Gasket decomposition of 3-colored planar maps

Juliette Schabanel

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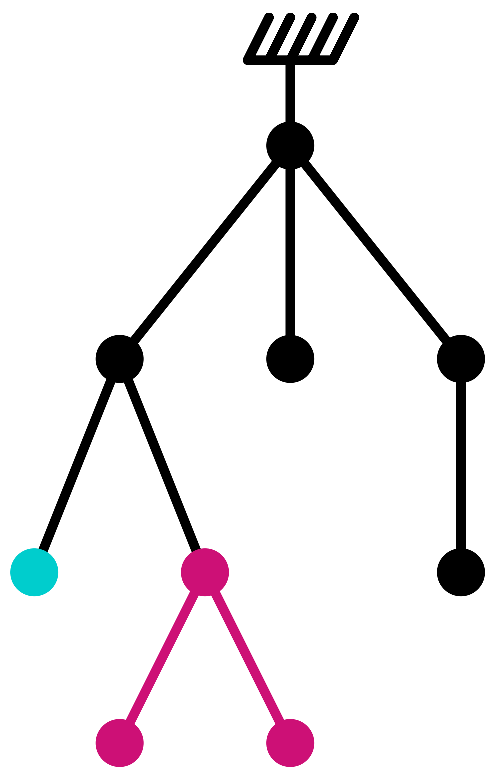
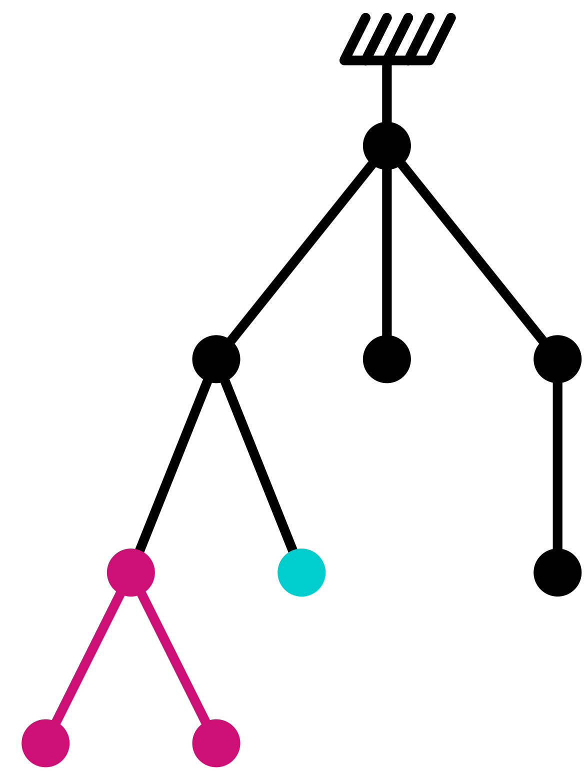
Introduction to Enumerative Combinatorics

First example: Trees

Plane tree = rooted tree with ordered children.
Size n = number of vertices.



$\neq$



$n = 9$

First example: Trees

Plane tree $=$ rooted tree with ordered children.

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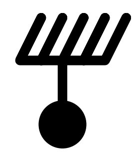
Binary trees: all vertices have 0 or 2 children.

size 1 :  $t_1 = 1$

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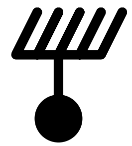
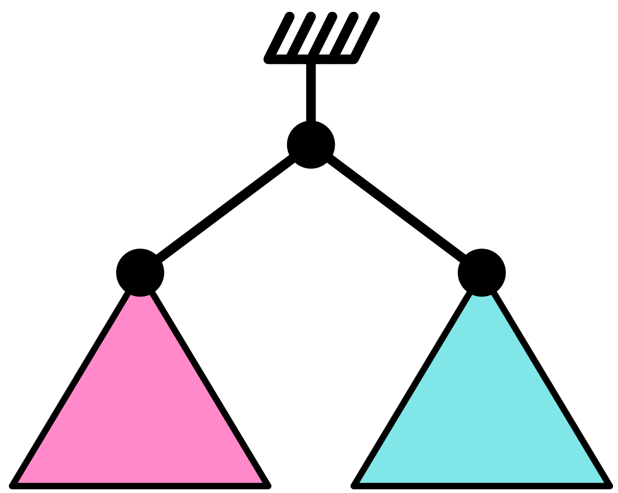
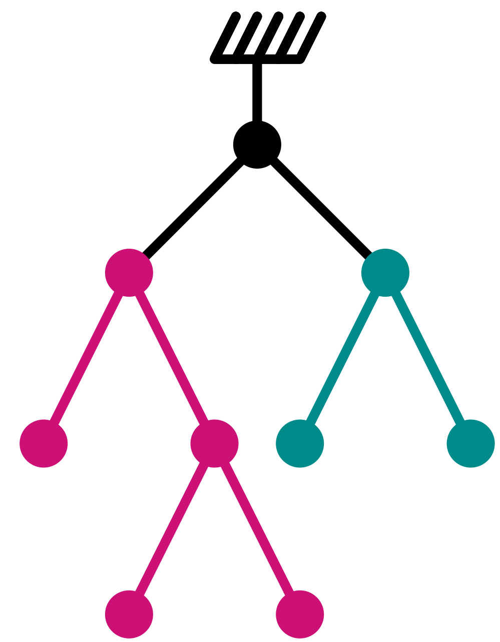


$t_1 = 1$

size $n > 1$:



k $\quad$ $n - k - 1$



$n = 9$

$k = 5$

$n - k - 1 = 3$

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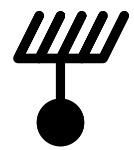

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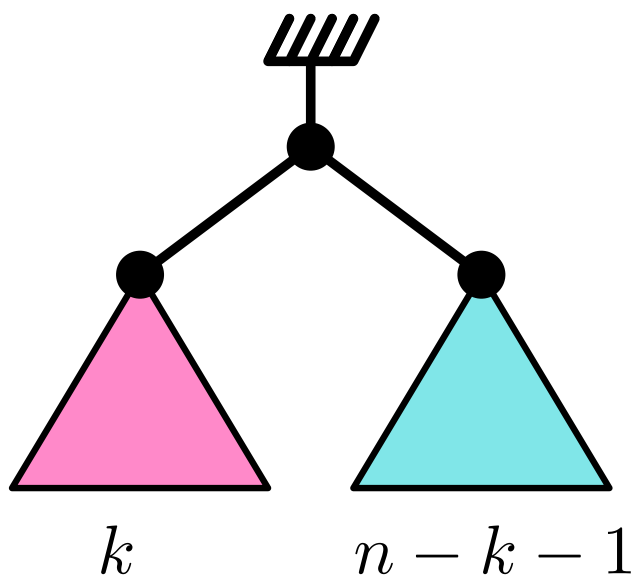

$$t_n = \sum_{k=1}^{n-1} t_k t_{n-k-1}$$

Solution : $t_n = \dfrac{1}{n+1}\dbinom{2n}{n}$

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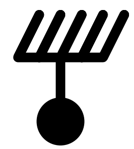

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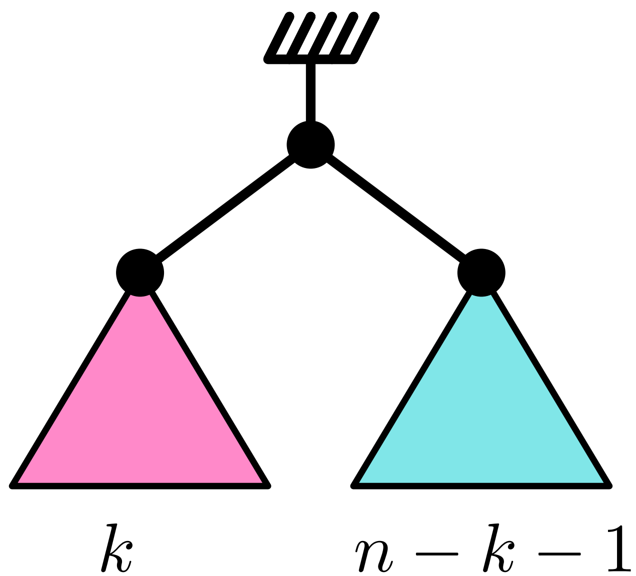

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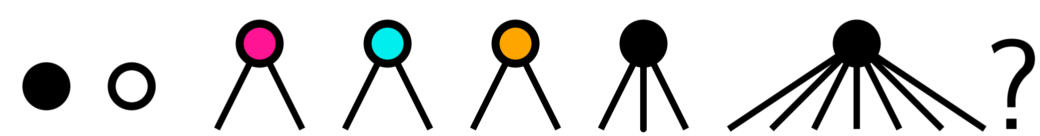
Trees with nodes



Generating functions

The **generating function** of a combinatorial class $\mathcal{C}$ is the series

$$C(t) = \sum_{\gamma \in \mathcal{C}} t^{|\gamma|} = \sum_{n \geqslant 0} c_n t^n \in \mathbb{Q}[[t]].$$

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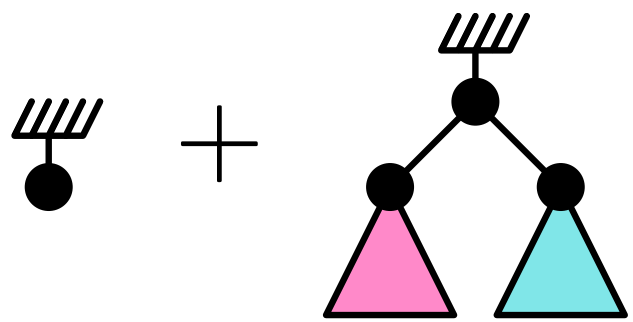

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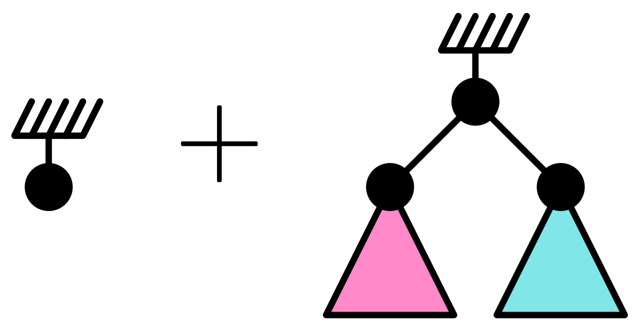

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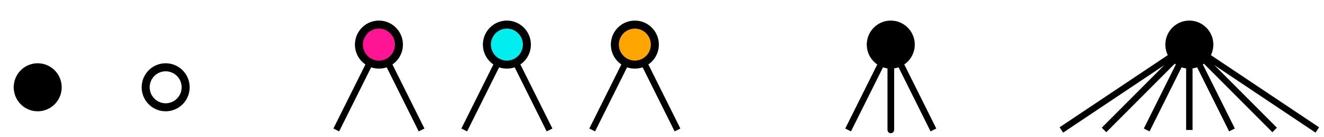

$$T(t) = 2t + 3tT(t)^2 + tT(t)^3 + tT(t)^7$$

Generating functions

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What to do with it ?

- If we can solve the equation and get a closed form for $C(t)$, we can recover the c_n through $C^{(n)}(0) = n! c_n$.
- Use the equation to compute asymptotics of c_n.
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The form of the equation gives information on the class:

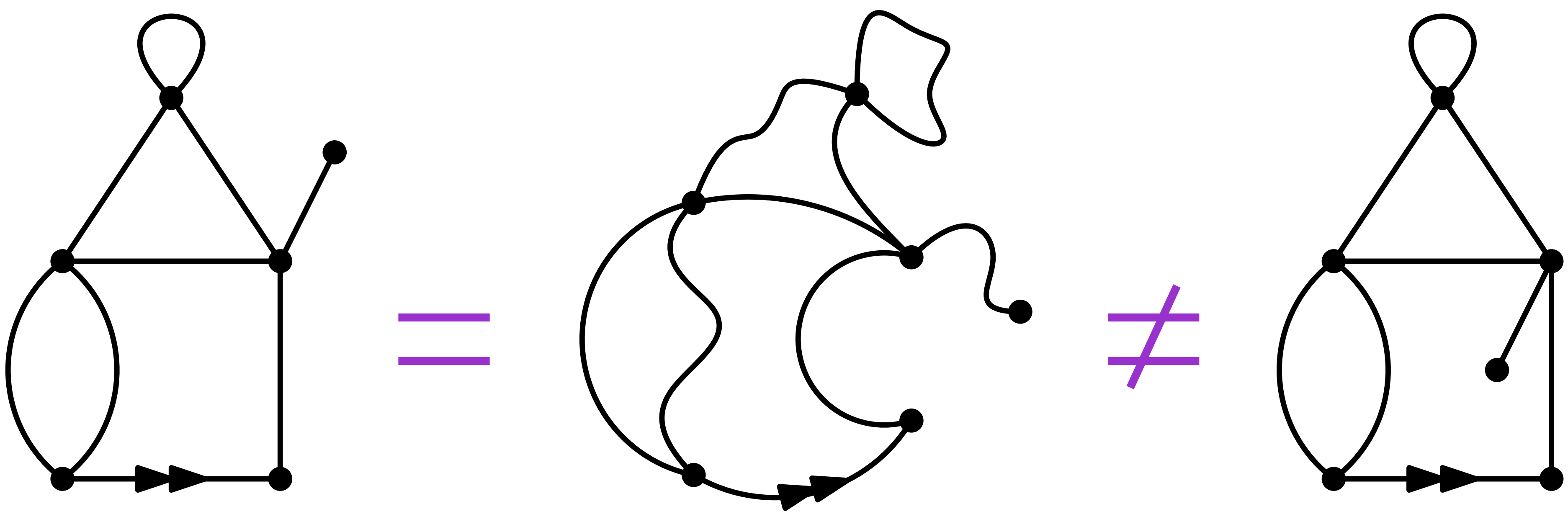
- Rational function: $C(t) = P(t) + Q(t)C(t)$, P, Q polynomials.
  $\implies$ $c_n \sim \kappa \lambda^n n^\alpha$, $\alpha \in \mathbb{N}$. $\quad \mathcal{C} \approx$ regular language?
- Algebraic function: $P(t, C(t)) = 0$, $P(t, C)$ polynomial.
  $\implies$ $c_n \sim \kappa \lambda^n n^\alpha$, $\alpha \in \mathbb{Q}$. $\quad \mathcal{C} \approx$ non-ambiguous grammar (trees)?

I- Recursive decomposition of colored planar maps

1) Definitions

Planar maps

A **planar map** is the embedding of a connected graph onto the sphere, up to orientation preserving homeomorphism.
Multi-edges and loops are allowed.



Planar map = planar graph + cyclic ordering of the edges around each vertex.

All maps are **rooted**, i.e. an oriented edge is marked.

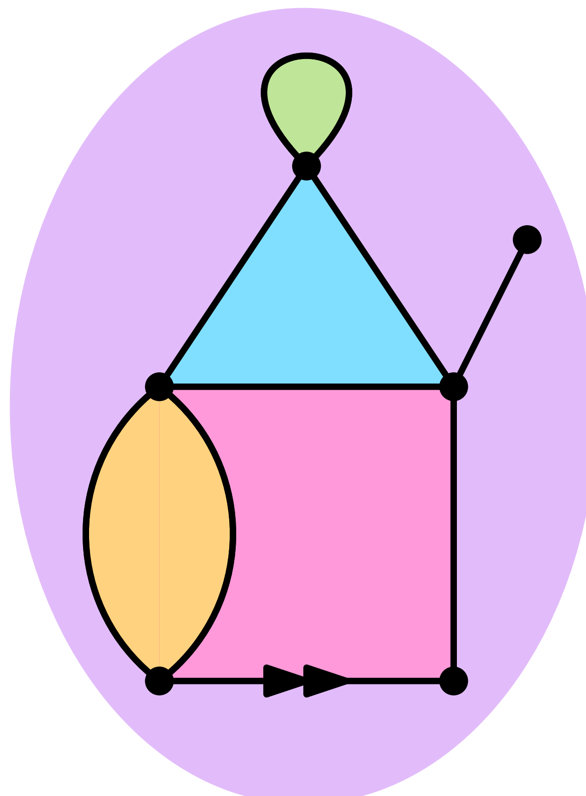
Planar maps

Planar map = planar graph + cyclic ordering of the edges around each vertex.

- **Vertices** and **edges** are inherited from the graph.
- **Faces** are the connected components of the sphere minus the map.

Size: number of edges n.

Here: 6 vertices, 9 edges et 5 faces.



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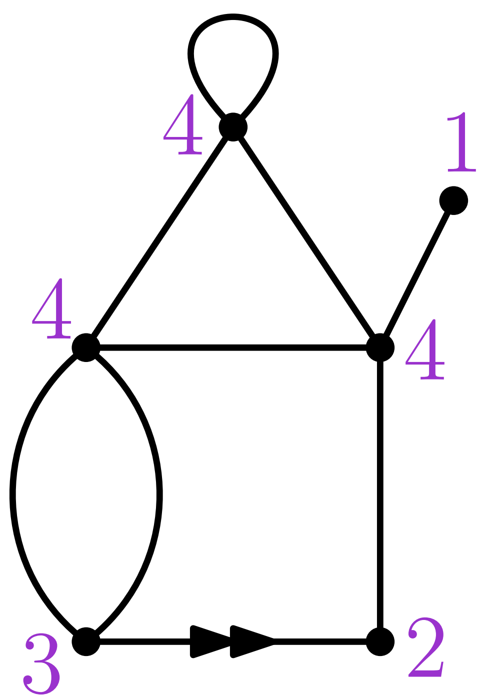
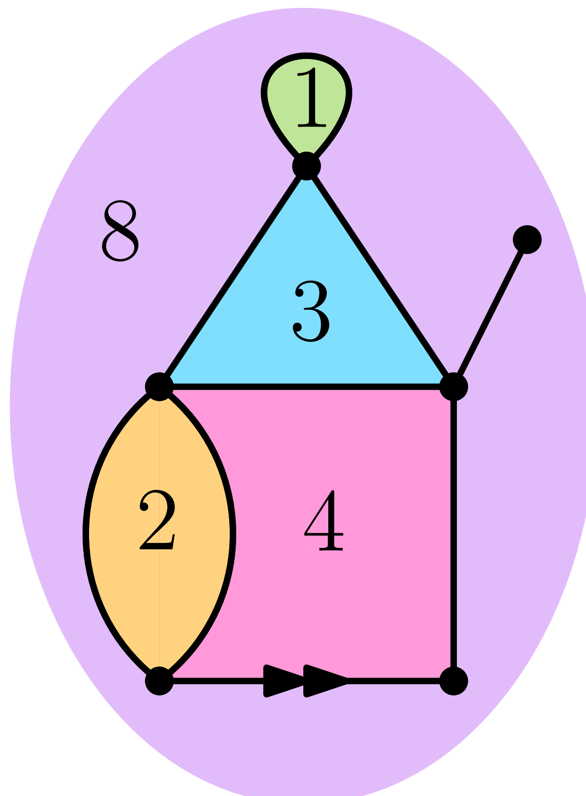
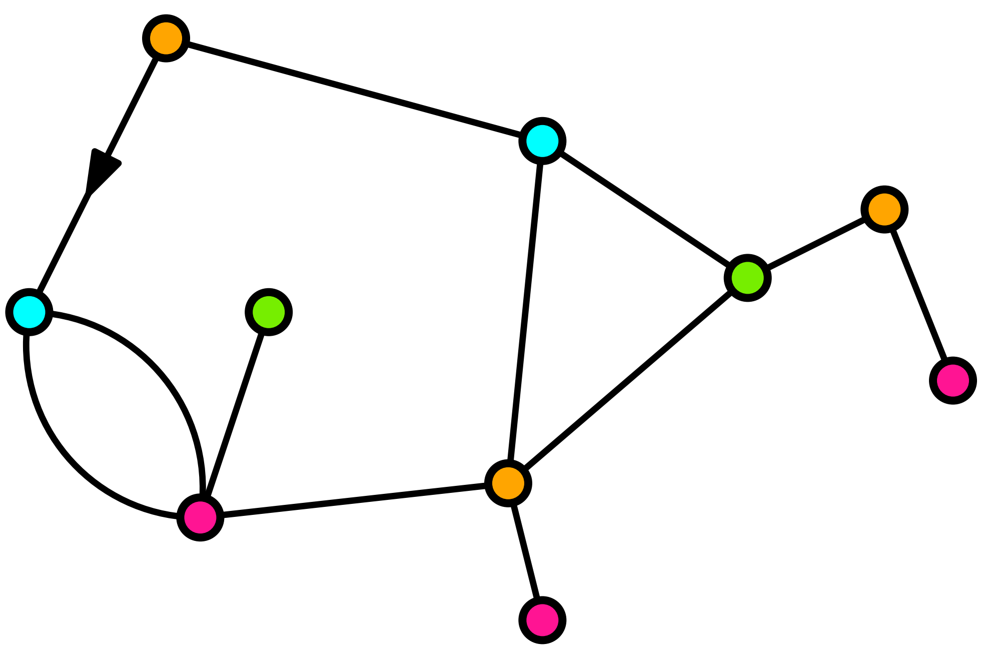
Degree (of a vertex or face) = number of incident half-edges.



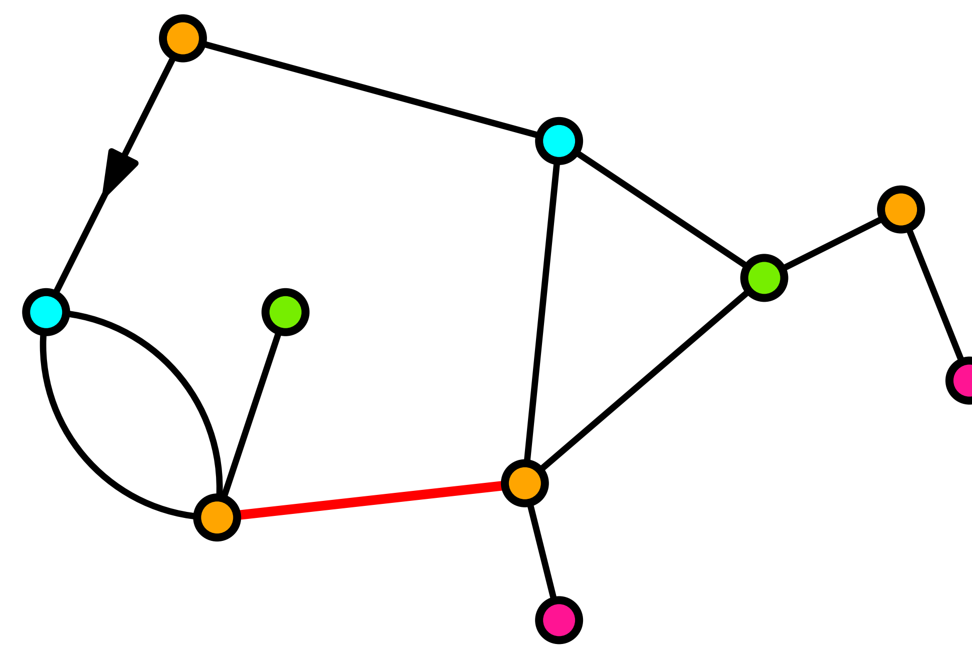


Colored maps

Colored map: assign a color to each vertex so that each pair of adjacent vertices receives different colors.



proper coloring



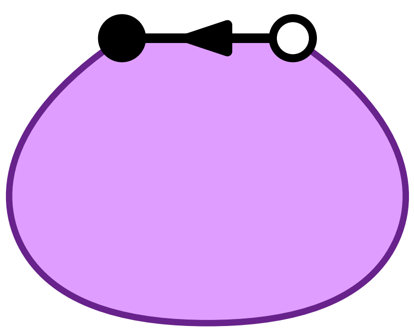
non-proper coloring

I- Recursive decomposition of colored planar maps

2) Tutte's decomposition

2-colored maps

$B(t)$: generating function of bicolored planar maps counted by number of edges.

 $=$

$$B(t) \quad =$$

2-colored maps

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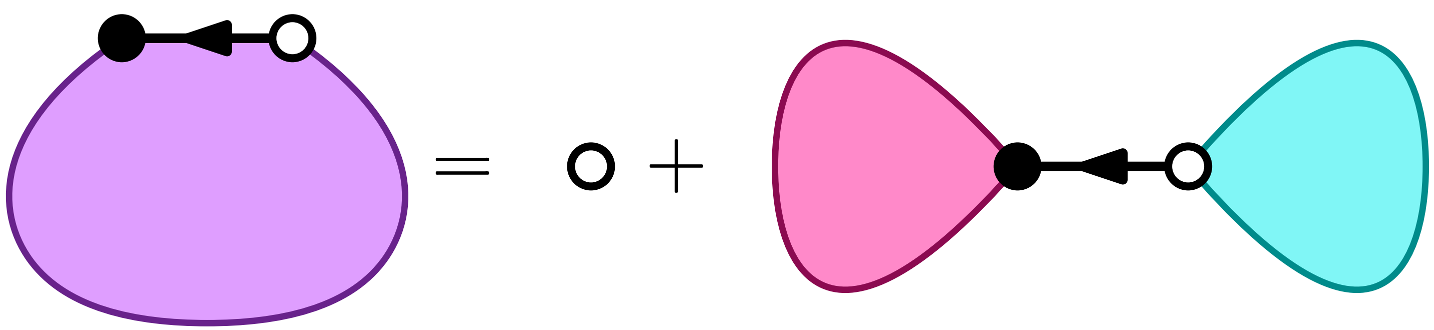

$$B(t) \quad = \quad 1 \quad + \quad tB(t)^2$$

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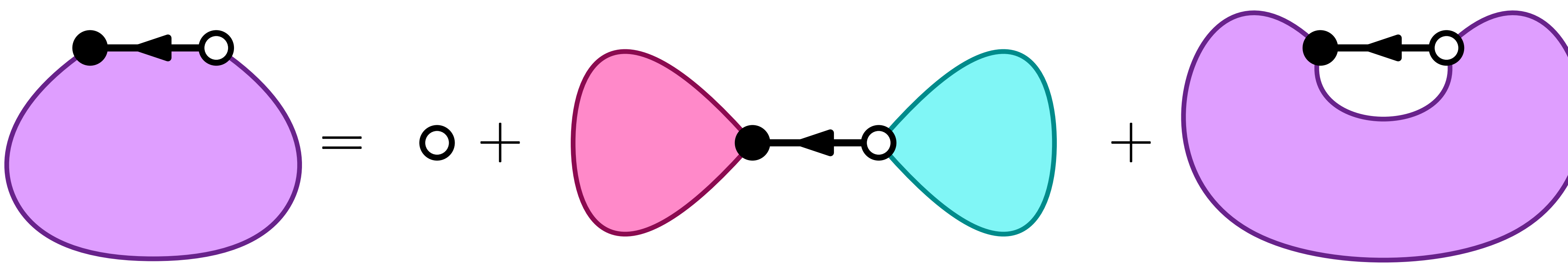

$$B(t) = 1 + tB(t)^2 + ???$$

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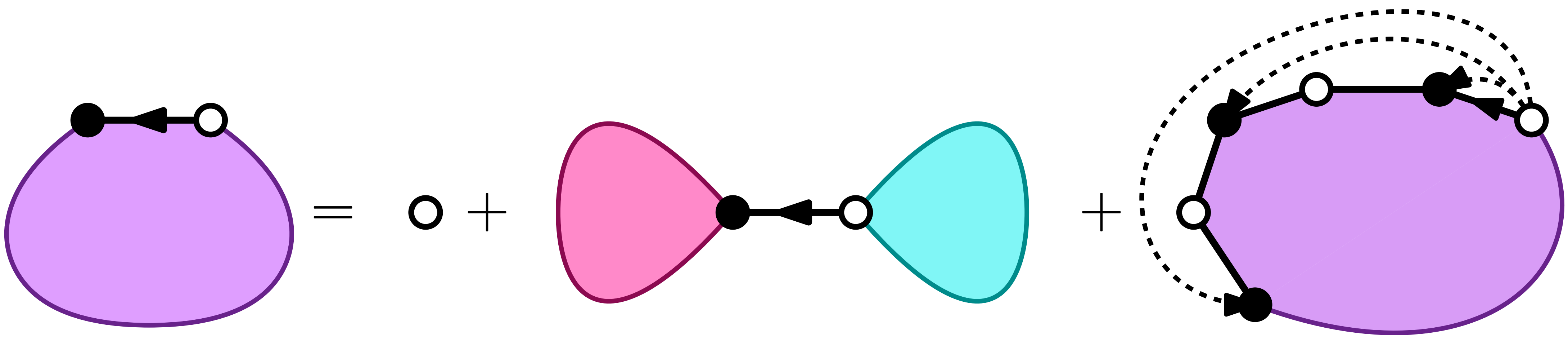

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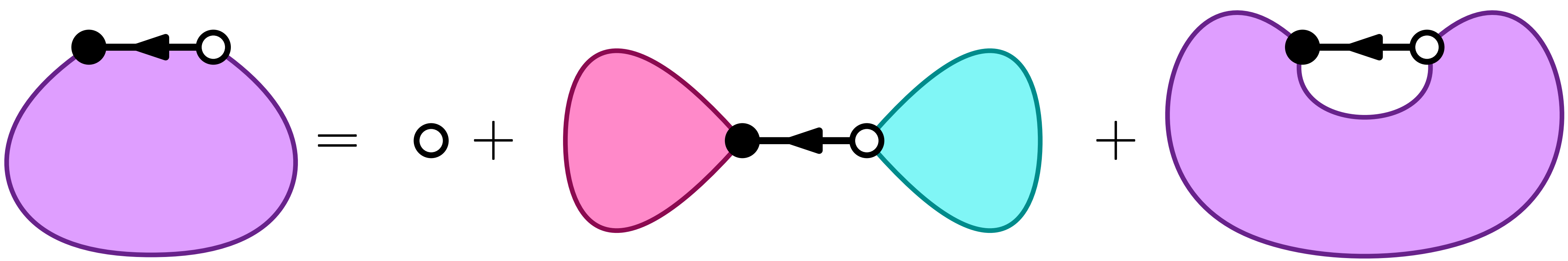
2-colored maps

$B(t)$: generating function of bicolored planar maps counted by number of edges.

$\Longrightarrow$ add a **catalytic variable** recording the half-length of the boundary

$$\hookrightarrow B(t, y) = \sum_{\mathfrak{b} \in \mathcal{B}} t^{e(\mathfrak{b})} y^{b(\mathfrak{b})}.$$



$$B(t, y) \quad = \quad 1 \quad + \qquad tyB(t, y)^2 \quad + ty \sum_{p \geqslant 1} \left(\sum_{k \geqslant 1}^{p} y^{p-k} \right) [y^p] B(t, y)$$

$$\hookrightarrow B(t, y) = 1 + tyB(t, y)^2 + ty \frac{B(t, y) - B(t, 1)}{y - 1}.$$

2-colored maps

$B(t,y)$: generating function of bicolored planar maps, $t \to$ edges, $y \to$ half degree of the outer face.

$$B(t,y) = 1 + tyB(t,y)^2 + ty\frac{B(t,y) - B(t,1)}{y-1}$$

$\hookrightarrow$ Can be solved with the quadratic method [Brown '60s].

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Theorem. [Bousquet-Mélou Jehanne '06] If $F(y) \equiv F(t, y) \in \mathbb{Q}[y][[t]]$ satisfies an equation

$$F(y) = P(y) + tQ(t, y, F(y), \Delta F(y), \ldots, \Delta^k F(y))$$

where P, Q are polynomials and $\Delta F(y) = \dfrac{F(y) - F(1)}{y - 1}$

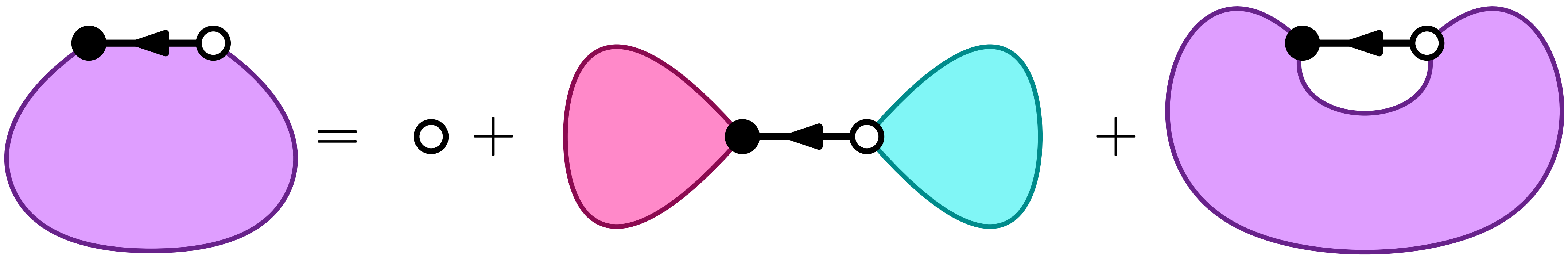
then $F(y)$ is an algebraic series.

$\hookrightarrow$ $B(t, y)$ is algebraic.

$+$ Explicit bijection with a family of trees [Schaeffer '97].

3-colored maps

$M(t, y)$: generating function of 3-colored planar maps counted by number of edges (variable t) and degree of the outer face (variable y).



$$M(t,y) = 1 + \color{red}{2}\color{black}{t y^2 M(t,y)^2} + ty???$$

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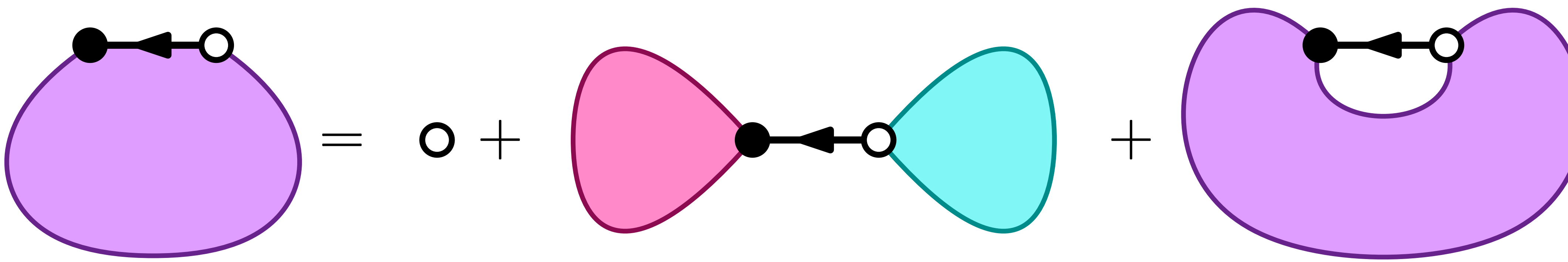
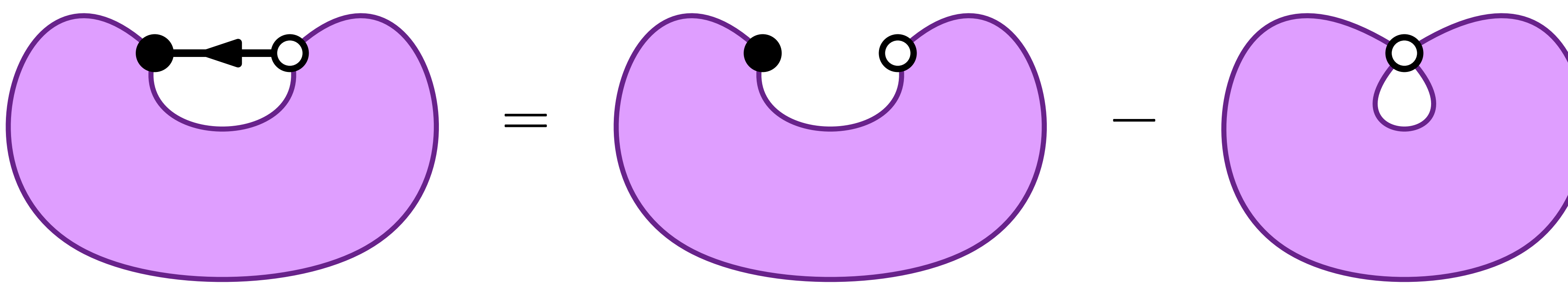


$$M(t,y) \quad = \quad 1 \quad + \quad 2ty^2 M(t,y)^2 \quad + \quad ty???$$

Solution:



$\Longrightarrow$ need a second catalytic variable recording the degree of the root vertex.

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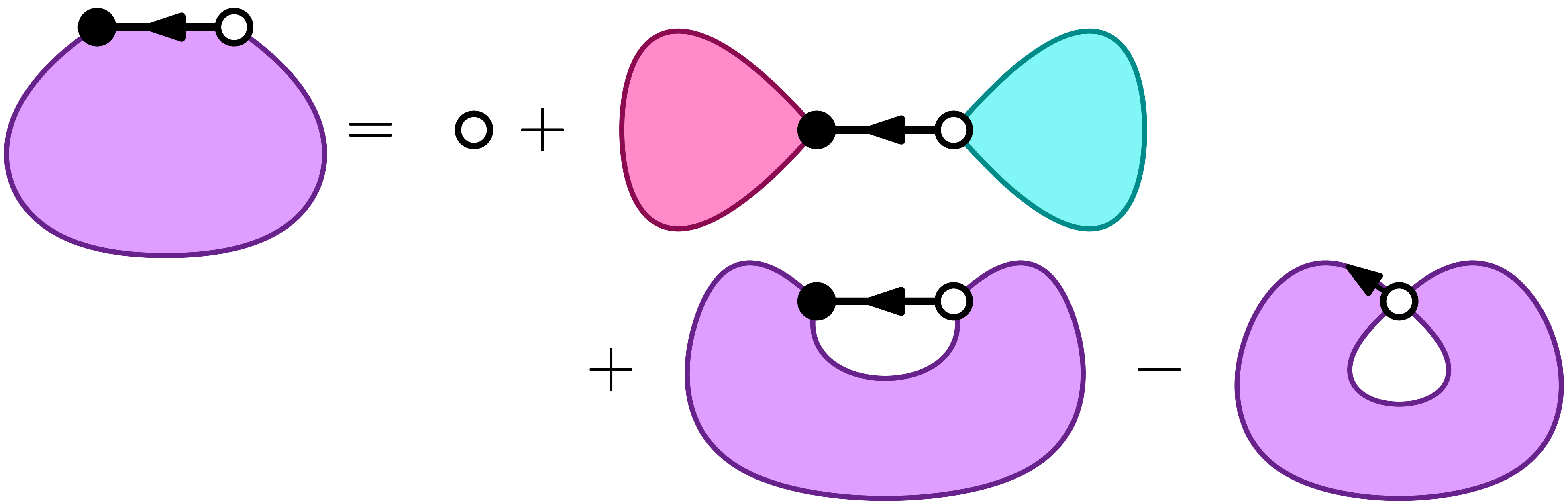
$M(t, x, y)$: generating function of 3-colored planar maps counted by number of edges (variable t), degree of the outer face (variable y) and degree of the root vertex (variable x).



Theorem. [Bernardi Bousquet-Mélou '11]

$$\begin{aligned} M(x,y) &= 1 + xyt(2y-1)M(x,y)M(1,y) - xytM(x,y)M(x,1) \\ &\quad - xyt\frac{xM(x,y) - M(1,y)}{x-1} + xyt\frac{yM(x,y) - M(x,1)}{y-1}. \end{aligned}$$

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Equation for 3-colored planar maps:

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Negative coefficients $\implies$ cannot build a random generator.

Not an algebraic/D-finite/... equation $\implies$ no toolbox ?

We have 2 catalytic variables so Bousquet-Mélou and Jehanne's theorem does not apply.

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Theorem. [Bernardi Bousquet-Mélou '11] The generating function of 3-colored planar map is algebraic and is equal to

$$M(1,1) = \frac{(1+2S)(1-2S^2-4S^3-4S^4)}{(1-2S^3)^2}$$

with S the solution of $t = \dfrac{S(1-S^3)}{(1+2S)^3}$ with constant term 0.

$\hookrightarrow$ Obtained via a reduction to 1-catalytic equation.

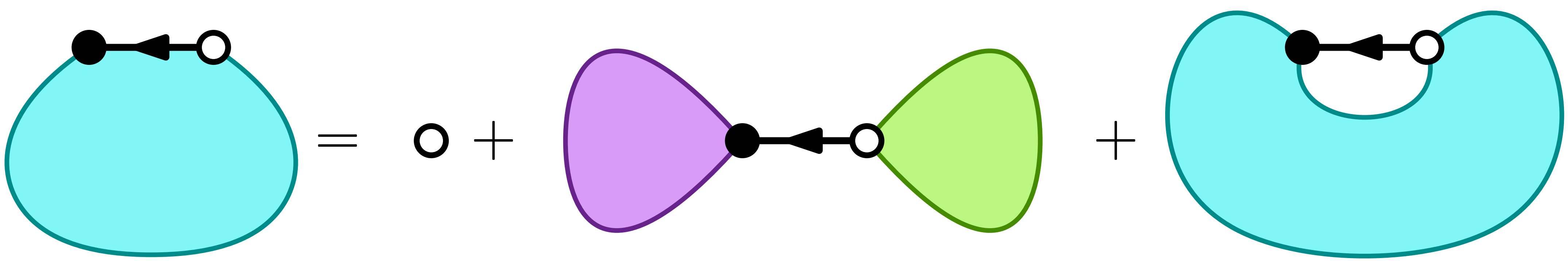
I- Recursive decomposition of colored planar maps

3) Gasket decomposition

Gasket decomposition

Idea: Use the structure from the coloring and get back to bipartite maps.

Let $\mathcal{M}$ be the class of 3-colored planar maps with a **black and white outer face**. Let $M(t, y)$ be their generating function with t counting the edges and y half the degree of the outer face.



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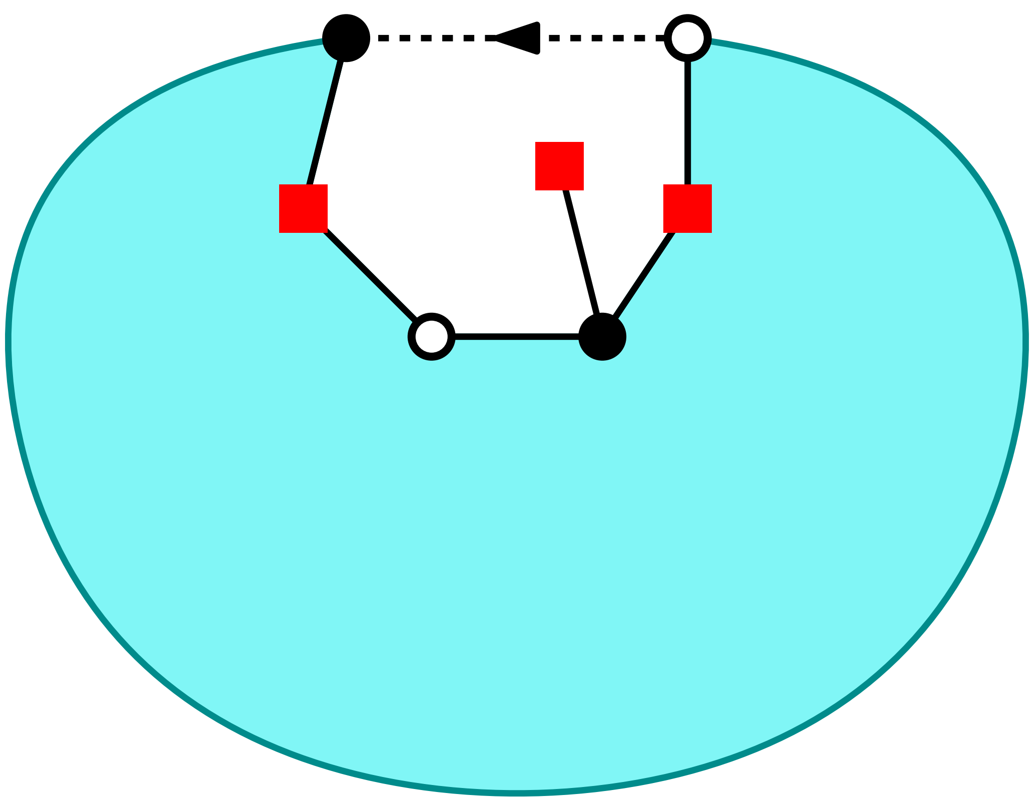

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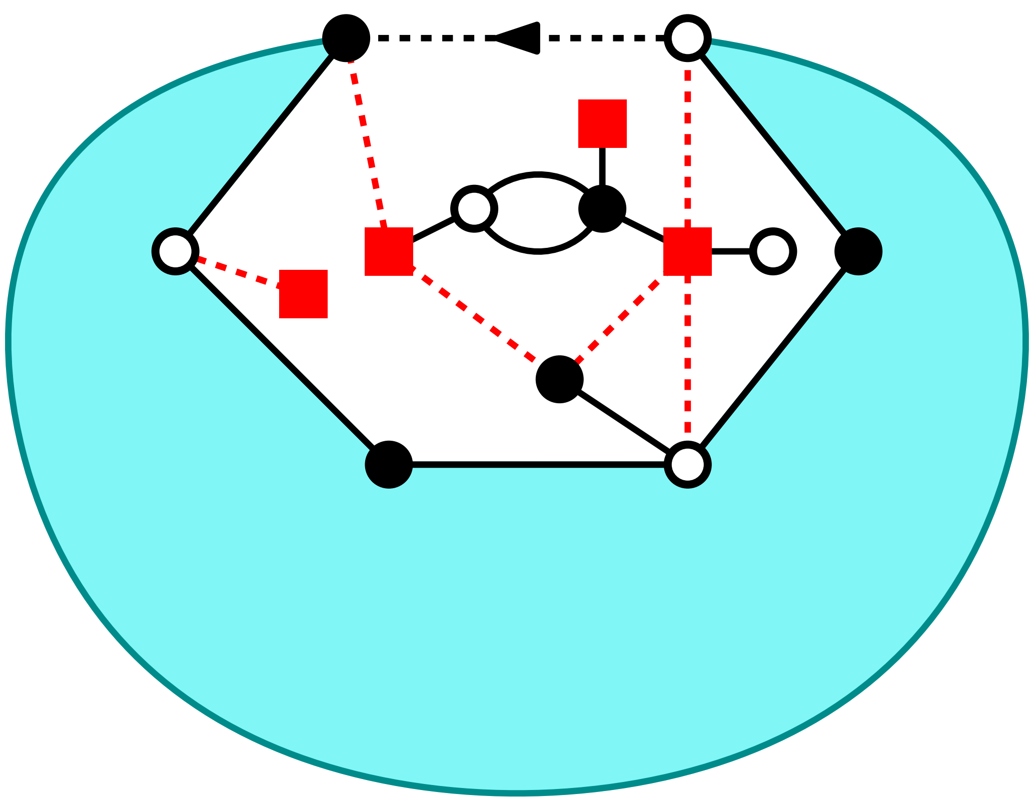

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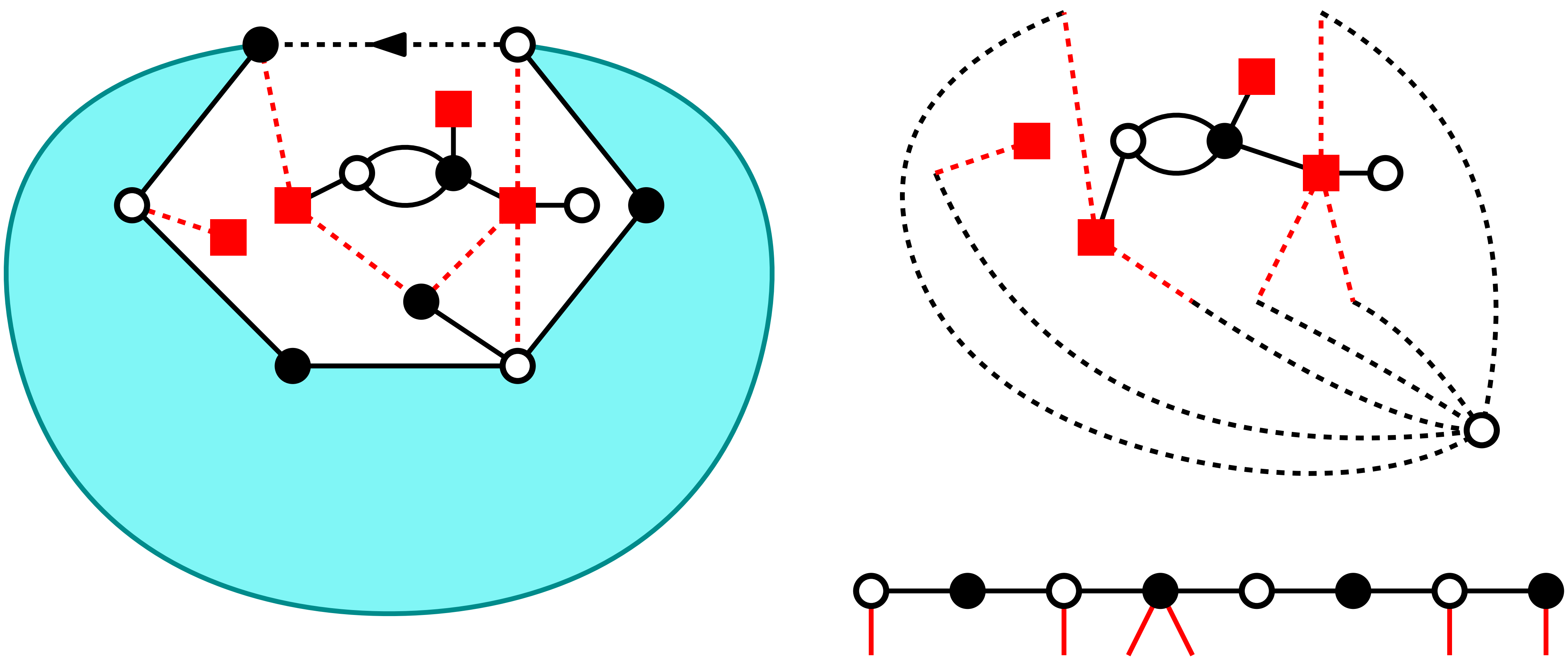

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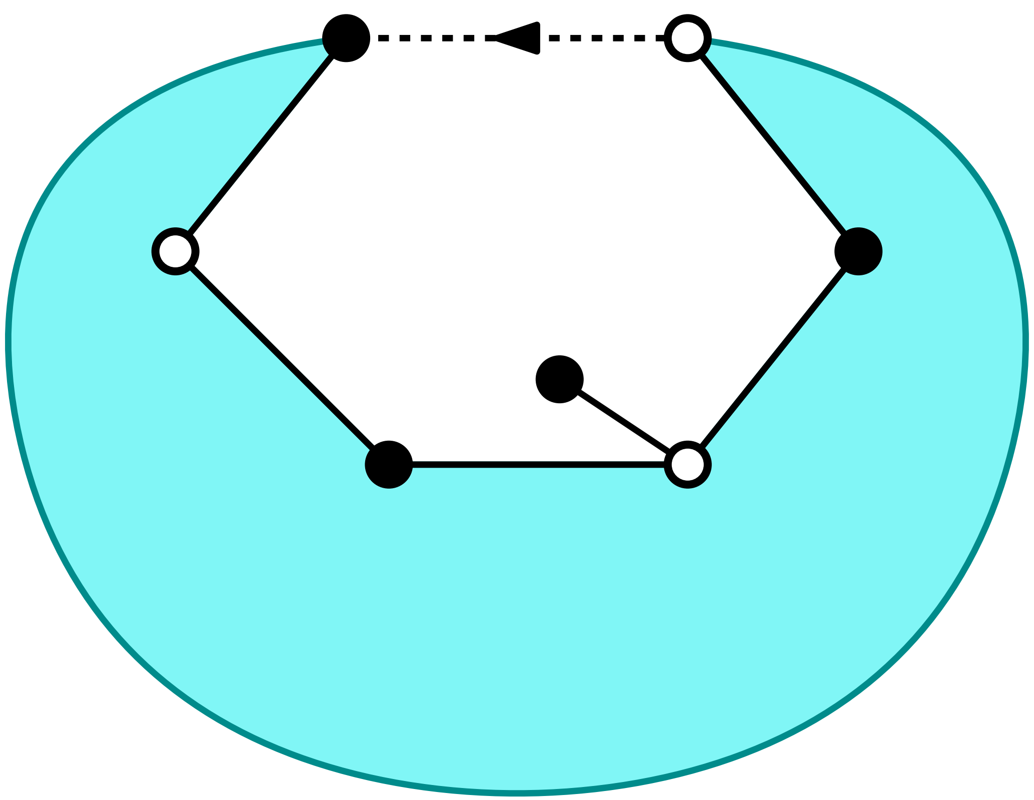

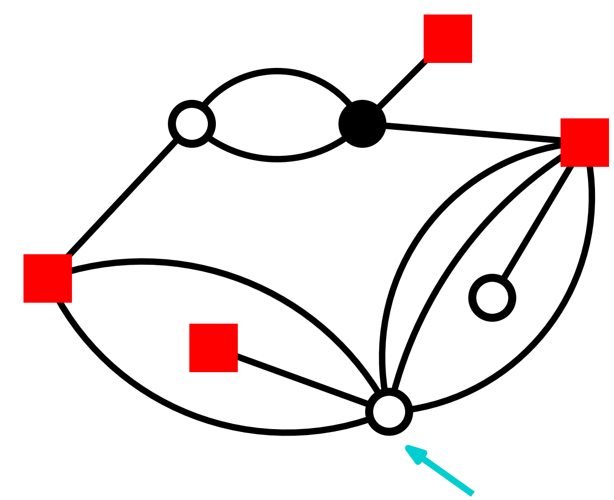

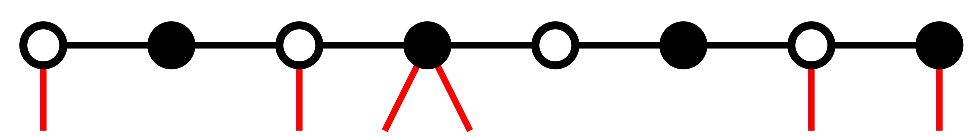

3-colored map where all neighbors of the root vertex are red.
$+$ connection caterpillar.

Gasket decomposition

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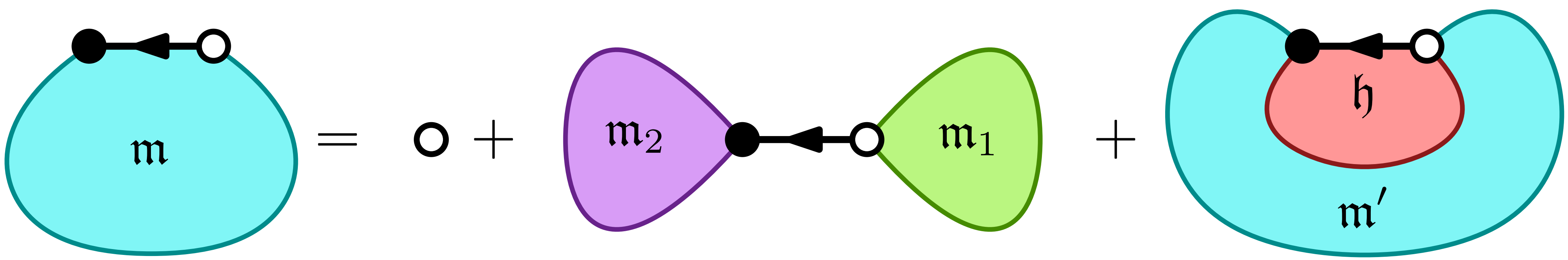

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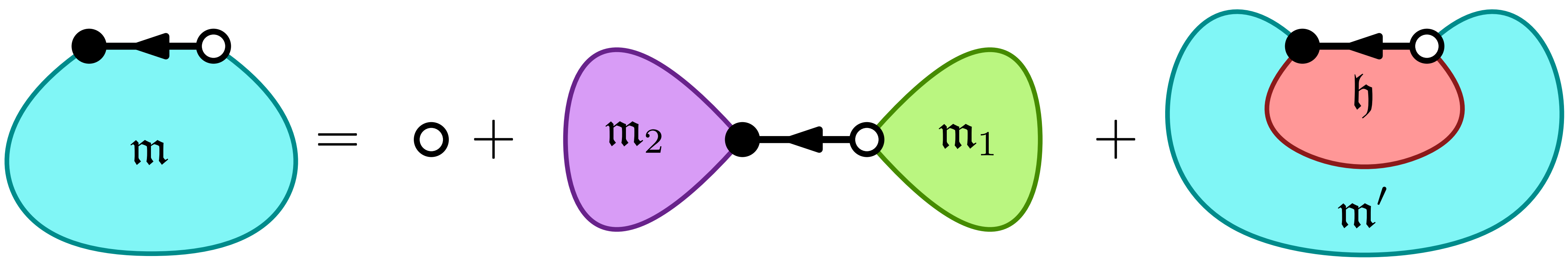

$$M(t,y) = 1 + tyM(t,y)^2 + ty\sum_{p\geqslant 1}\left(\sum_{k\geqslant 1}^{p} y^{p-k}h_k\right)[y^p]M(t,y)$$

$$h_k = \sum_{d\geqslant 0}\binom{2k+d-1}{2k-1}[x^d]H(t,x)$$

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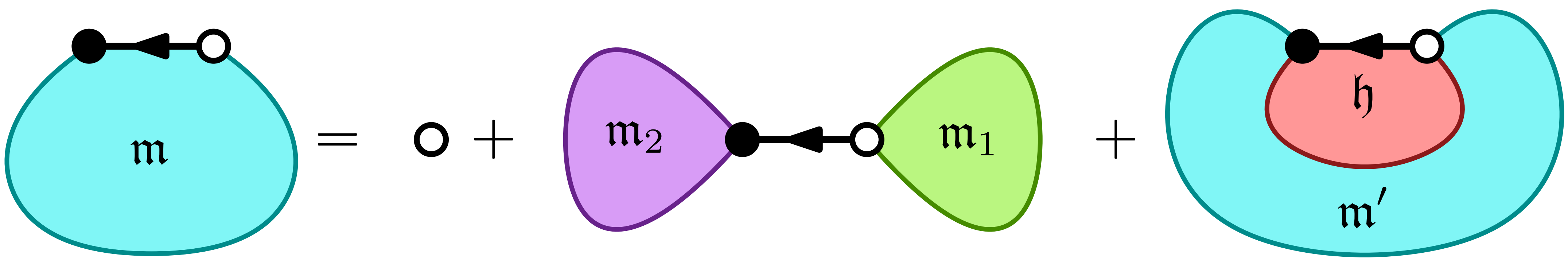
Let $\mathcal{M}$ be the class of 3-colored planar maps with a **black and white outer face**. Let $M(t,y)$ be their generating function with t counting the edges and y half the degree of the outer face.

Let $\mathcal{H}$ be the class of 3-colored planar maps where all the **neighbors** of the **root vertex** are **red**. Let $H(t,x)$ be their generating function with t counting the edges and x the degree of the root vertex.



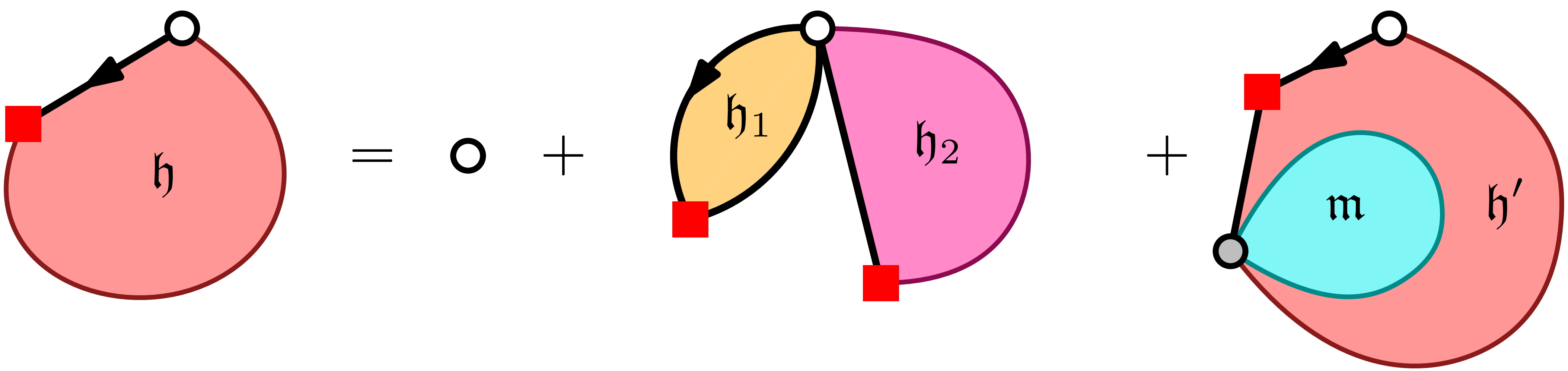
$$M(t,y) = 1 + tyM(t,y)^2 + ty[y^{\geqslant 0}](M(t,y)-1)\widetilde{H}(t,\overline{y}), \qquad \overline{y} = 1/y$$

$$\widetilde{H}(t,y) = \sum_{k\geqslant 1} y^k h_k = \frac{1}{2}\left[\frac{\sqrt{y}}{1-\sqrt{y}} H\left(t, \frac{1}{1-\sqrt{y}}\right) - \frac{\sqrt{y}}{1+\sqrt{y}} H\left(t, \frac{1}{1+\sqrt{y}}\right)\right].$$

Gasket decomposition

Let $\mathcal{M}$ be the class of 3-colored planar maps with a **black and white outer face**.

Let $\mathcal{H}$ be the class of 3-colored planar maps where all the **neighbors** of the **root vertex** are **red**. Let $H(t, x)$ be their generating function with t counting the edges and x the degree of the root vertex.



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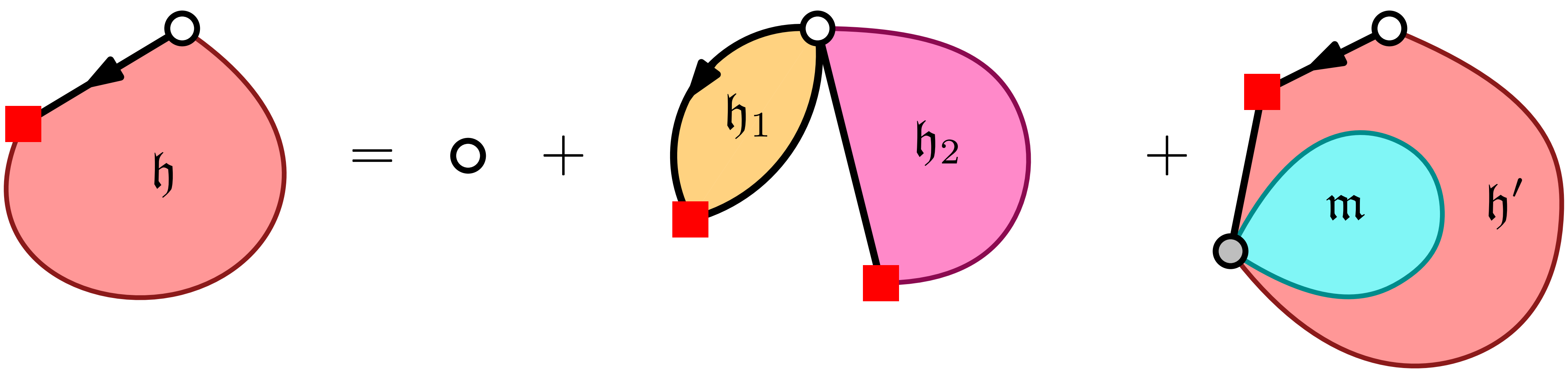

$$H(t,x) = 1 + txH(t,x)^2 + 2tx[x^{\geqslant 0}](H(t,x)-1)\widetilde{M}(t,\overline{x}).$$

$$\widetilde{M}(t,x) = \sum_{i\geqslant 1} x^i \sum_{p\geqslant 0} \binom{2p+i-1}{i-1}[y^p]M(t,y) = \frac{x}{(1-x)^2} M\Big(t, \frac{1}{(1-x)^2}\Big),$$

Gasket decomposition

Let $\mathcal{M}$ be the class of 3-colored planar maps with a **black and white outer face**.
Let $\mathcal{H}$ be the class of 3-colored planar maps where all the **neighbors** of the **root vertex** are **red**. Let $H(t,x)$ be their generating function with t counting the edges and x the degree of the root vertex.

Theorem. [S. '26]

$$\begin{cases} M(t,y) = 1 + tyM(t,y)^2 + ty[y^{\geqslant 0}](M(t,y)-1)\widetilde{H}(t,\overline{y}), \\ H(t,x) = 1 + txH(t,x)^2 + 2tx[x^{\geqslant 0}](H(t,x)-1)\widetilde{M}(t,\overline{x}), \end{cases}$$

with

$$\widetilde{M}(t,x) = \sum_{i\geqslant 1} x^i m_i = \frac{x}{(1-x)^2} M\left(t, \frac{1}{(1-x)^2}\right),$$

$$\widetilde{H}(t,y) = \sum_{i\geqslant 1} y^i h_i = \frac{1}{2}\left[\frac{\sqrt{y}}{1-\sqrt{y}} H\left(t, \frac{1}{1-\sqrt{y}}\right) - \frac{\sqrt{y}}{1+\sqrt{y}} H\left(t, \frac{1}{1+\sqrt{y}}\right)\right].$$

II- Algebraicity proof

$$P_M(S,y,M) = 4y^5(y-1)^2S^5(2S^3-1)^6M^6 + 24y^4(y-1)^2(1+2S)^3(2S^3-1)^5S^4M^5 - (1+2S)(2S^3-1)^4((64S^6+96S^5+88S^4+32S^3-2S+1)y^2 - (16S^4+456S^3+684S^2+334S+57)(1+2S)^2y + 56(1+2S)^5)S^3(y-1)y^3M^4 - 4(2S^3-1)^3(1+2S)^4((64S^6+96S^5+88S^4+32S^3-2S+1)y^2 - (16S^4+136S^3+204S^2+94S+17)(1+2S)^2y + 16(1+2S)^5)S^2(y-1)y^2M^3 - (1+2S)^2(2S^3-1)^2(2S(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3y^4 + (1+2S)(6144S^{10}+20928S^9+33600S^8+32064S^7+18380S^6+5244S^5-162S^4-536S^3-87S^2+18S+5)y^3 - 2(384S^6+1120S^5+1914S^4+1604S^3+706S^2+171S+23)(1+2S)^5y^2 + (96S^4+616S^3+924S^2+414S+77)(1+2S)^7y - 36(1+2S)^{10})SyM^2 - 2(2S^3-1)(1+2S)^5(2S(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3y^4 + (1+2S)(2048S^{10}+6592S^9+9536S^8+748\ldots$$ [illegible] $$\ldots-6S+1)y^3 - 2(128S^6+224S^5+266S^4+164S^3+50S^2+11S+3)(1+2S)^5y^2 + (32S^4+72S^3+108S^2+38S+9)(1+2S)^7y - 4(1+2S)^{10})M - (1+2S)^3(S(4S^2+2S+1)^3(S+1)^6y^4 + 2(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3(1+2S)^5y^3 + (1024S^9+3008S^8+3520S^7+1344S^6-1460S^5-2436S^4-1642S^3-616S^2-127S-12)(1+2S)^6y^2 - 2(64S^5+64S^4+14S^3-36S^2-34S-9)(1+2S)^{10}y + 8(2S^3-1)(1+2S)^{12})$$

$$P_H(S,\sqrt{\Delta},x,H) = S^3x^5(2S^3-1)^4(2x-1)H^4 - 2S^2x^3(2x-1)(x-2)(1+2S)^3(2S^3-1)^3H^3 + Sx(1+2S)(2S^3-1)^2(2S(2S^3+4S^2+2S+1)x^4 - 2(8S^4+8S^3+12S^2+2S+1)(1+2S)^2x^3 + (8S^4-24S^3-36S^2-22S-3)(1+2S)^2x^2 + 10(1+2S)^5x - 4(1+2S)^5)H^2 - 2(1+2S)(2S^3-1)(S((8S^5+28S^4+38S^3+26S^2+10S+2)\sqrt{\Delta} + 16S^6+56S^5+76S^4+58S^3+28S^2+8S+1)x^4 - 2(16S^6+48S^5+86S^4+76S^3+34S^2+9S+1)(1+2S)^3x^3 + (20S^4+56S^3+84S^2+32S+7)(1+2S)^5x^2 - (8S^4+56S^3+84S^2+38S+7)(1+2S)^5x + 2(1+2S)^8)H(1+2S)^3((32S^8+(16S^6+64S^5+104S^4+90S^3+46S^2+14S+2)\sqrt{\Delta}+176S^7+384S^6+452S^5+356S^4+182S^3+64S^2+15S+2)x^3 - 2(1+2S)(-64S^9-320S^8-656S^7+(8S^5+28S^4+38\sqrt{\Delta}S^3+26S^2+10S+2)\sqrt{\Delta} - 560S^6-72S^5+332S^4+354S^3+172S^2+44S+5)x^2 - 2(8S^7+96S^6+144S^5+60S^4-46S^3-80S^2-38S-7)(1+2S)^3x + 4(2S^3-1)(1+2S)^6)$$

The strategy: Guess and Check

$$\begin{cases} M(t,y) = 1 + tyM(t,y)^2 + ty[y^{\geqslant 0}](M(t,y)-1)\widetilde{H}(t,\overline{y}), \\ H(t,x) = 1 + txH(t,x)^2 + 2tx[x^{\geqslant 0}](H(t,x)-1)\widetilde{M}(t,\overline{x}). \end{cases}$$

Idea: The equations allow us to recursively compute all the coefficients of the series M and H

$\implies$ unique solution in $\mathbb{Q}[y][[t]] \times \mathbb{Q}[x][[t]]$.

Guess and Check:

1. Guess polynomial equations satisfied by our series.
2. Prove that the solutions of the polynomial equations satisfy the original equations.

Guess

$$\begin{cases} M(t,y) = 1 + tyM(t,y)^2 + ty[y^{\geqslant 0}](M(t,y)-1)\widetilde{H}(t,\overline{y}), \\ H(t,x) = 1 + txH(t,x)^2 + 2tx[x^{\geqslant 0}](H(t,x)-1)\widetilde{M}(t,\overline{x}). \end{cases}$$

Guess:

1. Use the equations to compute the first (250) terms of the series M and H.
2. Guess polynomial equations with `algeqtoseries` (package `gfun`).
3. Determine which solution of the polynomial equation is the candidate.

$\hookrightarrow$ We use the intermediate field $\mathbb{Q}(t,y) \to \mathbb{Q}(S,y) \to \mathbb{Q}(t,y,M(t,y))$ where $t = \dfrac{S(1-S^3)}{(1+2S)^3}$ to have smaller equations.

$\hookrightarrow$ Equations of degree 6 for $M(t,y)$ over $\mathbb{Q}(S,y)$ and of degree 8 for $H(t,x)$ over $\mathbb{Q}(S,x)$.

Guess

1. Use the equations to compute the first (250) terms of the series M and H.
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$\hookrightarrow$ Equations of degree 6 for $M(t,y)$ over $\mathbb{Q}(S,y)$ and of degree 8 for $H(t,x)$ over $\mathbb{Q}(S,x)$.

$$P_M(S,y,M) = 4y^5(y-1)^2S^5(2S^3-1)^6M^6 + 24y^4(y-1)^2(1+2S)^3(2S^3-1)^5S^4M^5 - (1+2S)(2S^3-1)^4((64S^6+96S^5+88S^4+32S^3-2S+1)y^2-(16S^4+456S^3+684S^2+334S+57)(1+2S)^2y+56(1+2S)^5)S^3(y-1)y^3M^4 - 4(2S^3-1)^3(1+2S)^4((64S^6+96S^5+88S^4+32S^3-2S+1)y^2-(16S^4+136S^3+204S^2+94S+17)(1+2S)^2y+16(1+2S)^5)S^2(y-1)y^2M^3 - (1+2S)^2(2S^3-1)^2(2S(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3y^4+(1+2S)(6144S^{10}+20928S^9+33600S^8+32064S^7+18380S^6+5244S^5-162S^4-536S^3-87S^2+18S+5)y^3-2(384S^6+1120S^5+1914S^4+1604S^3+706S^2+171S+23)(1+2S)^5y^2+(96S^4+616S^3+924S^2+414S+77)(1+2S)^7y-36(1+2S)^{10})SyM^2 - 2(2S^3-1)(1+2S)^5(2S(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3y^4+(1+2S)(2048S^{10}+6592S^9+9536S^8+7488S^7+2508S^6-900S^5-1346S^4-600S^3-119S^2-6S+1)y^3-2(128S^6+224S^5+266S^4+164S^3+50S^2+11S+3)(1+2S)^5y^2+(32S^4+72S^3+108S^2+38S+9)(1+2S)^7y-4(1+2S)^{10})M - (1+2S)^3(S(4S^2+2S+1)^3(S+1)^6y^4+2(64S^6+96S^5+94S^4+50S^3+18S^2+4S+1)(S+1)^3(1+2S)^5y^3+(1024S^9+3008S^8+3520S^7+1344S^6-1460S^5-2436S^4-1642S^3-616S^2-127S-12)(1+2S)^6y^2-2(64S^5+64S^4+14S^3-36S^2-34S-9)(1+2S)^{10}y+8(2S^3-1)(1+2S)^{12})$$

Guess

1. Use the equations to compute the first (250) terms of the series M and H.
2. Guess polynomial equations with `algeqtoseries` (package `gfun`).
3. Determine which solution of the polynomial equation is the candidate.

$\hookrightarrow$ Equations of degree 6 for $M(t,y)$ over $\mathbb{Q}(S,y)$ and of degree 8 for $H(t,x)$ over $\mathbb{Q}(S,x)$.

Both polynomial equations have only one solution that is a power series in t and they both have polynomial coefficients.
$\hookrightarrow$ We denote them M_c and H_c.

Check

Check: Prove that M_c and H_c satisfy

$$\begin{cases} M_c(t,y) = 1 + tyM_c(t,y)^2 + ty[y^{\geqslant 0}](M_c(t,y)-1)\widetilde{H}_c(t,\overline{y}), \\ H_c(t,x) = 1 + txH_c(t,x)^2 + 2tx[x^{\geqslant 0}](H_c(t,x)-1)\widetilde{M}_c(t,\overline{x}), \end{cases}$$

with $\widetilde{M}_c(t,x) = \dfrac{x}{(1-x)^2} M_c\left(t, \dfrac{1}{(1-x)^2}\right)$

and $\widetilde{H}_c(t,y) = \dfrac{1}{2}\left[\dfrac{\sqrt{y}}{1-\sqrt{y}} H_c\left(t, \dfrac{1}{1-\sqrt{y}}\right) - \dfrac{\sqrt{y}}{1+\sqrt{y}} H_c\left(t, \dfrac{1}{1+\sqrt{y}}\right)\right]$.

Check

Check: Prove that M_c and H_c satisfy

$$\begin{cases} M_c(t,y) = 1 + tyM_c(t,y)^2 + ty[y^{\geqslant 0}](M_c(t,y)-1)\widetilde{H}_c(t,\overline{y}), \\ H_c(t,x) = 1 + txH_c(t,x)^2 + 2tx[x^{\geqslant 0}](H_c(t,x)-1)\widetilde{M}_c(t,\overline{x}), \end{cases}$$

with $\widetilde{M}_c(t,x) = \dfrac{x}{(1-x)^2} M_c\left(t, \dfrac{1}{(1-x)^2}\right)$

and $\widetilde{H}_c(t,y) = \dfrac{1}{2}\left[\dfrac{\sqrt{y}}{1-\sqrt{y}} H_c\left(t, \dfrac{1}{1-\sqrt{y}}\right) - \dfrac{\sqrt{y}}{1+\sqrt{y}} H_c\left(t, \dfrac{1}{1+\sqrt{y}}\right)\right].$

1. Reformulate the equations:

$$\begin{cases} [y^{\geqslant 0}]R_M(y) = 0, \\ [x^{\geqslant 0}]R_H(x) = 0, \end{cases}$$

where $R_M(y) = (M_c(y)-1)\widetilde{H}_c(\overline{y}) - \dfrac{1}{ty}\left(M_c(y) - 1 - tyM_c(y)^2\right) \in \mathbb{Q}((\overline{y}))[[t]]$

and $R_H(x) = (H_c(x)-1)\widetilde{M}_c(\overline{x}) - \dfrac{1}{2tx}\left(H_c(x) - 1 - txH_c(x)^2\right) \in \mathbb{Q}((\overline{x}))[[t]].$

Check

Check: Prove that $[y^{\geqslant 0}]R_M(y) = [x^{\geqslant 0}]R_H(x) = 0$.

2. Compute minimal polynomials for R_M and R_H by eliminating M_c and H_c using resultants.

$\hookrightarrow$ An equation of degree 12 for R_M over $\mathbb{Q}(S, y)$ and one of degree 12 for R_H over $\mathbb{Q}(S, x)$.

Unfortunately, the equations have solutions with positive powers

$\Longrightarrow$ no "all solutions are negative" result

$\Longrightarrow$ getting something on R will be hard.

Check

Check: Prove that $[y^{\geqslant 0}]R_M(y) = [x^{\geqslant 0}]R_H(x) = 0$.

2. Compute minimal polynomials for R_M and R_H by eliminating M_c and H_c using resultants.

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Unfortunately, the equations have solutions with positive powers

$\Longrightarrow$ no "all solutions are negative" result

$\Longrightarrow$ getting something on R will be hard.

3. Find intermediate field extensions.

$$\mathbb{Q}(S,y) \xrightarrow{3} \mathbb{Q}(S,L) \xrightarrow{2} \mathbb{Q}(S,K) \xrightarrow{2} \mathbb{Q}(S,y,R_M)$$

$$L = -\frac{1}{y}\frac{(1+2S)^3(S+2S^2+4S^3-L)(4S^3-L)}{S(1+2S+2S^2+4S^3-L)^2},$$

$$K = \frac{L}{4S(4S^2+2S+1)(1-K)}.$$

Check

Check: Prove that $[y^{\leqslant 0}]R_M(y) = [x^{\leqslant 0}]R_H(x) = 0$.

3. Find intermediate field extensions.

$$\mathbb{Q}(S,y) \xrightarrow{3} \mathbb{Q}(S,L) \xrightarrow{2} \mathbb{Q}(S,K) \xrightarrow{2} \mathbb{Q}(S,y,R_M)$$

$$\begin{aligned} L &= -\frac{1}{y}\frac{(1+2S)^3(S+2S^2+4S^3-L)(4S^3-L)}{S(1+2S+2S^2+4S^3-L)^2}, \\ K &= \frac{L}{4S(4S^2+2S+1)(1-K)}. \end{aligned}$$

4. Solve the equation for R_M over $\mathbb{Q}(S,K)$.

$$\hookrightarrow \quad R_M(y) = \frac{B - C\sqrt{D}}{2A} \qquad \text{with } A,B,C,D \in \mathbb{Q}[S,K].$$

5. Prove that L then K then R_M is y-negative.

6. Apply the same method to R_H.

Algebraicity

a.
$$\begin{cases} M(t,y) = 1 + tyM(t,y)^2 + ty[y^{\geqslant 0}](M(t,y)-1)\widetilde{H}(t,\overline{y}), \\ H(t,x) = 1 + txH(t,x)^2 + 2tx[x^{\geqslant 0}](H(t,x)-1)\widetilde{M}(t,\overline{x}). \end{cases}$$

has a unique solution in $\mathbb{Q}[y][[t]] \times \mathbb{Q}[x][[t]]$.

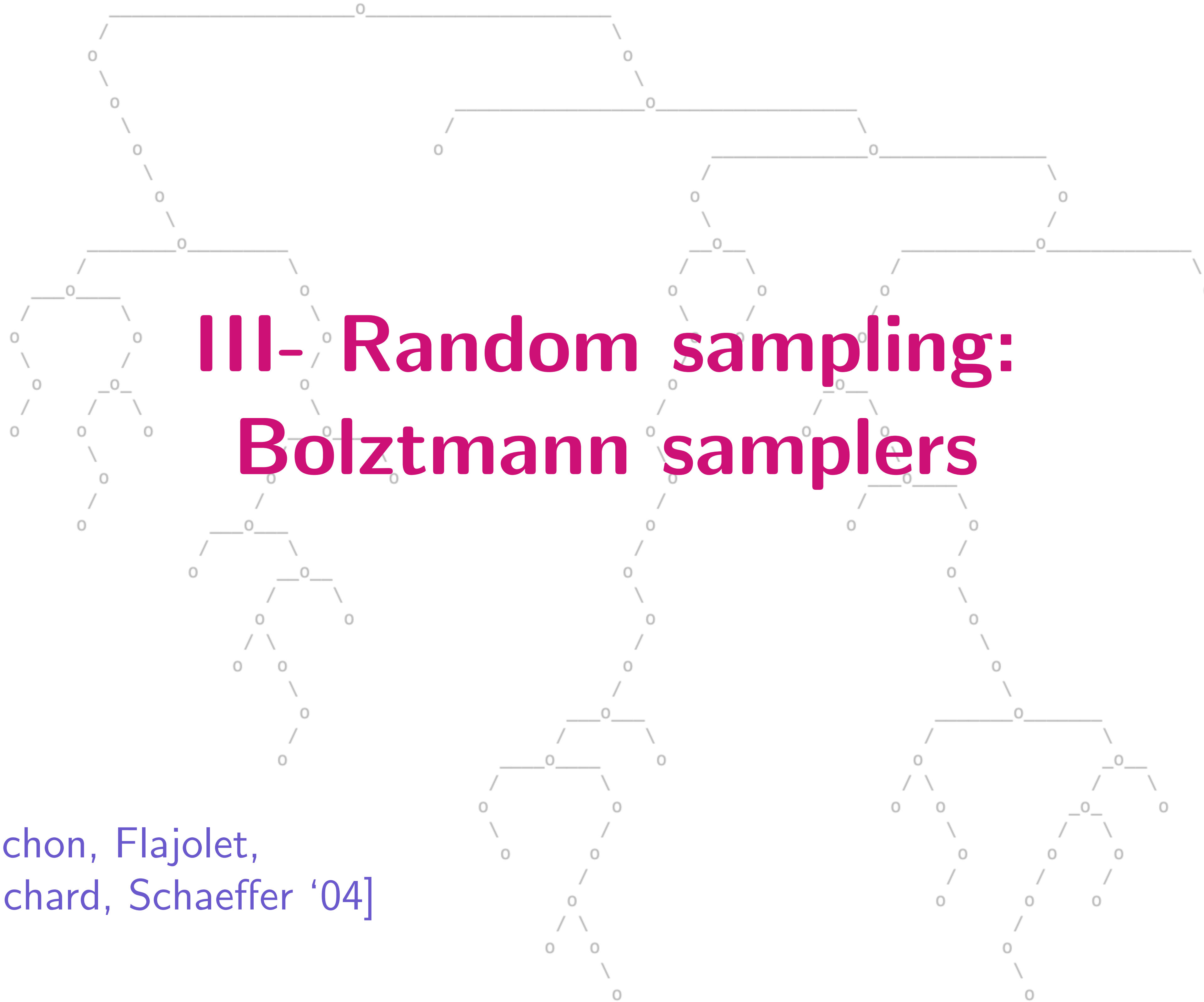
b. There are polynomials $P_M(S,y,M)$ and $P_H(S,x,H)$ whose only power series solution satisfy the system.

Theorem. [S. '26] The generating function $M(t,y)$ of 3-colored planar maps with black and white border is algebraic over $\mathbb{Q}(t,y)$ of degree 24.

- Complete map of the subfields of $\mathbb{Q}(t,y,M(y))$.

$$\mathbb{Q}(t,y) \xrightarrow{4} \mathbb{Q}(S,y) \xrightarrow{3} \mathbb{Q}(S,L) \xrightarrow{2} \mathbb{Q}(S,y,M(y))$$

- All $[y^p]M(S,y)$ are rational fractions in S.



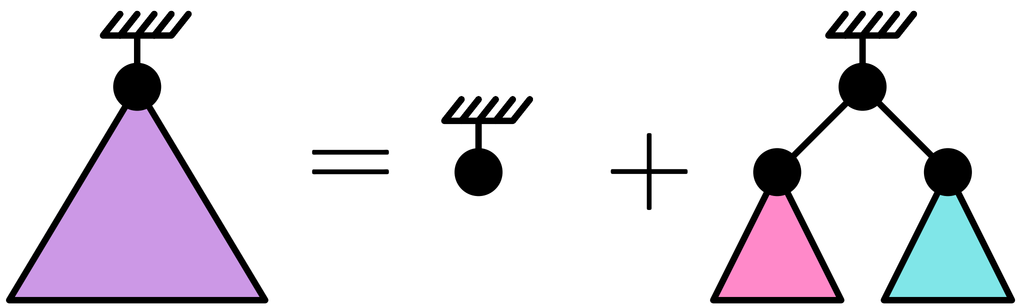
III- Random sampling: Bolztmann samplers

[Duchon, Flajolet, Louchard, Schaeffer ‘04]

Exact size sampling

Principle: use the recursive decomposition to sample uniformly objects of a given size n.

For binary trees :



$$t_1 = 1$$

$$t_n = \sum_{k=1}^{n-1} t_k t_{n-k-1}$$

BinTreeSampler`(n) =`
  if `n = 1` **then** •
  else `draw k with` $\mathbb{P}(k) = t_k t_{n-k-1}/t_n$
    `(BinTreeSampler(k), BinTreeSampler(n-1-k))`

Exact size sampling

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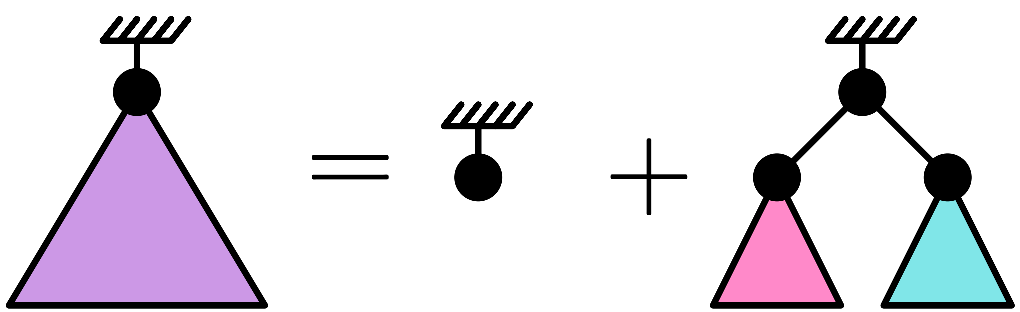


$$t_1 = 1$$

$$t_n = \sum_{k=1}^{n-1} t_k t_{n-k-1}$$

BinTreeSampler`(n) =`
  if `n = 1` **then** •
  else `draw k with` $\mathbb{P}(k) = t_k t_{n-k-1}/t_n$ ← very expensive
    `(BinTreeSampler(k), BinTreeSampler(n-1-k))`

- Need an explicit recurrence.
- Very slow for large size objects.

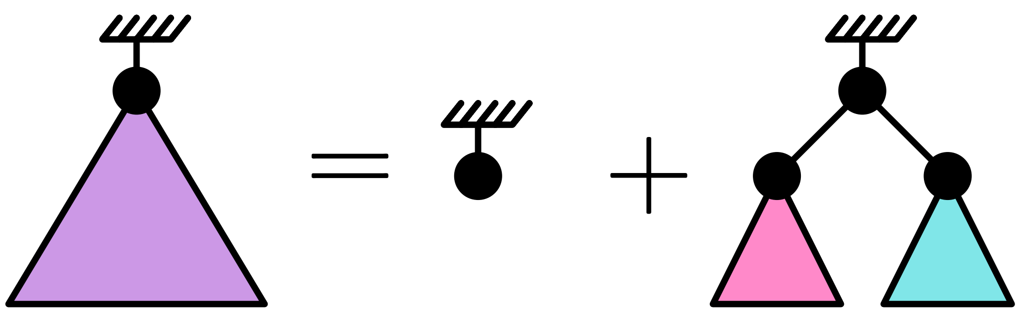
Boltzmann samplers

Principle: sample objects $\gamma \in \mathcal{C}$ according to the probability distribution

$$\mathbb{P}(\gamma) = \frac{t^{|\gamma|}}{C(t)}$$

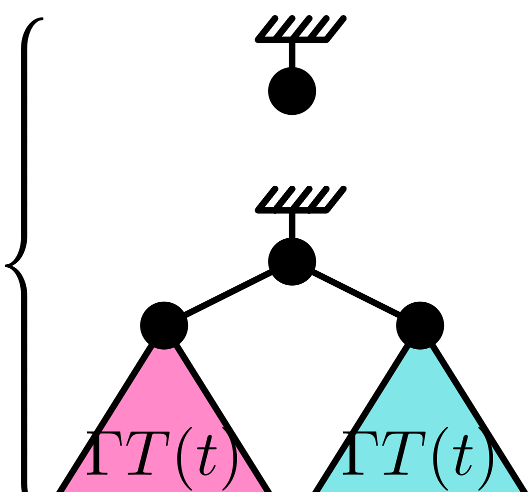
with t a parameter, by following a recursive decomposition of $\mathcal{C}$.

For binary trees :



$$T(t) = t + tT(t)^2$$

$$\Gamma T(t) = \begin{cases} \text{probability } \dfrac{t}{T(t)} \\ \text{probability } tT(t) \end{cases}$$



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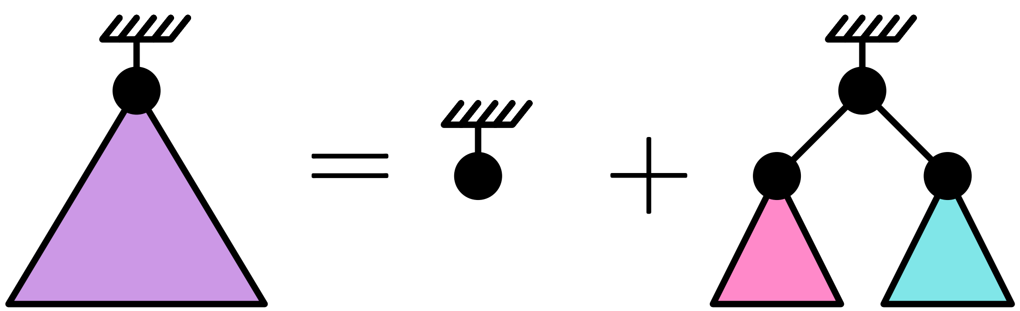
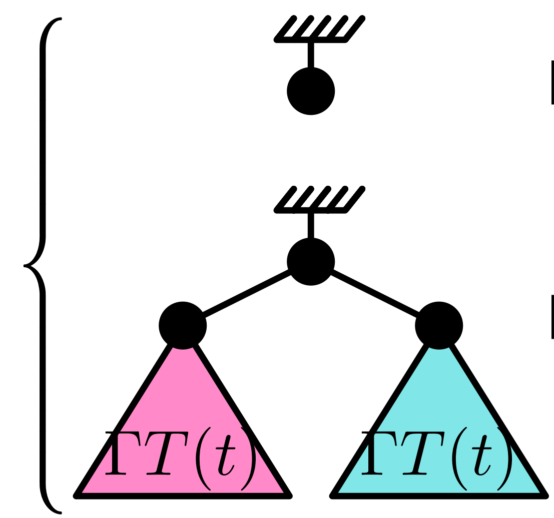


$$T(t) = t + tT(t)^2$$

$\Gamma T(t) =$



probability $\frac{t}{T(t)}$

probability $tT(t)$

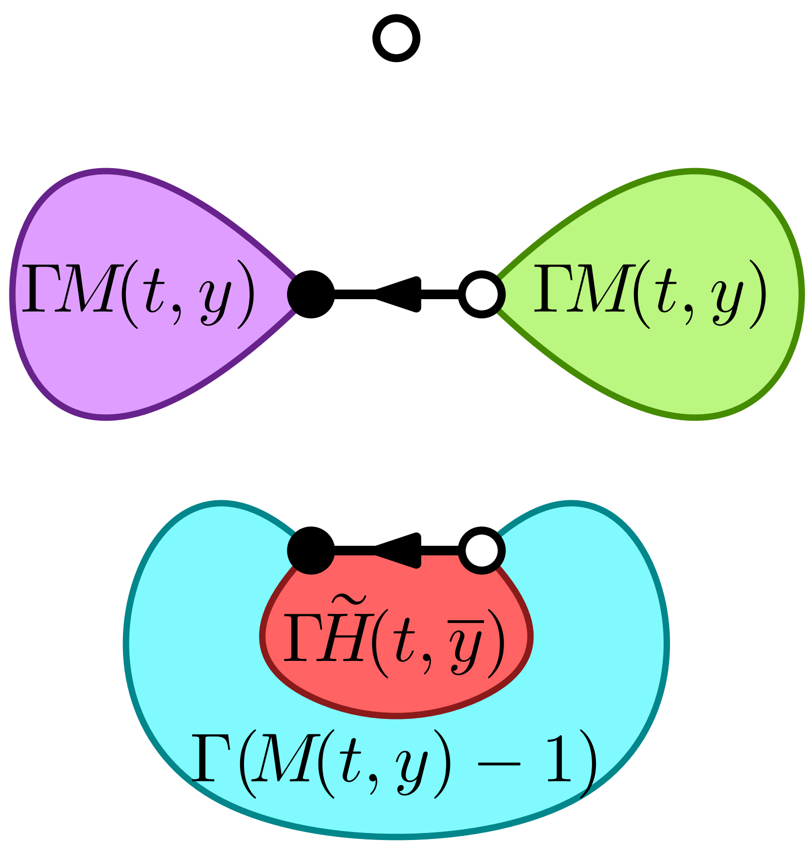
- Unfixed size (adjust the value of t to aim a given size).
- Uniform distribution when conditioned by the size of the object.
- Can be built as long as we have a positive combinatorial equation.
- Very efficient for large objects.

Boltzmann sampler for 3-colored maps

- The catalytic variables becomes a second parameter: $\mathbb{P}(\mathfrak{m}) = \dfrac{t^{e(\mathfrak{m})} y^{b(\mathfrak{m})}}{M(t,y)}$.
- We use rejection to handle the $[y^{\geqslant 0}]$ and $[x^{\geqslant 0}]$.

$$\begin{cases} M(t,y) = 1 + tyM(t,y)^2 + ty[y^{\geqslant 0}](M(t,y)-1)\widetilde{H}(t,\overline{y}), \\ H(t,x) = 1 + txH(t,x)^2 + 2tx[x^{\geqslant 0}](H(t,x)-1)\widetilde{M}(t,\overline{x}). \end{cases}$$

$\Gamma M(t,y) =$



- with probability $\dfrac{1}{M(t,y)}$
- with probability $tyM(t,y)$
- Otherwise. Reject while the lengths of the borders of the maps are incompatible.

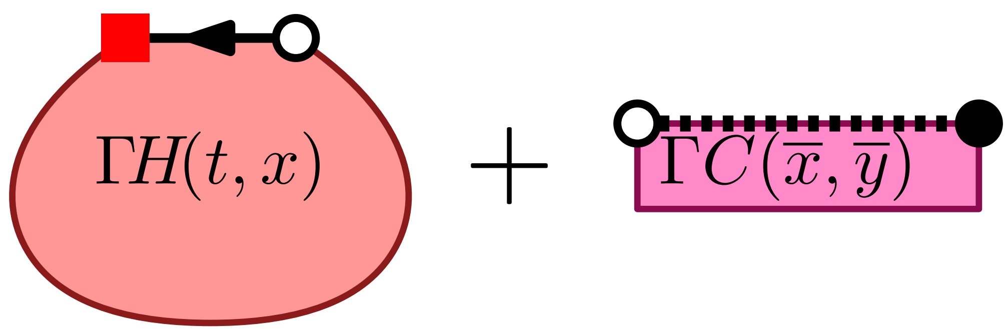
$\Gamma H(t,x)$ similarly.

Boltzmann sampler for 3-colored maps

- The catalytic variables becomes a second parameter: $\mathbb{P}(\mathfrak{m}) = \dfrac{t^{e(\mathfrak{m})} y^{b(\mathfrak{m})}}{M(t,y)}$.
- We use rejection to handle the $[y^{\geqslant 0}]$ and $[x^{\geqslant 0}]$.

$$\begin{cases} M(t,y) = 1 + tyM(t,y)^2 + ty[y^{\geqslant 0}](M(t,y)-1)\widetilde{H}(t,\overline{y}), \\ H(t,x) = 1 + txH(t,x)^2 + 2tx[x^{\geqslant 0}](H(t,x)-1)\widetilde{M}(t,\overline{x}). \end{cases}$$

$\Gamma\widetilde{H}(t,\overline{y}) =$



Reject while number of legs of $\mathfrak{c} \neq$ root degree of $\mathfrak{h}$.

C counts caterpillars of even length with $\overline{x} \to$ legs and $\overline{y} \to$ half length.

Boltzmann sampler for 3-colored maps

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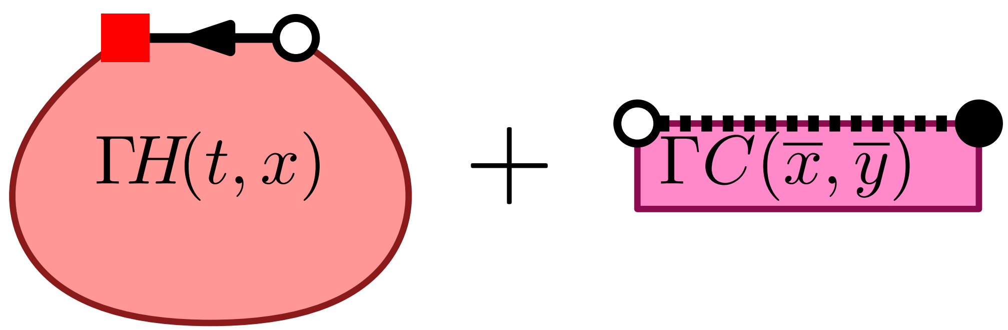

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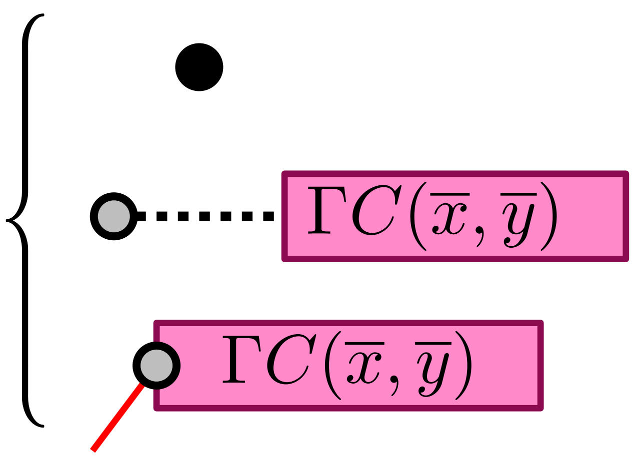
$\Gamma C(\overline{x},\overline{y}) =$



with probability $\sqrt{\overline{y}}/A(\overline{x},\overline{y})$,

with probability $\sqrt{\overline{y}}$,

with probability $\overline{x}$.

Reject if the length is odd.

Boltzmann sampler for 3-colored maps

Pretty image of a big random 3-colored map
that I'll have as soon as I find time to
implement this sampler.

Perspectives

- Adapt the gasket decomposition for others families of colored maps.
  - $\hookrightarrow$ Colorful quadrangulations [Budd '25].
  - $\hookrightarrow$ 4-colored triangulations.
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Thank you !