

# Predator–Prey Dynamics on Infinite Trees: A Branching Random Walk Approach

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**Abstract** We are interested in predator–prey dynamics on infinite trees, which can informally be seen as particular two-type branching processes where individuals may die (or be infected) only after their parent dies (or is infected). We study two types of such dynamics: the chase–escape process, introduced by Kordzakhia with a variant by Bordenave who sees it as a rumor propagation model, and the birth-and-assassination process, introduced by Aldous and Krebs. We exhibit a coupling between these processes and branching random walks killed at the origin. This sheds new light on the chase–escape and birth-and-assassination processes, which allows us to recover by probabilistic means previously known results and also to obtain new results. For instance, we find the asymptotic behavior of the tail of the number of infected individuals in both the subcritical and critical regimes for the chase–escape process and show that the birth-and-assassination process ends almost surely at criticality.

**Keywords** Chase–escape process · Birth-and-assassination process · Coupling · Killed branching random walks

**Mathematic Subject Classifications (2010)** 60J80 · 60J10 · 60F05

## 1 Introduction

We study the asymptotic behavior of two predator–prey dynamics on infinite trees. Let us first give an informal description of the chase–escape process on regular trees, which has been introduced by Kordzakhia [13]. Vertices can be of three types: predators, preys or vacant. At fixed rate  $\lambda > 0$ , preys may only spread to vacant nearest neighbors

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(when a prey spreads to a vacant neighbor, both vertices are then preys), while at fixed rate 1, predators may only spread to either vacant or prey nearest neighbors. The evolution starts with one predator at the root and one neighboring prey. Kordzakhia [13] identified the critical value of  $\lambda = \lambda_c$  that allows the preys to survive with positive probability. Later, a variant has been considered by Bordenave [6], where this time predators may only spread to nearest neighbors occupied by preys. In this context, for a certain class of trees, Bordenave [7] extended Kordzakhia's result, estimated the probability of extinction of the preys for  $\lambda > \lambda_c$ , and studied the moments of the final total number of predators.

Let us now informally describe the birth-and-assassination process, which was introduced by Aldous and Krebs [3] and is a system of evolving individuals. During its lifetime, each individual produces offspring at fixed rate  $\mu > 0$ . In addition, each individual  $u$  is equipped with a random timer of  $\mathcal{K}_u$  units of times, where the positive random variables  $(\mathcal{K}_u)$  are all independent identically distributed. The individual's timer is triggered as soon as its parent dies, and an individual dies when its timer reaches 0. The evolution starts with one individual with its timer triggered. Aldous and Krebs give the image of a finite collection of clans (which are the connected components of the living individuals in the genealogical trees) where only the heads of clans can be killed. Under certain conditions on  $\mathcal{K}_u$ , Aldous and Krebs identified the critical value of  $\mu = \mu_c$  that allows the evolution to survive indefinitely with positive probability (the motivation of Aldous and Krebs was to analyze a scaling limit of a queueing process with blocking, arising in database processing [19]). Later, Bordenave [6, 7] studied the birth-and-assassination process when  $\mathcal{K}_u$  is an exponential random variable, and Bordenave proved in particular that the evolution stops at  $\mu = \mu_c$  with probability 1.

In this work, we exhibit a coupling of these two models with branching random walks killed at the barrier 0, thus shedding new light on these models and giving an interesting application of the theory of killed branching random walks which do not have independent displacements. In the case of the chase–escape model on supercritical Galton–Watson trees, this enables us to make use of recent powerful results concerning killed branching random walks [2] and extend (under slightly more restrictive conditions on the structure of the tree) several results of Bordenave obtained in an analytic way and also to establish new results by probabilistic arguments. In particular, for  $\lambda \leq \lambda_c$ , we find the asymptotic behavior of the tail of the final number of predators. Bordenave [7] observed that the chase–escape process exhibits heavy-tail phenomena similar to those appearing in the Brunet–Derrida model of branching random walk killed below a linear barrier. Our coupling shows that this is not a coincidence. In the case of the birth-and-assassination process, we also show that in general case the evolution stops at  $\mu = \mu_c$  with probability 1, without assuming that  $\mathcal{K}_u$  is an exponential random variable.

### 1.1 The Chase–Escape Process

Following Bordenave [7], we now give a formal definition of the chase–escape process. For convenience of notation, we shall define it as a susceptible–infected–

recovered (SIR) dynamics, where preys (resp. predators and vacant sites) are infected (resp. recovered and susceptible) sites. As in [6], one may also think of susceptible/infected/recovered individuals as normal individuals/individuals propagating a rumor/individuals trying to scotch it.

Let  $G = (V, E)$  be a locally finite connected graph. Set  $\mathfrak{X} = \{S, I, R\}^V$  and for every  $v \in V$ , let  $I_v, R_v : \mathfrak{X} \rightarrow \mathfrak{X}$  be the maps defined by  $(I_v(x))_u = (R_v(x))_u = x_u$  if  $u \neq v$  and  $(I_v(x))_v = I$  and  $(R_v(x))_v = R$  with  $x = (x_u)_{u \in V}$ . The chase–escape process of infection intensity  $\lambda > 0$  is a Markov process taking values in  $\mathfrak{X}$  with transition rates

$$\begin{aligned} Q(x, I_v(x)) &= \lambda \cdot \mathbb{1}_{\{x_v=S\}} \cdot \sum_{\{u,v\} \in E} \mathbb{1}_{\{x_u=I\}}, & Q(x, R_v(x)) \\ &= \mathbb{1}_{\{x_v=I\}} \cdot \sum_{\{u,v\} \in E} \mathbb{1}_{\{x_u=R\}} & (v \in V, x \in \mathfrak{X}). \end{aligned}$$

This means that infected vertices spread with rate  $\lambda$  to neighboring susceptible vertices and that recovered vertices spread with rate 1 to neighboring infected vertices. Note that up to a time change there is no loss in generality in supposing that recovered vertices spread at rate 1. This dynamics differs from the classical SIR epidemics model, where infected vertices recover at a fixed rate (not depending on their neighborhood). Earlier, Kordzakhia [13] has considered a similar model where recovered vertices may spread with rate 1 to neighbors who are either infected or recovered. We say that the chase–escape process gets extinct if there exists a (random) time at which there are no more infected vertices.

If there are no recovered vertices present in the beginning, this dynamics is the so-called Richardson’s model [18]. Let us also mention that Häggström and Pemantle [10] and Kordzakhia and Lalley [14] have studied an extension of Richardson’s model with two species, in which infected and recovered vertices may only spread to susceptible vertices (in particular, infected and recovered vertices never change states). The chase–escape process is a variant of the famous and extensively studied Daley–Kendall model [8] for rumor propagation (in which, in addition, an infected individual may become recovered if it enters in contact with another infected individual or, in other words, when two individuals spreading the rumor meet, one of them stops spreading it). See also [6] for a directed version of the chase–escape process called the rumor-scotching process, and [15] for a study of the chase–escape process on large complete graphs.

In this work, we will be interested in the chase–escape process on (rooted) trees. To describe the initial condition which we will use, we introduce the following notation. First, let  $\mathcal{T}$  be a tree. If  $\mathcal{A} \subset \mathcal{T}$  is a subset of vertices and  $x \geq 0$ , we denote by  $\mathbf{C}(\mathcal{T}, \mathcal{A}, x)$  the chase–escape process on  $\mathcal{T}$  starting with the vertices of  $\mathcal{A}$  being infected and all the other vertices being susceptible, and with a recovered vertex attached to the root of  $\mathcal{T}$  (denoted by  $\emptyset$ ) at time  $x$ . To simplify, we shall say that  $\mathbf{C}(\mathcal{T}, \mathcal{A}, x)$  is the chase–escape process with initial condition  $\mathcal{A}$  and delay  $x$ . In particular, if  $\overline{\mathcal{T}}$  is the tree obtained from  $\mathcal{T}$  by adding a new parent to the root of  $\mathcal{T}$  and by rooting  $\overline{\mathcal{T}}$  at this new vertex, notice that the chase–escape process  $\mathbf{C}(\mathcal{T}, \{\emptyset\}, 0)$  may be seen as the chase–escape starting with the root of  $\overline{\mathcal{T}}$  being recovered, its child

being infected and all the other vertices being susceptible. This is the initial condition considered in [7].

### 1.2 Coupling the Chase–Escape Process with Branching Random Walks

We now introduce some background concerning one-dimensional discrete branching random walks on the real line  $\mathbb{R}$ . Let  $\mathcal{T}$  be a tree rooted at  $\emptyset$  and  $x \in \mathbb{R}$ . If  $u \in \mathcal{T}$ , let  $[[\emptyset, u]]$  denote the vertices belonging to the shortest path connecting  $\emptyset$  to  $u$ , set  $]\emptyset, u] = [[\emptyset, u] \setminus \{\emptyset\}$  and let  $|u| = \#]\emptyset, u]$  be the generation of  $u$ . Let  $(\xi_u; u \in \mathcal{T})$  be a collection of independent and identically distributed (i.i.d.) random variables. Then set

$$V(u) := x + \sum_{v \in ]\emptyset, u]} \xi_v, \quad u \in \mathcal{T}. \tag{1}$$

The collection  $(V(u); u \in \mathcal{T})$  will be called a branching random walk on  $\mathcal{T}$  with displacement distribution  $\xi_1$ , starting from  $x$ . If we view the tree  $\mathcal{T}$  as a genealogical tree and give each individual a displacement by saying that when a vertex  $u$  is born, its displacement is obtained by adding  $\xi_u$  to the displacement of its parent (with the displacement of the root being  $x$ ), then the displacement of  $u$  is exactly  $V(u)$ .

We are now ready to present the link between the chase–escape process on trees and branching random walks. Here and later,  $\text{Exp}(\lambda)$  denotes an exponential random variable of parameter  $\lambda > 0$ , independent of all the other mentioned random variables (in particular, different occurrences of  $\text{Exp}(1)$  denote different independent random variables).

Fix a tree  $\mathcal{T}$ . Let  $(R(u); u \in \mathcal{T})$  be a branching random walk on  $\mathcal{T}$  with displacement distribution  $\text{Exp}(1)$ , starting from a random point distributed according to an (independent)  $\text{Exp}(1)$  random variable. Let also  $(I(u); u \in \mathcal{T})$  be a branching random walk on  $\mathcal{T}$  with displacement distribution  $\text{Exp}(\lambda)$ , starting from 0. Finally, for every  $u \in \mathcal{T}$ , let  $\overleftarrow{u}$  be the parent of  $u$  and set  $W(u) = R(\overleftarrow{u}) - I(u)$ , with the convention  $R(\overleftarrow{\emptyset}) = 0$ .

**Theorem 1** (Coupling for the chase–escape process) *For every  $t \geq 0$  and  $u \in \mathcal{T}$ , set:*

$$X_t(u) = \begin{cases} I & \text{if } I(u) \leq t < R(u) \text{ and } W(v) \geq 0 \text{ for every } v \in ]\emptyset, u], \\ R & \text{if } R(u) \leq t \text{ and } W(v) \geq 0 \text{ for every } v \in ]\emptyset, u], \\ S & \text{otherwise.} \end{cases}$$

*Then  $(X_t)_{t \geq 0}$  has the same distribution as the chase–escape process  $\mathcal{C}(\mathcal{T}, \{\emptyset\}, 0)$ .*

Intuitively, if we let  $\emptyset = u_0, u_1, \dots, u_{|u|} = u$  be the vertices belonging to the shortest path connecting  $\emptyset$  to  $u \in \mathcal{T}$ , then  $u$  becomes infected if and only if, for every  $1 \leq i \leq |u|$ ,  $u_i$  has been infected before its parent has recovered, meaning that  $W(v) \geq 0$  for every  $v \in ]\emptyset, u]$ . In this case,  $I(u)$  is the first time when  $u$  becomes infected, while  $R(u)$  is the first time  $u$  recovers. Theorem 1 is then a simple consequence of the memoryless property of the exponential distribution. We omit a formal proof, since it would not be enlightening. This coupling, based on two independent branching

random walks, is reminiscent of the Athreya–Karlin coupling [4] used to study urn schemes by introducing independent branching processes.

Without further notice, we will assume that the chase–escape process is the process  $(X_t)_{t \geq 0}$  constructed in Theorem 1.

### 1.3 Coupling the Final State of the Chase–Escape Process with Killed Branching Random Walks

By taking the limit  $t \rightarrow \infty$  in Theorem 1, we get that the total number of infected individuals in the chase–escape process is equal to the number of elements of  $\{u \in \mathcal{T}; W(v) \geq 0 \text{ for every } v \in [\emptyset, u]\}$ . Unfortunately,  $W$  is *not* a branching random walk in the sense defined by (1). However, we still manage to represent the number of infected individuals as the total progeny of a killed branching random walk by allowing branching random walks with nonindependent increments.

More precisely, consider a single individual located at  $x \in \mathbb{R}$ . Its children are positioned on  $\mathbb{R}$  according to a certain point process  $\mathcal{L}$  on  $\mathbb{R}$  and represent the first generation. Each one of the individuals of the first generation independently gives birth to new individuals positioned (with respect to their birth places) according to an independent point process having the same law as  $\mathcal{L}$ , which represent the second generation. This process continues similarly for the next generation. The genealogy of the individuals forms a Galton–Watson tree denoted by  $\mathcal{T}$  and for every  $u \in \mathcal{T}$  we let  $V(u)$  be the position (or displacement) of  $u$ . We say that the collection  $(V(u); u \in \mathcal{T})$  is a branching random walk with displacement distribution given by the point process  $\mathcal{L}$ , starting from  $x$ . We also introduce a killing barrier at the origin: We imagine that any individual entering  $(-\infty, 0)$  is immediately killed and that its descendants is removed. Therefore, at every generation  $n \geq 0$ , only the individuals with a displacement that has always remained nonnegative up to generation  $n$  survive. In particular,  $\{u \in \mathcal{T}; V(v) \geq 0, \forall v \in [\emptyset, u]\}$  is the set of all the individuals that survive.

**Corollary 2** *Let  $V$  be the branching random walk produced with the point process*

$$\sum_{i=1}^U \delta_{\{\mathcal{E} - \text{Exp}_i(\lambda)\}},$$

*starting from  $x$ , where  $U$  is a nonnegative integer valued random variable, where  $\mathcal{E}$  is an independent exponential random variable of parameter 1 and  $(\text{Exp}_i(\lambda))_{i \geq 1}$  is an independent sequence of i.i.d. exponential random variables of parameter  $\lambda$ . Denote by  $\mathcal{T}$  the genealogical tree of this branching walk, set  $Z_n = \#\{u \in \mathcal{T}; |u| = n \text{ and } V(v) \geq 0 \text{ for every } v \in [\emptyset, u]\}$  and  $Z = \sum_{n \geq 0} Z_n$ . Then  $Z_n$  has the same distribution as the number of individuals at generation  $n$  in  $\mathcal{T}$  that have been infected until the chase–escape process  $\mathcal{C}(\mathcal{T}, \{\emptyset\}, x)$  has reached its absorbing state, and  $Z$  has the same distribution as the total number of individuals that have been infected in  $\mathcal{C}(\mathcal{T}, \{\emptyset\}, x)$ .*

This easily follows from Theorem 1 by observing that the displacement  $V(u)$  is the difference between the time when the parent of  $u$  recovers and the time when  $u$  gets

infected (without the presence of the recovered vertices). Note that Corollary 2 only covers the case of Galton–Watson trees, but its extension to any tree is straightforward. In future work, we hope to use similar couplings to study the chase–escape process on different kind of graphs such as other trees and  $\mathbb{Z}^2$ .

### 1.4 Chase–Escape Processes on Galton–Watson Trees

Our results concern the asymptotic behavior of the chase–escape process on supercritical Galton–Watson trees. Let  $\nu$  be a probability measure on  $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ . Recall that a Galton–Watson tree with offspring distribution  $\nu$  is a random rooted tree starting from one individual by supposing that each individual has an i.i.d. number of offspring distributed according to  $\nu$  (see e.g., [16, Section1] for a formal definition). If  $d \geq 2$  is an integer such that  $\nu(d) = 1$ ,  $\mathcal{T}$  is the infinite deterministic  $d$ -ary tree.

We henceforth assume that  $\mathcal{T}$  is a Galton–Watson tree with offspring distribution  $\nu$ , and are interested in the properties of the chase–escape process on  $\mathcal{T}$ . Let  $d = \sum_{i \geq 0} i \nu(i)$  be the expected number of offspring of an individual and suppose that  $d \in (1, \infty)$ , which implies that  $\#\mathcal{T} = \infty$  with positive probability.

If  $d \geq 2$  is an integer, Kordzakhia [13] identified

$$\lambda_c := 2d - 1 - 2\sqrt{d(d - 1)}$$

as the critical value of  $\lambda$  by showing that for  $\lambda > \lambda_c$ , the probability that the chase–escape process  $\mathcal{C}(\mathcal{T}, \{\emptyset\}, 0)$  gets extinct on the infinite  $d$ -ary tree is less than one and that for  $\lambda \in (0, \lambda_c)$  the latter probability is equal to one. Bordenave [7, Theorem1.1] extended this result to more general trees, such as supercritical Galton–Watson trees. By using heavy analytic tools, Bordenave treats the case  $\lambda = \lambda_c$  for Galton–Watson trees:

**Proposition 3** (Bordenave, Corollary 1.5 in [7]) *For  $\lambda = \lambda_c$ , for almost every  $\mathcal{T}$ , almost surely, the chase–escape process  $\mathcal{C}(\mathcal{T}, \{\emptyset\}, 0)$  gets extinct.*

We give a very short probabilistic proof of Proposition 3 based on Corollary 2 and on the almost sure convergence toward 0 of the Biggins’ martingale associated with our branching random walk (see Sect. 2.1).

Our first main result gives the asymptotic behavior of the tail of the number of infected individuals in the chase–escape process.

**Theorem 4** *Denote by  $Z$  the total number of vertices that have been infected until the chase–escape process  $\mathcal{C}(\mathcal{T}, \{\emptyset\}, 0)$  has reached its absorbing state. Let  $U$  denote a random variable distributed according to  $\nu$ .*

(i) *Suppose that  $\lambda = \lambda_c$  (critical case). Assume that there exists  $\alpha > 2$  such that  $\mathbb{E}[U^\alpha] < \infty$ . Then*

$$\mathbb{P}(Z > n) \underset{n \rightarrow \infty}{\sim} \left(1 + \sqrt{\frac{d}{d - 1}}\right) \cdot \frac{1}{n(\ln(n))^2}.$$

(ii) Suppose that  $\lambda \in (0, \lambda_c)$  (subcritical case). Assume that there exists  $\alpha > \frac{(1-\lambda+\sqrt{\lambda^2-2\lambda(2d-1)+1})^2}{2(d-1)\lambda}$  such that  $\mathbb{E}[U^\alpha] < \infty$ . There exists a constant  $C_1 > 0$ , depending only on  $\lambda$  and  $d$ , such that

$$\mathbb{P}(Z > n) \underset{n \rightarrow \infty}{\sim} C_1 \cdot n^{-\frac{(1-\lambda+\sqrt{\lambda^2-2\lambda(2d-1)+1})^2}{4(d-1)\lambda}}.$$

Theorem 4 is a consequence of analogous results obtained by Aïdékon, Hu and Zindy [2, Theorem 1] for killed branching random walks. In particular, in the critical case, note that  $\mathbb{E}[Z] < \infty$  while  $\mathbb{E}[Z \ln(Z)] = \infty$ . Let us mention that Theorem 4 (ii) is consistent with a result of Bordenave [7, Theorem 1.3], which states that, for every supercritical offspring distribution  $\nu$ ,

$$\begin{aligned} & \sup\{u \geq 1; \mathbb{E}[Z^u] < \infty\} \\ &= \min\left(\frac{(1-\lambda+\sqrt{\lambda^2-2\lambda(2d-1)+1})^2}{4(d-1)\lambda}, \sup\{u \geq 1; \mathbb{E}[U^u] < \infty\}\right). \end{aligned}$$

The fact that the tail of number of infected individuals in the chase–escape process  $\mathcal{C}(\mathcal{T}, \{\emptyset\}, 0)$  is regularly varying allows us to obtain analogous results for more general initial conditions, where we start the chase–escape process with any finite number of infected individuals. To state these results, we denote by  $\partial\mathcal{A}$  the vertices belonging to  $\mathcal{T} \setminus \mathcal{A}$  and at distance 1 from  $\mathcal{A}$ . Recall also that  $|u|$  denotes the generation of a vertex  $u \in \mathcal{A}$ .

**Theorem 5** Assume that  $\nu(d) = 1$  where  $d \geq 1$  is an integer, so that  $\mathcal{T}$  is the infinite  $d$ -ary tree. Let  $\mathcal{A} \subset \mathcal{T}$  be a finite connected subset of  $\mathcal{T}$  containing  $\emptyset$ . Denote by  $Z_{\mathcal{A}}$  the total number of vertices that have been infected until the chase–escape process  $\mathcal{C}(\mathcal{T}, \mathcal{A}, 0)$  has reached its absorbing state.

(i) Suppose that  $\lambda = \lambda_c$ . Then

$$\mathbb{P}(Z_{\mathcal{A}} > n) \underset{n \rightarrow \infty}{\sim} \left(1 + \sqrt{\frac{d}{d-1}}\right) \cdot \left(\sum_{u \in \partial\mathcal{A}} \frac{|u|}{d \cdot (d - \sqrt{d(d-1)})^{|u|-1}}\right) \cdot \frac{1}{n(\ln(n))^2}.$$

(ii) Suppose that  $\lambda \in (0, \lambda_c)$ . Then

$$\mathbb{P}(Z_{\mathcal{A}} > n) \underset{n \rightarrow \infty}{\sim} C_1 \cdot \frac{\lambda \sum_{u \in \partial\mathcal{A}} ((\rho_- + \lambda)^{-|u|} - (\rho_+ + \lambda)^{-|u|})}{\sqrt{\lambda^2 - 2\lambda(2d-1) + 1}} \cdot n^{-\frac{(1-\lambda+\sqrt{\lambda^2-2\lambda(2d-1)+1})^2}{4(d-1)\lambda}},$$

where  $\rho_-$  and  $\rho_+$  are defined in (2).

Theorem 5 also relies on results of [2], but its proof uses estimates for killed branching random walks starting from a random point. In particular, the main difficulty to obtain Theorem 5 is to establish uniform estimates in the starting point (see Proposition 10, which may be of independent interest). Many constants are explicit in Theorem 5: This stems from the remarkable fact that the renewal function of the random walk associated with the branching random walk  $V$  may be calculated explicitly.

Observe finally that if  $\emptyset \in \mathcal{A}$  and  $\mathcal{A}$  is not a connected subset of  $\mathcal{T}$ , then  $\mathbb{P}(Z_{\mathcal{A}} = \infty) > 0$ . Indeed, if  $u \in \mathcal{A}$  is such that  $\overleftarrow{u} \notin \mathcal{A}$ , then  $\overleftarrow{u}$  has a positive probability of never becoming infected and thus of never recovering, so that all the descendants of  $u$  will be infected with positive probability. Also, in the statement of Theorem 5, we assume that  $\nu(d) = 1$ , but the proof we give is actually valid when  $\mathcal{T}$  is more generally a  $\nu$ -Galton–Watson tree (under the same integrability assumption as in Theorem 4) and when  $\mathcal{A}$  is chosen in such a way that the trees  $(\mathcal{T}_u; u \in \partial\mathcal{A})$  are independent  $\nu$ -Galton–Watson trees, where  $\mathcal{T}_u$  denotes the tree formed by a vertex  $u$  and its descendants.

### 1.5 Birth-and-Assassination Process

We now turn our attention to the birth-and-assassination process, which we first formally define following Aldous and Krebs [3]. Let  $\mathbb{N}$  be the set of all the positive integers and let  $\mathcal{U}$  be the set of all labels defined by  $\mathcal{U} = \bigcup_{n=0}^{\infty} (\mathbb{N})^n$ , where by convention  $(\mathbb{N})^0 = \{\emptyset\}$ . An element of  $\mathcal{U}$  is a sequence  $u = u_1 \cdots u_k$  of positive integers and we set  $|u| = k$ , which represents the generation of  $u$ . If  $u = u_1 \cdots u_i$  and  $v = v_1 \cdots v_j$  belong to  $\mathcal{U}$ , we write  $uv = u_1 \cdots u_i v_1 \cdots v_j$  for the concatenation of  $u$  and  $v$ , and we set  $u \leq v$  if there exists  $w \in \mathcal{U}$  such that  $v = uw$ . In particular, we have  $u\emptyset = \emptyset u = u$ . Let  $(\mathcal{P}_u; u \in \mathcal{U})$  be a family of i.i.d. Poisson processes on  $\mathbb{R}_+$  with common arrival rate  $\lambda > 0$ , and let  $(\mathcal{K}_u; u \in \mathcal{U})$  be an independent collection of i.i.d. strictly positive random variables.

The process starts at time 0 with one individual, with label  $\emptyset$ , producing offspring at arrival times of  $\mathcal{P}_{\emptyset}$ , which enter the system with labels  $1, 2, \dots$  according to their birth order. Each new individual  $u$  entering the system immediately begins to produce offspring at the arrival times of  $\mathcal{P}_u$ , which enter the system with labels  $u1, u2, \dots$  according to their birth order. In addition, the ancestor  $\emptyset$  is *at risk* at time 0, and it only produces offspring until time  $T_{\emptyset} := \mathcal{K}_{\emptyset}$ , when it is removed from the system. In addition, if an individual  $u$  is removed from the system at time  $T_u$ , then, for every  $k \geq 0$ , such that the individual  $uk$  has been born,  $uk$  becomes at risk and continues to produce offspring until time  $T_{uk} := T_u + \mathcal{K}_{uk}$  when it is removed from the system.

The birth-and-assassination process can be equivalently seen as a Markov process on  $\{S, I, R\}^{\mathcal{U}}$ , where an individual at state R (resp. I and S) is a removed (resp. alive and not yet born) individual.

We will couple the birth-and-assassination process with a branching random walk where individuals have infinitely many offspring. Let  $(D_0(u); u \in \mathcal{U})$  be a collection of independent random variables distributed as  $\mathcal{K}_{\emptyset}$  and, for every  $u \in \mathcal{U}$ , set  $D(u) = \sum_{0 \leq v \leq u} D_0(v)$ . Let also  $(B(u); u \in \mathcal{U})$  be the branching random walk on  $\mathcal{U}$  with displacement distribution given by the point process

$$\delta_{\{\text{Exp}_1(\lambda)\}} + \delta_{\{\text{Exp}_1(\lambda)+\text{Exp}_2(\lambda)\}} + \delta_{\{\text{Exp}_1(\lambda)+\text{Exp}_2(\lambda)+\text{Exp}_3(\lambda)\}} + \dots,$$

where  $(\text{Exp}_i(\lambda))_{i \geq 1}$  is an i.i.d. collection of exponential random variables of parameter  $\lambda$ , with starting point 0 (the children are indexed by increasing new displacement). Finally, set  $V'(u) = D(\overleftarrow{u}) - B(u)$  for  $u \in \mathcal{T}$ , with the convention  $D(\overleftarrow{\emptyset}) = 0$ .

**Theorem 6** (Coupling for the birth-and-assassination process) *For every  $t \geq 0$  and  $u \in \mathcal{U}$ , we set:*

$$Y_t(u) = \begin{cases} I & \text{if } B(u) \leq t < D(u) \text{ and } V'(v) \geq 0 \text{ for every } v \in \llbracket \emptyset, u \rrbracket \\ R & \text{if } D(u) \leq t \text{ and } V'(v) \geq 0 \text{ for every } v \in \llbracket \emptyset, u \rrbracket \\ S & \text{otherwise.} \end{cases}$$

Then  $(Y_t)_{t \geq 0}$  has the same distribution as the birth-and-assassination process.

This is just a simple, yet useful, reformulation of the definition of the birth-and-assassination process. Without further notice, we will assume that the birth-and-assassination process is the process  $(Y_t)_{t \geq 0}$  appearing in Theorem 6. In particular, as for the chase–escape process, by letting  $t \rightarrow \infty$ , we see that the total progeny of this process is related to the total progeny of a certain killed branching walk:

**Corollary 7** *There exists a (random) time such that no individuals remain in the birth-and-assassination process if and only if there is a finite number of individuals in the branching random walk produced with the point process*

$$\delta_{\{\mathcal{K}_\emptyset - \text{Exp}_1(\lambda)\}} + \delta_{\{\mathcal{K}_\emptyset - (\text{Exp}_1(\lambda) + \text{Exp}_2(\lambda))\}} + \delta_{\{\mathcal{K}_\emptyset - (\text{Exp}_1(\lambda) + \text{Exp}_2(\lambda) + \text{Exp}_3(\lambda))\}} + \dots,$$

starting from 0 and killed at 0.

Following Aldous and Krebs, we say the process is *stable* if almost surely there exists a (random) time when no individuals remain in the system, and *unstable* otherwise. Let  $\phi(u) = \mathbb{E} [e^{u\mathcal{K}_\emptyset}]$  be the moment-generating function of  $\mathcal{K}_\emptyset$  for  $u \in \mathbb{R}$ . Under the assumption that  $\phi$  is finite on a neighborhood of the origin, Aldous and Krebs [3] proved that if  $\min_{u>0} \lambda u^{-1} \phi(u) < 1$  then the process is stable, and if  $\min_{u>0} \lambda u^{-1} \phi(u) > 1$  then the process is unstable. Later, in the particular case where  $\mathcal{K}_\emptyset$  is an exponential random variable of parameter 1, Bordenave [6, Corollary 2] proved that the process is stable if  $\min_{u>0} \lambda u^{-1} \phi(u) = 1$  (which corresponds to the case  $\lambda = 1/4$ ). Our last contribution is to establish that this fact holds more generally.

**Theorem 8** *Assume that  $\phi$  is finite on a neighborhood of the origin and that  $\min_{u>0} \lambda u^{-1} \phi(u) = 1$ . Then the birth-and-assassination process is stable.*

The proof is very similar to the one we give to Proposition 3 and is also based on the almost sure convergence toward 0 of the Biggins’ martingale associated with our branching random walk.

### 1.6 Structure of the Paper

We prove our results concerning the chase–escape process in Sect. 2, and the results concerning the birth-and-assassination process in Sect. 3. Finally, technical uniform estimates on branching random walks are established in Sect. 4.

The main goal of this paper is thus to present new couplings between branching random walks and the two models under consideration and to illustrate the strength of the couplings by showing how known results on branching random walks yield interesting new results on the chase–escape and birth-and-assassination models. In Sect. 2.4 and at the end of Sect. 3, we suggest other applications which would require new results on branching random walks that we hope to investigate in future work.

## 2 The Chase–Escape Process

In this section, we study the chase–escape process and prove in particular Proposition 3 and Theorems 4 and 5 by using branching random walks.

### 2.1 Branching Random Walks

We start by introducing some relevant quantities of the branching random walk involved in the coupling with the chase–escape process. For  $x \geq 0$ , let  $\mathbb{P}_x$  be a probability measure such that under  $\mathbb{P}_x$ ,  $(V(u); u \in \mathcal{T})$  is the law of a branching random walk produced with the point process

$$\mathcal{L} = \sum_{i=1}^U \delta_{\{\mathcal{E} - \text{Exp}_i(\lambda)\}},$$

starting from  $x$ , where  $U$  is a nonnegative integer valued random variable distributed according to  $\nu$ . We denote by  $\mathbb{E}_x$  the corresponding expectation. Recall that  $d > 1$  is the mean value of  $\nu$ . A straightforward computation yields the logarithmic generation function for the branching random walk:

$$\begin{aligned} \psi(t) &:= \ln \mathbb{E}_0 \left[ \sum_{|u|=1} e^{tV(u)} \right] = \ln \left( \sum_{k \geq 0} \mathbb{E}_0 \left[ \mathbb{1}_{\{U=k\}} \sum_{i=1}^k e^{t(\mathcal{E} - \text{Exp}_i(\lambda))} \right] \right) \\ &= \ln \left( \sum_{k \geq 0} \mathbb{P}(U = k) \mathbb{E}_0 \left[ \sum_{i=1}^k e^{t(\mathcal{E} - \text{Exp}_i(\lambda))} \right] \right) \\ &= \ln \left( \sum_{k \geq 0} \mathbb{P}(U = k) k \frac{\lambda}{(1-t)(\lambda+t)} \right) \\ &= \ln(d) + \ln \left( \frac{1}{1-t} \cdot \frac{\lambda}{\lambda+t} \right), \quad t \in (-\lambda, 1). \end{aligned}$$

In addition, there exists  $\rho_\star \in (0, 1)$  such that  $\psi(\rho_\star) = \rho_\star\psi'(\rho_\star)$ . Recalling that  $\lambda_c = 2d - 1 - 2\sqrt{d(d - 1)}$ , notice that

$$\psi'(\rho_\star) \begin{cases} < 0 & \text{if } \lambda \in (0, \lambda_c) \\ = 0 & \text{if } \lambda = \lambda_c \\ > 0 & \text{if } \lambda > \lambda_c. \end{cases}$$

In the terminology of branching random walks, this means that the branching random walk  $V$  is critical for  $\lambda = \lambda_c$ , subcritical for  $\lambda \in (0, \lambda_c)$  and supercritical for  $\lambda > \lambda_c$ . For  $\lambda = \lambda_c$ , note that  $\rho_\star = 1 + \sqrt{d(d - 1)} - d = (1 - \lambda_c)/2$ .

We are now ready to give an effortless proof of Proposition 3.

*Proof of Proposition 3* Assume that  $\lambda = \lambda_c$ , so that  $\psi'(\rho_\star) = 0$ . By [5, Lemma 5] (see also [17] for a probabilistic proof), under  $\mathbb{P}_0$ , the martingale  $W_n := \sum_{|u|=n} e^{\rho_\star V(u)}$  converges almost surely toward 0 as  $n \rightarrow \infty$ . This implies that, under  $\mathbb{P}_0$ ,

$$\max_{|u|=n} V(u) \rightarrow -\infty \quad a.s.$$

(See [11, Theorem 1.2] for a more precise rate of convergence under additional assumptions on the moments of  $v$ .) This implies that the number of surviving individuals in the branching random walk  $V$  killed at the barrier 0 is almost surely finite. Proposition 3 then simply follows from Corollary 2. □

We finally introduce some notation in the subcritical case. When  $\lambda < \lambda_c$ , we let  $\rho_-, \rho_+$  be such that  $0 < \rho_- < \rho_\star < \rho_+ < 1$  and  $\psi(\rho_-) = \psi(\rho_+) = 0$ . Setting  $\Delta = \lambda^2 - 2\lambda(2d - 1) + 1$ , one checks that

$$\rho_- = \frac{1}{2} \cdot (1 - \lambda - \sqrt{\Delta}), \quad \rho_+ = \frac{1}{2} \cdot (1 - \lambda + \sqrt{\Delta}). \tag{2}$$

### 2.2 Total Progeny of a Killed Branching Random Walk

In this section, we assume that  $\lambda \leq \lambda_c$  and shall prove Theorem 4. To this end, we begin by presenting results concerning the total progeny of branching random walks killed at the barrier 0 starting from a fixed initial point. We first need to introduce an auxiliary random walk  $(S_n)_{n \geq 0}$  defined as follows. Set  $\rho = \rho_\star$  if  $\lambda = \lambda_c$  (critical case) and  $\rho = \rho_+$  if  $\lambda \in (0, \lambda_c)$  (subcritical case). Then, for every  $x \geq 0$ , let  $(S_n)_{n \geq 0}$  be a random walk such that, under the probability measure  $\mathbb{Q}_x$ , we have  $S_0 = x$  and such that its step distribution is characterized by the fact that

$$\begin{aligned} & \mathbb{Q}_x [f(S_1 - S_0)] \\ &= \mathbb{E}_0 \left[ \sum_{|u|=1} f(V(u)) e^{\rho V(u)} \right] \quad \text{for every measurable function } f : \mathbb{R} \rightarrow \mathbb{R}_+. \end{aligned}$$

Next, let  $R$  be the renewal function associated with  $(S_n)_{n \geq 0}$  defined by

$$R(x) = \mathbb{Q}_0 \left[ \sum_{j=0}^{\tau^*-1} \mathbb{1}_{\{S_j \geq -x\}} \right], \quad \text{where } \tau^* = \inf\{j \geq 1; S_j \geq 0\},$$

and set  $\tau_0^- = \inf\{k \geq 0; S_k < 0\}$ .

We are now able to state a result obtained by Aïdékon et al. [2], which describes the asymptotic behavior of the total progeny  $Z$  of the branching random walk killed at the barrier 0 starting from a fixed initial point which is not necessarily the origin (this will be needed for the proof of Theorem 5).

**Theorem 9** (Theorem 1 in [2])

(i) Suppose that  $\lambda = \lambda_c$ . Then for every fixed  $x \geq 0$ ,

$$\mathbb{P}_x(Z > n) \underset{n \rightarrow \infty}{\sim} \frac{\mathbb{Q}_0 \left[ e^{-\rho_* S_{\tau_0^-}} \right] - 1}{d - 1} \cdot R(x) e^{\rho_* x} \cdot \frac{1}{n(\ln(n))^2}.$$

(ii) Suppose that  $\lambda \in (0, \lambda_c)$ . There exists a constant  $C_1 > 0$ , depending only on  $d$  and  $\lambda$ , such that for every  $x \geq 0$  and  $n \geq 1$ ,

$$\mathbb{P}_x(Z > n) \underset{n \rightarrow \infty}{\sim} C_1 \cdot R(x) e^{\rho_* x} \cdot n^{-\frac{\rho_*}{\rho_-}}.$$

*Proof of Theorem 4* (i) Suppose that  $\lambda = \lambda_c$ . We start by finding the step distribution of  $(S_n)_{n \geq 0}$  under  $\mathbb{Q}_x$ . If  $f : \mathbb{R} \rightarrow \mathbb{R}_+$  is a measurable function, as for the calculation of the logarithmic generating function of  $V$ , write

$$\begin{aligned} \mathbb{Q}_0[f(S_1)] &= \sum_{k \geq 0} \mathbb{P}(U = k) \sum_{i=1}^k \mathbb{E}_0 \left[ f(\mathcal{E} - \text{Exp}_i(\lambda_c)) e^{\rho(\mathcal{E} - \text{Exp}_i(\lambda_c))} \right] \\ &= \lambda_c d \int dx dy \mathbb{1}_{\{x, y \geq 0\}} f(x - y) e^{\rho_*(x-y)} e^{-x - \lambda_c y} \\ &= \frac{\lambda_c d}{\lambda_c + 1} \cdot \int_{\mathbb{R}} f(u) e^{-(\lambda_c + 1)|u|/2} du, \end{aligned} \tag{3}$$

where we have used the fact that  $\rho_* = (1 - \lambda_c)/2$  in the last equality (note that  $\mathbb{Q}_0$  is a probability distribution by definition of  $\lambda_c$ ). Hence, the step distribution of  $(S_n)_{n \geq 1}$  is a symmetric two-sided exponential distribution. This implies that for  $x \geq 0$ , under  $\mathbb{Q}_x$ , the random variable  $-S_{\tau_0^-}$  is distributed according to  $\text{Exp}((1 + \lambda_c)/2)$  (see e.g., Example (b) in [9, Sect. VI.8]). In particular,

$$\begin{aligned} \mathbb{Q} \left[ e^{-\rho_* S_{\tau_0^-}} \right] &= \frac{1 + \lambda_c}{2} \cdot \int_0^\infty dx e^{(1-\lambda_c)x/2} e^{-(1+\lambda_c)x/2} \\ &= \frac{1 + \lambda_c}{2} \cdot \int_0^\infty dx e^{-\lambda_c x} = \frac{1 + \lambda_c}{2\lambda_c}. \end{aligned} \tag{4}$$

Hence, by Theorem 9 (i),

$$\mathbb{P}_x(Z > n) \underset{n \rightarrow \infty}{\sim} \frac{1}{d-1} \cdot \left( \frac{1 + \lambda_c}{2\lambda_c} - 1 \right) \cdot R(x)e^{\rho_*x} \cdot \frac{1}{n(\ln(n))^2}. \tag{5}$$

Assertion (i) then follows by taking  $x = 0$  in (5) and noting that  $R(0) = 1$ . For the second assertion, it suffices to take  $x = 0$  in Theorem 9 (ii).  $\square$

In the subcritical case  $\lambda \in (0, \lambda_c)$ , the expression of the constant  $C_1$  appearing in Theorem 9 (ii) is much more complicated than in the critical case: See in particular Lemma 1 in [2] and the expression of the constant  $c_{\rho_-}$  in Eq. (8.18) in [2], which arises in the asymptotic tail behavior of the almost sure limit of the martingale  $\sum_{|u|=n} e^{\rho_- V(u)}$  as  $n \rightarrow \infty$ . Unfortunately, we have not managed to find a simple expression for  $C_1$  in our particular case.

### 2.3 Total Progeny of a Killed Branching Random Walk with a Random Starting Point

We keep here the notation introduced in the previous section. The proof of Theorem 5 requires the following uniform bounds on  $\mathbb{P}_x(Z > n)$ , which may be of independent interest.

**Proposition 10** *There exists a constant  $C_2(\lambda, d) > 0$  such that the following assertions hold.*

- (i) *Suppose that  $\lambda = \lambda_c$ . For every  $x \geq 0$  and  $n \geq 1$ ,  $\mathbb{P}_x(Z > n) \leq C_2(\lambda_c, d) \cdot (x + 1)e^{\rho_*x} \cdot \frac{1}{n(\ln(n))^2}$ .*
- (ii) *Suppose that  $\lambda \in (0, \lambda_c)$ . For every  $x \geq 0$  and  $n \geq 1$ ,  $\mathbb{P}_x(Z > n) \leq C_2(\lambda, d) \cdot e^{\rho_+x} \cdot n^{-\frac{\rho_+}{\rho_-}}$ .*

We postpone the proof of Proposition 10 to Sect. 4.

*Proof of Theorem 5* Recall that  $\mathcal{A} \subset \mathcal{T}$  is a finite connected subset of  $\mathcal{T}$  containing  $\emptyset$ . To simplify notation, denote by  $Z(\mathcal{T}, \mathcal{A}, x)$  the total number of infected individuals in the chase–escape process  $\mathbf{C}(\mathcal{T}, \mathcal{A}, x)$ . For  $u \in \partial\mathcal{A}$ , let  $R_u$  be the first time when  $\overleftarrow{u}$  recovers in the chase–escape process  $\mathbf{C}(\mathcal{T}, \mathcal{A}, 0)$ . In particular,  $R_u$  has the same distribution as the sum of  $|u|$  independent exponential random variables of parameter 1. Let also  $(\mathcal{E}_u; u \in \partial\mathcal{A})$  be a collection of independent exponential random variables of parameter  $\lambda$ , independent of  $(R_u; u \in \partial\mathcal{A})$ . If  $u \in \partial\mathcal{A}$ , without the presence of recovered vertices,  $\overleftarrow{u}$  would infect  $u$  after a time distributed as  $\mathcal{E}_u$ . Hence, if  $\mathcal{T}_u$  denotes the tree of descendants of  $u$  (including  $u$ ), then the number of individuals of  $\mathcal{T}_u$  that will be infected in the chase–escape process  $\mathbf{C}(\mathcal{T}, \mathcal{A}, 0)$  has the same distribution as  $Z(\mathcal{T}_u, \{\emptyset\}, R_u - \mathcal{E}_u) \mathbb{1}_{\{R_u > \mathcal{E}_u\}}$ , where the random variables  $(Z(\mathcal{T}_u, \{\emptyset\}, R_u - \mathcal{E}_u) \mathbb{1}_{\{R_u > \mathcal{E}_u\}}; u \in \partial\mathcal{A})$  are independent conditionally on  $(R_u; u \in \partial\mathcal{A})$ , and in addition this equality holds jointly in distribution for all  $u \in \partial\mathcal{A}$ . Now let  $u_1, \dots, u_K$  be an enumeration of the vertices of  $\partial\mathcal{A}$ . By the previous

discussion, we have

$$\begin{aligned} & \mathbb{P} \left( Z_{\mathcal{A}} > n \mid R_{u_1}, \dots, R_{u_K} \right) \\ &= \mathbb{P} \left( |\mathcal{A}| + \sum_{i=1}^K Z(\mathcal{T}_{u_i}, \{\emptyset\}, R_{u_i} - \mathcal{E}_{u_i}) \mathbb{1}_{\{R_{u_i} > \mathcal{E}_{u_i}\}} > n \mid R_{u_1}, \dots, R_{u_K} \right) \end{aligned} \tag{6}$$

where we denote by  $|\mathcal{A}|$  the cardinal of the set  $\mathcal{A}$ . Moreover, by Corollary 2,

$$\begin{aligned} & \mathbb{P} \left( Z(\mathcal{T}_{u_i}, \{\emptyset\}, R_{u_i} - \mathcal{E}_{u_i}) \mathbb{1}_{\{R_{u_i} > \mathcal{E}_{u_i}\}} > n \mid R_{u_1}, \dots, R_{u_K} \right) \\ &= \mathbb{E} \left[ \mathbb{P}_{R_{u_i} - \mathcal{E}_{u_i}} (Z > n) \mathbb{1}_{\{R_{u_i} > \mathcal{E}_{u_i}\}} \mid R_{u_i} \right] \end{aligned}$$

for every  $1 \leq i \leq K$ .

We start by proving (i), where  $\lambda = \lambda_c$ . Recall from (3) that under  $\mathbb{Q}_x$ ,  $(S_n)_{n \geq 0}$  is a random walk with step distribution a symmetric two-sided exponential distribution. The renewal function  $R$  is thus explicit:  $R(x) = 1 + (1 + \lambda_c)x/2$  for  $x \geq 0$  (see e.g., Eq. (4.3) of Chapter XII.4 in [9]). Hence, by Theorem 9 (i) and (4),

$$\begin{aligned} & n(\ln(n))^2 \cdot \mathbb{P}_{R_{u_i} - \mathcal{E}_{u_i}} (Z > n) \mathbb{1}_{\{R_{u_i} > \mathcal{E}_{u_i}\}} \\ & \xrightarrow{n \rightarrow \infty} \frac{1 - \lambda_c}{2\lambda_c(d - 1)} \cdot \left( 1 + \frac{(1 + \lambda_c)(R_{u_i} - \mathcal{E}_{u_i})}{2} \right) \\ & \quad e^{\rho_*(R_{u_i} - \mathcal{E}_{u_i})} \cdot \mathbb{1}_{\{R_{u_i} > \mathcal{E}_{u_i}\}}. \end{aligned}$$

Proposition 10 (i) allows us to use the conditioned dominated convergence theorem to deduce that

$$\mathbb{E} \left[ \mathbb{P}_{R_{u_i} - \mathcal{E}_{u_i}} (Z > n) \mathbb{1}_{\{R_{u_i} > \mathcal{E}_{u_i}\}} \mid R_{u_i} \right] \underset{n \rightarrow \infty}{\sim} G(R_{u_i}) \cdot \frac{1}{n(\ln(n))^2},$$

where  $G(x)$  is defined by

$$G(x) = \frac{1 - \lambda_c}{2\lambda_c(d - 1)} \cdot \mathbb{E} \left[ \left( 1 + \frac{(1 + \lambda_c)(x - \mathcal{E}_{u_i})}{2} \right) e^{\rho_*(x - \mathcal{E}_{u_i})} \mathbb{1}_{\{x > \mathcal{E}_{u_i}\}} \right], \quad x \geq 0.$$

But if  $X_1, \dots, X_K$  are independent random variables such that  $\mathbb{P}(X_i > n) \sim \kappa_i / (n \ln(n)^2)$  as  $n \rightarrow \infty$ , since  $1/\ln(n)^2$  is regularly varying, then  $\mathbb{P}(X_1 + \dots + X_K > n) \sim (\kappa_1 + \kappa_2 + \dots + \kappa_K) / (n \ln(n)^2)$  (see e.g., [9, Proposition in Sect. VIII.8]). Thus

$$\begin{aligned} & \mathbb{P} \left( |\mathcal{A}| + \sum_{i=1}^K Z(\mathcal{T}_{u_i}, \{\emptyset\}, R_{u_i} - \mathcal{E}_{u_i}) \mathbb{1}_{\{R_{u_i} > \mathcal{E}_{u_i}\}} > n \mid R_{u_1}, \dots, R_{u_K} \right) \underset{n \rightarrow \infty}{\sim} \\ & \quad \left( \sum_{i=1}^K G(R_{u_i}) \right) \cdot \frac{1}{n \ln(n)^2}. \end{aligned}$$

Observing that

$$\begin{aligned} & \mathbb{P}\left(|\mathcal{A}| + \sum_{i=1}^K Z(\mathcal{T}_{u_i}, \{\emptyset\}, R_{u_i} - \mathcal{E}_{u_i}) \mathbb{1}_{\{R_{u_i} > \mathcal{E}_{u_i}\}} > n \mid R_{u_1}, \dots, R_{u_K}\right) \\ & \leq \sum_{i=1}^K \mathbb{P}\left(Z(\mathcal{T}_{u_i}, \{\emptyset\}, R_{u_i} - \mathcal{E}_{u_i}) \mathbb{1}_{\{R_{u_i} > \mathcal{E}_{u_i}\}} > (n - |\mathcal{A}|)/K \mid R_{u_1}, \dots, R_{u_K}\right) \\ & = \sum_{i=1}^K \mathbb{E}\left[\mathbb{P}_{R_{u_i} - \mathcal{E}_{u_i}}\left(Z > (n - |\mathcal{A}|)/K\right) \mathbb{1}_{\{R_{u_i} > \mathcal{E}_{u_i}\}} \mid R_{u_i}\right], \end{aligned}$$

noting that  $r_n \ln^2(r_n)/(n \ln^2(n))$  is bounded as  $n \rightarrow \infty$  when  $r_n = (n - |\mathcal{A}|)/K$ , we combine once again Proposition 10 (i) with the dominated convergence theorem, to get, using (6), that

$$\mathbb{P}(Z_{\mathcal{A}} > n) = \mathbb{E}\left[\mathbb{P}(Z_{\mathcal{A}} > n \mid R_{u_1}, \dots, R_{u_K})\right] \underset{n \rightarrow \infty}{\sim} \left(\sum_{i=1}^K \mathbb{E}[G(R_{u_i})]\right) \cdot \frac{1}{n \ln(n)^2}.$$

It hence remains to compute  $\mathbb{E}[G(R_{u_i})]$ . Recall that  $R_{u_i}$  has the same distribution as the sum of  $|u_i|$  independent exponential random variables of parameter 1, and to simplify notation set  $k = |u_i|$ . Then using a change of variables and Fubini’s theorem, write for every measurable function  $F : \mathbb{R} \rightarrow \mathbb{R}_+$ :

$$\begin{aligned} \mathbb{E}\left[F(R_{u_i} - \mathcal{E}_{u_i}) \mathbb{1}_{\{R_{u_i} > \mathcal{E}_{u_i}\}}\right] &= \int dx dy F(x - y) \frac{x^{k-1} e^{-x}}{\Gamma(k)} \lambda e^{-\lambda y} \mathbb{1}_{\{x > y \geq 0\}} \\ &= \int_0^\infty du \frac{\lambda u^{k-1} e^{-u(1+\lambda)}}{\Gamma(k)} \int_0^u F(v) e^{\lambda v} dv. \end{aligned} \tag{7}$$

Recall that  $\rho_\star = (1 - \lambda_c)/2$ . Since  $\int_0^u (1 + (1 + \lambda_c)v/2) e^{(1-\lambda_c)v/2} e^{\lambda_c v} dv = u \cdot e^{(1+\lambda_c)u/2}$ , a straightforward calculation yields

$$\begin{aligned} \mathbb{E}[G(R_{u_i})] &= \mathbb{E}\left[F(R_{u_i} - \mathcal{E}_{u_i}) \mathbb{1}_{\{R_{u_i} > \mathcal{E}_{u_i}\}}\right] \\ &= \frac{1 - \lambda_c}{2\lambda_c(d - 1)} \cdot \int_0^\infty du \frac{\lambda_c u^{k-1} e^{-u(1+\lambda_c)}}{\Gamma(k)} u \cdot e^{(1+\lambda_c)u/2}. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E}[G(R_{u_i})] &= \left(1 + \sqrt{\frac{d}{d-1}}\right) \cdot \frac{\lambda_c |u_i|}{((1 + \lambda_c)/2)^{1+|u_i|}} \\ &= \left(1 + \sqrt{\frac{d}{d-1}}\right) \cdot \left(\frac{|u_i|}{(d \cdot (d - \sqrt{d(d-1)})^{|u_i|-1})}\right). \end{aligned}$$

This completes the proof of (i).

The proof of the second assertion, where  $\lambda \in (0, \lambda_c)$  is very similar, so we only sketch the main calculations. Under  $\mathbb{Q}_x$ ,  $(S_n)_{n \geq 0}$  is still a random walk with step distribution a (nonsymmetric) two-sided exponential distribution, with density  $\frac{\lambda d}{d+1} e^{-(1-\rho_+)u}$  on  $\mathbb{R}_+$  and density  $\frac{\lambda d}{d+1} e^{(\lambda+\rho_+)u}$  on  $\mathbb{R}_-$ . By [9, Example (a) (ii) in XII.4], the renewal function  $R$  is equal to

$$R(x) = \frac{\rho_+ + \lambda}{2\rho_+ + \lambda - 1} + \frac{\rho_+ - 1}{2\rho_+ + \lambda - 1} e^{-(2\rho_+ + \lambda - 1)x}.$$

(In our case, the value of  $\kappa$  defined in [9, Example (a) (ii) in XII.4] is  $\kappa = 2\rho_+ + \lambda - 1$ .) Hence,

$$\mathbb{P}(Z_{\mathcal{A}} > n) \underset{n \rightarrow \infty}{\sim} \left( \sum_{i=1}^K \mathbb{E}[H(Ru_i)] \right) \cdot n^{-\frac{\rho_+}{\rho_-}}$$

where  $H(x)$  is defined by

$$H(x) = C_1 \cdot \mathbb{E} \left[ \left( \frac{\rho_+ + \lambda}{2\rho_+ + \lambda - 1} + \frac{\rho_+ - 1}{2\rho_+ + \lambda - 1} e^{-(2\rho_+ + \lambda - 1)(x - \mathcal{E}_{u_i})} \right) e^{\rho_+(x - \mathcal{E}_{u_i})} \mathbb{1}_{\{x > \mathcal{E}_{u_i}\}} \right], \quad x \geq 0.$$

We compute  $\mathbb{E}[H(Ru_i)]$  by using (7): Since  $\int_0^u R(v) e^{\rho_+ v} \cdot e^{\lambda v} dv = (e^{(\lambda+\rho_+)u} - e^{u(1-\rho_+)}) / (2\rho_+ + \lambda - 1)$ , we get

$$\begin{aligned} \mathbb{E}[H(Ru_i)] &= C_1 \frac{\lambda \left( (1 - \rho_+)^{-k} - (\lambda + \rho_+)^{-k} \right)}{2\rho_+ + \lambda - 1} \\ &= C_1 \cdot \frac{\lambda}{\sqrt{\lambda^2 - 2\lambda(2d - 1) + 1}} \left( (\rho_- + \lambda)^{-|u|} - (\rho_+ + \lambda)^{-|u|} \right). \end{aligned}$$

This completes the proof. □

Observe that this proof shows that Theorem 5 holds when  $\mathcal{T}$  is a  $\nu$ -Galton–Watson tree (under the same integrability assumption as in Theorem 4) and when  $\mathcal{A}$  is chosen in such a way that the trees  $(\mathcal{T}_u; u \in \partial\mathcal{A})$  are independent  $\nu$ -Galton–Watson trees.

### 2.4 Reaching High Generations

In this section, we state a conjecture concerning the asymptotic behavior of the probability that the infection reaches high levels of the tree. Assume that  $\nu(d) = 1$  where  $d \geq 2$  is an integer, so that  $\mathcal{T}$  is the infinite  $d$ -ary tree. Recall that  $Z_n$  denotes the number of individuals of the  $n$ -th level of the tree  $\mathcal{T}$  that have been infected in the chase–escape process  $\mathbf{C}(\mathcal{T}, \{\emptyset\}, 0)$ . Finally, set  $\gamma = 4\lambda / (1 + \lambda)^2$ .

We believe that the following results hold:

- (i) If  $\lambda \in (0, \lambda_c)$ , there exists a constant  $C_3 > 0$ , depending only on  $\lambda$  and  $d$ , such that  $\mathbb{P}(Z_n > 0) \sim C_3 \cdot (\gamma d)^n n^{-3/2}$  as  $n \rightarrow \infty$ .
- (ii) If  $\lambda = \lambda_c$ , we have  $\ln \mathbb{P}(Z_n > 0) \sim - (3(1 - 1/d)\pi^2)^{1/3} \cdot n^{1/3}$  as  $n \rightarrow \infty$ .

Aidékon and Jaffuel [1] proved this result for branching random walks on  $\mathcal{T}$  with i.i.d. displacements. However, we believe that even in our case the displacements are not independent, an analog result should hold and that their proof could be adapted (see in particular [12], where the displacements are not supposed to be independent). We also believe that this should hold more generally for Galton–Watson trees under adequate integrability conditions.

### 3 Birth-and-Assassination Process

We are now interested in the asymptotic behavior of the birth-and-assassination process at criticality. Denote by  $\tilde{\mathbb{P}}$  the law of the branching random walk  $(\tilde{V}(u); u \in \mathcal{U})$  on  $\mathcal{U} = \cup_{n \geq 0} \mathbb{N}^n$  produced with the point process

$$\tilde{\mathcal{L}} = \delta_{\{\mathcal{K}_\emptyset - \text{Exp}_1(\lambda)\}} + \delta_{\{\mathcal{K}_\emptyset - (\text{Exp}_1(\lambda) + \text{Exp}_2(\lambda))\}} + \delta_{\{\mathcal{K}_\emptyset - (\text{Exp}_1(\lambda) + \text{Exp}_2(\lambda) + \text{Exp}_3(\lambda))\}} + \dots,$$

where  $(\text{Exp}_i(\lambda))_{i \geq 1}$  is an i.i.d. collection of exponential random variables of parameter  $\lambda$ , with starting point 0 and independent of  $\mathcal{K}_\emptyset$ .

Introduce the logarithmic generation function  $\tilde{\psi}(t) = \ln \mathbb{E}_0 \left[ \sum_{|u|=1} e^{t\tilde{V}(u)} \right]$  of this branching random walk. Recalling that  $\phi(t) = \mathbb{E} [e^{t\mathcal{K}_\emptyset}]$  is the moment-generating function of  $\mathcal{K}_\emptyset$ , a straightforward calculation yields

$$\begin{aligned} \tilde{\psi}(t) &= \ln \left( \sum_{i=1}^{\infty} \mathbb{E} [e^{t\mathcal{K}_\emptyset}] \mathbb{E} [e^{-t\text{Exp}_1(\lambda)}]^i \right) = \ln \left( \phi(t) \sum_{i=1}^{\infty} \left( \frac{\lambda}{\lambda + t} \right)^i \right) \\ &= \ln \left( \frac{\lambda\phi(t)}{t} \right), \quad t \geq 0. \end{aligned}$$

*Proof of Theorem 8* The proof is very similar to the one we gave for Proposition 3: Assuming that  $\min_{u>0} \lambda u^{-1} \phi(u) = 1$  and that this minimum is attained at  $u_\star > 0$ , we have  $\tilde{\psi}'(u_\star) = 0$ , so that by [5, Lemma 5] (see also [17]), the martingale  $W_n := \sum_{|u|=n} e^{u_\star \tilde{V}(u)}$  converges almost surely toward 0 as  $n \rightarrow \infty$ . This implies that under  $\tilde{\mathbb{P}}$ ,

$$\max_{|u|=n} \tilde{V}(u) \rightarrow -\infty \quad a.s.$$

Hence, the number of surviving individuals in the branching random walk  $\tilde{V}$  killed at the barrier 0 is almost surely finite. Theorem 8 then simply follows from Corollary 7. □

Now assume that  $\mathcal{K}_\emptyset$  is an exponential random variable of parameter 1, which implies that  $\tilde{\psi}(t) = \ln (\lambda/(t(1 - t)))$ , so that the critical parameter is  $\lambda = 1/4$ . In this

case, if  $N$  denotes the total number of born individuals in the birth-and-assassination process, using analytical methods Bordenave [6, Theorem2] showed that for every  $\lambda \in (0, 1/4)$ ,

$$\sup\{u \geq 0; \mathbb{E}[N^u] < \infty\} = \frac{1 + \sqrt{1 - 4\lambda}}{1 - \sqrt{1 - 4\lambda}} \tag{8}$$

and that  $\mathbb{E}[N] = 2$  for  $\lambda = 1/4$ . In this case, and also when  $\mathcal{K}_\emptyset$  is any positive random variable, it would be interesting to find whether similar phenomena as in the chase–escape process occur for the tail of the total progeny and the probability of having individuals born at large generations occur in the birth-and-assassination process. Unfortunately, we may not apply known results on killed branching random walks since here on the one hand each individual has an infinite number of offspring (the branching is infinite) and on the other hand the logarithmic generation function  $\psi$  is not finite on a neighborhood of the origin. However, it seems likely that this is not an obstacle to have an analog of Theorem 9 to hold in this case as well. It is plausible that, in general, if  $\min_{u>0} \lambda u^{-1} \phi(u) = 1$ , then  $\mathbb{P}(N > n) \sim C/(n \ln(n)^2)$  as  $n \rightarrow \infty$  for a certain  $C > 0$ , and if  $\min_{u>0} \lambda u^{-1} \phi(u) < 1$ , then  $\mathbb{P}(N > n) \sim C \cdot n^{-\tilde{\rho}_+/\tilde{\rho}_-}$  for a certain  $C > 0$ , where  $0 < \tilde{\rho}_- < \tilde{\rho}_+$  are such that

$$\lambda \frac{\phi(\tilde{\rho}_-)}{\tilde{\rho}_-} = \lambda \frac{\phi(\tilde{\rho}_+)}{\tilde{\rho}_+} = 1.$$

In this case, notice that by convexity of  $t \mapsto \mathbb{E}[e^{t\mathcal{K}_\emptyset}]/t$ , there exist exactly two values of  $t > 0$  such that  $\lambda t^{-1} \phi(t) = 1$ . In addition, when  $\mathcal{K}_\emptyset$  is an exponential random variable of parameter 1, we have  $\tilde{\rho}_\pm = (1 \pm \sqrt{1 - 4\lambda})/2$ , so that this prediction is consistent with (8).

### 4 Proof of the Technical Estimate

We first introduce some notation which appears in [2]. We shall refer the reader very often to [2] to stay as concise as possible.

Let  $\mathcal{L}[0]$  be the set of all the individuals (i.e., vertices of the associated genealogical tree) of the (nonkilled) branching random walk which lie below 0 for its first time, see [2, Eq. (1.8) and Fig. 1]. For  $L > 0$ , let  $H(L)$  be the number of individuals of the branching random walk on  $[0, L]$  with two killing barrier which were absorbed at level  $L$  (see [2, Eq. (1.10) and Fig. 2]). Then let  $Z[0, L]$  be the number of individuals of  $\mathcal{L}[0]$  which have not crossed level  $L$  (see [2, Eq. (1.12)]). These individuals are partitioned into *good* and *bad* individuals, whose number is denoted, respectively, by  $Z_g[0, L]$  and  $Z_b[0, L]$  (and taking the parameter  $\lambda = 1$  appearing in their definition, see [2, Eq. (7.2)] for the critical case and [2, Eq. (8.5)] for the subcritical case—roughly speaking, individuals are good if their children do not make extraordinary jumps, and the number of their children is not too big). Finally, for a random walk  $(S_n)_{n \geq 0}$  set  $\tau_a^+ = \inf\{k \geq 0; S_k > a\}$  and  $\tau_a^- = \inf\{k \geq 0; S_k < a\}$  for  $a \in \mathbb{R}$ .

*Proof of Proposition 10* The following proof is due to Elie Aidékon. A close inspection of the proof of Lemma 2 of [2] shows that it is sufficient to establish Proposition 10

when  $\mathbb{P}_x(Z > n)$  is replaced by  $\mathbb{P}_x(\#\mathcal{L}[0] > n)$ . In the sequel  $C$  denotes a positive constant which may change from line to line.

We first concentrate on the critical case  $\lambda = \lambda_c$ . By a linear transformation on  $V$ , we may assume that  $\rho_\star = 1$ . We first claim that there exists a constant  $C > 0$  such that for every  $L > 0$  and  $x \geq 0$ ,  $\mathbb{E}_x[H(L)] \leq Ce^x(1+x)e^{-L}/L$ . It is enough to check this inequality for  $0 < x < L$ . In this case, using Proposition 3 in [2] (applied with  $t = L$ ,  $\lambda = 1$  and  $h(u) = e^{V(u)}$  with  $\mathbb{Q}_x$  defined by [2, Eq. (5.16)]), we have

$$\mathbb{E}_x[H(L)] = e^x \mathbb{Q}_x \left[ e^{-\tau_L^+} \mathbb{1}_{\{\tau_L^+ < \tau_0^-\}} \right] \leq e^{x-L} \mathbb{Q}_x(\tau_L^+ < \tau_0^-) \leq e^{x-L} \frac{x+L}{L},$$

where we have used [2, Eq. (4.12)] for the last inequality. This yields our claim. In particular, taking  $L = L_n = \ln n + \ln \ln n$ , we get

$$\mathbb{P}_x(H(L_n) \geq 1) \leq C(1+x)e^x \frac{1}{n(\ln(n))^2}. \tag{9}$$

Next, by Lemma 13 in [2] we have  $\mathbb{E}_x[Z_b[0, L_n]] \leq C(1+x)e^x \cdot (\ln(n))^{-2}$  and by Lemma 14 in [2] we have  $\mathbb{E}_x[Z_g[0, L_n]^2] \leq C(1+x)e^x \cdot n(\ln(n))^{-2}$ . Hence, using Markov’s inequality we get

$$\begin{aligned} \mathbb{P}_x(Z_b[0, L_n] \geq n/2) &\leq C(1+x)e^x \cdot \frac{1}{n(\ln(n))^2}, \\ \mathbb{P}_x(Z_g[0, L_n] \geq n/2) &\leq C(1+x)e^x \cdot \frac{1}{n(\ln(n))^2}. \end{aligned} \tag{10}$$

The conclusion then follows by observing that  $\mathbb{P}_x(\#\mathcal{L}[0] > n) \leq \mathbb{P}_x(H(L_n) \geq 1) + \mathbb{P}_x(Z[0, L_n] > n)$  and using (9) and (10).

We now turn to the subcritical case. Using Proposition 3 in [2] (applied with  $t = L$ ,  $\lambda = 1$  and  $h(u) = e^{\rho_- V(u)}$  with  $\mathbb{Q}_x^{(\rho_-)}$  defined in the beginning of Sec. 8 in [2]) and Eq. (8.1) in [2], one similarly shows that

$$\mathbb{E}_x[H(L)] = e^{\rho_- x} \mathbb{Q}_x^{(\rho_-)} \left[ e^{-\rho_- \tau_L^+} \mathbb{1}_{\{\tau_L^+ < \tau_0^-\}} \right] \leq e^{\rho_- x - \rho_- L} \mathbb{Q}_x^{(\rho_-)}(\tau_L^+ < \tau_0^-) \leq e^{\rho_- x - \rho_+ L}.$$

Hence, taking  $L = L_n = \ln(n)/\rho_-$  we get  $\mathbb{P}_x(H(L_n) \geq 1) \leq e^{\rho_- x} n^{-\rho_+/\rho_-}$ . By Lemma 19 in [2],  $\mathbb{E}_x[Z_g[0, L_n]^{k_\star}] \leq Ce^{\rho_+ x} \cdot n^{k_\star - \rho_+/\rho_-}$  with  $k_\star = \lfloor \rho_+/\rho_- \rfloor + 1$ , and by Lemma 20 (i) in [2],  $\mathbb{E}_x[Z_b[0, L_n]] \leq Ce^{\rho_+ x} \cdot n^{1 - \rho_+/\rho_-}$ . Markov’s inequality then entails

$$\begin{aligned} \mathbb{P}_x(Z_b[0, L_n] \geq n/2) &\leq Ce^{\rho_+ x} \cdot n^{-\rho_+/\rho_-}, \\ \mathbb{P}_x(Z_g[0, L_n] \geq n/2) &\leq Ce^{\rho_+ x} \cdot n^{-\rho_+/\rho_-}. \end{aligned}$$

We conclude as in the critical case. □

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