipartition λ, we write $\lambda = (\lambda^\bullet, \lambda^\circ)$. The *size* of λ is $|\lambda| = (|\lambda^\bullet|, |\lambda^\circ|)$, and its *length* is $l(\lambda) = l(\lambda^\bullet) + l(\lambda^\circ)$.

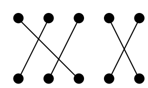
We define a partial order on the set of bipartitions, $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$. We also set $(\lambda^\bullet, \lambda^\circ)^* = (\lambda^\circ, \lambda^\bullet)$.

2.4.2 Symmetric group

Let $\mathfrak{S}_r$ denote the symmetric group on r elements, and $\mathbb{C}\mathfrak{S}_r$ its group algebra. Then $\mathbb{C}\mathfrak{S}_r$ is naturally isomorphic to $B_{r,0}$ and to $B_{0,r}$, via

$$\begin{array}{ccccc} B_{r,0} & \leftarrow & \mathfrak{S}_r & \rightarrow & B_{0,r} \\ \sigma^\bullet & \leftarrow\!\shortmid & \sigma & \mapsto & \sigma^\circ \end{array} \tag{2.13}$$

where $\sigma^\bullet$ is the $(w_{r,0}, w_{r,0})$-diagram in which the ith letter of the bottom row is adjacent to the $\sigma(i)$th letter of the upper row. The definition of σ° is exactly the same, with white letters instead of black ones. For example, if $\sigma = (123)(45) \in \mathfrak{S}_5$, then

$\sigma^\bullet =$ 

More generally, we have an inclusion

$$\begin{aligned} \mathbb{C}[\mathfrak{S}_r \times \mathfrak{S}_s] &\hookrightarrow B_{r,s} \\ (\sigma, \tau) &\mapsto \sigma^\bullet \otimes \tau^\circ \end{aligned}$$

For now on, we will consider $\mathbb{C}[\mathfrak{S}_r \times \mathfrak{S}_s]$ as a subalgebra of $B_{r,s}$.

We will admit the following result (see [4], Proposition 2.3, (2) for the proof).

Lemma 2.4.3. *There is also a surjection of algebra* $\pi : B_{r,s} \twoheadrightarrow \mathbb{C}[\mathfrak{S}_r \times \mathfrak{S}_s]$, *such that*

$$\pi(a) = a \quad \text{for all } a \in \mathbb{C}[\mathfrak{S}_r \times \mathfrak{S}_s] \subset B_{r,s}.$$

2.4.3 The primitive idempotent e_λ

It is a well known result that partitions of r are in bijection with primitive idempotents of $\mathbb{C}\mathfrak{S}_r$, up to conjugation. Let us write z_λ for the primitive idempotent of $\mathbb{C}\mathfrak{S}_r$ corresponding to the partition λ. If λ is a bipartition $\lambda = (\lambda^\bullet, \lambda^\circ)$, with $|\lambda^\bullet| = r$ and $|\lambda^\circ| = s$, then the idempotent of $\mathbb{C}[\mathfrak{S}_r \times \mathfrak{S}_s]$ corresponding to λ will be written $z_\lambda = z^\bullet_{\lambda^\bullet} \otimes z^\circ_{\lambda^\circ}$. z_λ is not necessarily primitive in the larger algebra $B_{r,s}$. Write a decomposition of z_λ is sum of primitive idempotents of $B_{r,s}$ mutually orthogonal $z_\lambda = e_1 + \cdots + e_k$. Then $\pi(e_1), \ldots, \pi(e_k)$ are mutually orthogonal idempotents of $\mathbb{C}[\mathfrak{S}_r \times \mathfrak{S}_s]$ such that $z_\lambda = \pi(e_1) + \cdots + \pi(e_k)$. As z_λ is a primitive idempotent of $\mathbb{C}[\mathfrak{S}_r \times \mathfrak{S}_s]$, there a unique $1 \leq i \leq k$ such that $\pi(e_i) \neq 0$. Let $e_\lambda = e_i$. It is a primitive idempotent of $B_{r,s}$, defined up to conjugation.

Proposition 2.4.4. *If* $\lambda \neq (\varnothing, \varnothing)$, *or* $t \neq 0$, *we can define, up to conjugation, a primitive idempotent* $e_\lambda^{(i)}$ *of* $B_{r+i,s+i}$, *for* $i > 0$, *such that* $\operatorname{im} e_\lambda$ *and* $\operatorname{im} e_\lambda^{(i)}$ *are isomorphic in* $\textbf{\textit{Rep}}(Gl_t)$.

This is the result of Section 4.4 of [3]. The proof is lengthy so we are not going to write is here, will we admit this result.

2.4.4 Classification of idempotents

We have a theorem of classification of conjugacy classes of idempotents in walled Brauer algebras, using the notations of Proposition 2.4.4. As stated in [3], the following result is merely a translation of the classification of simple modules for walled Brauer algebras (see [4], Theorem 2.7) to the language of primitive idempotents.

Theorem 2.4.5. *1. If $r \neq s$ or $t \neq 0$ then*

$$\{e_\lambda^{(i)} \mid |\lambda| = (r-i, s-i), 0 \leq i \leq \min(r,s)\}$$

is a complete set of pairwise non-conjugate primitive idempotents of $B_{r,s}$.

2. If $t = 0$ and $r > 0$ then

$$\{e_\lambda^{(i)} \mid |\lambda| = (r-i, r-i), 0 \leq i < r\}$$

is a complete set of pairwise non-conjugate primitive idempotents of $B_{r,r}$.

Now we come back to the case of indecomposable objects of $\mathbf{Rep}(Gl_t)$. Let λ be a bipartition of size (r,s), and e_λ its corresponding primitive idempotent in $B_{r,s}$. Let $L(\lambda) = \operatorname{im} e_\lambda$, the image of e_λ in $\mathbf{Rep}(Gl_t)$. Using Proposition 1.5.2, as e_λ is a primitive idempotent of $\mathbf{Rep}_0(Gl_t)$ defined up to conjugation, then $L(\lambda)$ is a indecomposable object, defined up to isomorphism.

Theorem 2.4.6. *The map $\lambda \mapsto L(\lambda)$ defines a bijection*

$$\left\{ \begin{array}{c} \textit{bipartitions of} \\ \textit{arbitrary size} \end{array} \right\} \xrightarrow{bij.} \left\{ \begin{array}{r} \textit{nonzero indecomposable objects in} \\ \textbf{\textit{Rep}}(Gl_t)\textit{, up to isomorphism} \end{array} \right\} \tag{2.14}$$

Proof. Let X be an indecomposable object in $\mathbf{Rep}(Gl_t)$, by Proposition 1.5.2, it is isomorphic to the image $\operatorname{im} e$ of a primitive idempotent $e \in \operatorname{End}(w)$ of $\mathbf{Rep}_0(Gl_t)$. But w is isomorphic to some $w_{r,s}$, with $r, s \geq 0$. Hence, without loss of generality, we can assume that $e \in B_{r,s}$, for some $r, s \geq 0$. Thus by Theorem 2.4.5, $e = e_\lambda^{(i)}$, for a certain bipartition λ and i an integer. Finally X is isomorphic to $\operatorname{im} e = \operatorname{im} e_\lambda^{(i)} = \operatorname{im} e_\lambda = L(\lambda)$. We have proven that $\lambda \mapsto L(\lambda)$ is surjective.

Let λ, μ be bipartitions. We write $|\lambda| = (r,s)$ and $|\mu| = (r', s')$. Suppose $L(\lambda) \cong L(\mu)$. We have $\operatorname{Hom}(L(\lambda), L(\mu)) = \operatorname{Hom}(\operatorname{im} e_\lambda, \operatorname{im} e_\mu) \subset \operatorname{Hom}(w_{r,s}, w_{r',s'})$. By Proposition 2.1.3, this hom-space is nonzero if and only if $r + s' = r' + s$. Suppose that $r \geq r'$, then there is an integer $i \geq 0$ such that $(r,s) = (r'+i, s'+i)$. Let us put aside the case when $\mu = (\varnothing, \varnothing)$ and $t = 0$, for which it can be shown that $\lambda = (\varnothing, \varnothing)$ also. If $\mu \neq (\varnothing, \varnothing)$ or $t \neq 0$ then $e_\mu^{(i)}$ is defined, and $\operatorname{im} e_\mu^{(i)} \cong \operatorname{im} e_\mu \cong \operatorname{im} e_\lambda$. By Proposition 1.5.2, this implies that $e_\mu^{(i)}$ and e_λ are conjugate in $B_{r,s}$. By Theorem 2.4.5, $e_\mu^{(i)} = e_\lambda$ and thus $\mu = \lambda$.

□

2.4.5 Analogy with the classical case

It is well known that in the classical case of finite-dimensional representations of the general linear group $Gl_N(\mathbb{C})$, irreducible representations are in correspondence with dominant weights of $Gl_N(\mathbb{C})$. There is a link between this classic correspondence and the study we just did.

As in 2.2.2, let $\underline{\mathrm{Rep}}(Gl_N)$ denote the category of complex finite-dimensional representation of Gl_N. For $1 \leq i \leq N$, let ϵ_i denote the function that maps any matrix to its (i,i) entry, and let Γ denote the set of dominant weights of $Gl_N(\mathbb{C})$.

$$\Gamma = \{\gamma = \sum_{i=1}^{N} \gamma_i \epsilon_i \mid \gamma_i \in \mathbb{Z}, \gamma_1 \geq \cdots \geq \gamma_N\}. \tag{2.15}$$

The weights of Γ are in bijection with the bipartitions λ of size $l(\lambda) \leq N$, via

$$\begin{aligned} \{\text{bipartitions of length } \leq N\} &\xrightarrow{\sim} \Gamma \\ \lambda = (\lambda^\bullet, \lambda^\circ) &\mapsto wt(\lambda) = \sum_{i=1}^{N} \lambda^\bullet \epsilon_i - \sum_{j=1}^{N} \lambda^\circ \epsilon_{N-j+1} \end{aligned} \tag{2.16}$$

When $N \geq r + s$, the bipartition $\lambda = (\lambda^\bullet, \lambda^\circ)$, where $\lambda^\bullet = \lambda_1^\bullet \geq \cdots \geq \lambda_r^\bullet$ and $\lambda^\circ = \lambda_1^\circ \geq \cdots \geq \lambda_s^\circ$, can be written in terms of a weight $(\lambda_1^\bullet, \ldots, \lambda_r^\bullet, \underbrace{0 \ldots, 0}_{N-r-s}, -\lambda_s^\circ, \ldots, -\lambda_1^\circ)$. Reciprocally, if $\gamma = \sum_{i=1}^{N} \gamma_i \epsilon_i \in \Gamma$, let $\lambda^\bullet = \{\gamma_1 \geq \gamma_2 \geq \cdots \geq 0\}$ and $\lambda^\circ = \{-\gamma_N \geq -\gamma_{N-1} \geq \cdots \geq 0\}$, and $\lambda = (\lambda^\bullet, \lambda^\circ)$, then clearly $wt(\lambda) = \gamma$.

As every finite-dimensional indecomposable representation L of Gl_N is isomorphic to a highest weight module of Gl_N, V_γ, for some $\gamma \in \Gamma$, hence if we write V_λ for $V_{wt(\lambda)}$, then $L \cong V_\lambda$ for some bipartition λ. Hence, isomorphism classes of indecomposable object of $\mathrm{Rep}(Gl_N)$, up to isomorphism, are in bijection with bipartitions of size $l(\lambda) \leq N$. The situation is the same as for $\mathbf{Rep}(Gl_t)$.

But in the case when N is an integer, we can give explicit expressions of representations of fundamental weights. If $V = \mathbb{C}^N$ (the tautological object for $\mathrm{Rep}(Gl_N)$) and $k \leq N$, then $\Lambda^k V$ is an irreducible highest weight representation of weight $(\underbrace{1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{N-k})$, and $\Lambda^k V^*$ is an irreducible highest weight representation of Gl_N of weight $(\underbrace{0, \ldots, 0}_{N-k}, \underbrace{-1, \ldots, -1}_{k})$. With the bijection (2.16), $\Lambda^k V$ corresponds to the bipartition $(\lambda^\bullet, \lambda^\circ)$, with $\lambda^\bullet = \underbrace{1 + 1 + \cdots + 1}_{k}$ and $\lambda^\circ = 0$, and $\Lambda^k V^*$ corresponds to its dual $(\lambda^\circ, \lambda^\bullet)$.

More generally, if N is large enough, the representation $\bigotimes_{i=1}^{m} \Lambda^i V^{\otimes r_i} \otimes \bigotimes_{j=1}^{n} \Lambda^j V^{*\otimes s_j}$ is of highest weight

$$(r_1 + \cdots + r_m, r_2 + \cdots + r_m, \ldots, r_m, 0, \ldots, 0, -s_n, -s_n - s_{n-1}, \ldots, -s_n - \cdots - s_1). \tag{2.17}$$

Hence, if w_r denotes the partition $(\underbrace{1, \ldots, 1}_{r}, 0, \ldots, 0)$, and

$$\lambda^\bullet = \sum_{i=1}^{m} r_i w_i, \quad \lambda^\circ = \sum_{j=1}^{n} s_j w_j, \tag{2.18}$$

then the irreducible representation of Gl_N corresponding to this bipartition $(\lambda^\bullet, \lambda^\circ)$ is the highest weight representation of weight (2.17). Thus it is a quotient of $\bigotimes_{i=1}^{m} \Lambda^i V^{\otimes r_i} \otimes \bigotimes_{j=1}^{n} \Lambda^j V^{*\otimes s_j}$.

As every bipartition can be written in the form (2.18), every irreducible finite-dimensional representation of Gl_N is a quotient of such a tensor product.

2.5 Generic semisimplicity of $\mathbf{Rep}(Gl_t)$

In this section we are going to see for which t the category $\mathbf{Rep}(Gl_t)$ is a semisimple category. We have the following theorem.

Theorem 2.5.1. *The category **Rep**(Gl_t) is semisimple if and only if $t \notin \mathbb{Z}$.*

Remark 2.5.2. If $\mathcal{C}$ is a $\mathbb{C}$-linear Krull-Schmidt category, it is semisimple if and only if $\mathrm{End}_{\mathcal{C}}(L) \cong \mathbb{C}$ for any indecomposable object L of $\mathcal{C}$, and $\mathrm{Hom}_{\mathcal{C}}(L, L') = 0$ if L, L' are non isomorphic indecomposable objects.

Proof. We are going to use the criteria in Remark 2.5.2 to prove the result. First we have to look at the walled Brauer algebras $B_{r,s} = \mathrm{End}(w_{r,s})$, as in 2.4. Let us admit that if $t \in \mathbb{Z}$ then $B_{r,s}$ is not semisimple for some $r, s > 0$ (see [4] for a complete proof). Then, in that case, with Theorem 2.4.5, there exists two different bipartitions λ, μ, with $l(\lambda) = (r-i, s-i), l(\mu) = (r-j, s-j)$, with $i, j \geq 0$, such that $e_\lambda^{(i)} B_{r,s} e_\mu^{(j)} \neq 0$. But $e_\lambda^{(i)} B_{r,s} e_\mu^{(j)} \cong \mathrm{Hom}(\mathrm{im}\, e_\mu^{(j)}, \mathrm{im}\, e_\lambda^{(i)})$, by $e_\lambda^{(i)} f e_\mu^{(j)} = \iota_{e_\lambda^{(i)}} \pi_{e_\lambda^{(i)}} f \iota_{e_\mu^{(j)}} \pi_{e_\mu^{(j)}} \mapsto \pi_{e_\lambda^{(i)}} f \iota_{e_\mu^{(j)}} \in \mathrm{Hom}(\mathrm{im}\, e_\mu^{(j)}, \mathrm{im}\, e_\lambda^{(i)})$, and $g \mapsto \iota_{e_\lambda^{(i)}} g \pi_{e_\mu^{(j)}}$; by the splitting relations, these two maps are reciprocally inverse. Hence, if $t \in \mathbb{Z}$, $\mathrm{Hom}(L(\mu), L(\lambda)) = \mathrm{Hom}(\mathrm{im}\, e_\mu^{(j)}, \mathrm{im}\, e_\lambda^{(i)}) \neq 0$, and $\mathbf{Rep}(Gl_t)$ is not a semisimple category.

Let us suppose now that $t \notin \mathbb{Z}$, then for all r, s, $B_{r,s}$ is a semisimple algebra. Let λ, μ be bipartitions such that $\mathrm{Hom}(L(\mu), L(\lambda)) \neq 0$. If we write $l(\lambda) = (r, s)$ and $l(\mu) = (r', s')$, then as in proof of Theorem 2.4.6, this implies that $(r', s') = (r+i, s+i)$. Let us suppose that $i \geq 0$. Then take $e_\lambda^{(i)} \in B_{r+i,s+i} = B_{r',s'}$. We have $\mathrm{Hom}(L(\mu), L(\lambda)) = \mathrm{Hom}(\mathrm{im}\, e_\mu, \mathrm{im}\, e_\lambda^{(i)}) \neq 0$. But as $B_{r',s'}$ is semisimple, this imply that $e_\lambda^{(i)}$ and e_μ are conjugate, and by Theorem 2.4.5, that $\lambda = \mu$. The semisimplicity of $B_{r,s}$ also implies that $\mathrm{End}(L(\lambda)) \cong e_\lambda B_{r,s} e_\lambda = \mathbb{C}$. □

Chapter 3

Yangian of $\mathfrak{g}$

As we said in the introduction, a Yangian is a type of quantum group. For any finite-dimensional semisimple Lie algebra $\mathfrak{a}$, the *Yangian* $Y(\mathfrak{a})$ is a Hopf algebra deformation of $\mathcal{U}(\mathfrak{a}[z])$. See [7] for a construction of the Yangian of a simple Lie algebra with an invariant scalar product, $\mathfrak{a}$. The Yangian is there defined by taking $h = 1$ in a quantization of the polynomial current Lie algebra $\mathfrak{g} = \mathfrak{a}[u]$.

3.1 Classical Yangian of $\mathfrak{gl}_N$

There are different ways to define to define the Yangian of a semisimple Lie algebra. In [2], there are two different presentations of the Yangian of a simple finite-dimensional Lie algebra, and the isomorphisms between these presentations.

In the case of $\mathfrak{gl}_N$ there exists a third presentation of $Y(\mathfrak{g})$, called the Faddeev-Reshetikhin-Takhtajan presentation, or RTT presentation, and it the one we are going to use.

3.1.1 Motivation

The definition of the Yangian $Y(\mathfrak{gl}_N)$ is modeled on the structure of the universal enveloping algebra $U(\mathfrak{gl}_N)$.

Consider the general linear $\mathfrak{gl}_N$, and its basis vectors E_{ij}, for $1 \leq i, j \leq N$. We have the following relations.

$$[E_{ij}, E_{kl}] = \delta_{kj} E_{il} - \delta_{il} E_{kj}, \quad \forall 1 \leq i, j, k, l \leq N \tag{3.1}$$

Now, introduce the matrix E of size $N \times N$, with coefficients in $\mathfrak{gl}_N$ whose ij-th entry is E_{ij}. Then, for $p \geq 1$, and $1 \leq i, j \leq N$, $(E^p)_{ij} = \sum E_{ik_1} E_{k_1 k_2} \cdots E_{k_{s-1} l} \in U(\mathfrak{gl}_N)$. Then, we can prove the following result by induction, using (3.1).

$$[E_{ij}, (E^p)_{kl}] = \delta_{kj} (E^p)_{il} - \delta_{il} (E^p)_{kj}, \quad \forall 1 \leq i, j, k, l \leq N. \tag{3.2}$$

With (3.2), and another induction, we get the following equation, for $p, q \geq 0$ ($E^0 = 1$ by convention).

$$[(E^{p+1})_{ij}, (E^q)_{kl}] - [(E^p)_{ij}, (E^{q+1})_{kl}] = (E^p)_{kj} (E^q)_{il} - (E^q)_{kj} (E^p)_{il} \tag{3.3}$$

The algebra $Y(\mathfrak{gl}_N)$ is the one obtained by replacing $(E^p)_{ij}$ by an abstract generator $t_{ij}^{(p)}$.

3.1.2 Definition

Definition 3.1.1. The *Yangian* $Y(\mathfrak{gl}_N)$ of the general linear algebra $\mathfrak{gl}_N$, for N an integer, is a unitary associative algebra over $\mathbb{C}$ with generators $t_{ij}^{(m)}$, where $i, j = 1, \ldots, N$ and $m \in \mathbb{N}^*$, and relations

$$[t_{ij}^{(p+1)}, t_{kl}^{(q)}] - [t_{ij}^{(p)}, t_{kl}^{(q+1)}] = t_{kj}^{(q)} t_{il}^{(q)} - t_{kj}^{(p)} t_{il}^{(q)} \tag{3.4}$$

where $p, q \in \mathbb{N}$, and $t_{ij}^{(0)} = \delta_{ij}$ by convention.

Remark 3.1.2. The relation we use to define $Y(\mathfrak{gl}_N)$ in (3.4) is actually not the same as (3.3), the left-hand side is multiplied by -1. We use here this this other definition of $Y(\mathfrak{gl}_N)$ because it will be convenient afterwards, but the two definitions give isomorphic algebras, the isomorphism being defined by $t_{ij}^{(p)} \mapsto t_{ji}^{(p)}$.

Proposition 3.1.3. *The system of the defining relations (3.4) of the Yangian is equivalent to the following system, where $i, j = 1, \dots, N$*

$$[t_{ij}^{(p)}, t_{kl}^{(q)}] = \sum_{a=1}^{\min\{p,q\}} \left(t_{kj}^{(p+q-a)} t_{il}^{(a-1)} - t_{kj}^{(a-1)} t_{il}^{(p+q-a)} \right) \tag{3.5}$$

Proof. Let us suppose that $p < q$. We are going to prove the result by induction on p.

$$\begin{aligned}
[t_{ij}^{(1)}, t_{kl}^{(q)}] &= [t_{ij}^{(0)}, t_{kl}^{(q+1)}] + t_{kj}^{(q)} t_{il}^{(0)} - t_{kj}^{(0)} t_{il}^{(q)} \\
&= \delta_{il} t_{kj}^{(q)} - \delta_{kj} t_{il}^{(q)}
\end{aligned}$$

So (3.5) is satisfied for $p = 1$. Let us suppose that it is satisfied up to a certain rank p. Then

$$\begin{aligned}
[t_{ij}^{(p+1)}, t_{kl}^{(q)}] &= [t_{ij}^{(p)}, t_{kl}^{(q+1)}] + t_{kj}^{(q)} t_{il}^{(p)} - t_{kj}^{(p)} t_{il}^{(q)} \\
&= \sum_{a=1}^{p} \left(t_{kj}^{(p+q-a)} t_{il}^{(a-1)} - t_{kj}^{(a-1)} t_{il}^{(p+q-a)} \right) + t_{kj}^{(q)} t_{il}^{(p)} - t_{kj}^{(p)} t_{il}^{(q)} \\
&= \sum_{a=1}^{p+1} \left(t_{kj}^{(p+q-a)} t_{il}^{(a-1)} - t_{kj}^{(a-1)} t_{il}^{(p+q-a)} \right)
\end{aligned}$$

□

Let us define the formal generating series

$$t_{ij}(u) := \delta_{ij} + \sum_{k \in \mathbb{N}^*} t_{ij}^{(k)} u^{-k} \quad \in Y(\mathfrak{gl}_N)[[u^{-1}]]. \tag{3.6}$$

Then the relation (3.4) can be written in the form

$$(u - v)[t_{ij}(u), t_{kl}(v)] = t_{kj}(v) t_{il}(u) - t_{kj}(u) t_{il}(v) \tag{3.7}$$

where the indeterminates u and v are considered to be commuting with each other and with the elements of the Yangian.

3.1.3 Matrix form of the defining relations

Let us define $T(u)$, an element of $\mathfrak{gl}_N \otimes Y(\mathfrak{gl}_N)[[u^{-1}]]$.

$$T(u) := \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u) \tag{3.8}$$

where $e_{ij} \in \mathfrak{gl}_N$ are the standard basis matrices, with 1 in position (i, j) and 0 elsewhere.

We can translate the defining relations (3.4) of the Yangian with only one equation on $T(u)$. First we define the permutation

$$\sigma : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N \otimes \mathbb{C}^N, \quad x \otimes y \mapsto y \otimes x. \tag{3.9}$$

As an element of $\mathrm{End}(\mathbb{C}^N) = \mathfrak{gl}_N \otimes \mathfrak{gl}_N$, it can be written $\sigma = \sum_{i,j=1}^{N} e_{ij} \otimes e_{ji}$.

Then we introduce

$$R(u) := 1 + \frac{\sigma}{u} \in \mathfrak{gl}_N^{\otimes 2}$$

We write 1 instead of $1 \otimes 1$ for brevity.

Remark 3.1.4. R is called the *Yang R-matrix* and it satisfies the ***Yang-Baxter equation***.

$$R^{12}(u)R^{13}(u+v)R^{23}(v) = R^{23}(v)R^{13}(u+v)R^{12}(u) \tag{3.10}$$

where $R^{12}(u)$ denotes $1 + \frac{1}{u}\sum_{i,j=1}^{N} e_{ij} \otimes e_{ji} \otimes 1 = 1 + \frac{(12)}{u}$, and the same goes for $R^{13}(u)$ and $R^{23}(u)$. All are elements of $\mathfrak{gl}_N^{\otimes 3}$.

Proof.

$$\begin{aligned}
&R^{12}(u)R^{13}(u+v)R^{23}(v) - R^{23}(v)R^{13}(u+v)R^{12}(u) = \\
&\quad \left(1 + \frac{(12)}{u}\right)\left(1 + \frac{(13)}{u+v}\right)\left(1 + \frac{(23)}{v}\right) - \left(1 + \frac{(23)}{v}\right)\left(1 + \frac{(13)}{u+v}\right)\left(1 + \frac{(12)}{u}\right) \\
&\quad = 1 + \frac{(12)}{u} + \frac{(13)}{u+v} + \frac{(23)}{v} + \frac{(132)}{u(u+v)} + \frac{(123)}{uv} + \frac{(132)}{v(u+v)} + \frac{(13)}{uv(u+v)} \\
&\quad - \left(1 + \frac{(12)}{u} + \frac{(13)}{u+v} + \frac{(23)}{v} + \frac{(123)}{u(u+v)} + \frac{(132)}{uv} + \frac{(123)}{v(u+v)} + \frac{(13)}{uv(u+v)}\right) \\
&\quad = \frac{1}{uv}((132) + (123)) - \frac{1}{uv}((123) + (132)) = 0
\end{aligned}$$

□

Finally, as before, let $T^{13}(u)$ and $T^{23}(u)$ respectively denote $\sum_{i,j=1}^{N} e_{ij} \otimes 1 \otimes t_{ij}(u)$ and $\sum_{i,j=1}^{N} 1 \otimes e_{ij} \otimes t_{ij}(u)$, and $R^{12}(u)$ denote $1 + \frac{1}{u}\sum_{i,j=1}^{N} e_{ij} \otimes e_{ji} \otimes \mathbb{1}$, where $\mathbb{1}$ is the unit of $Y(\mathfrak{gl}_N)$ (and 1 is short for $1 \otimes 1 \otimes \mathbb{1}$).

These are all elements of $\mathfrak{gl}_N^{\otimes 2} \otimes Y(\mathfrak{gl}_N)[[u^{-1}]]$.

Proposition 3.1.5. *The definition relations of the algebra $Y(\mathfrak{gl}_N)$ are equivalent to the following single relation in $\mathfrak{gl}_N^{\otimes 2} \otimes Y(\mathfrak{gl}_N)[[u^{-1}]]$*

$$R^{12}(u-v)T^{13}(u)T^{23}(v) = T^{23}(v)T^{13}(u)R^{12}(u-v). \tag{3.11}$$

Proof. We have

$$\begin{aligned}
\sigma^{12}T^{13}(u)T^{23}(v) &= \left(\sum e_{ij} \otimes e_{ji} \otimes \mathbb{1}\right)\left(\sum e_{kl} \otimes 1 \otimes t_{kl}(u)\right)(1 \otimes e_{pq} \otimes t_{pq}(v)) \\
&= \left(\sum e_{il} \otimes e_{ji} \otimes t_{jl}(u)\right)(1 \otimes e_{pq} \otimes t_{pq}(v)) \\
&= \sum_{i,j,l,q} e_{il} \otimes e_{jq} \otimes t_{jl}(u)t_{ip}(v) = \sum_{i,j,k,l} e_{ij} \otimes e_{kl} \otimes t_{kj}(u)t_{il}(v)
\end{aligned}$$

Whereas

$$T^{23}(v)T^{13}(u)\sigma^{12} = \sum_{i,j,k,l} e_{ij} \otimes e_{kl} \otimes t_{kj}(v)t_{il}(u)$$

Hence

$$\sigma^{12}T^{13}(u)T^{23}(v) - T^{23}(v)T^{13}(u)\sigma^{12} = \sum_{i,j,k,l} e_{ij} \otimes e_{kl} \otimes (t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u))$$

We recognize the left-hand side of the relation (3.7) (up to a sign). Next we have

$$(u-v)(\big(T^{13}(u)T^{23}(v) - T^{23}(v)T^{13}(u)\big) =$$

$$(u-v)\big[\Big(\sum e_{ij}\otimes 1\otimes t_{ij}(u)\Big)\Big(\sum 1\otimes e_{kl}\otimes t_{kl}(v)\Big) - \Big(\sum 1\otimes e_{kl}\otimes t_{kl}(v)\Big)\Big(\sum e_{ij}\otimes 1\otimes t_{ij}(u)\Big)\big]$$
$$= (u-v)\sum_{i,j,k,l} e_{ij}\otimes e_{kl}\,(t_{ij}(u)t_{kl}(v) - t_{kl}(v)t_{ij}(u))$$

If we combine everything we have, we get

$$(u-v)\left(R^{12}(u-v)T^{13}(u)T^{23}(v) - T^{23}(v)T^{13}(u)R^{12}(u-v)\right) =$$
$$\sum_{i,j,k,l} e_{ij}\otimes e_{kl}\otimes((u-v)[t_{ij}(u),t_{kl}(v)] + t_{kj}(u)t_{il}(v) - t_{kj}(v)t_{il}(u))$$

It is clear that (3.11) is equivalent to (3.7), which is equivalent to (3.4), the defining relations of the Yangian.

□

Remark 3.1.6. The relation (3.11) is usually called the *RTT relation*. We can note that it resembles the relation of the R-matrix (3.10).

3.2 Yangian in complex rank

3.2.1 Definition

We are going to define the Yangian of $\mathfrak{g}$, which we defined in Section 2.3, and is the equivalent of $\mathfrak{gl}_N$ in Deligne's category $\mathbf{Rep}(Gl_t)$. We are going to follow the definition of $Y(\mathfrak{g})$ given in Section 7 of [8], which is an interpolation of the RTT presentation, in complex rank.

Let us recall that $\mathfrak{g}$ is the object $\mathfrak{g} = V \otimes V^* = \bullet \otimes \circ = \bullet\circ$, of the category $\mathbf{Rep}(Gl_t)$.

First, let us define the tensor algebra $A := \mathbf{T}(\bigoplus_{i\geq 0}(V \otimes V^*)_{(i)})$, which is not technically an object of the category $\mathbf{Rep}(Gl_t)$, but a Ind-object. Then let T_i be the coevaluation map

$$T_i = \mathbb{1} \longrightarrow (V^* \otimes V) \otimes (V \otimes V^*)_{(i)} \in \mathrm{Hom}(\mathbb{1}, (V^* \otimes V) \otimes A).$$

Then let $T(u) := 1 + T_0u^{-1} + T_1u^{-2} + \cdots$ be the generating function of T_i. Let σ be the permutation $\sigma : V \otimes V \to V \otimes V$ and let $R(u) := 1 + \frac{\sigma}{u}$, just as before (but in the general case we do not have basis to give another definition of σ).

Let us consider the series

$$Q(u,v) := (u-v)(R^{12}(u-v)T^{13}(u)T^{23}(v) - T^{23}(v)T^{13}(u)R^{12}(u-v))$$
$$= \sum Q_{i,j}u^iv^j$$

The coefficients $Q_{i,j}$ are elements of $\mathrm{Hom}(\mathbb{1}, (V^* \otimes V) \otimes (V^* \otimes V) \otimes A)$, but they can be seen as morphisms

$$Q_{ij} = (V \otimes V^*) \otimes (V \otimes V^*) \longrightarrow A$$

Let J be the ideal of A generated by the images of all the Q_{ij}.

Definition 3.2.1. The algebra $Y(\mathfrak{g}) := A/J$ is called the *Yangian of* $\mathfrak{g}$.

Remark 3.2.2.

- We can see that it is the same definition as in the classical case, but without using basis vectors, which we do not have in complex rank.
- $Y(\mathfrak{g})$ is generated by the images of the coefficients T_i of $T(u)$, seen as morphisms $V \otimes V^* \longrightarrow A$. Thus from now on, for all the morphisms of A, we are going to study only their actions on $T(u)$.

3.2.2 Hopf algebra structure

As promised before, the Yangian is supposed to be an Hopf algebra. $Y(\mathfrak{g})$ as we defined has indeed an Hopf algebra structure.

Proposition 3.2.3. *The Yangian $Y(\mathfrak{g})$ is an Hopf algebra with*

$$\textbf{\textit{Coproduct}}\ \Delta : T(u) \mapsto T^{12}(u)T^{13}(u),$$
$$\textbf{\textit{Antipode}}\ S : T(u) \mapsto T^{-1}(u),$$
$$\textbf{\textit{Counit}}\ \epsilon : T(u) \mapsto 1.$$

Remark 3.2.4. If we write the coproduct using the definition in the classical case we get

$$\Delta : t_{ij}^{(r)} \mapsto \sum_k \sum_{s=0}^{r} t_{ik}^{(s)} \otimes t_{kj}^{(r-s)} \tag{3.12}$$

Indeed, $T^{12}(u)T^{13}(u) = \left(\sum e_{ij} \otimes t_{ij}^{(p)} u^{-p} \otimes 1\right)\left(\sum e_{kl} \otimes 1 \otimes t_{kl}^{(q)} u^{-q}\right)$
$= \sum e_{il} \otimes t_{ij}^{(p)} \otimes t_{jl}^{(q)} u^{-q-p} = \sum_{i,l} e_{il} \otimes \sum_k \left(\sum_{r=0}^{+\infty} \left(\sum_{s=0}^{r} t_{ik}^{(s)} \otimes t_{kl}^{(r-s)}\right) u^{-r}\right),$
which is exactly the result we wanted.

The counit is defined by $\epsilon : t_{ij}^{(r)} \mapsto 0$ for $r \geq 1$.

Proof. First of all, we have to verify that $(Y(\mathfrak{g}), \Delta, \epsilon)$ is a bialgebra.

For the associativity of the coproduct, it is enough to see that

$$((\Delta \otimes \mathrm{id}) \circ \Delta)T(u) = ((\mathrm{id} \otimes \Delta) \circ \Delta)T(u) = T^{12}(u)T^{13}(u)T^{14}(u)$$

where the definition of $T^{14}(u)$ is the obvious one.

Furthermore $((\epsilon \otimes \mathrm{id}) \circ \Delta)T(u) = (\epsilon \otimes \mathrm{id})T^{12}(u)T^{13}(u) = T(u)$, and the same goes for $\mathrm{id} \otimes \epsilon$. Hence the counit axiom is satisfied, and $Y(\mathfrak{g})$ is a bialgebra.

Next, in order for $Y(\mathfrak{g})$ to be an Hopf algebra, we also have to check that the following diagram commutes (where m is the product and $\eta : \mathbb{C} \to Y(\mathfrak{g})$ is the unit $1 \mapsto \mathbb{1}$)

$$\begin{array}{ccccccc} & & Y(\mathfrak{g}) \otimes Y(\mathfrak{g}) & \xrightarrow{S\otimes \mathrm{id}} & Y(\mathfrak{g}) \otimes Y(\mathfrak{g}) & & \\ & \overset{\Delta}{\nearrow} & & & & \overset{m}{\searrow} & \\ Y(\mathfrak{g}) & & \xrightarrow{\quad \epsilon \quad} & \mathbb{C} & \xrightarrow{\quad \eta \quad} & & Y(\mathfrak{g}) \\ & \underset{\Delta}{\searrow} & & & & \underset{m}{\nearrow} & \\ & & Y(\mathfrak{g}) \otimes Y(\mathfrak{g}) & \xrightarrow{\mathrm{id}\otimes S} & Y(\mathfrak{g}) \otimes Y(\mathfrak{g}) & & \end{array}$$

We have $T(u) \overset{\Delta}{\longmapsto} T^{12}(u)T^{13}(u) \overset{S\otimes\mathrm{id}}{\longmapsto} (T^{12})^{-1}(u)T^{13}(u) \overset{m}{\longmapsto} \mathbb{1} = (\eta \circ \epsilon)T(u)$, and the same goes for the lower part of the diagram. Thus $(Y(\mathfrak{g}), \Delta, S, \epsilon)$ is indeed an Hopf algebra.

$\square$

Chapter 4

Representations of the Yangian $Y(\mathfrak{gl}_N)$, where N is an integer

In this chapter we consider the Yangian of $\mathfrak{gl}_N$, $Y(\mathfrak{gl}_N)$, where N is an integer, with classical definitions of $\mathfrak{gl}_N$ and $\underline{\mathrm{R}}\mathrm{ep}(Gl_N)$. We are going to study the representations L of $Y(\mathfrak{gl}_N)$. If L and M are representations of $Y(\mathfrak{gl}_N)$, then their tensor product space $L \otimes M$ can be equipped with the structure of a $Y(\mathfrak{gl}_N)$-module also, using the coproduct Δ, as $Y(\mathfrak{gl}_N)$ is an Hopf algebra.

In the theory of irreducible representations of $Y(\mathfrak{gl}_N)$, as in the case of representations of reductive Lie algebras, there is a classification of finite-dimensional representations, using *highest weights*. See [13] or [2] for further reference.

4.1 Highest weight representations

Definition 4.1.1. A representation L of $Y(\mathfrak{gl}_N)$ is called a *highest weight representation* if there exist a cyclic vector ζ in L (such that $Y(\mathfrak{gl}_N).\zeta = L$) which satisfies

$$\begin{array}{ll} t_{ij}(u).\zeta = 0 & \text{for } 1 \leq j < i \leq N \\ t_{ii}(u).\zeta = \lambda_i(u)\zeta & \text{for } 1 \leq i \leq N \end{array}$$

where $\lambda_i(u)$ is a formal series

$$\lambda_i(u) = 1 + \lambda_i^{(1)} u^{-1} + \lambda_i^{(2)} u^{-2} + \cdots \in 1 + u^{-1}\mathbb{C}[[u^{-1}]].$$

The vector ζ is called the *highest weight vector* of L and the tuple $\lambda(u) := (\lambda_1(u), \lambda_2(u), \ldots, \lambda_N(u))$ is called the *highest weight* of L.

This definition is inspired from the classical representation theory of semi-simple Lie algebras, and as in that case, we have can define the Verma module.

Definition 4.1.2. Let $\lambda(u) := (\lambda_1(u), \lambda_2(u), \ldots, \lambda_N(u))$ be a tuple of formal series $\lambda_i(u) \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$. The *Verma module* $M(\lambda(u))$ is the quotient $Y(\mathfrak{gl}_n)/J$, where J is the left ideal of $Y(\mathfrak{gl}_N)$ generated by all the coefficients of the series $t_{ij}(u)$ for $1 \leq j < i \leq N$ and $t_{ii}(u) - \lambda_i(u)$ for $1 \leq i \leq N$.

Lemma 4.1.3. *Let $\mathbb{1}_{\lambda(u)}$ be the image of $\mathbb{1} \in Y(\mathfrak{gl}_N)$ in the quotient. Given any order on the set of generators $t_{ij}^{(r)}$, with $1 \leq i < j \leq N$ and $r \geq 1$, the elements*

$$t_{i_1 j_1}^{(r_1)} \cdots t_{i_m j_m}^{(r_m)} \mathbb{1}_{\lambda(u)}, \quad m \geq 0,$$

with ordered products of the generators, form a basis of $M(\lambda(u))$.

Proof. This follows from the Poincaré-Birkhoff-Witt theorem in the $Y(\mathfrak{gl}_N)$ case : given an arbitrary linear order on the set of generators $t_{ij}^{(r)}$, any element of the algebra $Y(\mathfrak{gl}_N)$ can be uniquely written as a linear combination of ordered monomials in these generators. See [13] for details. $\square$

As with a semi-simple Lie algebra, $M(\lambda(u))$ possesses a unique maximal proper submodule, the sum of all proper submodules. Hence we can define the irreducible highest weight representations.

Definition 4.1.4. The *irreducible highest weight representation* $L(\lambda(u))$ of $Y(\mathfrak{gl}_n)$ is defined as the quotient of $M(\lambda(u))$ by its unique maximal proper submodule.

Proposition 4.1.5. *Every finite-dimensional irreducible representation L of the Yangian $Y(\mathfrak{gl}_n)$ is isomorphic to a unique irreducible highest weight representation $L(\lambda(u))$.*

Proof. First of all, we can see that with the embedding

$$E_{ij} \longmapsto t_{ji}^{(1)}, \tag{4.1}$$

every representation of $Y(\mathfrak{gl}_N)$ is also a $\mathfrak{gl}_N$-module.

Remark 4.1.6. This is indeed an algebra morphism, as $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}$ and by (3.5) $[t_{ji}^{(1)}, t_{lk}^{(1)}] = \delta_{jk}E_{li} - \delta_{il}E_{jk}$.

Let us recall some notations regarding representations of $\mathfrak{gl}_N$. Let $\mathfrak{h}$ be the diagonal Cartan subalgebra of $\mathfrak{gl}_N$, and $\mathfrak{h}^*$ its dual space. Let $\epsilon_1, \dots, \epsilon_N$ be the basis vectors of $\mathfrak{h}^*$ dual to the basis vectors of $\mathfrak{h}, E_{11}, \dots, E_{NN}$, respectively. Every weight (in the classical sense) μ of L can be written $\mu = \mu_1\epsilon_1 + \dots + \mu_N\epsilon_N$. If α and β are two weights of L, we say that α *precedes* β, and write $\alpha < \beta$, if $\beta - \alpha$ is nonzero, and can be written as a $\mathbb{Z}_+$-linear combination of the simple roots $\epsilon_i - \epsilon_j$, with $i < j$.

Let L be a finite-dimensional irreducible representation of $Y(\mathfrak{gl}_N)$. Let us introduce the following subspace of L,

$$L^0 := \{\, \zeta \in L \mid t_{ij}(u).\zeta = 0, \quad 1 \le j < i \le N \,\}.$$

The first step of the proof of Proposition 4.1.5 is to show that $L^0 \neq \{0\}$.

As we said before, L is also a representation of $\mathfrak{gl}_N$, let us consider the set of weights of L. It is a finite set as L is finite-dimensional, and has a maximal weight μ with respects to the ordering introduced above. The corresponding weight vector ζ belongs to L^0. Indeed, for $1 \le j < i \le N$ and $k = 1, \dots, N$

$$E_{kk}t_{ij}(u)\zeta = [E_{kk}, t_{ij}(u)]\zeta + t_{ij}(u)\,(E_{kk}\zeta)$$

By identifying the elements E_{ij} of $\mathfrak{gl}_N$ with their images via the embedding (4.1) we have

$$\begin{aligned}[E_{kk}, t_{ij}(u)] &= [t_{kk}^{(1)}, t_{ij}(u)] \\ &= \delta_{kj}t_{ik}(u) - \delta_{ik}t_{kj}(u) \qquad \text{by taking the coefficient of } v^0 \text{ in (3.7)}\end{aligned}$$

Thus $E_{kk}t_{ij}(u)\zeta = (\delta_{kj}t_{ik}(u) - \delta_{ik}t_{kj}(u))\zeta + \mu(E_{kk})t_{ij}(u)\zeta = (\mu + \epsilon_j - \epsilon i)(E_{kk})t_{ij}(u)\zeta$

Hence, if $t_{ij}(u)\zeta \neq 0$ then it is a non zero vector of L of weight $\mu + \epsilon_j - \epsilon i$, which is $> \mu$ with the ordering defined above (if j<i). This is a contradiction with the maximality of μ. Thus $\zeta \in L^0$.

Next step is to show that L^0 is invariant under the action of all elements $t_{kk}(u)$, and consequently under the action of all elements $t_{kk}^{(r)}$. Let ζ be an element of L^0.

$$\begin{aligned}[t_{ij}(u), t_{kk}(v)]\zeta &= \frac{1}{u-v}(t_{kj}(v)t_{ik}(u) - t_{kj}(u)t_{ik}(v))\zeta = 0 \text{ if } i < k \\ &= -[t_{kk}(v), t_{ij}(u)]\zeta = \frac{1}{u-v}(t_{ik}(u)t_{kj}(v) - t_{ik}(v)t_{kj}(u))\zeta = 0 \text{ if } k < j\end{aligned}$$

Thus if $i < j$, $t_{ij}(u)t_{kk}(v)\zeta = [t_{ij}(u), t_{kk}(v)]\zeta = 0$, and $t_{kk}(v)\zeta \in L^0$.

Furthermore, (3.7) implies that

$$[t_{ii}(u), t_{ii}(v)] = 0$$

And

$$(u - v)[t_{ii}(u), t_{jj}(v)] = t_{ji}(v)t_{ij}(u) - t_{ji}(u)t_{ij}(v)$$

Hence, as operators on L^0, the elements $t_{kk}(u)$, for $k = 1, \dots, N$ are pairwise commutative, and so are the elements t^r_{kk}, for $k = 1, \dots, N$ and $r \geq 1$.

Let ζ be a simultaneous eigenvector of L^0 for these operators. ζ generates L because L is supposed to be irreducible. So ζ is a highest weight vector with respects to definition 4.1.1, with a certain weight $\lambda(u)$.

By Lemma 4.1.3, the vector space L is spanned by the elements

$$t^{(r_1)}_{i_1 j_1} \cdots t^{(r_m)}_{i_m j_m} \zeta, \quad m \geq 0, i_k < j_k$$

with ordered products of the generators. Based on what we did earlier we know that $t^{(r)}_{ij}\zeta$ is of weight $\mu + \epsilon_j - \epsilon_i$, which is $< \mu$ with respects to the partial order we defined above, if $i < j$. This implies that the weight space L_μ is one-dimensional and spanned by the vector ζ. Moreover, if ν is a weight of L and $\nu \neq \mu$ then $\nu < \mu$. Hence the highest vector ζ of L is determined uniquely, up to a constant factor.

Hence we have shown that L is a highest weight representation. In order to prove that it is isomorphic to a unique $L(\lambda(u))$, one must use the same arguments as for a semisimple Lie algebra. As it is not the center of the discussion, we will not give details about that. □

We now have that every finite-dimensional irreducible representation of $Y(\mathfrak{gl}_N)$ is isomorphic to a unique $L(\lambda(u))$. Thus to obtain a classification of irreducible representations of $Y(\mathfrak{gl}_N)$, we only have to know which $L(\lambda(u))$ is finite-dimensional. We will begin by studying the case of representations of $Y(\mathfrak{gl}_2)$, and in the following section we will give the criteria for $L(\lambda(u))$ to be finite-dimensional.

4.2 Representations of $Y(\mathfrak{gl}_2)$

4.2.1 A word on representations of $\mathfrak{gl}_2$

Let α, β be complex numbers and let $L(\alpha, \beta)$ denote the irreducible highest weight representation of the Lie algebra $\mathfrak{gl}_2$ of weight (α, β). Let ζ denote the highest vector of $L(\alpha, \beta)$, then we have

$$E_{11}\zeta = a\zeta, \quad E_{22}\zeta = \beta\zeta, \quad E_{21}\zeta = 0$$

It is well known that if $\alpha - \beta \in \mathbb{Z}_+$, then $L(\alpha, \beta)$ is finite-dimensional and that the vectors $(E_{21})^r\zeta$, for $r = 0, 1, \dots, \alpha_\beta$ form a basis of $L(\alpha, \beta)$. If $\alpha - \beta \notin \mathbb{Z}_+$, then the vectors $(E_{21})^r\zeta$, where r runs over all nonnegative integers, are linearly independent and form a basis of $L(\alpha, \beta)$, which is not finite-dimensional.

It can be equipped with a $Y(\mathfrak{gl}_2)$-module structure via some *evaluation morphisms*, for an $z \in \mathbb{C}$, $\mathrm{ev}_z : Y(\mathfrak{gl}_2) \to \mathcal{U}(\mathfrak{gl}_2)$ is defined (on the formal series (3.6))by

$$t_{ij}(u) \mapsto \delta_{ij} + \frac{E_{ji}}{u - z}. \tag{4.2}$$

As stated, ev_z is actually a morphism from $Y(\mathfrak{gl}_2)[[u^1]]$ to $\mathcal{U}(\mathfrak{gl}_2)[[u^{-1}]]$, but by taking each component of the formal series in u^{-1}, we get a morphism form $Y(\mathfrak{gl}_2)$ to $\mathcal{U}(\mathfrak{gl}_2)$. Indeed,

$$\mathrm{ev}_z(t_{ij}(u)) = \delta_{ij} + \frac{E_{ji}}{u - z} = \delta_{ij} + \frac{E_{ji}}{u} \frac{1}{1 - \frac{z}{u}} = \delta_{ij} + E_{ji} \sum_{k=0}^{+\infty} z^k u^{-k-1} \tag{4.3}$$

Hence the formula (4.2) is just a simpler way to write the following morphism

$$\begin{array}{rrcl} \mathrm{ev}_z : & Y(\mathfrak{gl}_2) & \to & \mathcal{U}(\mathfrak{gl}_2), \\ & t_{ij}^{(r)} & \mapsto & E_{ji} z^{r-1} \text{ if } r \geq 1, \\ & t_{ij}^{(0)} = \delta_{ij} & \mapsto & \delta_{ij}. \end{array} \tag{4.4}$$

Lemma 4.2.1. *The formula (4.2) defines an algebra morphism from $Y(\mathfrak{gl}_2)$ to $\mathcal{U}(\mathfrak{gl}_2)$.*

Proof. We have to check that, for any $z \in \mathbb{C}$, the definition of ev_z is consistent with the defining relations of the Yangian $Y(\mathfrak{gl}_2)$. We are going to show that the definition (4.2) is consistent with the relation (3.7) of the Yangian.

$$\begin{aligned} \mathrm{ev}_z([t_{ij}(u), t_{kl}(v)]) &= \mathrm{ev}_z\left(\frac{1}{u-v}(t_{kj}(v)t_{il}(u) - t_{kj}(u)t_{il}(v)\right) \\ &= \frac{1}{u-v}\left(\left(\delta_{kj} + \frac{E_{jk}}{v-z}\right)\left(\delta_{il} + \frac{E_{li}}{u-z}\right) - \left(\delta_{kj} + \frac{E_{jk}}{u-z}\right)\left(\delta_{il} + \frac{E_{li}}{v-z}\right)\right) \\ &= \frac{1}{u-v}\left(\delta_{kj}E_{li}\left(\frac{1}{u-z} - \frac{1}{v-z}\right) + \delta_{il}E_{jk}\left(\frac{1}{v-z} - \frac{1}{u-z}\right)\right) \\ &= \frac{1}{u-v}(\delta_{il}E_{jk} - \delta_{kj}E_{li})\frac{u-v}{(u-z)(v-z)} = (\delta_{il}E_{jk} - \delta_{kj}E_{li})\frac{1}{(u-z)(v-z)} \\ [\mathrm{ev}_z(t_{ij}(u)), \mathrm{ev}_z(t_{kl}(v))] &= \left[\delta_{ij} + \frac{E_{ji}}{u-z}, \delta_{kl} + \frac{E_{lk}}{v-z}\right] = [E_{ji}, E_{lk}]\frac{1}{(u-z)(v-z)} \end{aligned}$$

□

Definition 4.2.2. Let V be a representation of $\mathcal{U}(\mathfrak{gl}_2)$. The representation of $Y(\mathfrak{gl}_2)$ obtained by taking the composition of the evaluation morphism ev_z and of the action of $\mathcal{U}(\mathfrak{gl}_2)$ on V is called the *evaluation representation* of V and is denoted $\mathrm{ev}_z^*(V)$. It is also called a pull-back representation.

Remark 4.2.3. This definition can be generalized. The formula (4.2) also defines an algebra morphism from $Y(\mathfrak{gl}_N)$ to $\mathcal{U}(\mathfrak{gl}_N)$, for $N \geq 3$. Hence every $\mathcal{U}(\mathfrak{gl}_N)$-module can be endowed with the structure of a $Y(\mathfrak{gl}_N)$-module. The evaluation representations are denoted the same.

Let us note that via the evaluation morphism, $L(\alpha, \beta)$ is an irreducible highest weight representation, with highest vector ζ and highest weight $(\alpha_i(u), \beta_i(u))$, where

$$\begin{aligned} \alpha(u) &= 1 + \frac{\alpha}{u^- z} \\ \beta(u) &= 1 + \frac{\beta}{u-z}. \end{aligned}$$

Hence the notation $L(\alpha, \beta)$ or $L(\lambda)$ where $\lambda = (\alpha, \beta)$ is consistent, as a highest weight representation of both $\mathfrak{gl}_2$ and $Y(\mathfrak{gl}_2)$.

4.2.2 Prerequisites for the classification theorem

In the rest of the section, we are going to consider an irreducible highest weight representation $L(\lambda(u))$ of $Y(\mathfrak{gl}_2)$, with weight $(\lambda_1(u), \lambda_2(u))$.

Proposition 4.2.4. *If $L(\lambda(u))$ is finite-dimensional, then there is a formal series of the form*

$$f(u) = 1 + f_1 u^{-1} + f_2 u^{-2} + \cdots, \quad f_k \in \mathbb{C}$$

such that $f(u)\lambda_1(u)$ and $f(u)\lambda_2(u)$ are polynomials in u^{-1}.

Proof. The mapping $T(u) \mapsto \lambda_2(u)^{-1}T(u)$ is an automorphism of $Y(\mathfrak{gl}_2)$ (the image satisfies the RTT relation). By twisting the action of $Y(\mathfrak{gl}_2)$ on $L(\lambda(u))$ by this automorphism, we get a irreducible highest weight representation of weight $(\nu(u),1)$, with $\nu(u) = \lambda_1(u)/\lambda_2(u)$, hence isomorphic to $L(\nu(u),1)$. Thus, we can consider that the highest weight of $L(\lambda(u))$ as the form $\lambda(u) = (\nu(u),1)$, without loss of generality.

Let ζ denote the highest weight vector of $L(\nu(u),1)$. Since this representation is finite-dimensional, the vectors $t_{12}^{(r)}\zeta, r \geq 1$ are linearly dependent. Hence there is a integer m and $c_1,\dots,c_m \in \mathbb{C}$, such that $c_m \neq 0$ and $\sum_{r=1}^{m} c_r t_{12}^{(r)}\zeta = 0$. According to Lemma 4.1.3, the vector

$$\xi = \sum_{r=1}^{m} c_r t_{12}^{(r)} \mathbb{1}_{\lambda(u)}$$

is non-zero, and its image in the quotient $L(\nu(u),1)$ is zero, hence ξ is a non-zero vector of K, the unique maximal proper submodule of $M(\nu(u),1)$.

Using (3.5), in $M(\nu(u),1)$, we have

$$t_{21}^{(k)}t_{12}^{(r)}\mathbb{1}_{\lambda(u)} = \sum_{a=1}^{\min\{k,r\}} \left(t_{11}^{(k+r-a)}t_{22}^{(a-1)} - t_{11}^{(a-1)}t_{22}^{(k+r-a)}\right)\mathbb{1}_{\lambda(u)} = \nu^{(k+r-1)}\mathbb{1}_{\lambda(u)}$$

So $\quad t_{21}^{(k)}\xi = \sum_{r=1}^{m} c_r t_{21}^{(k)}t_{12}^{(r)}\mathbb{1}_{\lambda(u)} = \sum_{r=1}^{m} c_r \nu^{(k+r-1)}\mathbb{1}_{\lambda(u)} \quad \in \mathbb{C}\mathbb{1}_{\lambda(u)}$.

Hence, $t_{21}^{(k)}\xi = 0$ for all $k \geq 1$, otherwise $\mathbb{1}_{\lambda(u)}$ would be in K, and we have the relations, for all $k \geq 1$

$$\sum_{r=1}^{m} c_r \nu^{(k+r-1)} = 0$$

They imply

$$\nu(u)\left(\sum_{r=1}^{m} c_r u^{r-1}\right) = \sum_{k=0}^{+\infty} c_r \nu^{(k)} u^{r-k-1} = \sum_{s=0}^{m-1} u^s \underbrace{\sum_{r=1}^{m} c_r \nu^{(r-s-1)}}_{=0 \text{ if } s\leq -1 \text{ or } r-s-1<0} = \sum_{s=0}^{m-1} u^s \sum_{r=s+1}^{m} c_r \nu^{(r-s-1)}$$

So $\nu(u)(c_1 + c_2u + \dots + c_m u^m) = b_1 + b_2 u + \dots + b_m u^{m-1}$, with $b_m = c_m$. Thus, taking

$$f(u) := c_m^{-1}\sum_{r=1}^{m} c_r u^{-m-r}$$

we get that $f(u)\nu(u)$ and $f(u)1$ are monic polynomials in u^{-1}. $\square$

This proposition implies that by taking the composition of the action of $Y(\mathfrak{gl}_2)$ on $L(\lambda(u))$ with an appropriate automorphism, we get another highest weight representation of $Y(\mathfrak{gl}_2)$ for which both components of the highest weight are polynomials in u^{-1}. From now on, we are going to assume that this is the case.

We write the decompositions

$$\lambda_1(u) = (1+\alpha_1 u^{-1})\dots(1+\alpha_m u^{-1})$$
$$\lambda_2(u) = (1+\beta_1 u^{-1})\dots(1+\beta_m u^{-1})$$

where $\{\alpha_i\}_{1\leq i\leq m}, \{\beta_i\}_{1\leq i\leq m}$ are complex numbers.

Lemma 4.2.5. *Suppose that for every $1 \leq i \leq m-1$ the following condition holds : if the multiset $\{\alpha_p - \alpha_q \mid i \leq p, q \leq m\}$ contains non-negative integers, then $\alpha_i - \beta_i$ is minimal amongst them. Then the representation $L(\lambda_1(u), \lambda_2(u))$ of $Y(\mathfrak{gl}_2)$ is isomorphic to the tensor product module*

$$L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \ldots \otimes L(\alpha_m, \beta_m).$$

We admit this result. The complete proof can be found in [13]. What is interesting about this lemma is that the condition is not restrictive. Indeed, if some of the differences $\alpha_p - \beta_q$ are nonnegative integers, choose a minimal amongst them and re-enumerate it $\alpha_1 - \beta_1$, then proceed by induction, considering all differences $\alpha_p - \beta_q$.

The following Lemma is also going to be useful in the proof of the classification theorem.

Lemma 4.2.6. *The tensor product module $L = L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \cdots \otimes L(\alpha_p, \beta_p)$ has a submodule which is a highest weight module of weight $(\alpha(u), \beta(u))$, with*

$$\alpha(u) = (1 + \alpha_1 u^{-1})(1 + \alpha_2 u^{-1}) \cdots (1 + \alpha_p u^{-1}),$$
$$\beta(u) = (1 + \beta_1 u^{-1})(1 + \beta_2 u^{-1}) \cdots (1 + \beta_p u^{-1}).$$

Proof. For $1 \leq i \leq p$, let ζ_i be the highest weight vector of $L(\alpha_i, \beta_i)$. We form the element $\zeta = \zeta_1 \otimes \zeta_2 \cdots \otimes \zeta_p \in L$. Let L' be the $Y(\mathfrak{gl}_2)$-submodule of L generated by ζ.

$$L' = Y(\mathfrak{gl}_2).\zeta \tag{4.5}$$

Then, by the definition of the coproduct Δ in $Y(\mathfrak{gl}_2)$, we have for $i, j \in \{1, 2\}$,

$$t_{ij}(u).\zeta = \sum_{(i_1, \ldots, i_p) \in \{1,2\}^p} t_{1i_1}(u).\zeta_1 \otimes t_{i_1 i_2}(u).\zeta_2 \otimes \cdots \otimes t_{i_p j}(u).\zeta_p. \tag{4.6}$$

Hence $t_{12}(u).\zeta = 0$, because we can not have $2 \leq i_1 \leq i_2 \leq \cdots \leq i_p \leq 1$. And

$$t_{11}(u).\zeta = t_{11}(u).\zeta_1 \otimes t_{11}(u).\zeta_2 \otimes \cdots \otimes t_{11}(u).\zeta_p = \prod_{i=1}^{p} (1 + \alpha_i u^{-1})$$
$$t_{22}(u).\zeta = t_{22}(u).\zeta_1 \otimes t_{22}(u).\zeta_2 \otimes \cdots \otimes t_{22}(u).\zeta_p = \prod_{i=1}^{p} (1 + \beta_i u^{-1})$$

Thus, L' is indeed a submodule of L which is a highest weight module of weight $(\alpha(u), \beta(u))$. □

4.2.3 The classification theorem

Theorem 4.2.7. *The irreducible highest weight representation $L(\lambda_1(u), \lambda_2(u))$ of $Y(\mathfrak{gl}_2)$ is finite-dimensional if and only if there exists a monic polynomial $P(u)$ in u, such that*

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{P(u+1)}{P(u)} \tag{4.7}$$

And in this case, P is unique.

Proof. Suppose that $L(\lambda_1(u), \lambda_2(u))$ is finite-dimensional. By proposition 4.2.4, there is a formal series $f(u)$ such that

$$f(u)\lambda_1(u) = 1 + a_1 u^{-1} + \cdots + a_m u^{-m} = (1 + \alpha_1 u^{-1}) \ldots (1 + \alpha_m u^{-1})$$
$$f(u)\lambda_2(u) = 1 + b_1 u^{-1} + \cdots + b_m u^{-m} = (1 + \beta_1 u^{-1}) \ldots (1 + \beta_m u^{-1})$$

for some $m \geq 0$, where $\{\alpha_i\}_{1\leq i\leq m}, \{\beta_i\}_{1\leq i\leq m}$ are complex numbers. As we saw before, we can assume that the condition of lemma 4.2.5 is satisfied, by re-enumerating these parameters if necessary.

Then $L(\lambda_1(u), \lambda_2(u))$ is isomorphic to $L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \ldots \otimes L(\alpha_m, \beta_m)$. As it is finite-dimensional, necessarily $\alpha_i - \beta_i$ are nonnegative integers for all $1 \leq i \leq m$ (see introduction of the section).

Then the polynomial

$$P(u) := \prod_{i=1}^{m} (u+\beta_i)(u+\beta_i+1)\ldots(u+\alpha_i-1)$$

is well defined and satisfies (4.7). Indeed,

$$\frac{P(u+1)}{P(u)} = \prod_{i=1}^{m} \frac{(u+\beta_i+1)(u+\beta_i+2)\ldots(u+\alpha_i)}{(u+\beta_i)(u+\beta_i+1)\ldots(u+\alpha_i-1)} = \frac{\prod_{i=1}^{m}(u+\alpha_i)}{\prod_{i=1}^{m}(u+\beta_i)} = \frac{f(u)\lambda_1(u)}{f(u)\lambda_2(u)} = \frac{\lambda_1(u)}{\lambda_2(u)}$$

For the reverse implication, suppose that the condition (4.7) holds for a polynomial $P(u) = (u+\gamma_1)\ldots(u+\gamma_p)$. Set

$$\mu_1(u) = (1+(\gamma_1+1)u^{-1})\ldots(1+(\gamma_p+1)u^{-1}), \quad \mu_2(u) = (1+\gamma_1 u^{-1})\ldots(1+\gamma_p u^{-1})$$

and consider the tensor product module

$$L := L(\gamma_1+1, \gamma_1) \otimes L(\gamma_2+1, \gamma_2) \otimes \ldots \otimes L(\gamma_p+1, \gamma_p).$$

L is obviously finite-dimensional. From Lemma 4.2.6, we know that under these conditions, L has a submodule which is a highest weight module of weight $(\mu_1(u), \mu_2(u))$. This submodule is then finite-dimensional and so is its irreducible quotient $L(\mu_1(u), \mu_2(u))$. Since

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{P(u+1)}{P(u)} = \frac{\mu_1(u)}{\mu_2(u)}$$

Consider the automorphism of $Y(\mathfrak{gl}_2)$, $T(u) \mapsto f(u)T(u)$, where $f(u) = \lambda_1/\mu_1 = \lambda_2/\mu_2$. Its composition with the representation $L(\mu_1(u), \mu_2(u))$ is isomorphic to $L(\lambda_1(u), \lambda_2(u))$, which is then finite-dimensional.

The only thing left to prove is that the polynomial P is unique. Take another monic polynomial Q which satisfies (4.7). Then we have

$$\frac{P(u+1)}{P(u)} = \frac{Q(u+1)}{Q(u)}$$

which means that the fraction $P(u)/Q(u)$ is periodic in u, which is only possible if it is a constant, hence if $P = Q$.

□

Definition 4.2.8. The polynomial P is called the *Drinfeld Polynomial* of the finite-dimensional representation $L(\lambda_1(u), \lambda_2(u))$.

The relation (4.7) is the key of Theorem 4.2.7, hence we introduce a simpler notation. From now on, we will write

$$\lambda_1(u) \longrightarrow \lambda_2(u) \tag{4.8}$$

if there exists a monic polynomial $P(u)$ such that (4.7) is satisfied.

4.3 Irreducible representations of $Y(\mathfrak{gl}_N)$

4.3.1 A criteria for finite-dimensional representations

We now have the tools we need to prove the classification theorem for irreducible representations of $Y(\mathfrak{gl}_N)$. Let us recall that every irreducible representation of $Y(\mathfrak{gl}_N)$ is isomorphic to a highest weight representation $L(\lambda(u))$, where $\lambda(u)$ is a N-tuple of formal series $\lambda(u) = (\lambda_1(u), \dots, \lambda_N(u))$, where each of the formal series $\lambda_i(u)$ is of the form

$$\lambda_i(u) = 1 + \lambda_i^{(1)} u^{-1} + \lambda_i^{(2)} u^{-2} + \dots \in \mathbb{C}[[u^{-1}]]$$

Theorem 4.3.1. *The irreducible highest weight representation $L(\lambda(u))$ of the Yangian $Y(\mathfrak{gl}_N)$ is finite-dimensional if and only if the following relations are satisfied*

$$\lambda_1(u) \longrightarrow \lambda_2(u) \longrightarrow \cdots \longrightarrow \lambda_N(u). \tag{4.9}$$

Proof. First, suppose that $\dim L(\lambda(u)) < +\infty$.

For $0 \leq k \leq N-2$, let us consider the morphism

$$\begin{aligned} Y(\mathfrak{gl}_2) &\longrightarrow Y(\mathfrak{gl}_N) \\ t_{ij}(u) &\longmapsto t_{i+k,j+k}(u), \quad \text{for } i,j = 1,2 \end{aligned}$$

$Y(\mathfrak{gl}_2)$ acts on $L(\lambda(u))$ via this morphism and the action of $Y(\mathfrak{gl}_N)$. Let ζ be the highest weight vector of $L(\lambda(u))$ (as a $Y(\mathfrak{gl}_N)$-module). The $Y(\mathfrak{gl}_2)$-module $Y(\mathfrak{gl}_2)\zeta$ is a highest weight module of weight $(\lambda_{k+1}(u), \lambda_{k+2}(u))$. It is finite-dimensional, as a submodule of $L(\lambda(u))$. Hence, its irreducible quotient is also finite-dimensional, and by Theorem 4.2.7, we have $\lambda_{k+1}(u) \longrightarrow \lambda_{k+2}(u)$. This being true for all $k \in \{0, \dots, N-2\}$, we have proven the first implication.

Conversely, suppose that we have $\lambda_1(u) \longrightarrow \lambda_2(u) \longrightarrow \cdots \longrightarrow \lambda_N(u)$. Then, for $1 \leq i \leq N-1$, there exists a monic polynomial P_i, such that

$$\frac{\lambda_i(u)}{\lambda_{i+1}(u)} = \frac{P_i(u+1)}{P_i(u)} \tag{4.10}$$

We introduce some notations. Let k_i be the degree of the polynomial P_i. We write

$$P_i(u) = (u + \gamma_i^{(1)})(u + \gamma_i^{(2)}) \dots (u + \gamma_i^{(k_i)}), \quad \gamma_i^{(l)} \in \mathbb{C}. \tag{4.11}$$

For $1 \leq i \leq N$, let

$$\begin{aligned} \mu_i(u) &= u^{-k} P_1(u) \cdots P_{i-1}(u) P_i(u+1) \cdots P_{N-1}(u+1) & (4.12) \\ &= (1 + \mu_i^{(1)} u^{-1})(1 + \mu_i^{(2)} u^{-1}) \cdots (1 + \mu_i^{(k)} u^{-1}) & (4.13) \end{aligned}$$

where $k = k_1 + \dots + k_{N-1}$. It is defined in order to have

$$\frac{\mu_i(u)}{\mu_{i+1}(u)} = \frac{P_i(u+1)}{P_i(u)} \tag{4.14}$$

Consider $L(\mu(u))$, the highest weight representation of $Y(\mathfrak{gl}_N)$ of weight $\mu(u) = (\mu_1(u), \dots, \mu_N(u))$. We have

$$\begin{aligned} \mu_i(u) = u^{-k} &(u + \gamma_1^{(1)}) \dots (u + \gamma_1^{(k_1)})(u + \gamma_2^{(1)}) \dots (u + \gamma_2^{(k_2)}) \dots (u + \gamma_{i-1}^{(k_{i-1})})(u + 1 + \gamma_i^{(1)}) \dots \\ &(u + 1 + \gamma_i^{(k_i)}) \dots (u + 1 + \gamma_N^{(k_N)}) \\ &= (1 + \gamma_1^{(1)} u^{-1}) \dots (1 + \gamma_{i-1}^{(k_{i-1})} u^{-1})(1 + u^{-1} + \gamma_i^{(1)} u^{-1}) \dots (1 + u^{-1} + \gamma_N^{(k_N)} u^{-1}) \end{aligned}$$

Hence for $1 \leq j \leq N-1$ and $1 \leq l \leq k_j$, we have

$$\mu_i^{(k_1+\cdots+k_{j-1}+l)} = \begin{cases} \gamma_j^{(l)} & \text{if } j < i \\ \gamma_j^{(l)} + 1 & \text{if } j \geq i \end{cases}$$

Let us consider $\mu^{(r)} = (\mu_1^{(r)}, \ldots, \mu_N^{(r)}) = \left(\underbrace{\gamma_j^{(l)} + 1, \ldots, \gamma_j^{(l)} + 1}_{j \text{ times}}, \underbrace{\gamma_j^{(l)}, \ldots, \gamma_j^{(l)}}_{r-j \text{ times}} \right)$, for $r = k_1 + \cdots + k_{j-1} + l$ $(l < k_j)$. The $\mathfrak{gl}_N$-module $L(\mu^{(r)})$ is finite-dimensional, as $\mu_i^{(r)} - \mu_{i+1}^{(r)} \in \mathbb{Z}_+$, for $1 \leq i \leq N-1$. Hence

$$L(\mu^{(1)}) \otimes L(\mu^{(2)}) \otimes \cdots \otimes L(\mu^{(k)})$$

is also finite-dimensional. It has the structure of a $Y(\mathfrak{gl}_N)$-module, via the evaluation morphism ev_0 (4.2) for example, as in 4.2.1. As in Lemma 4.2.6, the tensor product of the highest weight vectors of $L(\mu^{(r)})$ generates a highest weight representation of $Y(\mathfrak{gl}_N)$ with highest weight $\mu(u)$ (the proof of Lemma 4.2.6 extends clearly to the case of $Y(\mathfrak{gl}_N)$-modules). This representation is finite-dimensional, and so is its irreducible highest weight representation, which is isomorphic to $L(\mu(u))$. Thus we conclude that $L(\mu(u))$ is finite-dimensional.

But we have, by definition, (4.14). Hence, for $1 \leq i \leq N-1$

$$\frac{\mu_i(u)}{\mu_{i+1}(u)} = \frac{\lambda_i(u)}{\lambda_{i+1}(u)}. \tag{4.15}$$

If we let $f(u)$ be the formal series

$$f(u) = \frac{\lambda_1(u)}{\mu_1(u)} = \cdots = \frac{\lambda_N(u)}{\mu_N(u)} \tag{4.16}$$

then the composition of the representation of $Y(\mathfrak{gl}_N)$ on $L(\mu(u))$ and the automorphism of $Y(\mathfrak{gl}_N)$ defined by $T(u) \mapsto f(u)T(u)$ is isomorphic to $L(\lambda(u))$. Thus $L(\lambda(u))$ is also finite-dimensional.

□

4.3.2 Structure of finite-dimensional irreducible representations

The following result states that with only the evaluation representations, we can get all the irreducible representations of $Y(\mathfrak{g})$, and so all the representations of $Y(\mathfrak{g})$.

Let V denote the $\mathfrak{gl}_n$-module $\mathbb{C}^N$. If $\Lambda^i V$ is the ith exterior product of V, and $z \in \mathbb{C}$, let $\Lambda^i V(z)$ denote the pull-back $\mathrm{ev}_z^*(\Lambda^i V)$, via the evaluation morphism ev_z defined in 4.2.1.

Proposition 4.3.2. *Every finite-dimensional irreducible representation of $Y(\mathfrak{gl}_N)$ is isomorphic to a subquotient of a tensor product of evaluation representations, of the form*

$$\bigotimes_{i=1}^{m} \Lambda^i V(z_i^{(1)}) \otimes \cdots \otimes \Lambda^i V(z_i^{(r_i)}) \otimes \bigotimes_{j=1}^{n} \Lambda^j V^*(w_j^{(1)}) \otimes \cdots \otimes \Lambda^j V^*(w_j^{(s_j)}) \tag{4.17}$$

where $m, n \geq 0$ and $r_1, \ldots, r_m, s_1, \ldots, s_n \geq 0$ are integers, and $z_i^{(k)}, w_j^{(l)} \in \mathbb{C}$.

If $\vec{r} = (r_1, \ldots, r_m)$ and $\vec{s} = (s_1 \ldots, s_n)$, $z = \{z_i^{(k)}\}_{1 \leq i \leq m, 1 \leq k \leq r_i}$ and $w = \{w_j^{(l)}\}_{1 \leq j \leq n, 1 \leq l \leq s_j}$, then let us introduce the notation

$$\vec{E}(\vec{r}, \vec{s}, z, w) := \bigotimes_{i=1}^{m} \Lambda^i V(z_i^{(1)}) \otimes \cdots \otimes \Lambda^i V(z_i^{(r_i)}) \otimes \bigotimes_{j=1}^{n} \Lambda^j V^*(w_j^{(1)}) \otimes \cdots \otimes \Lambda^j V^*(w_j^{(s_j)}). \tag{4.18}$$

Proof. Via the evaluation morphism (4.2), $\Lambda^i V(z)$ is an irreducible representation of $Y(\mathfrak{gl}_N)$ of weight $\lambda(u) = (\underbrace{1+\frac{1}{u-z},\dots,1+\frac{1}{u-z}}_{i},1\dots,1)$, and $\Lambda^j V^*(w)$ is an irreducible representation of $Y(\mathfrak{gl}_N)$ of weight $\mu(u) = (1\dots,1,\underbrace{1-\frac{1}{u-w},\dots,1-\frac{1}{u-w}}_{i})$.

Using the coproduct Δ of the Hopf algebra $Y(\mathfrak{gl}_N)$, given by (3.12), we can compute the action of $Y(\mathfrak{gl}_N)$ on $\zeta_k\otimes\zeta_l$, seen as an element of $\Lambda^k V(z_1)\otimes\Lambda^l V(z_2)$. Suppose $k\le l$.

$$\begin{aligned}
t_{ij}(u)\zeta_k\otimes\zeta_l = \sum_{l=1}^{N} t_{il}(u)\zeta_k\otimes t_{lj}(u)\zeta_l &= 0 \quad \text{if } i<j\\
&= \left(1+\frac{1}{u-z_1}\right)\left(1+\frac{1}{u-z_2}\right) \quad \text{if } i=j\le k\le l\\
&= \left(1+\frac{1}{u-z_2}\right) \quad \text{if } k<i=j\le l\\
&= 1 \quad \text{if } i=j>l
\end{aligned}$$

Hence, $Y(\mathfrak{gl}_n)\zeta_k\otimes\zeta_l$ is a highest weight representation of $Y(\mathfrak{gl}_n)$ of weight

$$\Bigg(\left(1+\frac{1}{u-z_1}\right)\left(1+\frac{1}{u-z_2}\right),\dots,\left(1+\frac{1}{u-z_1}\right)\left(1+\frac{1}{u-z_2}\right),\\
\left(1+\frac{1}{u-z_2}\right),\dots,\left(1+\frac{1}{u-z_2}\right),1,\dots,1\Bigg).$$

Following the same logic, the action of $Y(\mathfrak{gl}_n)$ on $\bigotimes_{i=1}^{m}\zeta_i^{\otimes r_i}\otimes\bigotimes_{j=1}^{n}\xi_j^{\otimes s_j}$, seen as an element of $\vec{E}(\vec{r},\vec{s},z,w)$ spans a highest weight representation of weight (if N is a larger integer)

$$\Bigg(\prod_{i=1}^{m}\prod_{k=1}^{r_i}\left(1+\frac{1}{u-z_i^{(k)}}\right),\prod_{i=2}^{m}\prod_{k=1}^{r_i}\left(1+\frac{1}{u-z_i^{(k)}}\right),\dots,\prod_{k=1}^{r_m}\left(1+\frac{1}{u-z_m^{(k)}}\right),1,\dots,1,\\
\prod_{l=1}^{s_n}\left(1-\frac{1}{u-w_n^{(l)}}\right),\prod_{l=1}^{s_n}\left(1-\frac{1}{u-w_n^{(l)}}\right)\prod_{l=1}^{s_{n-1}}\left(1-\frac{1}{u-w_{n-1}^{(l)}}\right),\dots,\prod_{j=1}^{n}\prod_{l=1}^{s_j}\left(1-\frac{1}{u-w_j^{(l)}}\right)\Bigg). \tag{4.19}$$

Hence, there is a quotient of $Y(\mathfrak{gl}_n)\bigotimes_{i=1}^{m}\zeta_i^{\otimes r_i}\otimes\bigotimes_{j=1}^{n}\xi_j^{\otimes s_j}$ which is an irreducible highest weight representation of weight (4.19). As $Y(\mathfrak{gl}_n)\bigotimes_{i=1}^{m}\zeta_i^{\otimes r_i}\otimes\bigotimes_{j=1}^{n}\xi_j^{\otimes s_j}$ is a submodule of $\vec{E}(\vec{r},\vec{s},z,w)$, there is a subquotient (quotient of a submodule) of $\vec{E}(\vec{r},\vec{s},z,w)$ which is an irreducible highest weight representation of weight (4.19).

We have the criteria of Theorem 4.3.1 to see if is it a finite-dimensional representation. If $\lambda(u)$ denotes the weight (4.19), then $\lambda_k(u)/\lambda_{k+1}(u)$ is of the following form.

$$\frac{\lambda_k(u)}{\lambda_{k+1}(u)} = \prod_{k=1}^{r_i}\left(1+\frac{1}{u-z_i^{(k)}}\right) = \frac{\prod_{k=1}^{r_i}(u+1-z_i^{(k)})}{\prod_{k=1}^{r_i}(u-z_i^{(k)})}, \quad \text{or} \tag{4.20}$$

$$= \frac{\prod_{k=1}^{r_i}\left(1+\frac{1}{u-z_i^{(k)}}\right)}{\prod_{l=1}^{s_j}\left(1-\frac{1}{u-w_j^{(l)}}\right)} = \frac{\prod_{k=1}^{r_i}(u+1-z_i^{(k)})\prod_{l=1}^{s_j}(u-w_j^{(l)})}{\prod_{l=1}^{s_j}(u-1-w_j^{(l)})\prod_{k=1}^{r_i}(u-z_i^{(k)})}, \quad \text{or} \tag{4.21}$$

$$= \prod_{l=1}^{s_j}\left(1-\frac{1}{u-w_j^{(l)}}\right) = \frac{\prod_{l=1}^{s_j}(u-w_j^{(l)})}{\prod_{l=1}^{s_j}(u-1-w_j^{(l)})}. \tag{4.22}$$

We have explicit monic polynomials $P_k(u)$ such that

$$\frac{\lambda_k(u)}{\lambda_{k+1}(u)} = \frac{P_k(u+1)}{P_k(u)}.$$

For (4.20), $P_k(u) = \prod_{k=1}^{r_i}(u-z_i^{(k)})$, for (4.21), $P_k(u) = \prod_{l=1}^{s_j}(u-1-w_j^{(l)})\prod_{k=1}^{r_i}(u-z_i^{(k)})$, and for (4.22), $P_k(u) = \prod_{l=1}^{s_j}(u-1-w_j^{(l)})$. Hence $\vec{E}(\vec{r},\vec{s},z,w)$ has a subquotient which is a finite-dimensional irreducible representation of $Y(\mathfrak{gl}_n)$.

Conversely, $L(\lambda(u))$ is the finite-dimensional irreducible representation of $Y(\mathfrak{gl}_n)$ of weight $\lambda(u)$. Then by Theorem 4.3.1, there are monic polynomials $P_k(u)$ such that $\lambda_k(u)/\lambda_{k+1}(u) = P_k(u+1)/P_k(u)$. Which means that $\lambda_k(u)/\lambda_{k+1}(u)$ is of the form (4.20) (the three results (4.20),(4.21) and (4.22) are of the same form). Thus $\lambda(u)$ is of the form (4.19) and $L(\lambda(u))$ is a subquotient of a certain $\vec{E}(\vec{r},\vec{s},z,w)$. □

Chapter 5

Representations of $Y(\mathfrak{gl}_t)$

First of all, to be able to study them, we have to know what is a representation of $Y(\mathfrak{gl}_t)$ (or $Y(\mathfrak{g})$ for short), when t is not necessarily an integer. It cannot be a vector space on which $Y(\mathfrak{g})$ acts because $Y(\mathfrak{g})$ is not an algebra in the usual sense. In the last chapter, we saw that there was an embedding $E_{ij} \mapsto t_{ji}^{(1)}$ thanks to which every $Y(\mathfrak{gl}_N)$-module was also a $\mathfrak{gl}_N$-module. It is then logical to look for representations of $Y(\mathfrak{g})$ as objects of the category $\mathbf{Rep}(Gl_t)$. Hence we can give a definition similar to what exists for $\mathbf{Rep}(Gl_t)$, as seen in 2.3.3.

Definition 5.0.1. A *representation* of $Y(\mathfrak{g})$ is an object E of the category $\mathbf{Rep}(Gl_t)$, together with a morphism of $\mathrm{Hom}(Y \otimes E, E)$.

We cannot easily adapt the method we have seen in the last chapter to the general case of $Y(\mathfrak{g})$. The main reason is that we lack an equivalent for the notion of weight. $Y(\mathfrak{gl}_t)$ has only one type of generator, $T(u)$, so we cannot imitate the $\mathfrak{gl}_N$ case with (E_i, F_i, H_i)-type basis.

Nevertheless, there would be a way to introduce a highest weight theory on $\mathbf{Rep}(\mathfrak{gl}_t)$, via an inclusion of the type

$$S_t \ltimes \underbrace{(\mathbb{C}^*)^t}_{\in \mathbf{Rep}(S_t)} \subset Gl_t.$$

And then we would use the functors

$$\mathbf{Rep}(Y(\mathfrak{gl}_t)) \longrightarrow \mathbf{Rep}(Gl_t) \longrightarrow \mathbf{Rep}(S_t \ltimes (\mathbb{C}^*)^t)$$

The problem with this method is that the theory is not built yet on $\mathbf{Rep}(Gl_t)$, so it is not ready to be used in a more complex case. Hence it is not an approach we chose to investigate any further.

We are now going to consider different approaches to the problem, from giving explicit examples of representations of $Y(\mathfrak{g})$, to the beginning of a more detailed study.

5.1 Evaluation representations

We can easily obtain a family of representations of $Y(\mathfrak{g})$, but taking the pull-backs of the representations of $\mathfrak{g}$, hence elements of $\mathbf{Rep}(Gl_t)$, as in 4.2.1.

If we translate (4.2) in the matrix form, with the notations of 3.1.3, we obtain

$$T(u) = \sum_{i,j=1}^{N} e_{ij} \otimes t_{ij}(u) \mapsto 1 + \frac{\sigma}{u-z} = R(u-z) \tag{5.1}$$

As all these objects exist for $Y(\mathfrak{gl}_t)$, we can use the same definition, in a more general setting.

Definition 5.1.1. The *evaluation morphism* $\mathrm{ev}_z : Y(\mathfrak{gl}_t) \to U(\mathfrak{gl}_t)$ is defined by

$$T(u) \mapsto R(u-z). \tag{5.2}$$

As the Yang R-matrix R satisfies the Yang-Baxter equation (3.10), ev_z defines an algebra morphism.

Remark 5.1.2. $\sigma : V \otimes V \to V \otimes V$ can also be seen as a element of $\mathrm{Hom}_{\mathbf{Rep}(Gl_t)}(\mathbb{1}, V \otimes V^* \otimes V \otimes V^*) = \mathrm{Hom}(\mathbb{1}, \mathfrak{g} \otimes \mathfrak{g})$ hence, the images of the coefficients of $R(u-z)$ are indeed elements of $U(\mathfrak{gl}_t)$.

Definition 5.1.3. Let X be an object of $\mathbf{Rep}(Gl_t)$, and $z \in \mathbb{C}$. The *evaluation representation* $X(z)$ is the pullback $\mathrm{ev}_z^* X$ of X, via the evaluation morphism ev_z, to a representation of $Y(\mathfrak{g})$. We have morphisms

$$Y(\mathfrak{g}) \otimes X \xrightarrow{\mathrm{ev}_z} U(\mathfrak{gl}_t) \otimes X \to X. \tag{5.3}$$

If $X_1, \ldots, X_k$ are objects of $\mathbf{Rep}(Gl_t)$, and $z_1, \ldots, z_k$ are complex numbers, we can construct the tensor product $X_1(z_1) \otimes \cdots \otimes X_k(z_k)$, which is also a representation of $Y(\mathfrak{g})$,using the coproduct Δ associated to the Hopf algebra structure of $Y(\mathfrak{g})$.

Remark 5.1.4. Let us recall here that the two meanings of the notation $L(\lambda)$, first the indecomposable object of $\mathbf{Rep}(Gl_t)$ corresponding to the bipartition λ by (2.14), if λ is a partition, and then the irreducible highest weight representation of $\mathfrak{gl}_N$ of weight λ, if λ is a tuple $\lambda = (\lambda_1, \ldots, \lambda_N)$, actually refers to the same object, if $t = N$ is an integer, thanks to 2.4.5.

Hence, if λ be a bipartition of arbitrary size, and $L(\lambda)$ denotes the highest weight representation of $\mathfrak{gl}_N$ of weight $\lambda = (\lambda_1, \ldots, \lambda_N)$ then, as in the particular case of $\mathfrak{gl}_2$,the evaluation representation of $Y(\mathfrak{gl}_N)$ obtained via the composition of (4.2) and the the action of $\mathfrak{gl}_N$ is an irreducible highest weight representation of $Y(\mathfrak{gl}_N)$ of weight $\lambda(u) = (\lambda_1(u), \ldots, \lambda_N)$, where, for $i \in \{1, \ldots, N\}$

$$\lambda_i(u) = 1 + \frac{\lambda_i}{u-z} \quad \in \mathbb{C}[[u^{-1}]]. \tag{5.4}$$

We will write $L(\lambda)_z$ for this evaluation representation, and also more generally if $L(\lambda)$ is the indecomposable object of $\mathbf{Rep}(Gl_t)$ corresponding to the bipartition λ.

5.2 Two conjectures

We are now going to present two results which show that evaluation representations are the key to understand the classification of irreducible representations of the Yangian $Y(\mathfrak{g})$.

Unfortunately, these results will not be proven in this paper, but they could be ideas to expand the work that we started here.

5.2.1 First conjecture

Let $\lambda_1, \ldots, \lambda_k$ be bipartitions of arbitrary size, and $L(\lambda_1), \ldots, L(\lambda_k)$ the corresponding indecomposable objects of $\mathbf{Rep}(Gl_t)$. For $z_1, \ldots, z_k \in \mathbb{C}$, we are going to consider the tensor product of evaluation representations of indecomposables $L(\lambda_1)_{z_1} \otimes \cdots \otimes L(\lambda_k)_{z_k}$.

Conjecture 5.2.1. *For $\lambda_1, \ldots, \lambda_k$ bipartitions of arbitrary size, the representation of $Y(\mathfrak{g})$,*

$$L(\lambda_1)_{z_1} \otimes \cdots \otimes L(\lambda_k)_{z_k}$$

is irreducible for generic parameters values of $z_1, \ldots, z_k$.

This result was stated in [8]. It should not be too difficult to prove. We would first see that with the action of the loop algebra, it is already a irreducible module.

The question of knowing for exactly which values of $z_1, \ldots, z_k$ the $Y(\mathfrak{g})$-module $L(\lambda_1)_{z_1} \otimes \cdots \otimes L(\lambda_k)_{z_k}$ is irreducible is a much more difficult question.

This result is classic in the context of representations of the Yangian $Y(\mathfrak{gl}_N)$, where N is an integer, see [13] for a detailed proof.

5.2.2 Second conjecture

The second conjecture is the fact that Proposition 4.3.2 stays true in the case of the Yangian $Y(\mathfrak{g})$.

Conjecture 5.2.2. *Every finite-dimensional irreducible representation of $Y(\mathfrak{g})$ is isomorphic to a subquotient of a tensor product of evaluation representations, of the form*

$$\bigotimes_{i=1}^{m} \Lambda^i V(z_i^{(1)}) \otimes \cdots \otimes \Lambda^i V(z_i^{(r_i)}) \otimes \bigotimes_{j=1}^{n} \Lambda^j V^*(w_j^{(1)}) \otimes \cdots \otimes \Lambda^j V^*(w_j^{(s_j)}) \tag{5.5}$$

where $m, n \geq 0$ and $r_1, \ldots, r_m, s_1, \ldots, s_n \geq 0$ are integers, and $z_i^{(k)}, w_j^{(l)} \in \mathbb{C}$.

5.3 The invariant algebra $Y(\mathfrak{g})^{\mathfrak{g}}$

Another idea to study $Y(\mathfrak{g})$ and its representations is to look at the invariant algebra $Y(\mathfrak{g})^{\mathfrak{g}}$.

Definition 5.3.1. Let $Y(\mathfrak{g})^{\mathfrak{g}}$, or simply Y^{inv}, denote the invariant algebra $\text{Hom}_{\mathbf{Rep}(Gl_t)}(\mathbb{1}, Y(\mathfrak{g}))$.

Y^{inv} is a simpler object to study than $Y(\mathfrak{g})$ because it is an ordinary algebra, thanks to the definition of the hom-spaces of $\mathbf{Rep}(Gl_t)$.

5.3.1 Representations of the invariant algebra

The following Proposition explains how to link representations of Y and representations of Y^{inv}.

Proposition 5.3.2. *If Y acts on E then Y^{inv} acts on* $\text{Hom}(X, E)$*, for any another object X in* $\boldsymbol{Rep}(Gl_t)$.

Proof. Let E be a representation of Y, then there is a morphism $Y \otimes E \to E$. $\mathbf{Rep}(Gl_t)$ is a semisimple category, as seen in 2.5 (we consider the case where $t \notin \mathbb{Z}$ here), so E, as an object of $\mathbf{Rep}(Gl_t)$, can be uniquely written as the sum $E = \bigoplus_{\lambda \text{ bipartition}} L(\lambda) \otimes E_\lambda$, where E_λ are vectors spaces, all but finitely many of them are zero (actually, as in a more usual framework, $E_\lambda = \text{Hom}(L(\lambda), E)$). We have also $Y = \bigoplus L(\lambda') \otimes Y_{\lambda'}$.

Then there is a morphism

$$\bigoplus L(\lambda) \otimes L(\lambda') \otimes Y_\lambda \otimes E_{\lambda'} \to \bigoplus L(\lambda'') \otimes E_{\lambda''}. \tag{5.6}$$

If we restrict at the source to $L(\lambda) = \mathbb{1}$, then $Y_\lambda = \text{Hom}(\mathbb{1}, Y) = Y^{\text{inv}}$, and we get a morphism

$$\bigoplus L(\lambda) \otimes Y^{\text{inv}} \otimes E_\lambda \to \bigoplus L(\lambda') \otimes E_{\lambda'}. \tag{5.7}$$

$L(\lambda)$ and $L(\lambda')$ being irreducible objects, according to Schur's Lemma, there are morphisms $L(\lambda) \to L(\lambda')$ if and only if $\lambda = \lambda'$, and if so the morhism is a homothety. Then for every bipartition λ, we have a morphism

$$Y^{\text{inv}} \otimes E_\lambda \to E_\lambda \tag{5.8}$$

Hence we have shown that Y^{inv} acts on $\text{Hom}(L(\lambda), E)$, for every bipartition λ. As every object X can be written $X = \bigoplus L(\lambda) \otimes X_\lambda$, we can easily build an action

$$Y^{\text{inv}} \otimes \text{Hom}(X, E) \to \text{Hom}(X, E). \tag{5.9}$$

□

Remark 5.3.3. In particular, Y^{inv} acts on $\text{Hom}(\mathbb{1}, E)$.

5.3.2 Generators of the invariant algebra

Let us recall that in the Yangian $Y(\mathfrak{g})$, we have the generating function

$$T(u) = 1 + T_0 u^{-1} + T_1 u^{-2} + \cdots, \tag{5.10}$$

with $T_i \in \mathrm{Hom}(\mathbb{1}, (V^* \otimes V) \otimes Y) = \mathrm{Hom}(\mathbb{1}, \mathfrak{g}^* \otimes Y) \cong (\mathfrak{g} \otimes Y)^{\mathrm{inv}}$. Indeed, $\mathfrak{g}^* \cong \mathfrak{g}$ are clearly isomorphic in the category.

Let $r \in \mathbb{N}$ and $i_1, \ldots, i_r \geq 0$ be a r-tuple of integers. We will use the following notation.

$$C_{i_1 \cdots i_r} := \mathrm{Tr}_V(T_{i_1} \cdots T_{i_r}) \tag{5.11}$$

$\mathrm{Tr}_V : \mathfrak{g} \to \mathbb{C}$ is a morphism of the category $\mathbf{Rep}(Gl_t)$, represented by . Here we only apply Tr_V to the first component, in order to get an element of Y^{inv}.

Remark 5.3.4. It is well known that in the case of regular vector spaces, using the definition $\mathfrak{g} = V \otimes V^*$, the trace map can be simply written

$$\begin{array}{rrl} \mathrm{Tr:} & V \otimes V^* \to & \mathbb{C} \\ & v \otimes \phi \mapsto & \phi(v). \end{array} \tag{5.12}$$

$C_{i_1 \cdots i_r}$ is called a *cyclic word*. It can also be written using the generating functions

$$C_{i_1 \cdots i_r} = \mathrm{Tr}_V(\mathrm{coeff}_{u_1^{-i_1-1} \cdots u_r^{-i_r-1}} T(u_1) \cdots T(u_r)). \tag{5.13}$$

Theorem 5.3.5. *The cyclic words $C_{i_1 \cdots i_r}$, for $i_1, \ldots, i_r \geq 0$ generate Y^{inv}.*

It works exactly the same way as in a more classical setting of polynomial invariants of matrices, more precisely

Lemma 5.3.6. *(Procesi's Theorem) The ring of invariants of k $n \times n$ matrices, $A_1, \ldots, A_k$ is generated by $\mathrm{tr}(A_{i_1} \cdots A_{i_r})$, $A_{i_1} \cdots A_{i_r}$ running over all possible (noncommutative) monomials.*

Proof. See [14]. □

With the same reasoning as before, the space of $Gl(V)$-invariant vectors of $V^{\otimes r} \otimes V^{*\otimes r}$ is the space of linear maps $V^{*\otimes r} \otimes V^{\otimes r} \to \mathbb{C}$ invariant under $Gl(V)$, which identifies naturally to the space of $Gl(V)$-invariants of $\mathrm{End}(V)^{\otimes r}$. From Schur-Weyl duality, we know that this space is generated by elements indexed by permutations of S_r. The isomorphism can be easily written, as follows.

$$\begin{array}{rccl} \Phi: & S_k & \to & \{V^{*\otimes k} \otimes V^{\otimes k} \to \mathbb{C}\}^{Gl(V)}, \\ & \sigma & \mapsto & \left(\phi_1 \otimes \ldots \otimes \phi_k \otimes v_1 \otimes \ldots \otimes v_k \mapsto \prod_{i=1}^{k} \phi_{\sigma(i)}(v_i)\right). \end{array}$$

We are going to see how the images of this morphism can be obtain by products of traces, as in 5.3.6. First of all, if we work in $\mathfrak{g}^* = V^* \otimes V$, the product is

$$(\phi \otimes v) \cdot (\psi \otimes w) = \phi \otimes \psi(v) w. \tag{5.14}$$

Hence,

$$\begin{aligned} (\phi_1 \otimes v_1) \cdot (\phi_2 \otimes v_2) \cdot \cdots \cdot (\phi_k \otimes v_k) &= (\phi_1 \otimes \phi_2(v_1) v_2) \cdot (\phi_3 \otimes v_3) \cdot \cdots \cdot (\phi_k \otimes v_k) \\ &= (\phi_1 \otimes \phi_2(v_1)\phi_3(v_2) v_3) \cdot \cdots \cdot (\phi_k \otimes v_k) \\ &= \phi_1 \otimes \phi_2(v_1)\phi_3(v_2) \cdots \phi_k(v_{k-1}) v_k \end{aligned}$$

As $\mathrm{tr}(\phi \otimes v) = \phi(v)$, $\quad \mathrm{tr}((\phi_{i_1} \otimes v_{i_1}) \cdots (\phi_{i_r} \otimes v_{i_r})) = \phi_{i_1}(v_{i_r})\phi_{i_2}(v_{i_2}) \cdots \phi_{i_r}(v_{i_{r-1}})$.

Thus, let σ be a permutation of S_k, if we decompose σ in disjoint cycles

$$\sigma = (i_1^1 i_2^1 \cdots i_{r_1}^1)(i_1^2 i_2^2 \cdots i_{r_2}^2) \cdots (i_1^m i_2^m \cdots i_{r_m}^m),$$

Then,

$$\Phi(\sigma)(\phi_1 \otimes \ldots \otimes \phi_k \otimes v_1 \otimes \ldots v_k) = \mathrm{tr}((\phi_{i_1^1} \otimes v_{i_1^1}) \cdots (\phi_{i_{r_1}^1} \otimes v_{i_{r_1}^1}))\, \mathrm{tr}((\phi_{i_1^2} \otimes v_{i_1^2}) \cdots (\phi_{i_{r_2}^2} \otimes v_{i_{r_2}^2})) \cdots \\ \mathrm{tr}((\phi_{i_1^m} \otimes v_{i_1^m}) \cdots (\phi_{i_{r_m}^m} \otimes v_{i_{r_m}^m})) \quad (5.15)$$

We have shown that with products of traces of products, we can obtain all the invariants. The Theorem 5.3.5 follows from the same reasoning.

5.4 Action of the invariant algebra

What we want to do now is to compute values of the action of the generators $C_{i_1 \cdots i_r}$ on certain vector spaces.

5.4.1 Action on an evaluation representation

By the evaluation representations, $T(u)$ acts on $V(z)$ as

$$T(u)_{|V(z)} = R(u - z) = 1 + \frac{\sigma}{u - z}. \quad (5.16)$$

If we look at the classical case, $\sigma = \sum e_{ij} \otimes e_{ji}$, as an element of $\mathcal{U}(\mathfrak{gl}_N)$, acts as the identity on $V(z)$, the same goes in the complex case.

If we generalize this result we get, for $n \geq 1$,

$$\mathrm{Tr}(T(u_1)T(u_2) \cdots T(u_n))_{|V(z)} = \mathrm{tr}\left(\left(1 + \frac{\sigma}{u_1}\right)\left(1 + \frac{\sigma}{u_2}\right) \cdots \left(1 + \frac{\sigma}{u_n}\right)\right), \quad (5.17)$$

with $\sigma^2 = 1$, $\mathrm{tr}(1) = t$ and $\mathrm{tr}(\sigma) = 1$.

Remark 5.4.1. Recall that we are only taking the trace on the first component, hence if we compute $\mathrm{tr}(\sigma)$ in the classical case, we get

$$\mathrm{tr}(\sigma) = \mathrm{tr}\left(\sum e_{ij} \otimes e_{ji}\right) = \sum e_{ii} = 1. \quad (5.18)$$

We will admit that the result in the same in our situation (see below for details on how to compute traces of permutations).

Proposition 5.4.2. *We have the following result, for V the tautological object of **Rep**(Gl_t) and $z \in \mathbb{C}$.*

$$Tr(T(u_1)T(u_2) \cdots T(u_n))_{|V(z)} = \frac{1}{2}\left((t+1)\prod_{i=1}^{n}\left(1 + \frac{1}{u_i}\right) + (t-1)\prod_{i=1}^{n}\left(1 - \frac{1}{u_i}\right)\right) \quad (5.19)$$

Proof.

$$\begin{aligned}\mathrm{Tr}(T(u_1)T(u_2)\cdots T(u_n))_{|V(z)} &= \mathrm{tr}\left(\left(1+\frac{\sigma}{u_1}\right)\left(1+\frac{\sigma}{u_2}\right)\cdots\left(1+\frac{\sigma}{u_n}\right)\right)\\ &= \mathrm{tr}\left(1+\sigma\left(\frac{1}{u_1}+\cdots+\frac{1}{u_n}\right)+\frac{1}{u_1u_2}+\frac{1}{u_1u_3}+\cdots\frac{1}{u_{n-1}u_n}+\right.\\ &\left.\sigma\left(\frac{1}{u_1u_2u_3}+\cdots+\frac{1}{u_{n-2}u_{n-1}u_n}\right)+\cdots+\sigma^n\left(\frac{1}{u_1u_2\cdots u_n}\right)\right)\\ &= t\left(\sum_{k=0}^{[n/2]}\sum_{1\le i_1<\cdots<i_{2k}}\frac{1}{u_1\cdots u_{2k}}\right)+\left(\sum_{l=1}^{[n+1/2]}\sum_{1\le i_1<\cdots<i_{2l-1}}\frac{1}{u_1\cdots u_{2l-1}}\right)\end{aligned}$$

Let us give name to these sums

$$S_e=\sum_{k=0}^{[n/2]}\sum_{1\le i_1<\cdots<i_{2k}}\frac{1}{u_1\cdots u_{2k}}\quad\text{and}\quad S_o=\sum_{l=1}^{[n+1/2]}\sum_{1\le i_1<\cdots<i_{2l-1}}\frac{1}{u_1\cdots u_{2l-1}}.$$

We have

$$\prod_{i=1}^{n}\left(1+\frac{1}{u_i}\right)=S_e+S_o\quad\text{and}\quad\prod_{i=1}^{n}\left(1-\frac{1}{u_i}\right)=S_e-S_o$$

Hence

$$\begin{aligned}\mathrm{Tr}(T(u_1)T(u_2)\cdots T(u_n))_{|V(z)} = tS_e+S_o &= t\left(\frac{1}{2}\left(\prod_{i=1}^{n}\left(1+\frac{1}{u_i}\right)+\prod_{i=1}^{n}\left(1-\frac{1}{u_i}\right)\right)\right)+\\ &\frac{1}{2}\left(\prod_{i=1}^{n}\left(1+\frac{1}{u_i}\right)-\prod_{i=1}^{n}\left(1-\frac{1}{u_i}\right)\right)\\ &=\frac{1}{2}\left((t+1)\prod_{i=1}^{n}\left(1+\frac{1}{u_i}\right)+(t-1)\prod_{i=1}^{n}\left(1-\frac{1}{u_i}\right)\right)\end{aligned}$$

□

Remark 5.4.3. The result is a scalar, but it was expected. As we saw in 5.3.2, invariant functions are generated by pairings of the V and V^* (and so by permutations). But there is only one pairing of $V\otimes V^*$ (or equivalently, one permutation of the set of one element).

Also, $V(z)$ being an irreducible representation of $Y(\mathfrak{g})$ (as V in an irreducible representation of $\mathbf{Rep}(Gl_t)$), by Schur's Lemma, it was expected that the action would be scalar.

The action on one $V(z)$, being scalar, is not the most interesting case. The situation becomes more interesting when we look at the action on a tensor product of evaluation representations, thanks to the coproduct Δ.

5.4.2 Action on a tensor product of evaluation representations

Let us see how the invariant algebra acts on a tensor product of evaluation representations, of the form $V(z_1)\otimes V(z_2)\otimes\cdots\otimes V(z_n)$.

$$T(u)_{|V(z_1)\otimes V(z_2)\otimes\cdots\otimes V(z_n)}=R^{12}(u-z_1)R^{13}(u-z_2)\cdots R^{1(n+1)}(u-z_n)\qquad(5.20)$$

Indeed, the coproduct on $Y(\mathfrak{g})$ is defined by $\Delta(T(u)) = T^{12}(u)T^{13}(u)$.

Hence the action of the invariant algebra is of the form

$$\operatorname{tr}\left(T(u_1)T(u_2)\cdots T(u_r)\right)_{|V(z_1)\otimes V(z_2)\otimes\cdots\otimes V(z_n)} =$$
$$\operatorname{tr}\left(\prod_{i=2}^{n+1} R^{1i}(u_1 - z_{i-1}) \prod_{i=2}^{n+1} R^{1i}(u_2 - z_{i-1}) \cdots \prod_{i=2}^{n+1} R^{1i}(u_r - z_{i-1})\right) = \operatorname{tr}\left(\prod_{i=1}^{r}\prod_{j=1}^{n}\left(1 + \frac{\sigma^{1(j+1)}}{u_i - z_j}\right)\right)$$

where $\sigma^{1(j+1)}$ denotes the permutation between the first and the $(j+1)$th component.

Remark 5.4.4. The products written here are **ordered**. As we are computing products of permutations, the order matters.

5.4.3 Traces of permutations

As we saw above, we needed to know how to compute the traces of different permutations, in order to get a result for the action of Y^{inv} on a tensor product of evaluation representations.

Example 5.4.5. First, let us look at a simple example. Let θ be the permutation $(\,1\,2\,3\,)$,

$$\theta(x \otimes y \otimes z) = z \otimes x \otimes y. \tag{5.21}$$

If we look at the classical case, this permutation would be written $\theta = \sum e_{ij} \otimes e_{jk} \otimes e_{ki}$.

Hence $\operatorname{tr}\theta = \sum e_{ik} \otimes e_{ki} = \sigma$.

Example 5.4.6. Now we are going to look at a more complex example, $\theta = (\,1\,2\,4\,)(\,3\,5\,)$, which sends $x \otimes y \otimes z \otimes u \otimes v$ to $u \otimes x \otimes v \otimes y \otimes z$. It can be written

$$\theta = \sum e_{ij} \otimes e_{jm} \otimes e_{kn} \otimes e_{mi} \otimes e_{nk}. \tag{5.22}$$

Hence, $\operatorname{tr}\theta = \sum e_{im} \otimes e_{kn} \otimes e_{mi} \otimes e_{nk} = (\,1\,3\,)(\,2\,4\,)$.

As we see that the first element plays a different part, we are going to denote it 0 instead of 1. Hence

$$\operatorname{tr}(\,0\,1\,2\,) = (\,1\,2\,) \quad \text{and} \quad \operatorname{tr}((\,0\,1\,3\,)(\,2\,4\,)) = (\,1\,3\,)(\,2\,4\,).$$

We begin to see the result appearing.

Proposition 5.4.7. *Let θ be a permutation of $\{0, 1, \ldots, n\}$. We write the decomposition of θ in disjoint cycles, beginning by the cycle containing 0.*

$$\theta = (\,0\,i_1\,\ldots\,i_p\,)(\,j_1\,\ldots\,j_q\,)\cdots(\,k_1\,\ldots\,k_r\,).$$

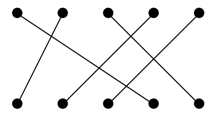
*Then, if pgeq*1 *the trace of θ is*

$$\operatorname{tr}\theta = (\,i_1\,\ldots\,i_p\,)(\,j_1\,\ldots\,j_q\,)\cdots(\,k_1\,\ldots k_r\,). \tag{5.23}$$

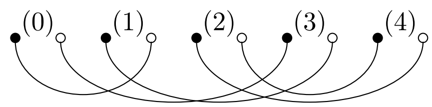
And if $p = 0$,

$$\operatorname{tr}\theta = t(\,j_1\,\ldots\,j_q\,)\cdots(\,k_1\,\ldots k_r\,). \tag{5.24}$$

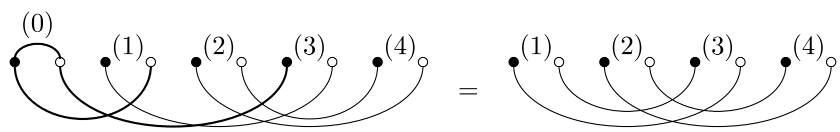
The best way to understand this result is to look at some figures. Let us represent the permutation $(\,0\,1\,3\,)(\,2\,4\,)$ with the same types of schemes as in 2.1.1

$(\,0\,1\,3\,)(\,2\,4\,) =$ 

This figure can be translated as a $(\mathbb{1}, \bullet \circ \bullet \circ \bullet \circ \bullet \circ \bullet\circ)$-diagram, hence as an invariant on $(V \otimes V^*)_{(0)} \otimes (V \otimes V^*)_{(1)} \otimes (V \otimes V^*)_{(2)} \otimes (V \otimes V^*)_{(3)} \otimes (V \otimes V^*)_{(4)}$.

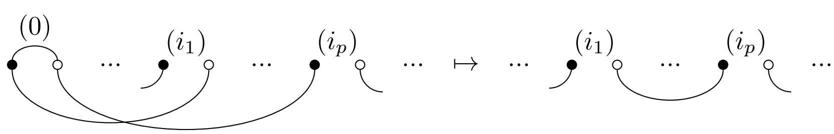
$(\,0\,1\,3\,)(\,2\,4\,) =$ 

Then, as we said in 5.3.2, the trace on $V \otimes V^*$ is the morphism  . Hence, by taking the trace on the first component, we are going to get a $(\mathbb{1}, \bullet \circ \bullet \circ \bullet \circ \bullet\circ)$-diagram, or an invariant on $(V \otimes V^*)_{(1)} \otimes (V \otimes V^*)_{(2)} \otimes (V \otimes V^*)_{(3)} \otimes (V \otimes V^*)_{(4)}$. We obtain

$\operatorname{tr}(\,0\,1\,3\,)(\,2\,4\,) =$ 

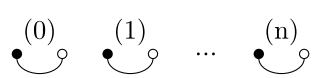
We have the same result as in the classical case, that $\operatorname{tr}(\,0\,1\,3\,)(\,2\,4\,) = (\,1\,3\,)(\,2\,4\,)$.

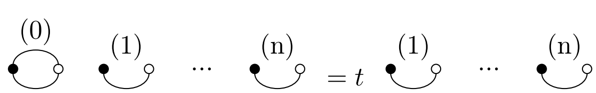
Proof. From these figures, it is clear that the cycles where 0 does not appear do not change under the trace, so we can suppose that θ is a $(p+1)$-cycle, $\theta = (\,0\,i_1\,\ldots\,i_p\,)$. First, let us study the case when $p \geq 1$. We follow the same reasoning as before



Hence, i_p is send to i_1 and the rest is unchanged. Thus $\operatorname{tr}(\,0\,i_1\,\ldots\,i_p\,) = (\,i_1\,\ldots\,i_p\,)$.

Let us now study the case when $p = 0$, i.e. when 0 is unchanged by the permutation. As we saw above, the other cycles are not disrupted by taking the trace on the first component, thus we can suppose that θ is the identity.

$\theta =$ 

$\operatorname{tr}\theta =$ 

Thus $\operatorname{tr}(\mathrm{id}_{\{0,1,\ldots,n\}}) = t\,\mathrm{id}_{\{1,\ldots,n\}}$.

□

5.4.4 Action on the evaluation of an exterior product representation

To study a simple case, we are going to compute the action of specific cyclic words $C_{i_1 i_2} = \operatorname{tr}_V(T_{i_1} T_{i_2})$ on an evaluation representation $\Lambda^n V(z)$.

Moreover, according to the Second Conjecture 5.2.2, these are the types of representations which generates the finite-dimensional irreducible representations, together with $\Lambda^n V^*(z)$, hence it is important to know how the invariant algebra Y^{inv} acts on them.

Theorem 5.4.8.

$$\operatorname{tr}(T(u)T(v))_{|\Lambda^n V(z)} = t + 2 - \left(1 - \frac{1}{u-z}\right)^n - \left(1 - \frac{1}{v-z}\right)^n + \sum_{k,l=1}^{n} \frac{(-1)^{k+l}}{u^k v^l}\left((t+1)\binom{n}{k+l-1} - \binom{n}{k}\binom{n}{l}\right)$$

Proof. We are going to use the same notations as in Section 5.4.3, by denoting by 0 the first elements of the permuted set for example. From Section 5.4.2, we know that

$$\operatorname{tr}\left(T(u)T(v)\right)_{|V(z)^{\otimes n}} = \operatorname{tr}\left(\prod_{i=1}^{n}\left(1 + \frac{(\,0\,i\,)}{u-z}\right)\prod_{j=1}^{n}\left(1 + \frac{(\,0\,j\,)}{v-z}\right)\right). \tag{5.25}$$

As we are looking at an action on an exterior product, only the signatures of the resulting permutations will matter. And, as we have product of transpositions, we only need to keep track of the parity of the number of transpositions we multiply.

As we saw in Section 5.4.3, if θ is permutation leaving 0 unchanged, the resulting permutation will have the same signature, and a factor t will appear. If 0 is permuted though, the signature will change. Hence, the question is to know which permutation will stabilize 0, and which will not, and count them.

First of all,

$$\operatorname{tr}\left(T(u)T(v)\right)_{|V(z)^{\otimes n}} = \operatorname{tr}\left(\left(\sum_{k=0}^{n}\frac{1}{(u-z)^k}\sum_{1\le i_1<\cdots<i_k\le n}(\,0\,i_k\,\cdots\,i_1\,)\right)\left(\sum_{l=0}^{n}\frac{1}{(v-z)^l}\sum_{1\le j_1<\cdots<j_l\le n}(\,0\,j_l\,\cdots\,j_1\,)\right)\right)$$

As, $(\,0\,i_1\,)(\,0\,i_2\,)\cdots(\,0\,i_k\,) = (\,0\,i_k\,\cdots\,i_1\,)$, for $1 \le i_1 < \cdots < i_k \le n$, and the same goes for the $(j_m)_{1\le m\le l}$. Hence,

$$\operatorname{tr}\left(T(u)T(v)\right) = \sum_{k,l=0}^{n}\frac{1}{(u-z)^k(v-z)^l}\operatorname{tr}\left(\sum_{1\le i_1<\cdots<i_k\le n}(\,0\,i_k\,\cdots\,i_1\,)\sum_{1\le j_1<\cdots<j_l\le n}(\,0\,j_l\,\cdots\,j_1\,)\right) \tag{5.26}$$

A product of two permutations of the form $(\,0\,i_k\,\cdots\,i_1\,)(\,0\,j_l\,\cdots\,j_1\,)$, with $1 \le i_1 < \cdots < i_k \le n$ and $1 \le j_1 < \cdots < j_l \le n$ leaves 0 unchanged if and only if

- either $k = l = 0$, and the result is the identity,
- or k **and** l are positive, and $j_l = i_1$.

Hence, if $k, l \ge 1$, the products of permutations $(\,0\,i_k\,\cdots\,i_1\,)(\,0\,j_l\,\cdots\,j_1\,)$ leaving 0 unchanged are in bijection with the sequences $1 \le j_1 < \cdots < j_l = i_1 < \cdots i_k \le n$, of length $k+l-1$ in $\{1,\ldots,n\}$. Thus their number is $\binom{n}{k+l-1}$.

The total number of products of permutations $(\,0\,i_k\,\cdots\,i_1\,)(\,0\,j_l\,\cdots\,j_1\,)$ is $\binom{n}{k}\binom{n}{l}$. Recall that if $(\,0\,i_k\,\cdots\,i_1\,)(\,0\,j_l\,\cdots\,j_1\,)$ leaves 0 unchanged, then

$$\operatorname{tr}((\,0\,i_k\,\cdots\,i_1\,)(\,0\,j_l\,\cdots\,j_1\,)) = \epsilon((\,0\,i_k\,\cdots\,i_1\,)(\,0\,j_l\,\cdots\,j_1\,)) = t(-1)^{k+l}. \tag{5.27}$$

And if $(\,0\,i_k\,\cdots\,i_1\,)(\,0\,j_l\,\cdots\,j_1\,)$ disrupt 0, then

$$\operatorname{tr}((\,0\,i_k\,\cdots\,i_1\,)(\,0\,j_l\,\cdots\,j_1\,)) = -\epsilon((\,0\,i_k\,\cdots\,i_1\,)(\,0\,j_l\,\cdots\,j_1\,)) = (-1)^{k+l-1}. \tag{5.28}$$

Thus,

$$\operatorname{tr}(T(u)T(v)) = \sum_{k,l=1}^{n} \frac{1}{(u-z)^k (v-z)^l} \left(t(-1)^{k+l} \binom{n}{k+l-1} \right.$$
$$\left. +(-1)^{k+l-1} \left(\binom{n}{k}\binom{n}{l} - \binom{n}{k+l-1} \right) \right) + \sum_{k=1}^{n} \frac{1}{(u-z)^k} \operatorname{tr} \left(\sum_{1\le i_1<\cdots<i_k\le n} (\, 0\, i_k\, \cdots\, i_1\,) \right)$$
$$+ \sum_{l=1}^{n} \frac{1}{(v-z)^l} \operatorname{tr} \left(\sum_{1\le j_1<\cdots<j_l\le n} (\, 0\, j_l\, \cdots\, j_1\,) \right) + \operatorname{tr}(\mathrm{id})$$

But

$$t(-1)^{k+l}\binom{n}{k+l-1} + (-1)^{k+l-1}\left(\binom{n}{k}\binom{n}{l} - \binom{n}{k+l-1} \right) =$$
$$(-1)^{k+l-1}\left(\binom{n}{k+l-1}(t+1) - \binom{n}{k}\binom{n}{l} \right)$$

And

$$\sum_{k=1}^{n} \frac{1}{(u-z)^k} \operatorname{tr} \left(\sum_{1\le i_1<\cdots<i_k\le n} (\, 0\, i_k\, \cdots\, i_1\,) \right) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(u-z)^k}\binom{n}{k} = -\sum_{k=1}^{n} \binom{n}{k}\left(\frac{-1}{u-z}\right)^k$$
$$= -\left(\left(1 - \frac{1}{u-z}\right)^n - 1 \right) = 1 - \left(1 - \frac{1}{u-z}\right)^n.$$

The same goes for $\sum\limits_{l=1}^{n} \frac{1}{(v-z)^l} \operatorname{tr}\left(\sum\limits_{1\le j_1<\cdots<j_l\le n} (\, 0\, j_l\, \cdots\, j_1\,) \right)$, and we have the result we predicted.

□

Remark 5.4.9. The exact same reasoning could be used to compute $\operatorname{tr}(T(u_1)T(u_2)\cdots T(u_r))_{|\Lambda^n V(z)}$, but we would need the number of products of r permutations of the form $(\, 0\, i_k\, \cdots\, i_1\,)(\, 0\, j_l\, \cdots\, j_1\,)$, with $1 \le i_1 < \cdots < i_k \le n$ leaving 0 unchanged, the rest of the arguments are the same.

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