Slit-slide-sew bijections for bipartite planar maps with prescribed degrees

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LaBRI

Introduction

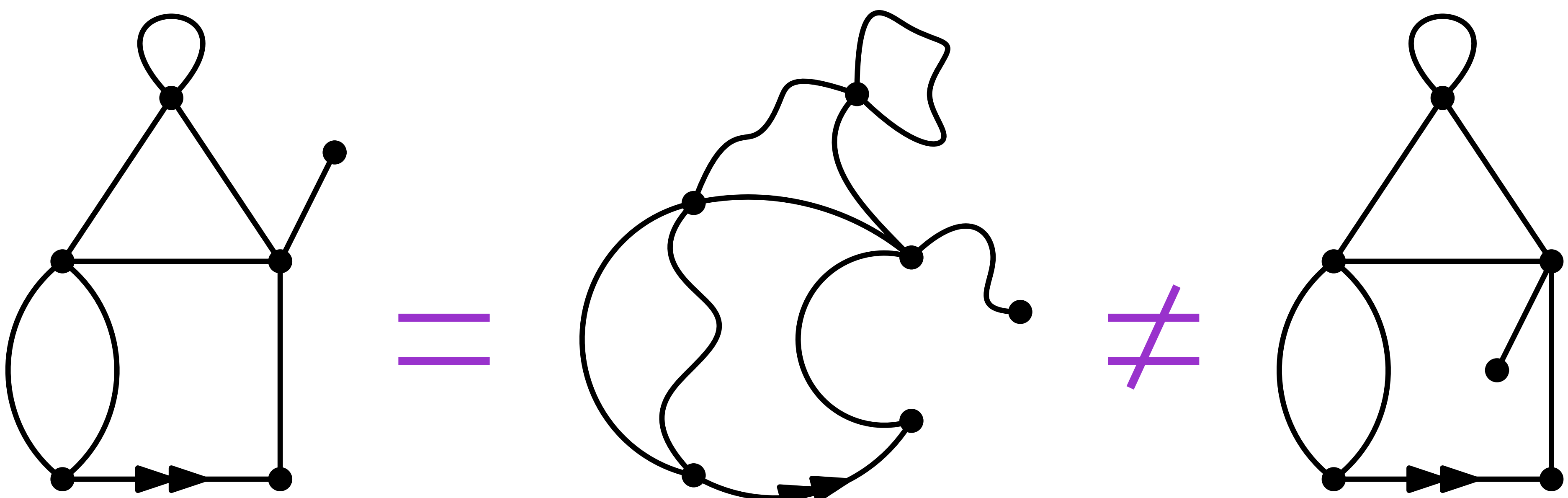
Bipartite planar maps

I- Enumeration of maps & Integrable Hierarchies

II- Bijections

Planar maps

A **planar map** is the embedding of a connected graph onto the sphere, up to orientation preserving homeomorphism.
Multi edges and loops are allowed.



Planar map = planar graph + cyclic ordering of the edges around each vertex.

All maps are **rooted**, i.e. an oriented edge is marked.

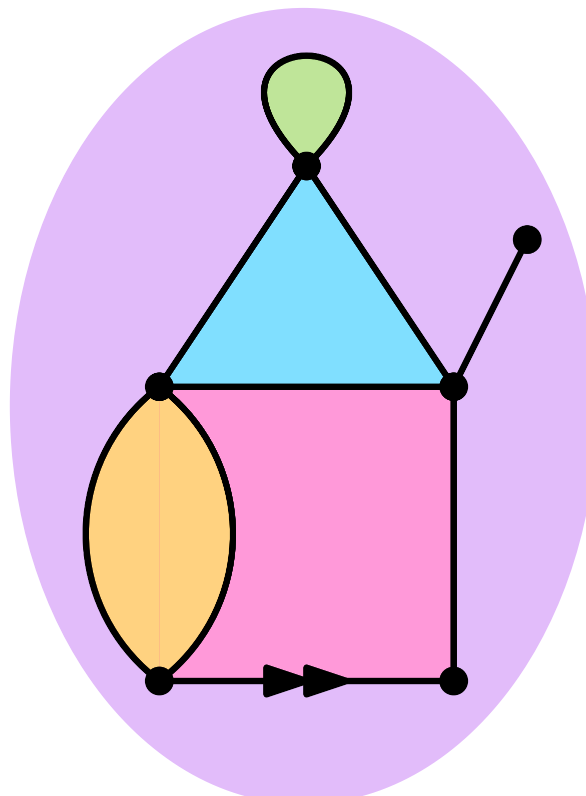
Map of genus $g > 0$ = graph embedded on a g-torus.

Planar maps

Planar map = planar graph + cyclic ordering of the edges around each vertex.

- **Vertices** and **edges** are inherited from the graph.(Counted by v and n)
- **Faces** are the connected components of the sphere minus the map. (Counted by f.)

Here: 6 vertices, 9 edges et 5 faces.



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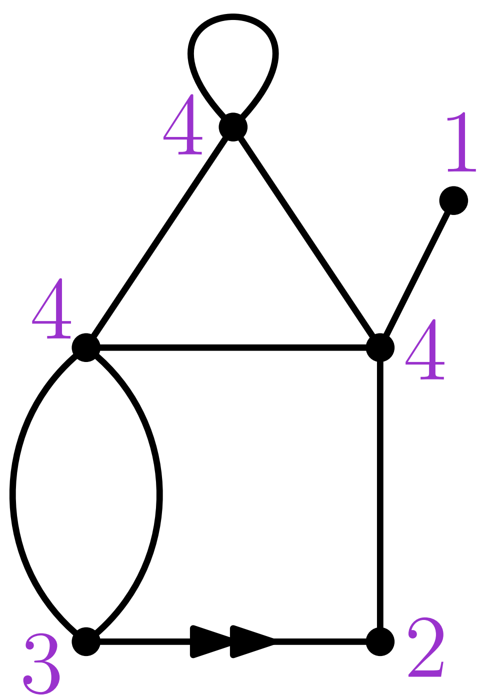
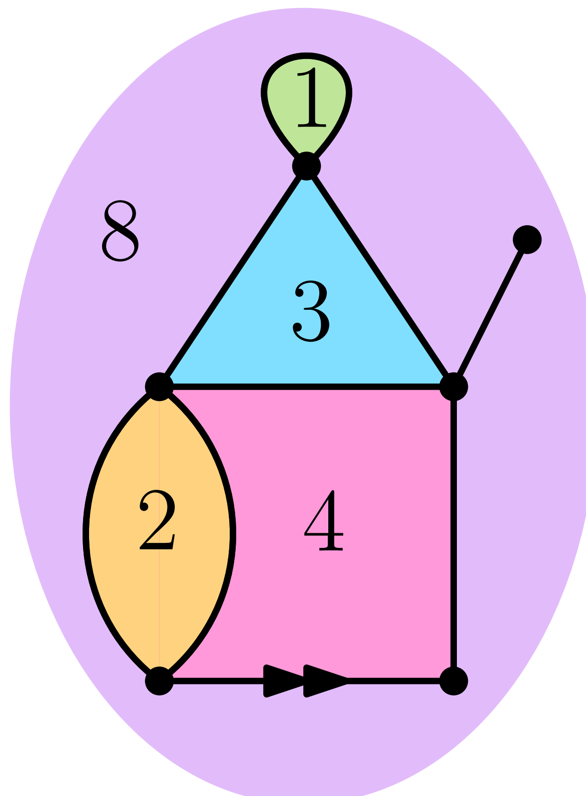
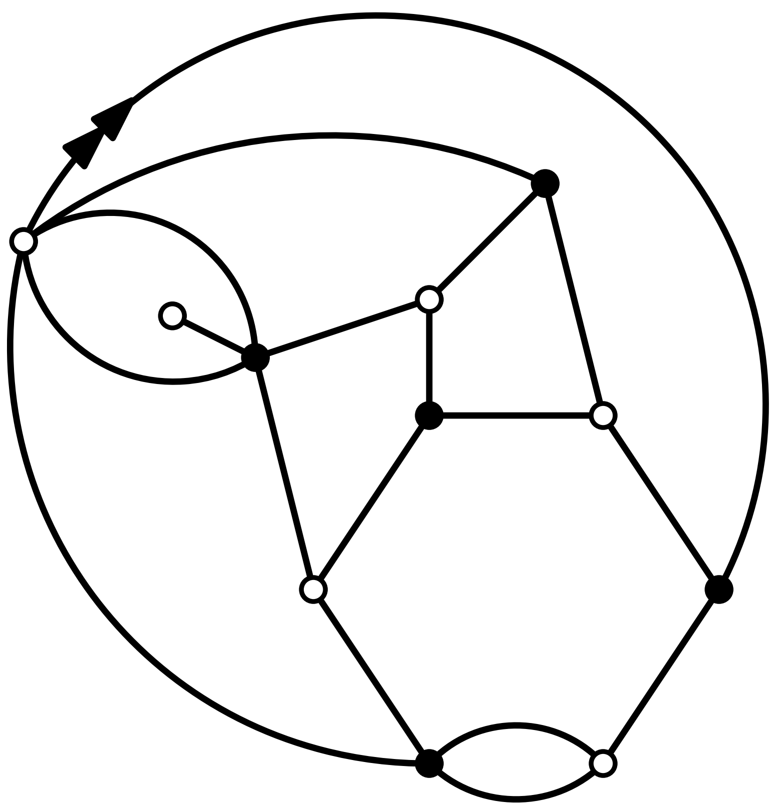
Degree (of a vertex or face) = number of incident half edges.





Bipartite maps

A planar map is **bipartite** if one can properly color its vertices in black and white $\iff$ all the faces have even degree.



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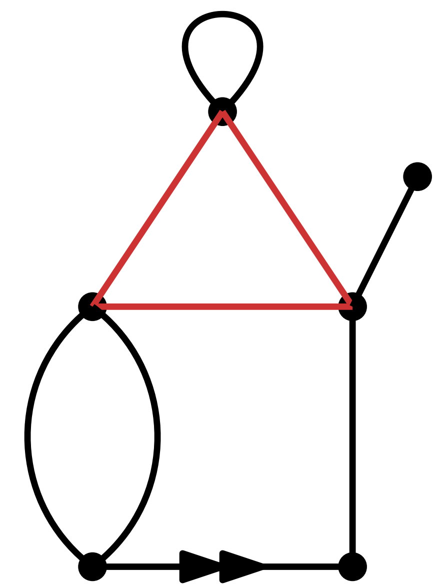

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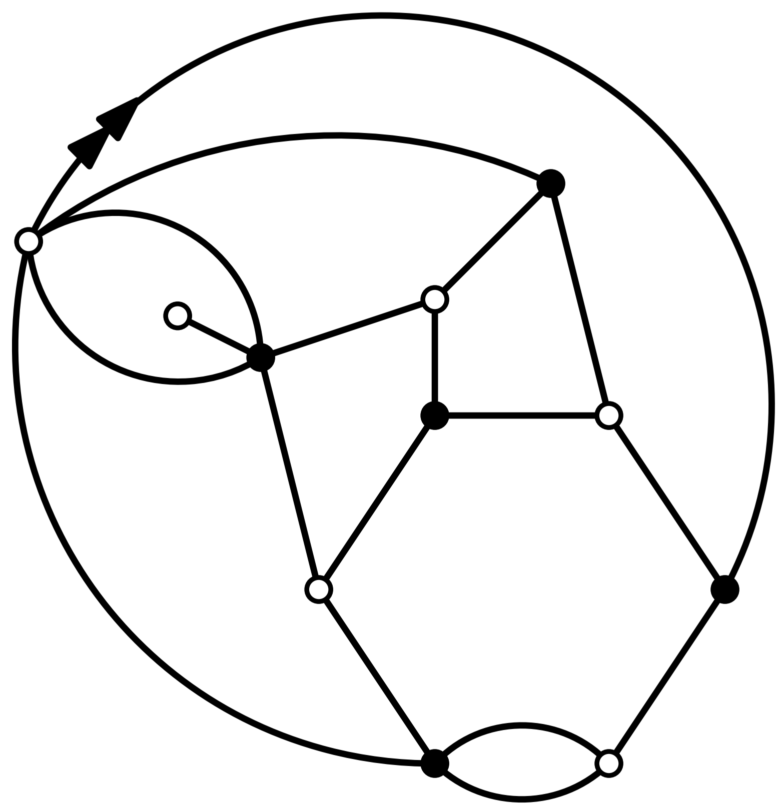

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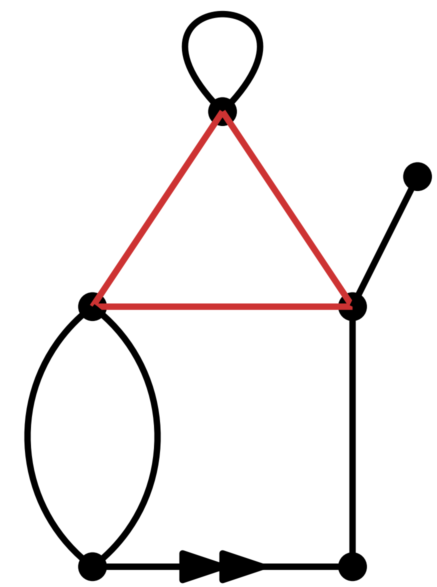
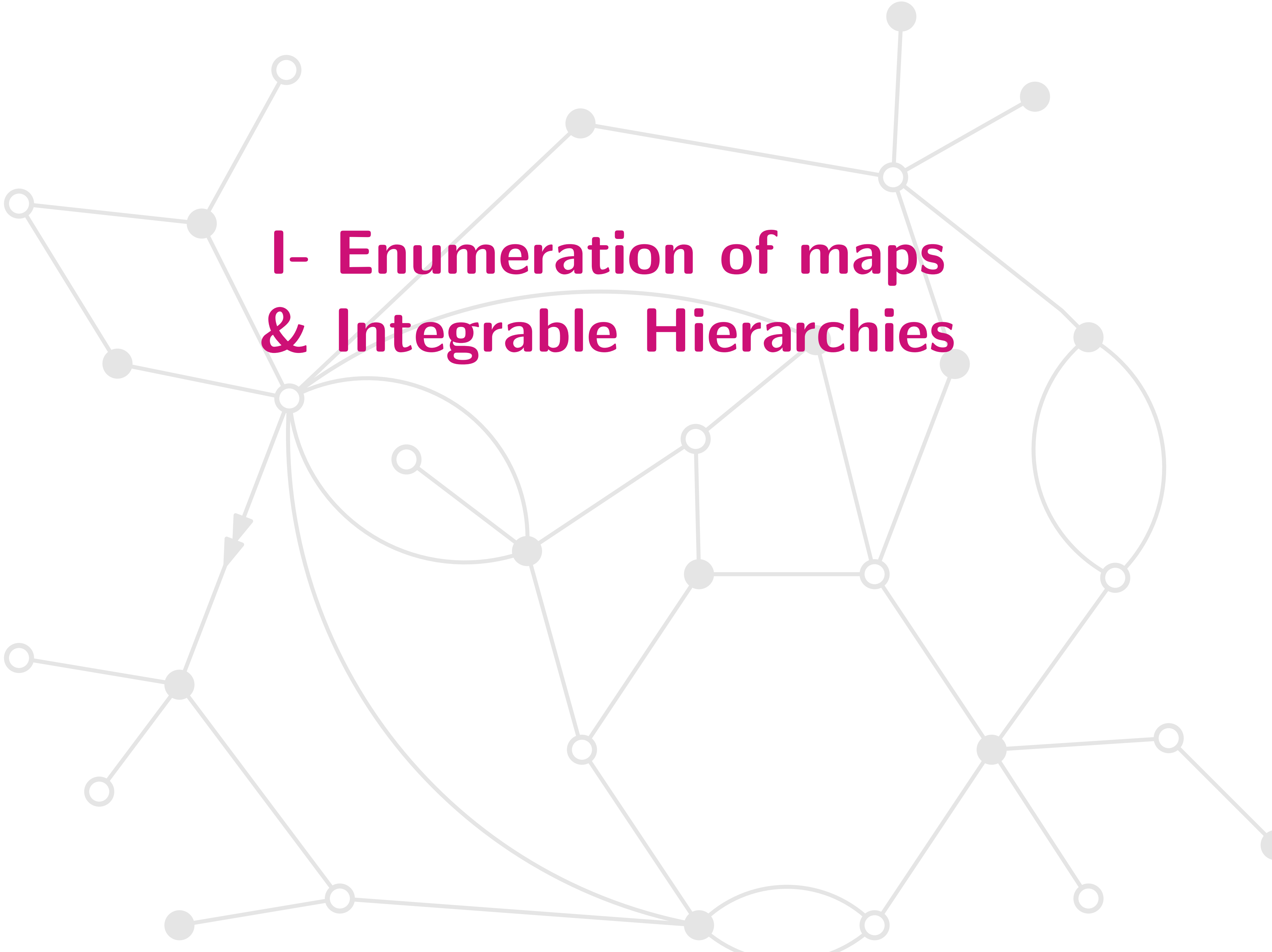


Not bipartite

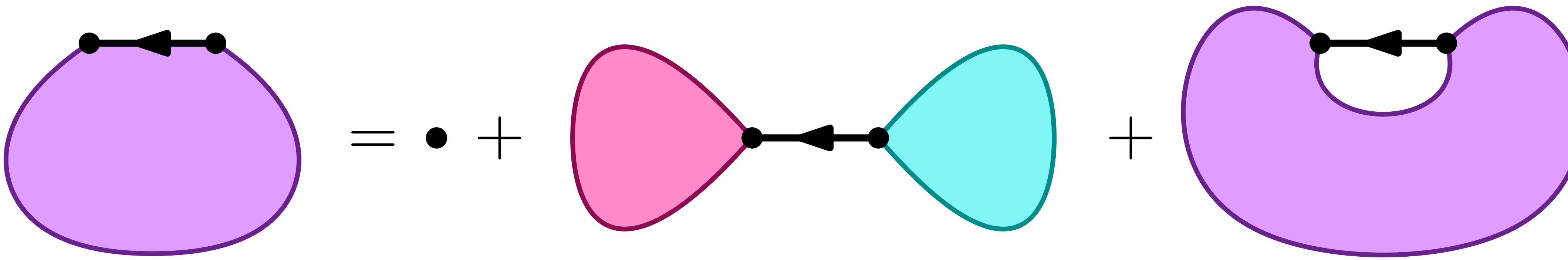
Canonical coloring: root edge from white to black. 



I- Enumeration of maps
& Integrable Hierarchies

Recursive approach

1) Find a decomposition of the map and write the associated equation on the generating series.



$$M(t,y) = 1 + ty^2 M(t,y)^2 + ty\frac{yM(t,y) - M(t,1)}{y-1}$$

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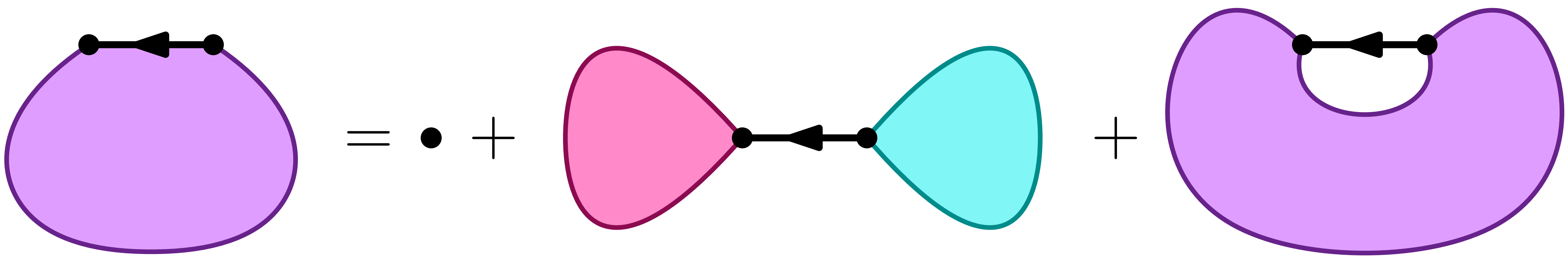

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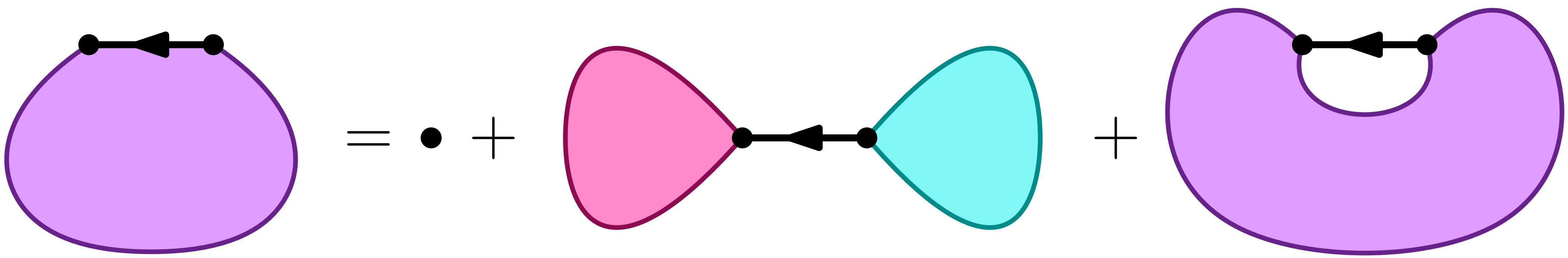

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Number of planar trees with n edges.

- Robust method.
- Bijections revelling hidden structure of maps. [Cori-Vauquelin-Schaeffer '98]

Integrable hierarchies

$$\partial_x(\partial_t u + u\partial_x u + \varepsilon^2 \partial_{xxt} u) + \lambda \partial_{yy} u = 0$$

KP hierarchy:

Infinite set of PDEs with an infinite set of variables $(p_1, p_2, \dots)$.

Obtained by studying the symmetries of the Kadomtsev-Petviashvili equation.

$$F_{3,1} = F_{2,2} + \frac{1}{2} F_{1,1}^2 + \frac{1}{12} F_{1,1,1,1}$$

$$F_{4,1} = F_{3,2} + F_{1,1} F_{2,1} + \frac{1}{6} F_{1,1,1,2}$$

$$\dots$$

2-Toda hierarchy: Extension of KP with two infinite sets of variables.

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2-Toda hierarchy: Extension of KP with two infinite sets of variables.

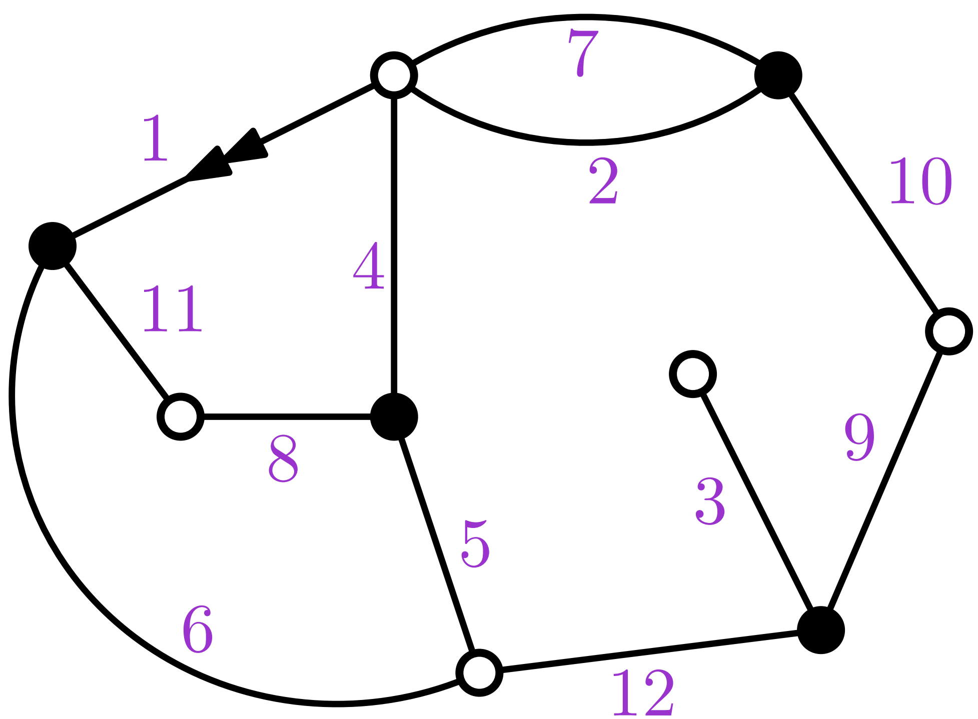
Some families of solutions of the integrable hierarchies are known.

$\Longrightarrow$ If a series is of that form then it satisfies the equations!

Algebraic combinatorics and KP/Toda

Maps = permutations factorisation!

Representation of **bipartite maps** (edge labeled, not planar nor connected) : a cycle of a permutation = the edges around a vertex/face.
Degree = cycle length.



$$\sigma_\circ = (1,4,2,7)(3)(5,6,12)(8,11)(9,10)$$

$$\sigma_\bullet = (1,6,11)(2,10,7)(3,12,9)(4,8,5)$$

$$\varphi = (1,12,10)(2,9,3,5)(4,11)(6,8)(7)$$

$$\sigma_\bullet\sigma_\circ = \varphi$$

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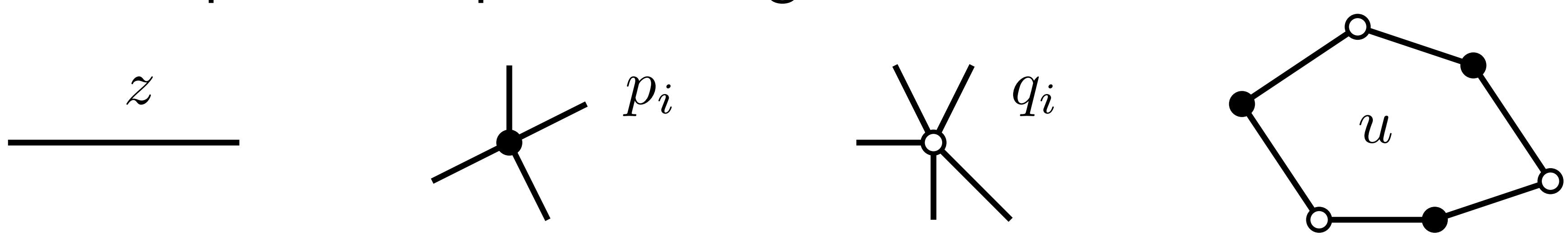
Using permutation encoding and representation theory, one can show that the series B of bipartite maps with weights



is an **hypergeometric tau** function, i.e. solution to the KP/Toda hierarchy.

$\hookrightarrow$ the PDEs translate as recurrence equations for given specialisations.

Combinatorial formulas from KP/Toda

Theorem. [Goulden-Jackson '08] Let $T_g(n)$ be the number of triangulations of genus g with $2n$ faces.

$$(n+1)T_g(n) = 4n(3n-2)(3n-4))T_{g-1}(n-2) + 4(3n-1))T_g(n-1) + 4\sum_{i+j=n-2}\sum_{g_1+g_2=g}(3i+2)(3j+2))T_{g_1}(i)T_{g_2}(j)$$

Theorem. [Louf '21] Let $B_g(\mathbf{f})$ be the number of bipartite maps of genus g with degree sequence $\mathbf{f}$.

$$\binom{n+1}{2}B_g(\mathbf{f}) = \sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f}\\ g_1+g_2+g^*=g}}(1+n_1)\binom{v_2}{2g^*+2}B_{g_1}(\mathbf{s})B_{g_2}(\mathbf{t}) + \sum_{g^*\geqslant 0}\binom{v+2g^*}{2g^*+2}B_{g-g^*}(\mathbf{f})$$

- Very fast counting.
- Lots of parameters.
- New bijections explaining the structure of maps.

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- Single vertex case [Chapuy-Féray-Fusy '13]
- Planar case [Louf '19]

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- Planar case [S. '24]

II- Bijections

What does the formula say?

Bipartite maps with prescribed degrees

Let $\mathbf{d} = (d_1, d_2, \ldots)$ be a sequence of integers. A bipartite map M has **degree sequence** $\mathbf{d}$ if it has exactly d_i faces with degree $2i$ for all i.
Denote by $\mathcal{B}(\mathbf{d})$ their set and $B(\mathbf{d})$ their number.

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Euler formula: $v + f = n + 2$ for all planar maps.

A map $M \in \mathcal{B}(\mathbf{d})$ has :

- $f(\mathbf{d}) = \sum_{i \geqslant 1} d_i$ faces
- $n(\mathbf{d}) = \sum_{i \geqslant 1} i d_i$ edges
- $v(\mathbf{d}) = n(\mathbf{d}) + 2 - f(\mathbf{d})$ vertices

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Theorem. [Louf '21]

$$\left(\binom{n(\mathbf{d})+1}{2} - \binom{v(\mathbf{d})}{2} \right) B(\mathbf{d}) = \sum_{\mathbf{s}+\mathbf{t}=\mathbf{d}} (1 + n(\mathbf{s})) \binom{v(\mathbf{t})}{2} B(\mathbf{s}) B(\mathbf{t}).$$

$s_i + t_i = d_i \forall i$

Louf's formula: planar case

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$$\left((f(\mathbf{d})-1)v(\mathbf{d}) + \binom{f(\mathbf{d})-1}{2}\right) B(\mathbf{d}) = \sum_{\mathbf{s}+\mathbf{t}=\mathbf{d}} (v(\mathbf{s})+f(\mathbf{s})-1)\binom{v(\mathbf{t})}{2} B(\mathbf{s})B(\mathbf{t}).$$

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Theorem. [S. '24] The number of bipartite planar maps with profile $\mathbf{d}$ satisfy :

$$2v(\mathbf{d})(f(\mathbf{d})-1)B(\mathbf{d}) = \sum_{\mathbf{s}+\mathbf{t}=\mathbf{d}} v(\mathbf{s})v(\mathbf{t})(v(\mathbf{t})-1)B(\mathbf{s})B(\mathbf{t})$$

and

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this talk's bijection

and

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Simplest proof: eulerian trees

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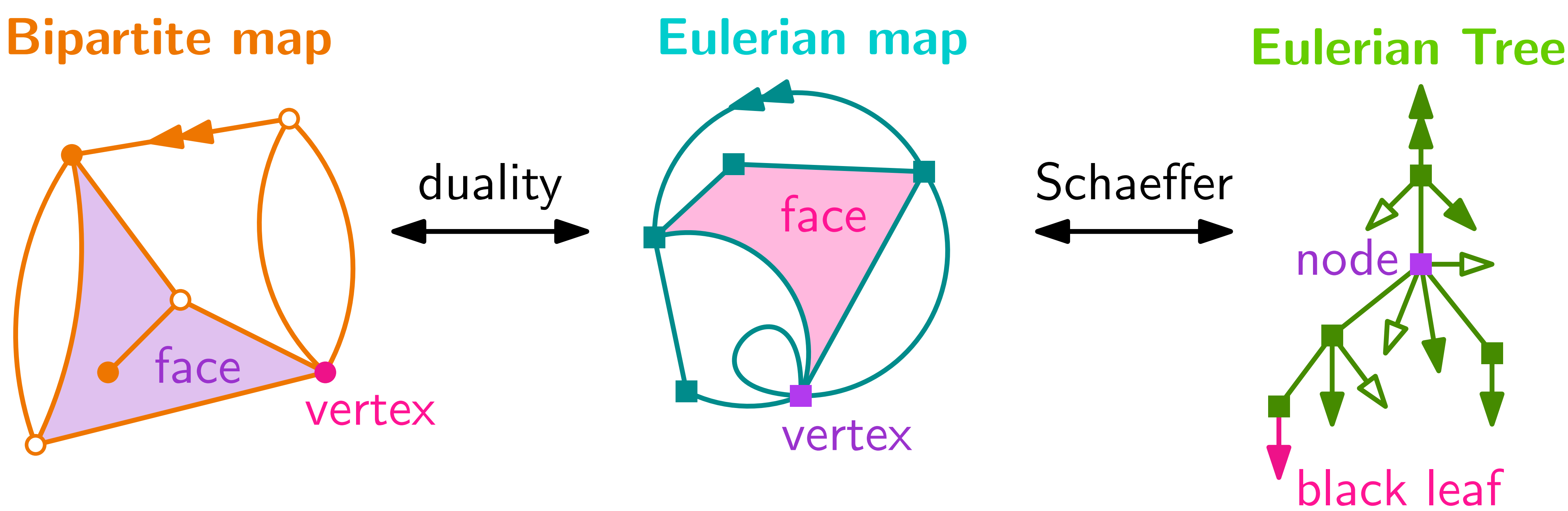

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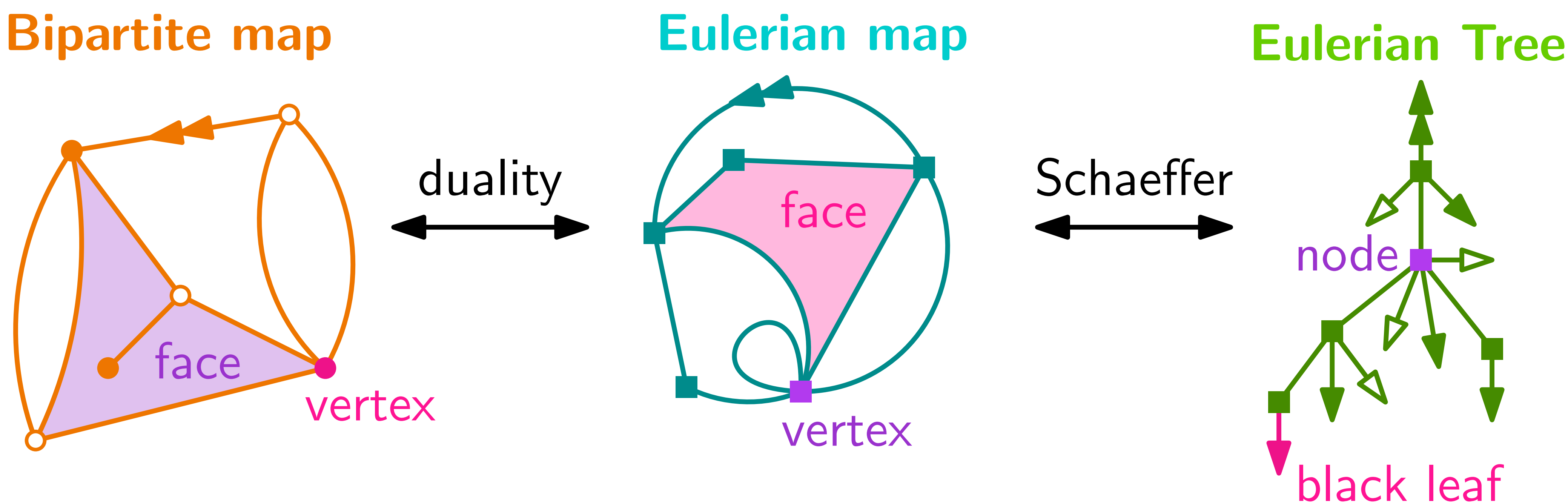

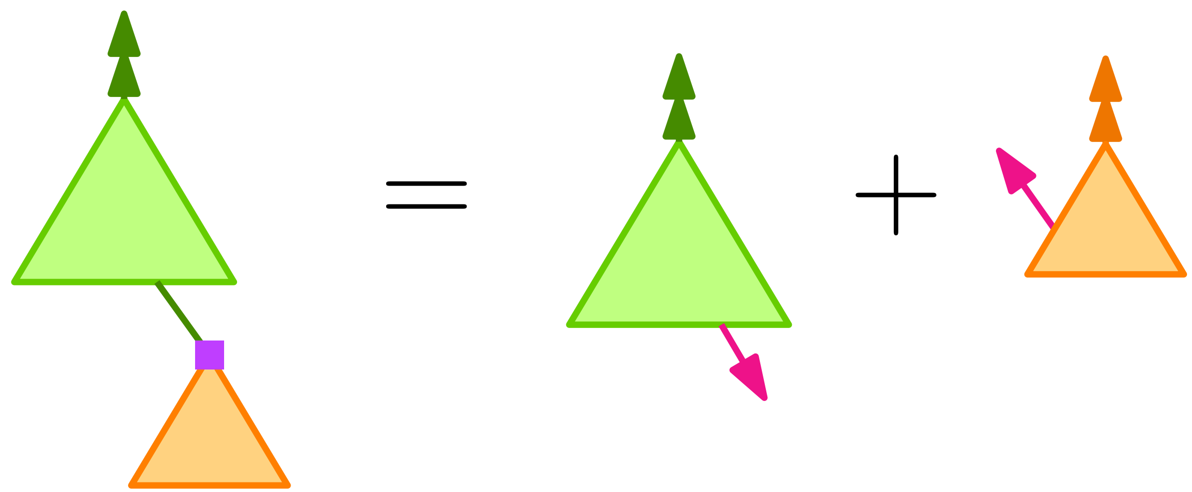

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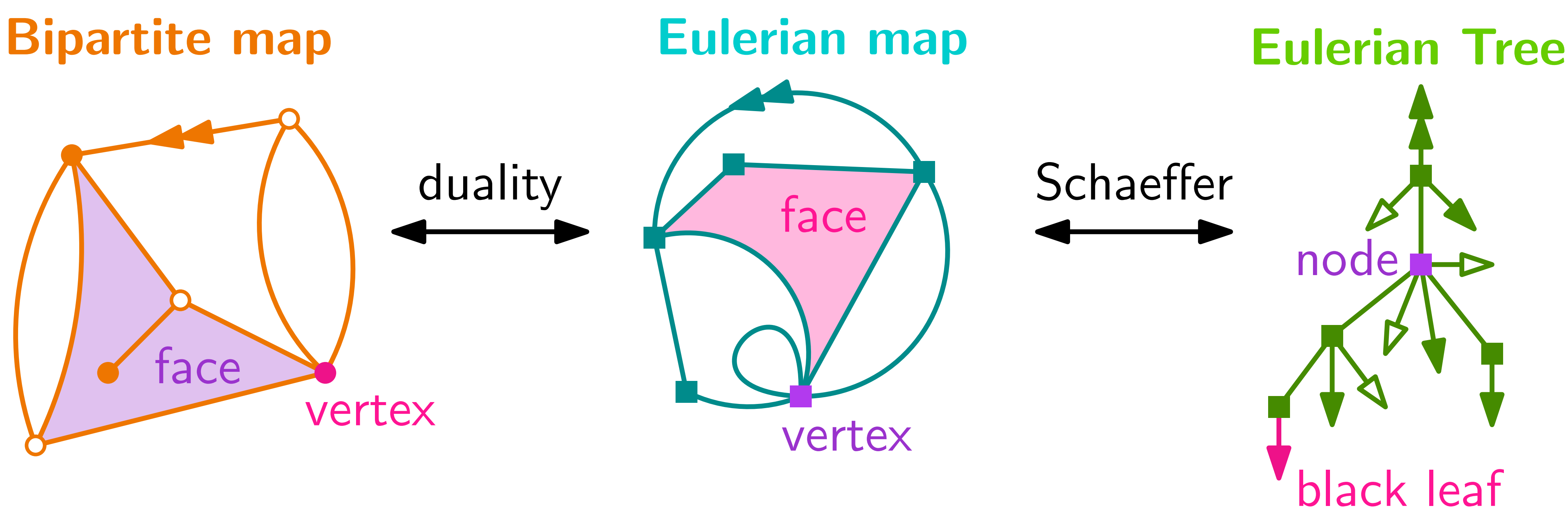

$$(M_1, v_1) + (M_2, v_2) \overset{duality}{\longleftrightarrow} (M_1^*, v_1^*) + (M_2^*, v_2^*) \overset{\text{Schaeffer}}{\longleftrightarrow} (T_1^*, l_1) + (T_2^*, l_2)$$

$$\tilde{\varphi} \downarrow \uparrow \tilde{\psi} \qquad\qquad \varphi \downarrow \uparrow \psi$$

Not really enlightening ...

$$(M, e) \overset{duality}{\longleftrightarrow} (M^*, e^*) \overset{\text{Schaeffer}}{\longleftrightarrow} (T^*, e^*)$$

What happens here?

II- Bijections

Warm-up: The 2-faces case.

The 2 faces case

The two faces case of the original formula [Bouttier-Guitter-Miermont, '22].

$$4(f(\mathbf{d}) - 1)B(\mathbf{d}) = \sum_{\mathbf{s}+\mathbf{t}=\mathbf{d}} v(\mathbf{s})v(\mathbf{t})B(\mathbf{s})B(\mathbf{t})$$

$$\mathbf{d} = \mathbb{1}_{i=m_1} + \mathbb{1}_{i=m_2} \qquad \mathbf{s} = \mathbb{1}_{i=m_1} \qquad \mathbf{t} = \mathbb{1}_{i=m_2}$$

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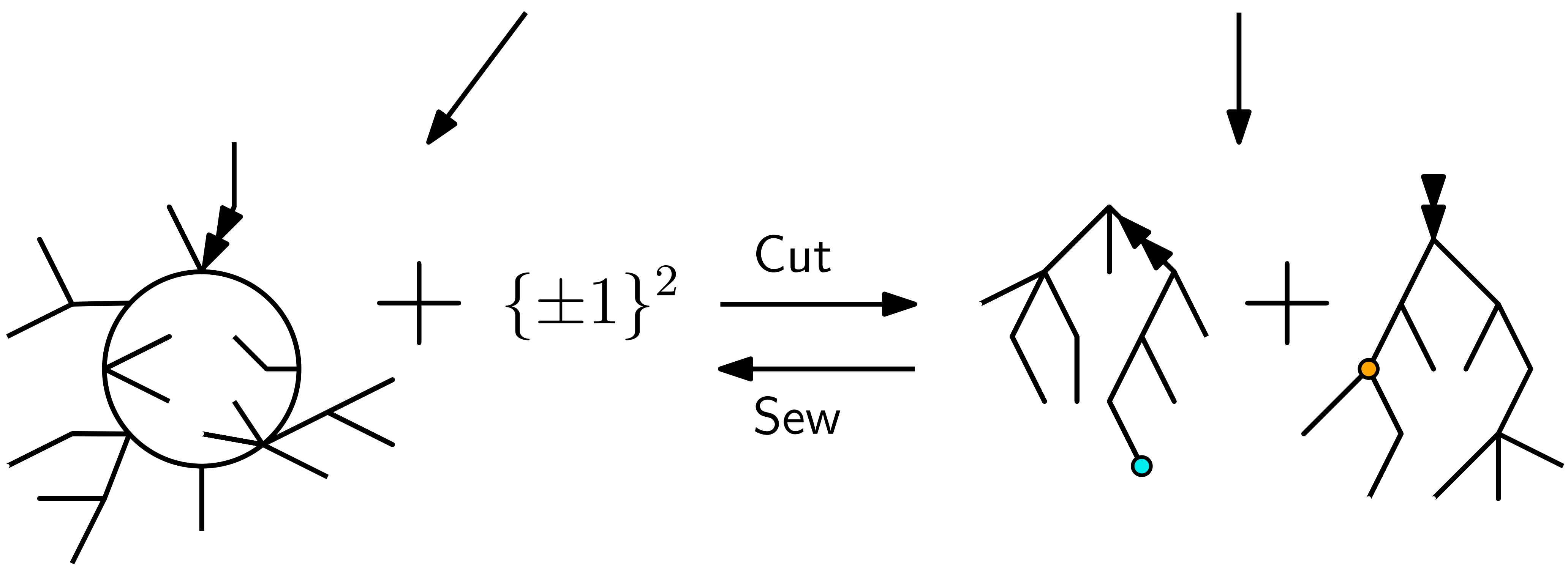
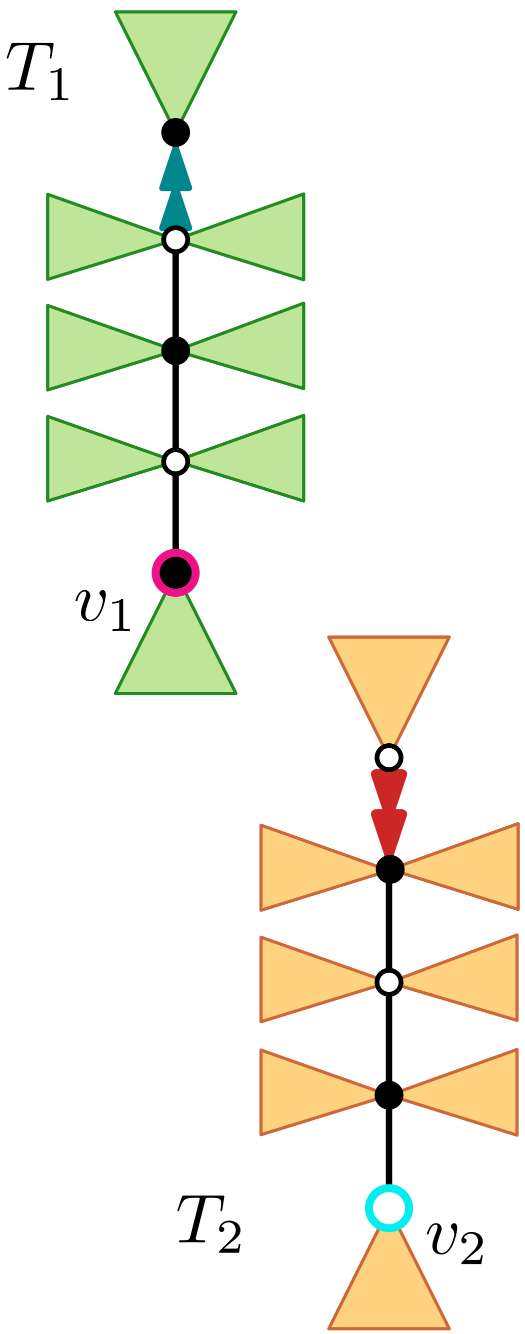


The 2 faces case: Slit and Sew



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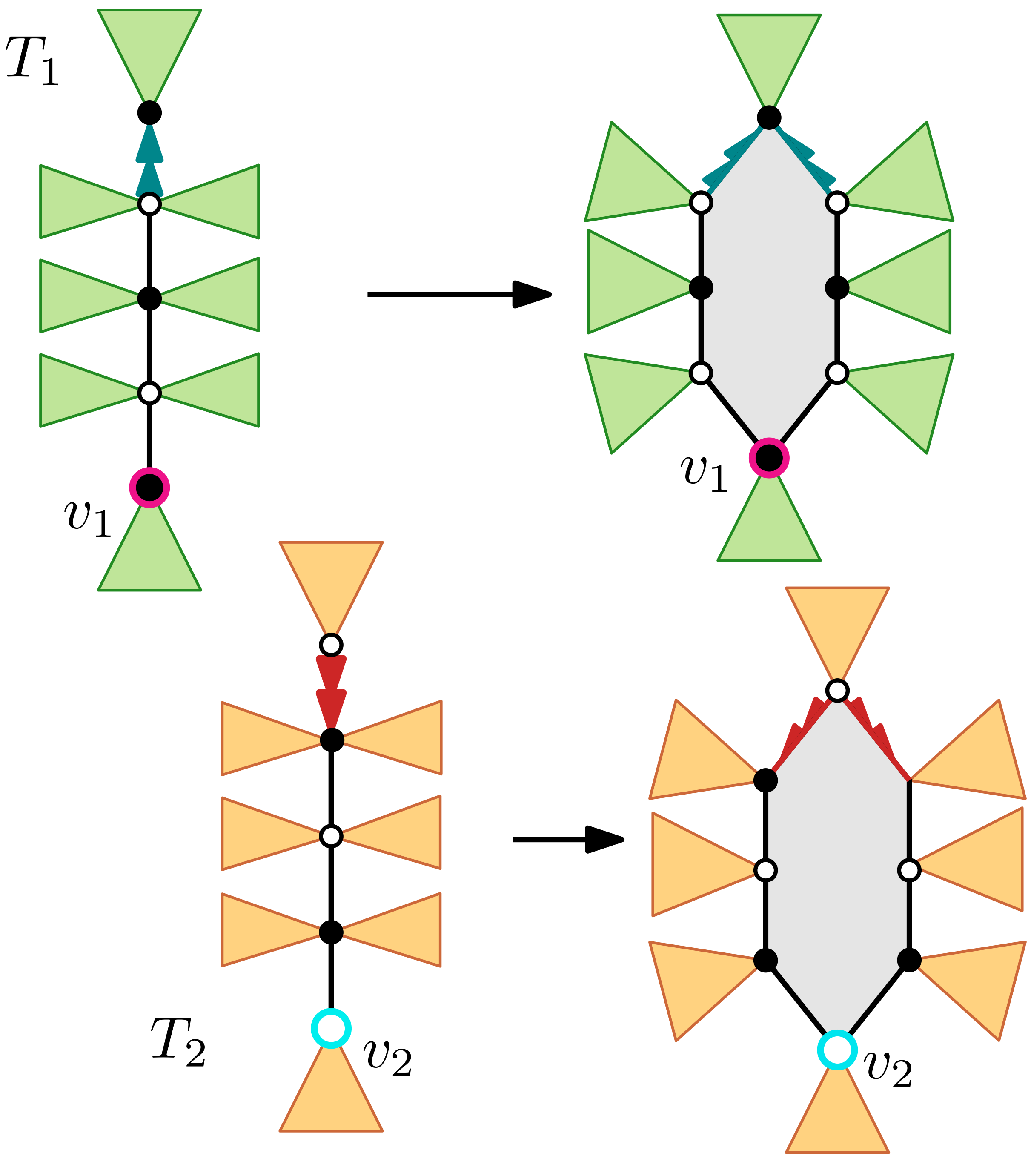

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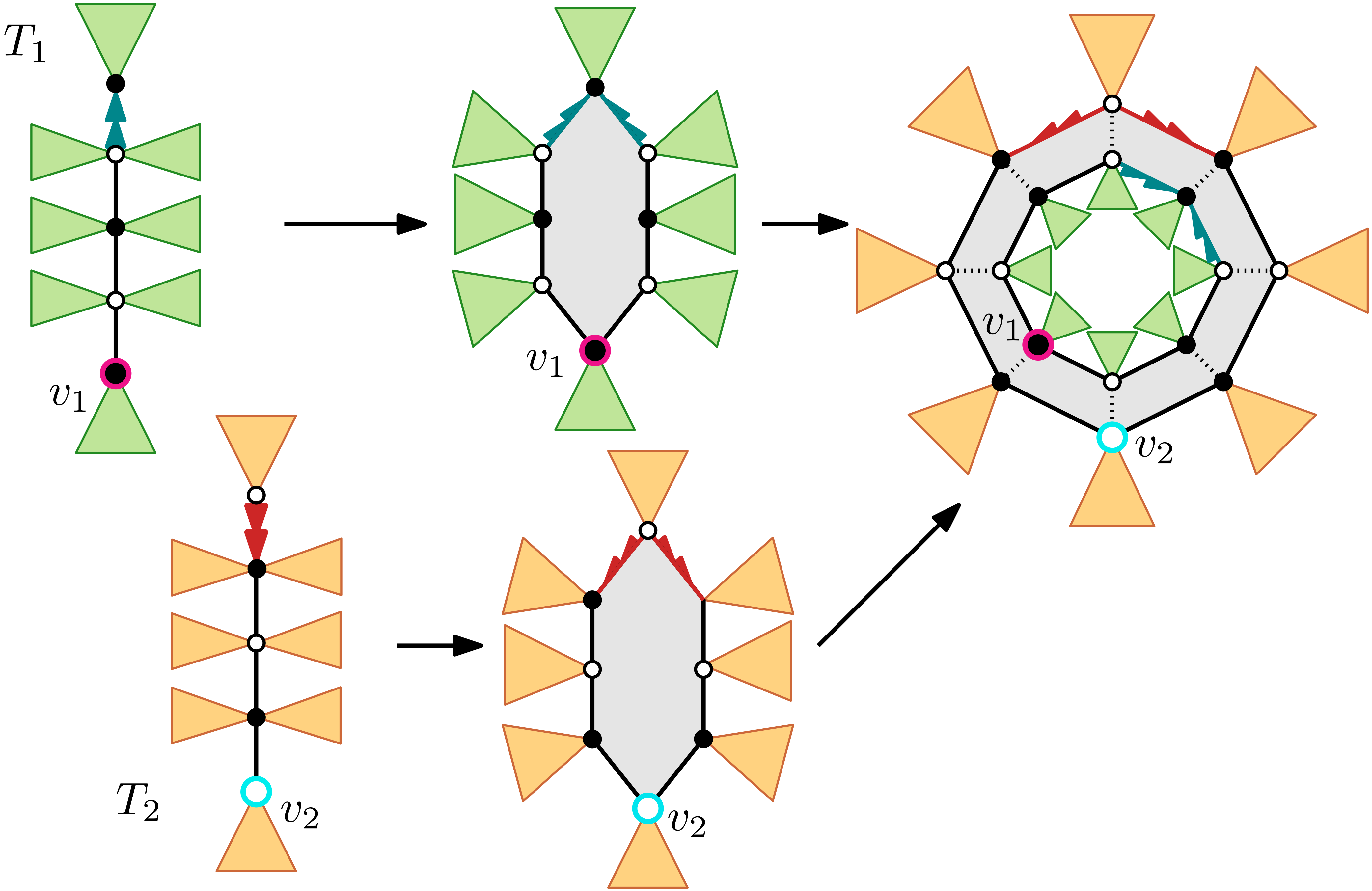

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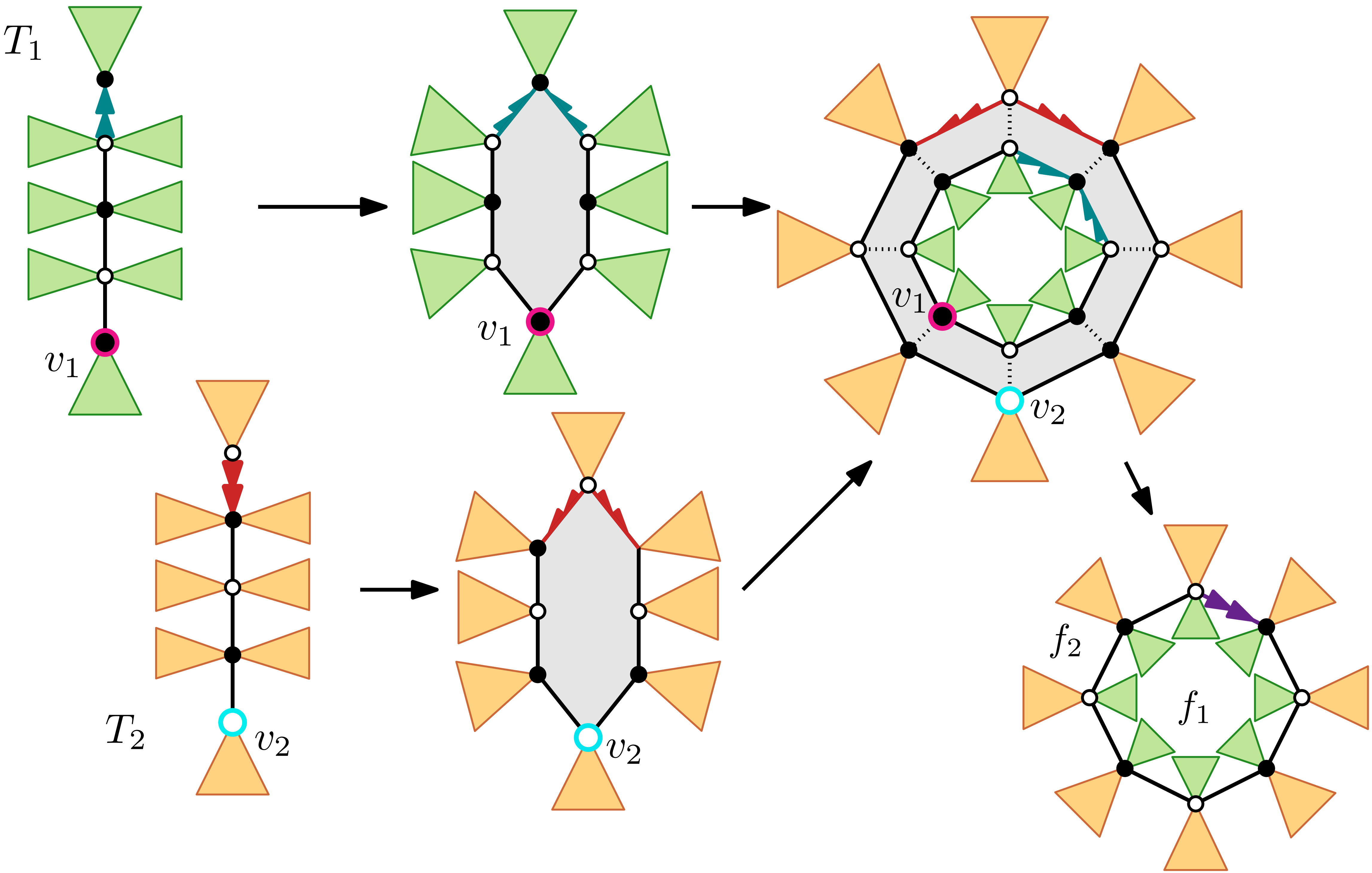

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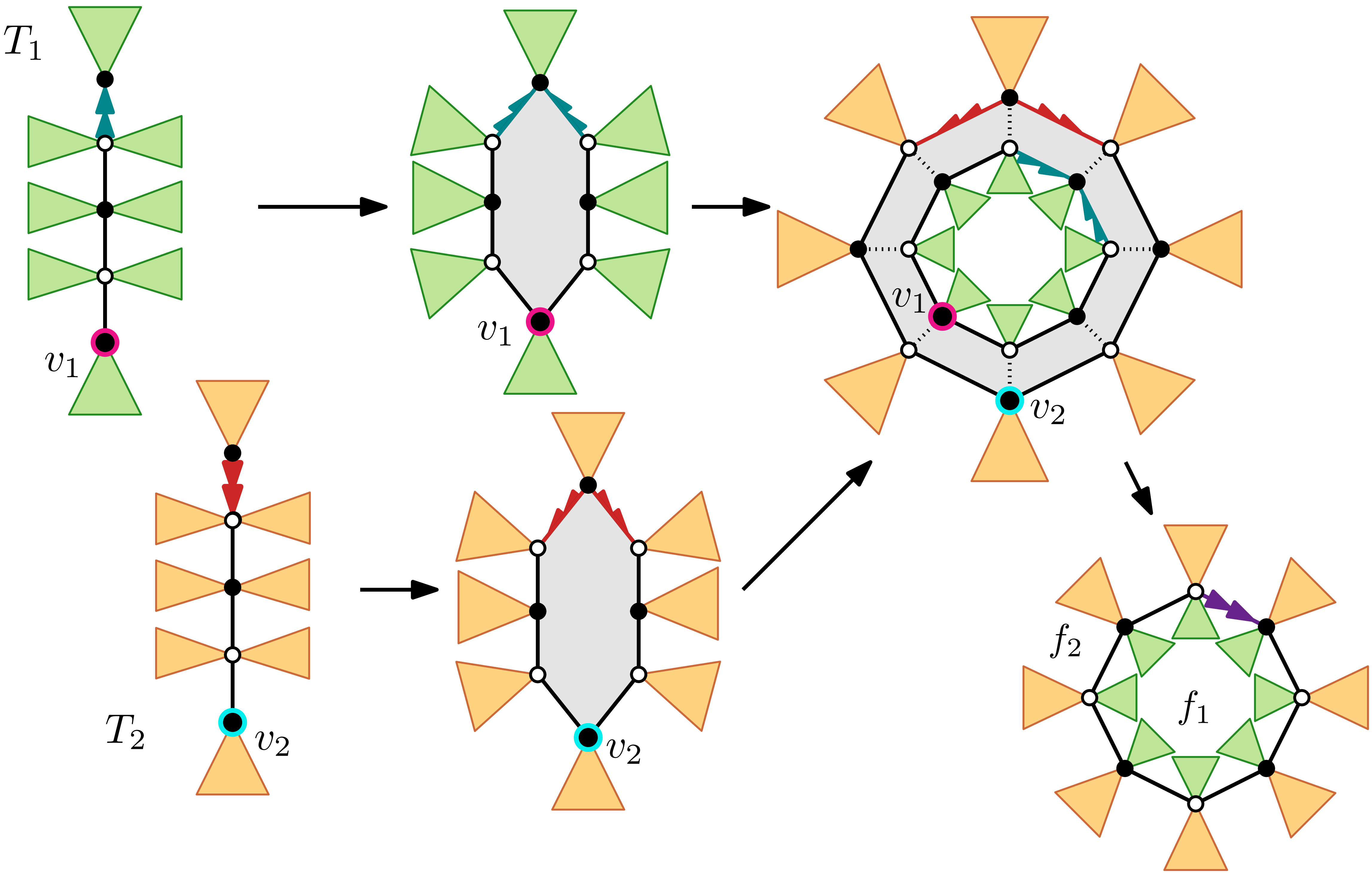

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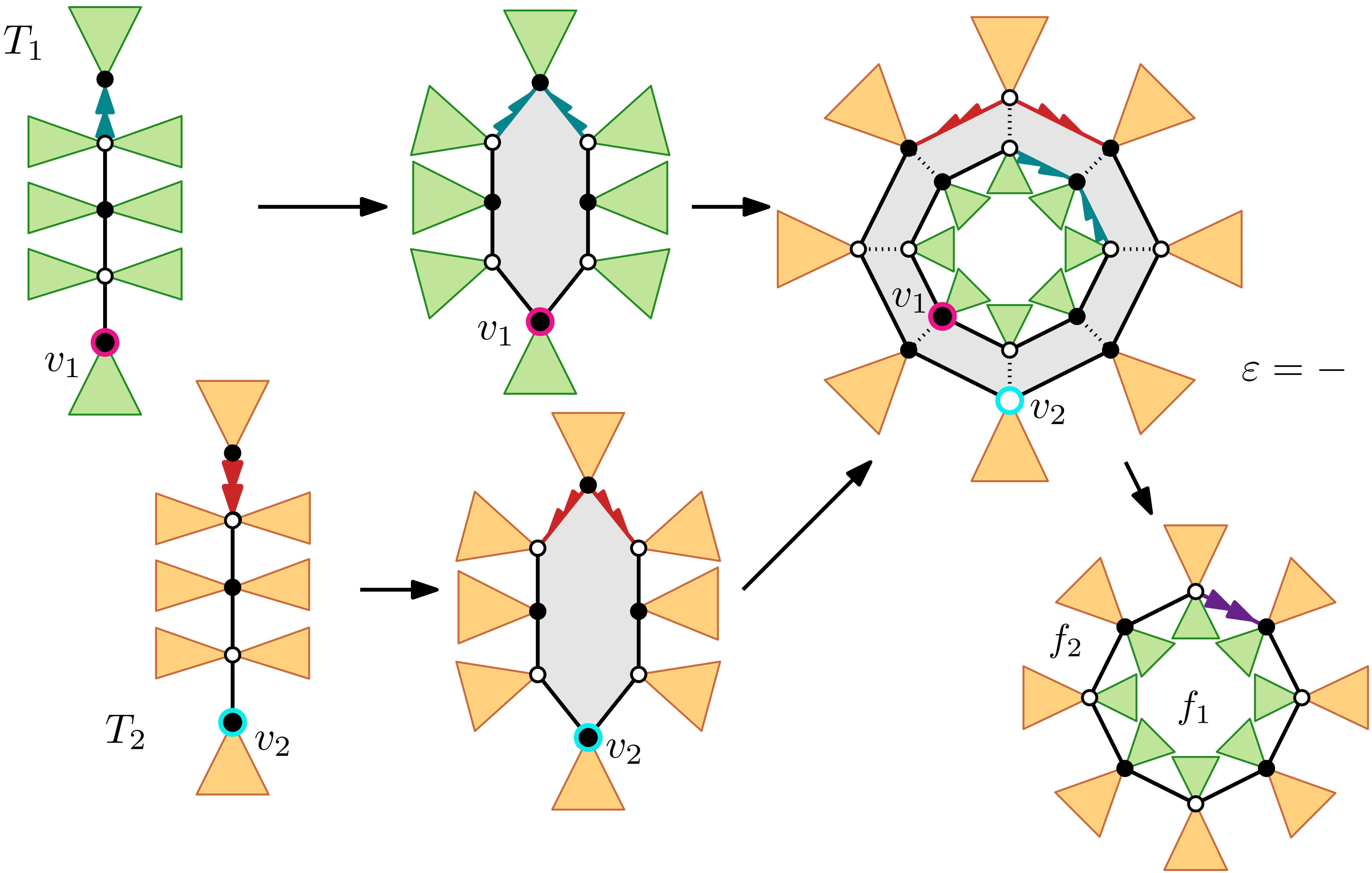

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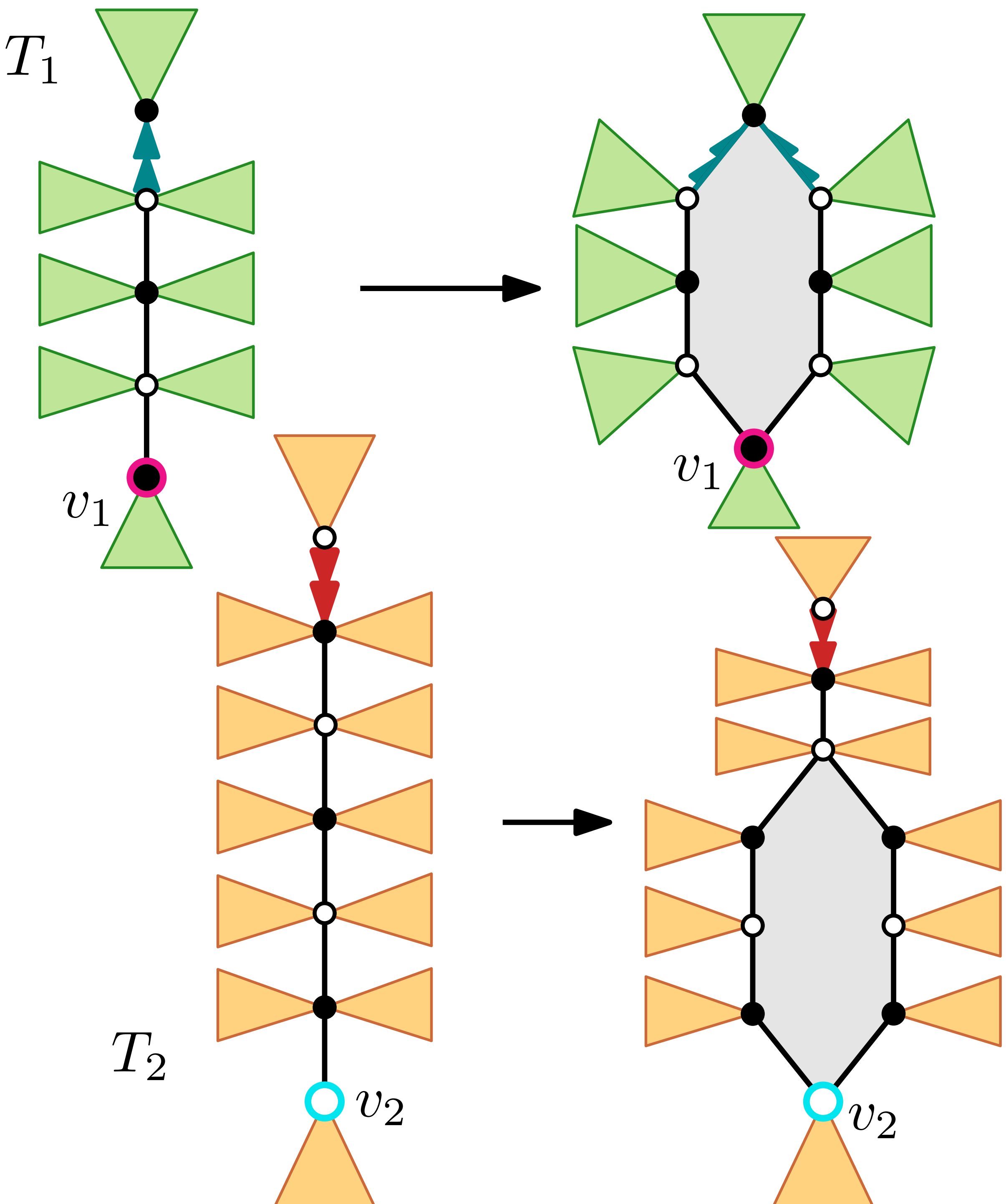

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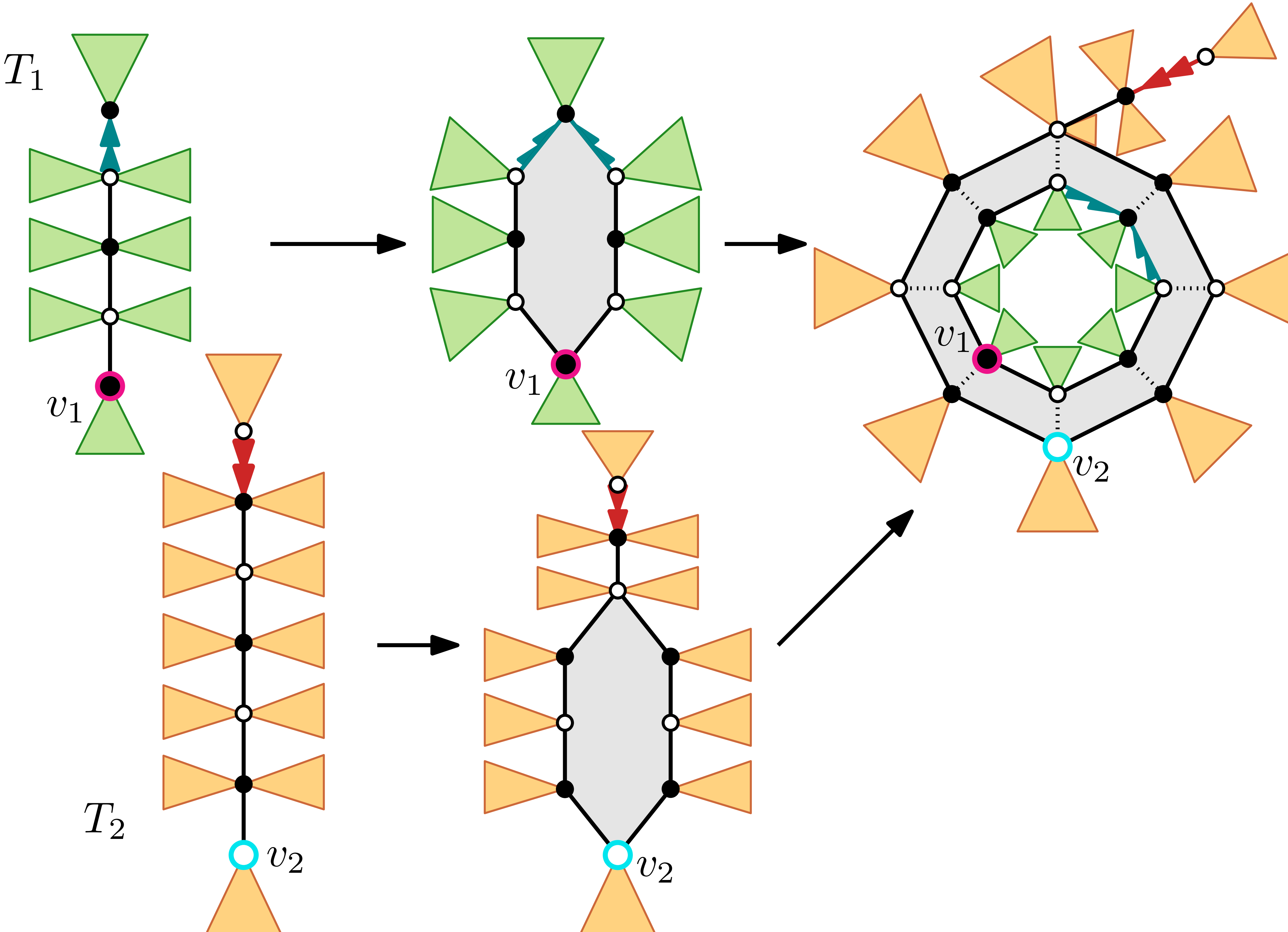

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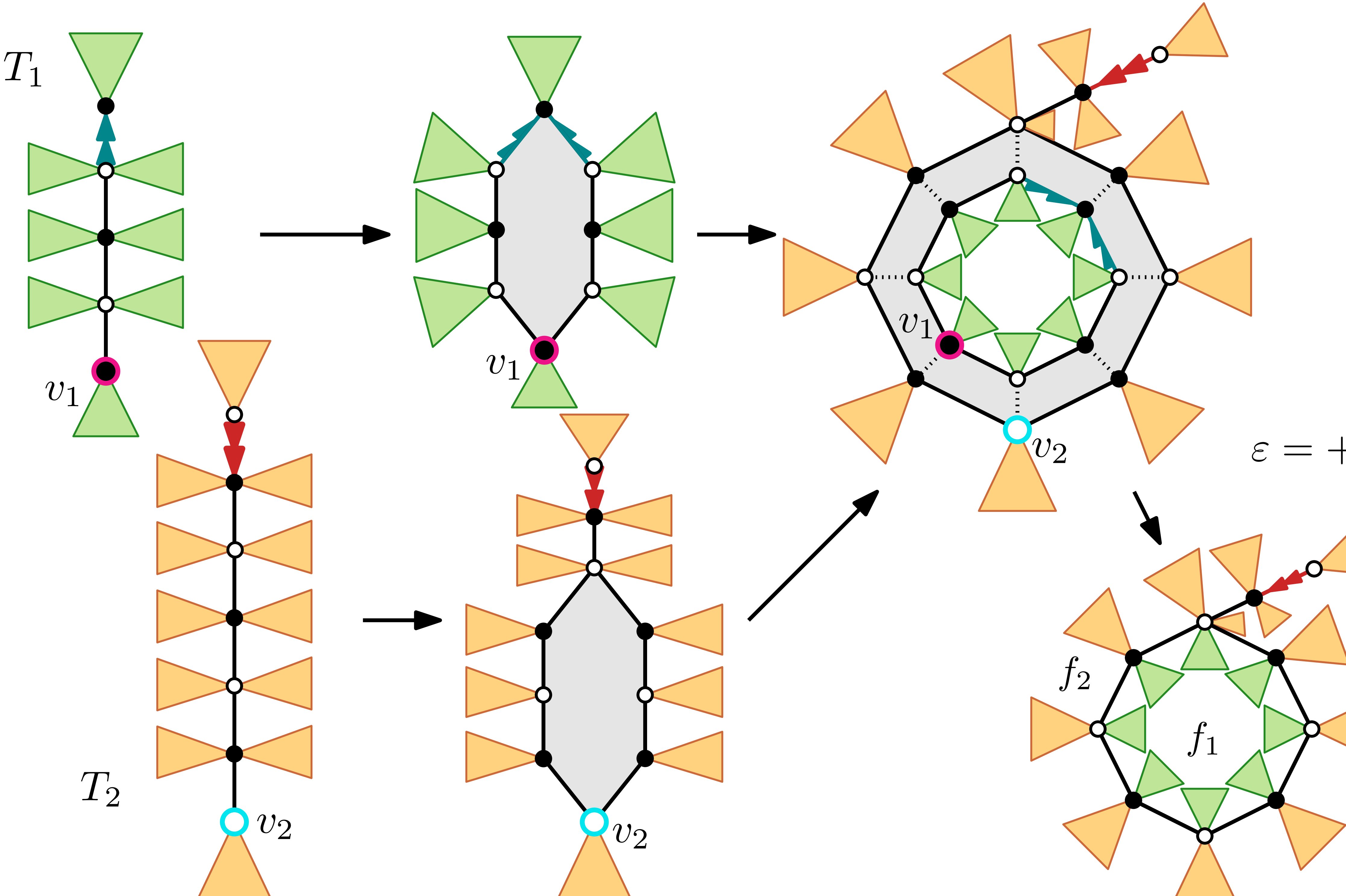
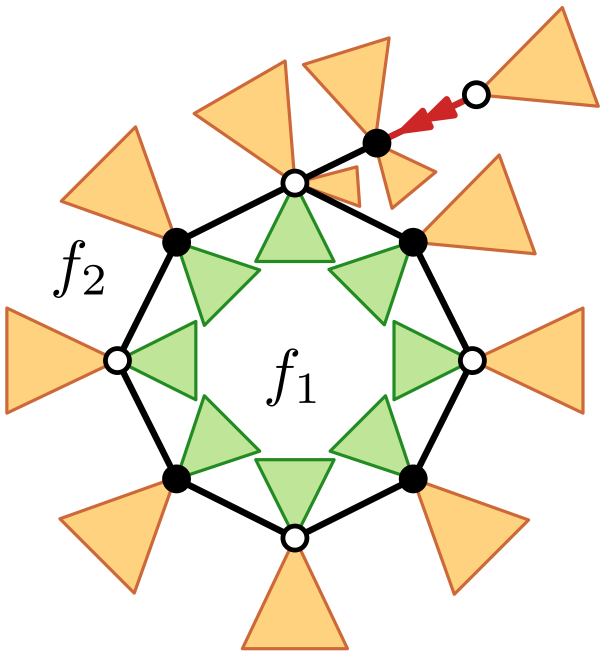
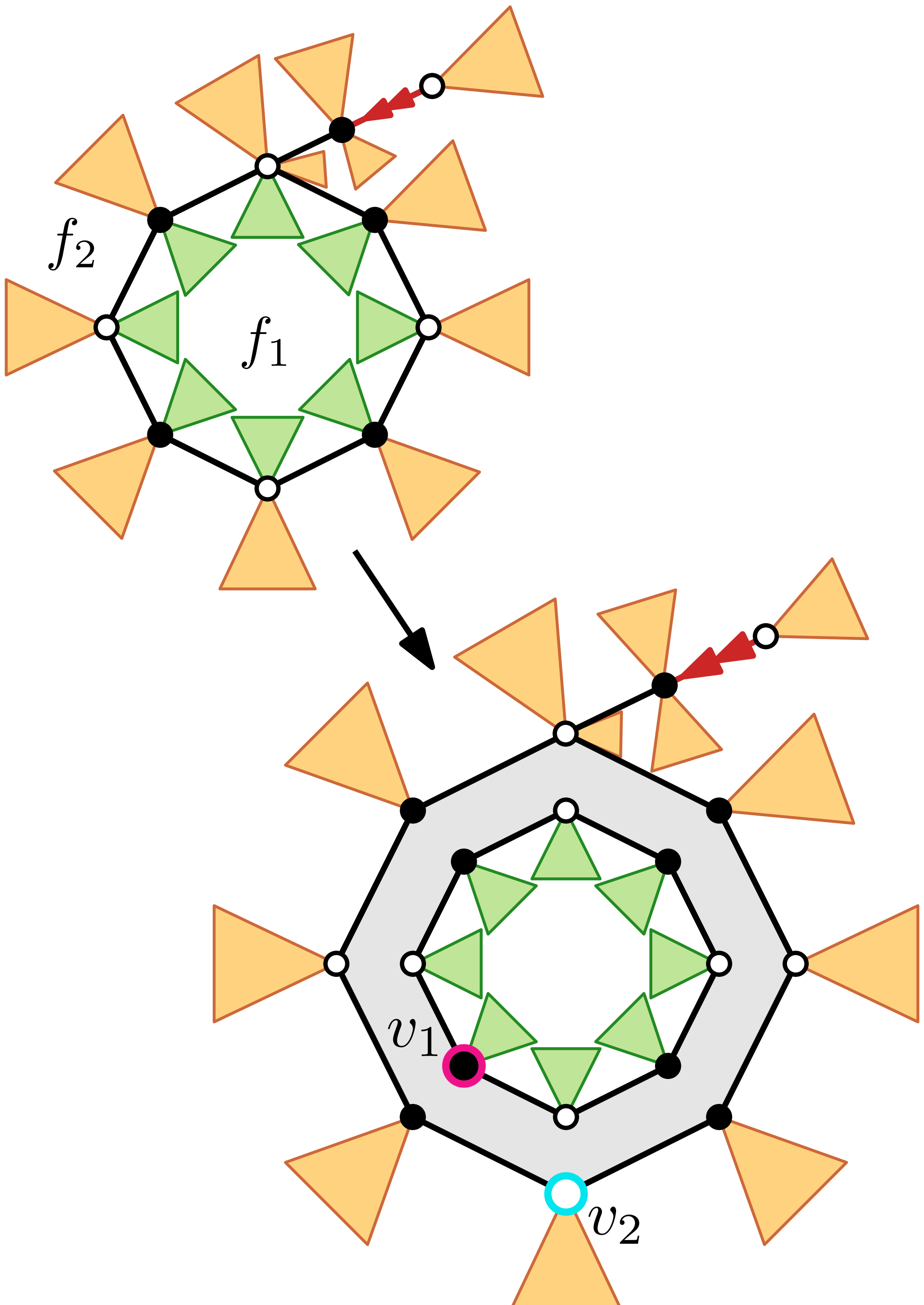


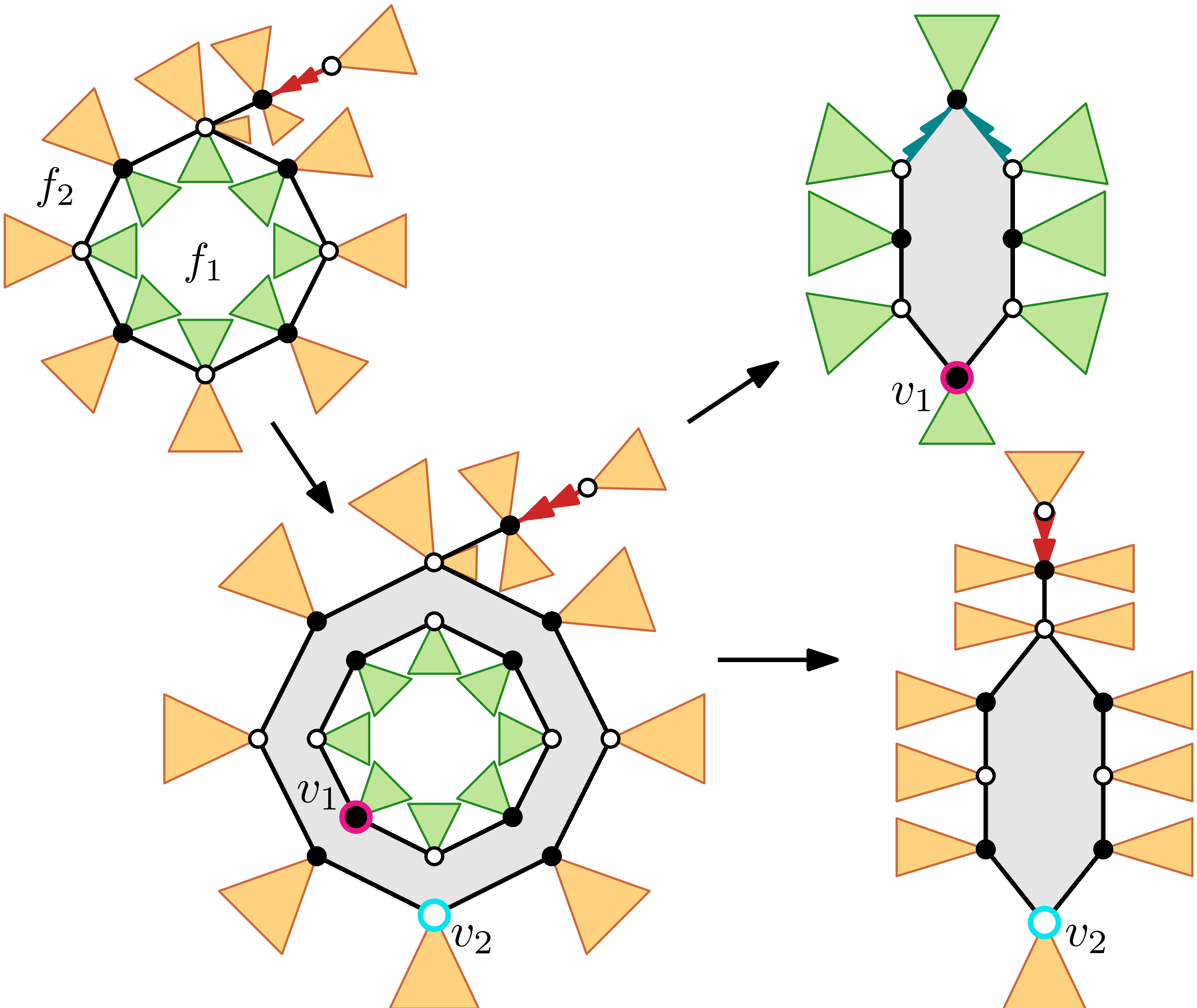
The 2 faces case: Cut and Close



The 2 faces case: Cut and Close



The 2 faces case: Cut and Close



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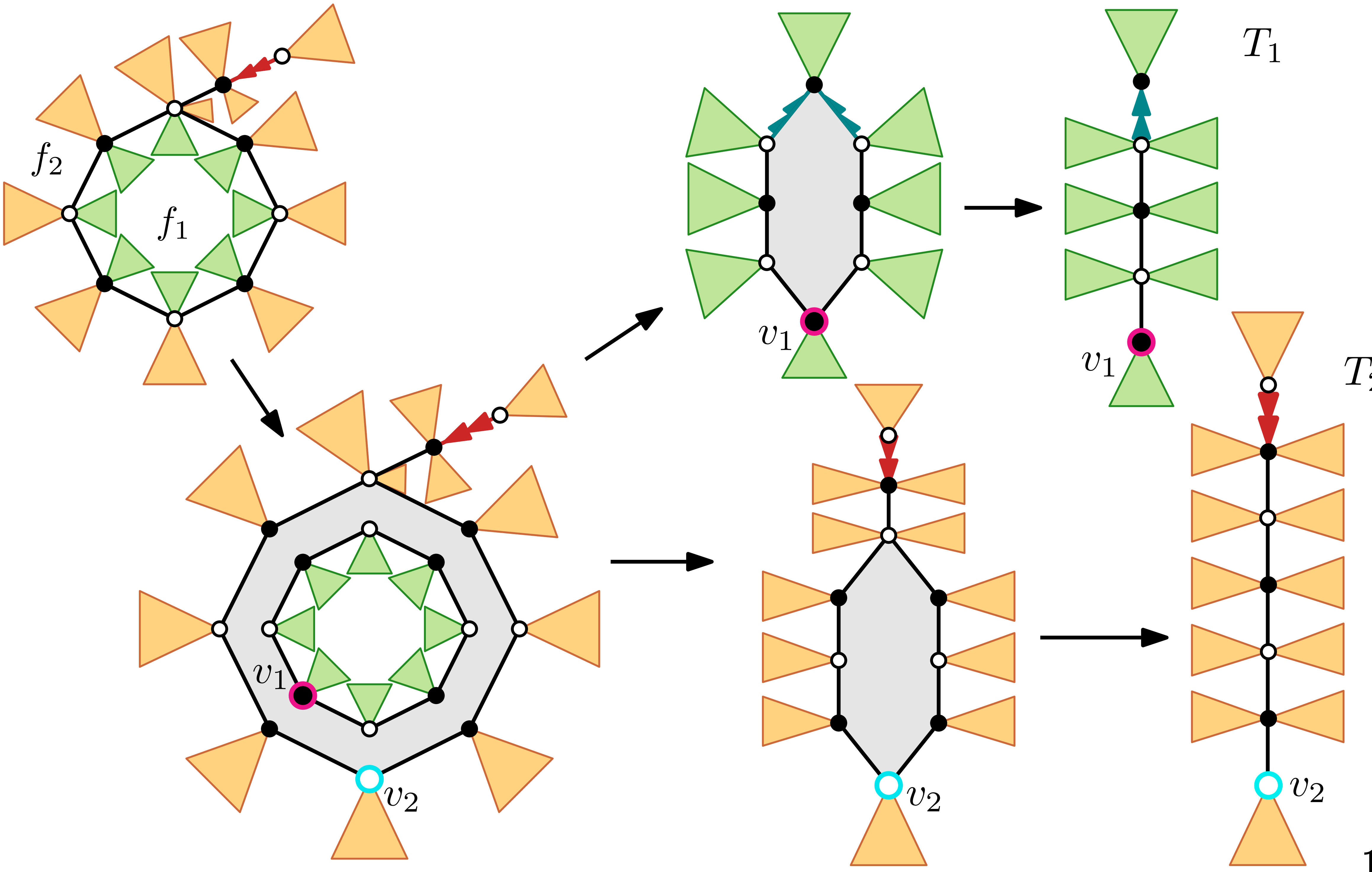
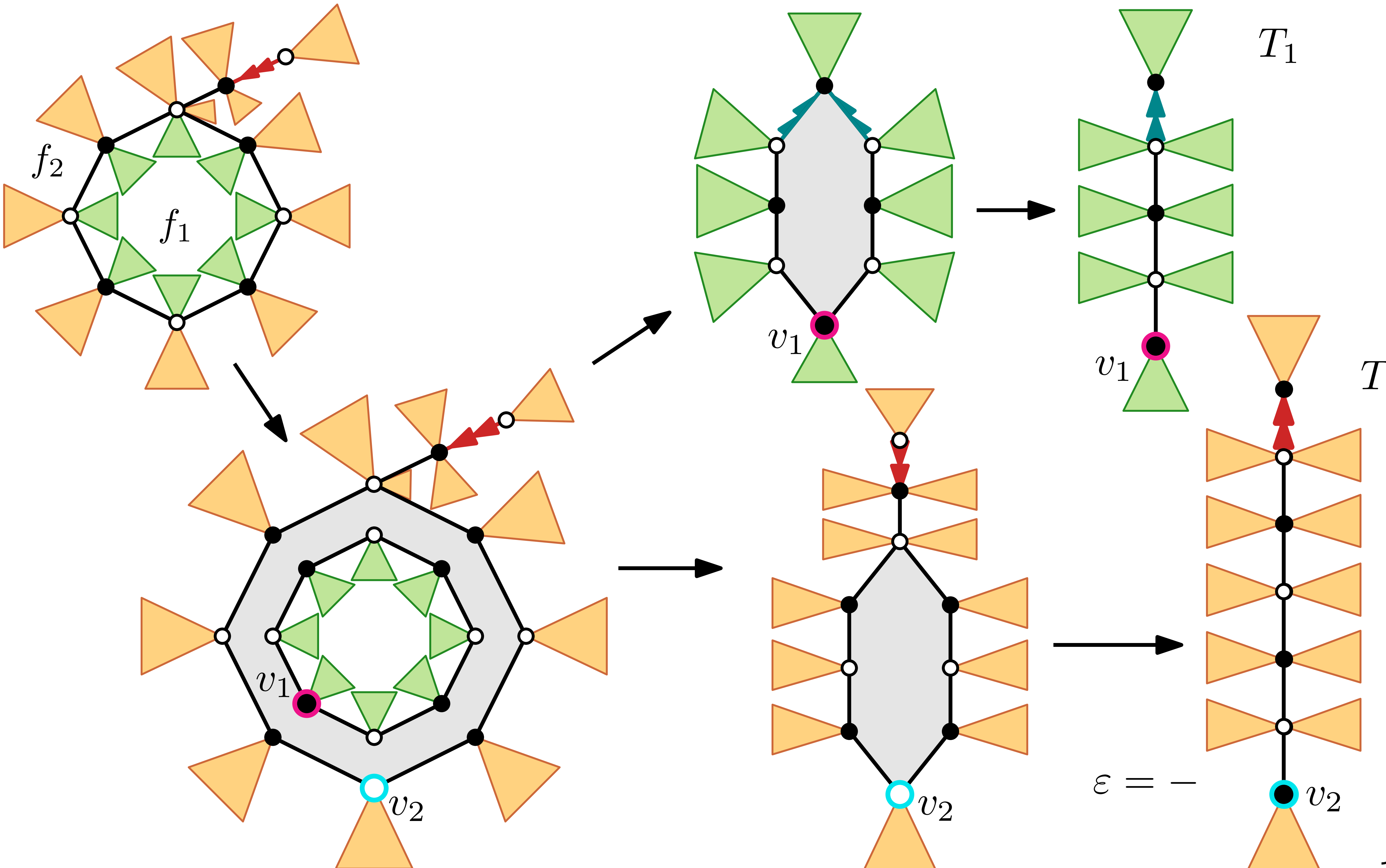
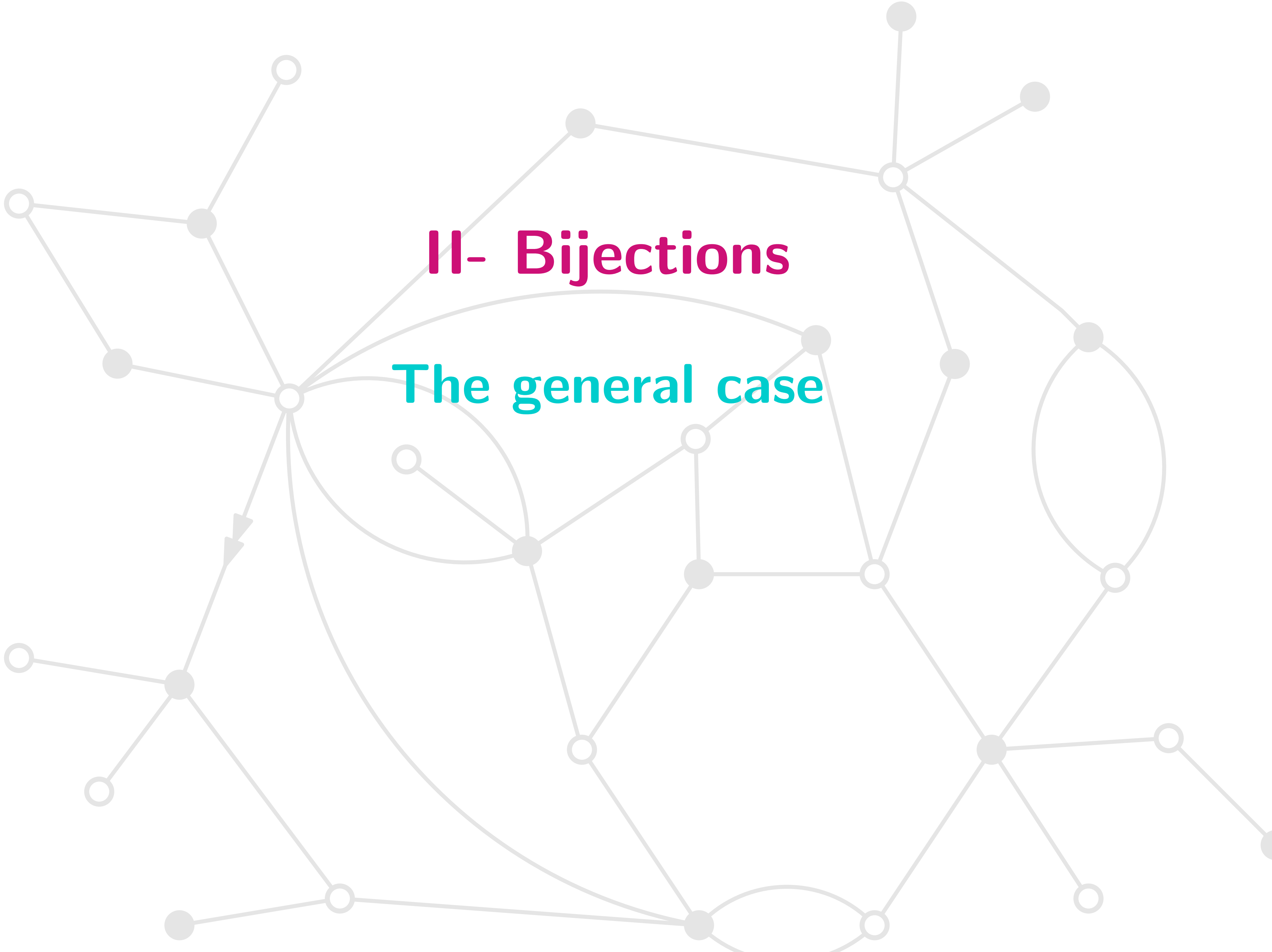


The 2 faces case: Cut and Close



II- Bijections

The general case



General case

$$4(f(\mathbf{d})-1)B(\mathbf{d}) = \sum_{\mathbf{s}+\mathbf{t}=\mathbf{d}} v(\mathbf{s})v(\mathbf{t})B(\mathbf{s})B(\mathbf{t})$$

Problems:

- Sew: Which path to slit?
- Cut: Which cycle to cut?

General case

$$4(f(\mathbf{d})-1)B(\mathbf{d}) = \sum_{\mathbf{s}+\mathbf{t}=\mathbf{d}} v(\mathbf{s})v(\mathbf{t})B(\mathbf{s})B(\mathbf{t})$$

Problems:

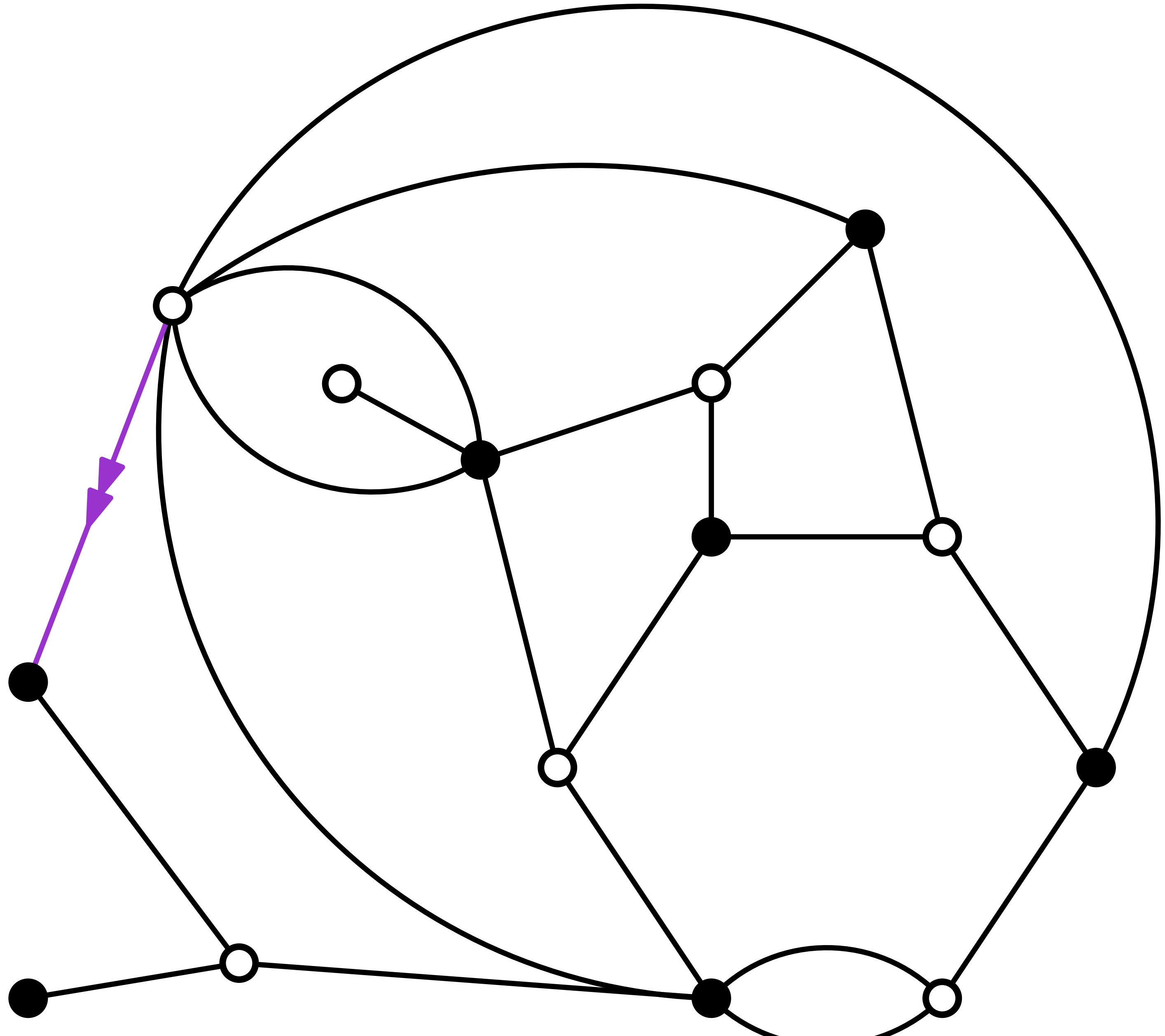
- Sew: Which path to slit?
- Cut: Which cycle to cut?

$\Rightarrow$ Take a spanning tree!

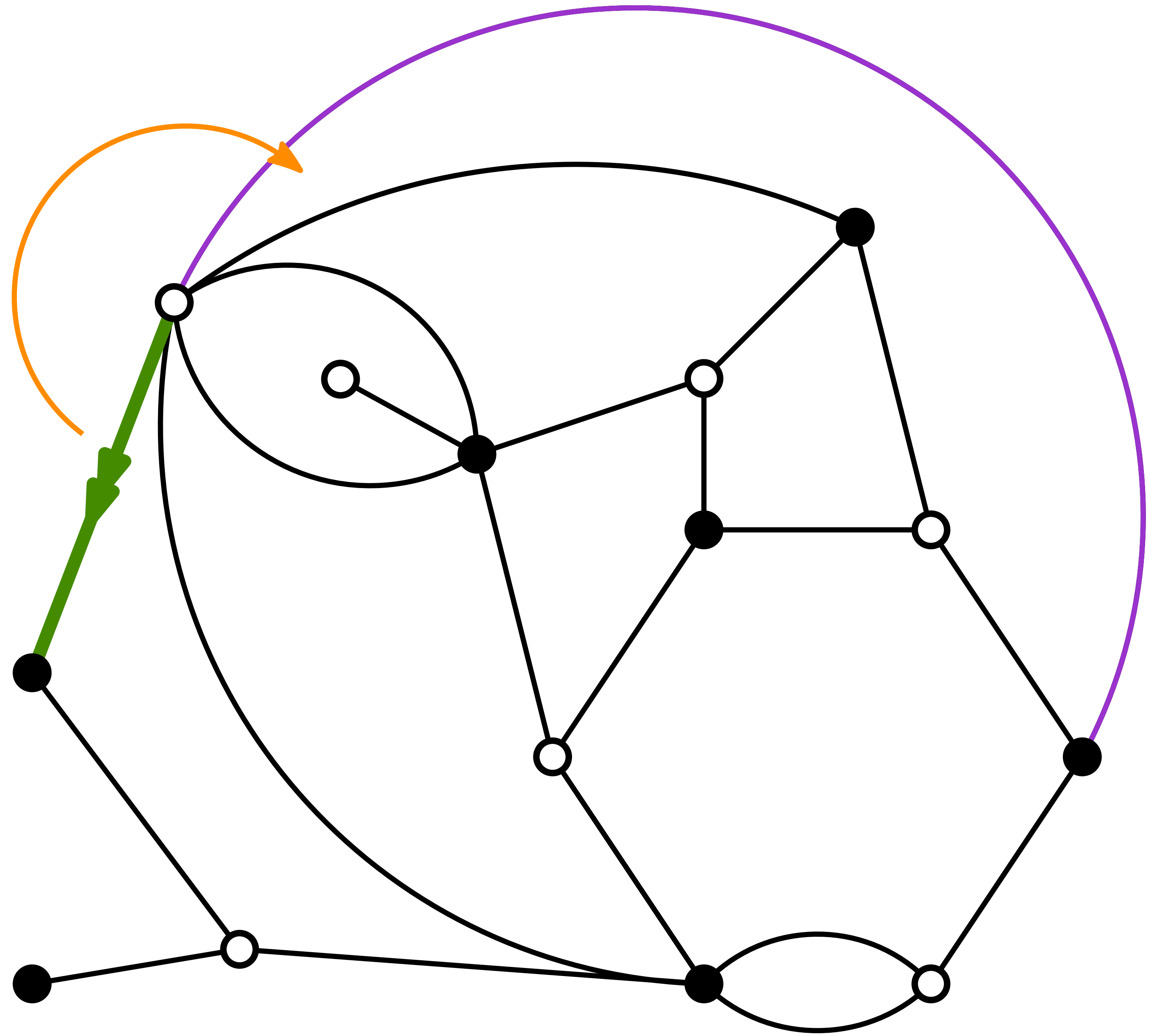
$\hookrightarrow$ Canonical path to slit.

$\hookrightarrow$ $2(f(\mathbf{d})-1)$ = an oriented edge outside of the spanning tree.

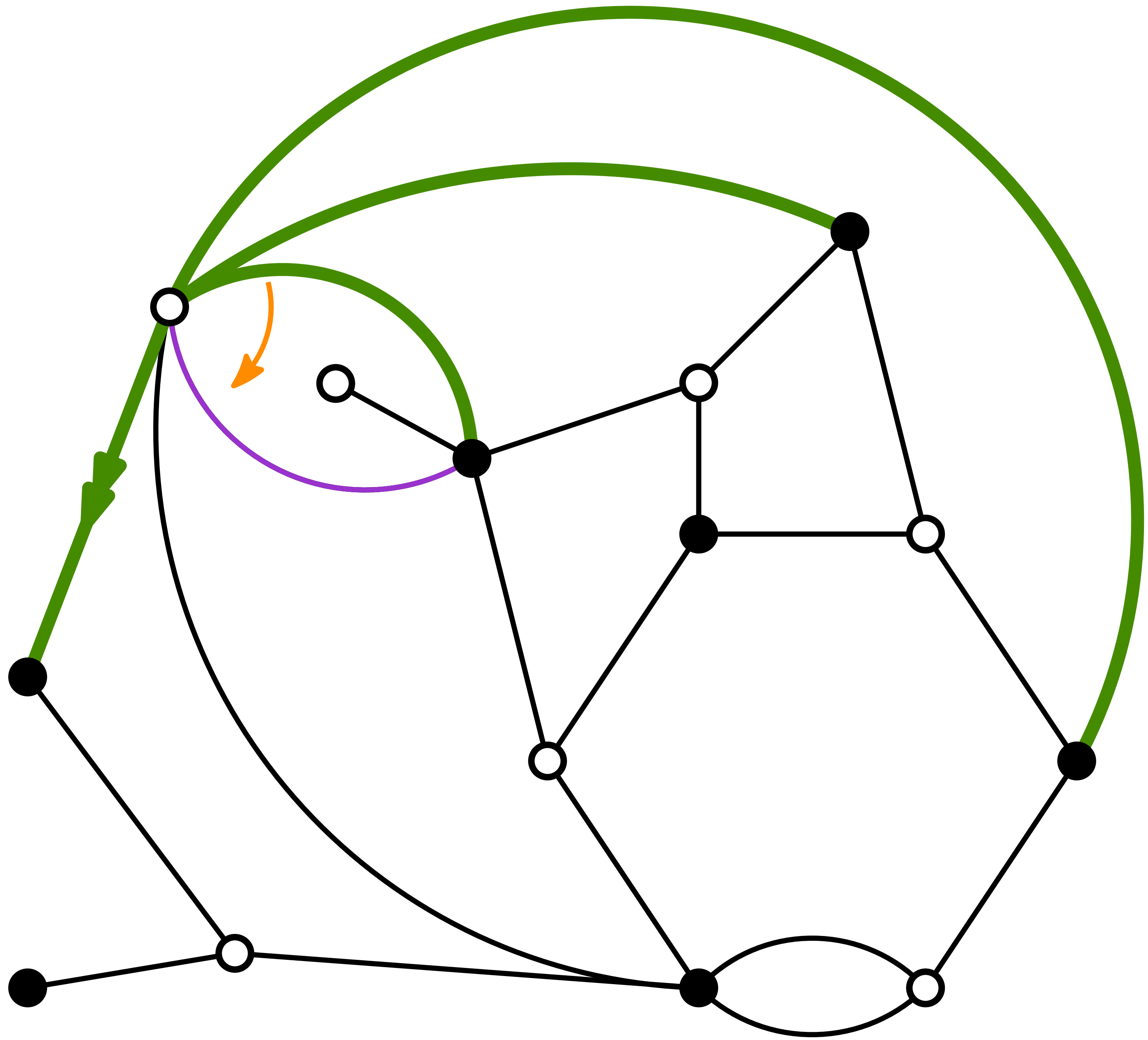
Breadth-first search tree



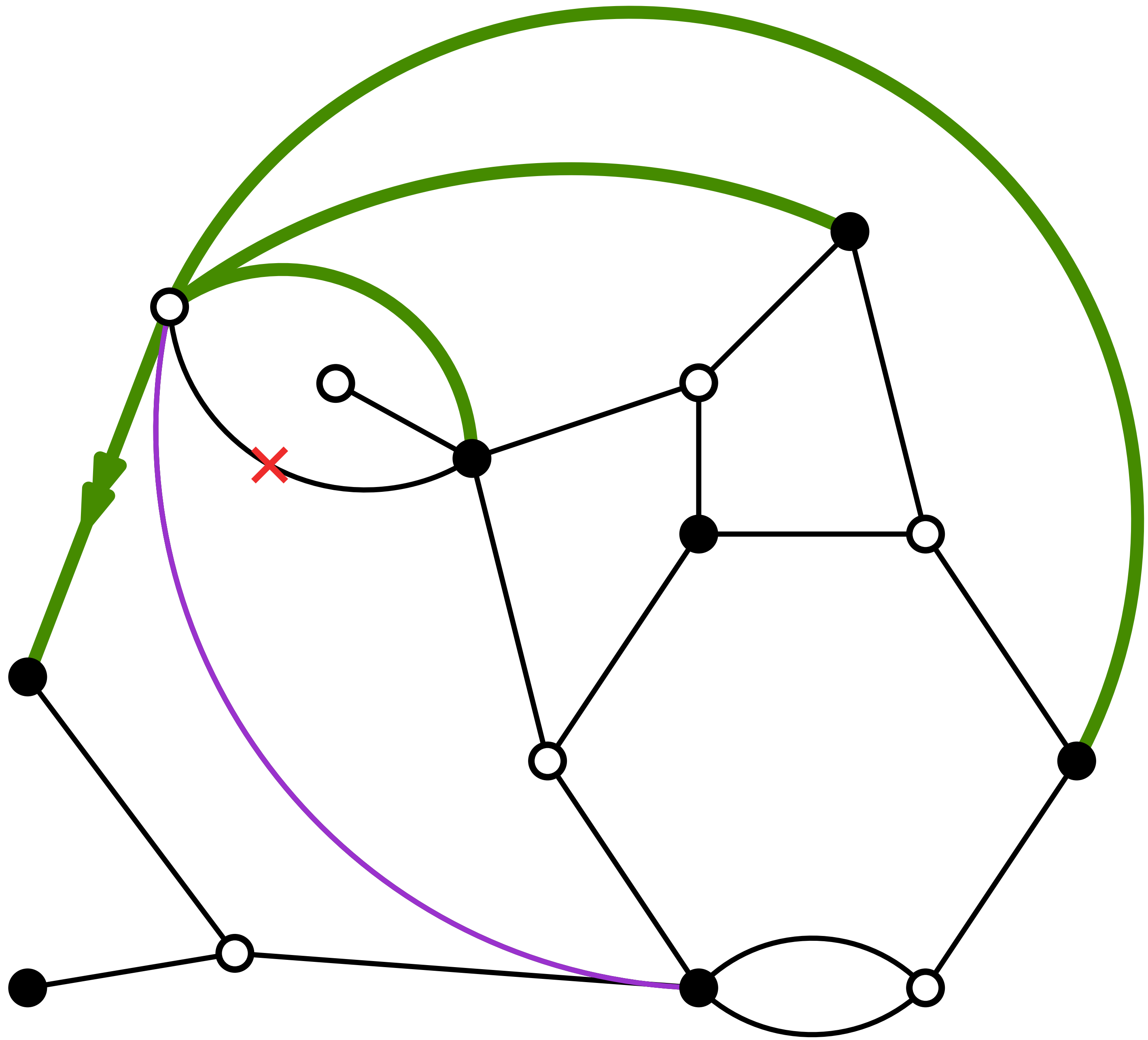
Breadth-first search tree



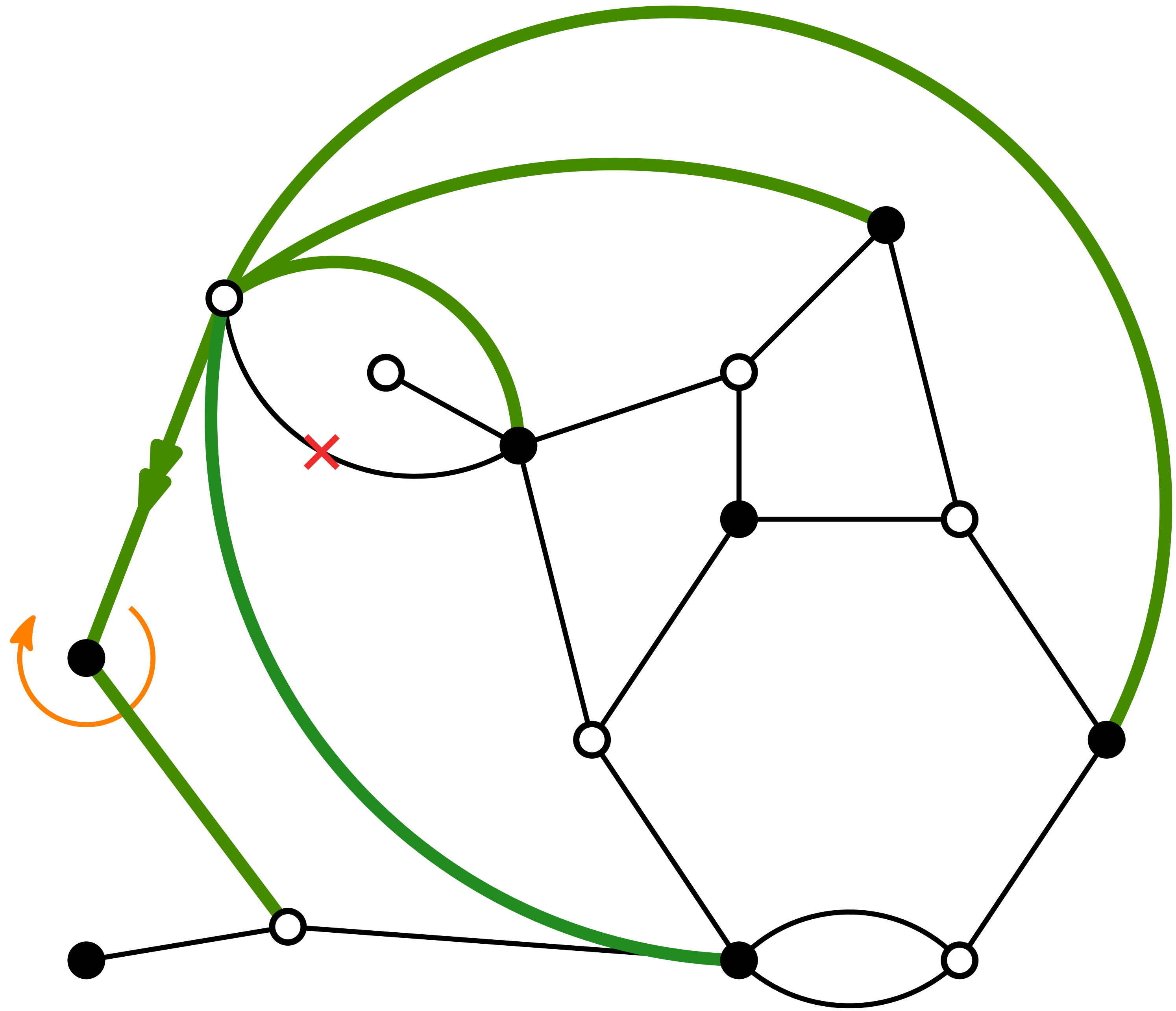
Breadth-first search tree



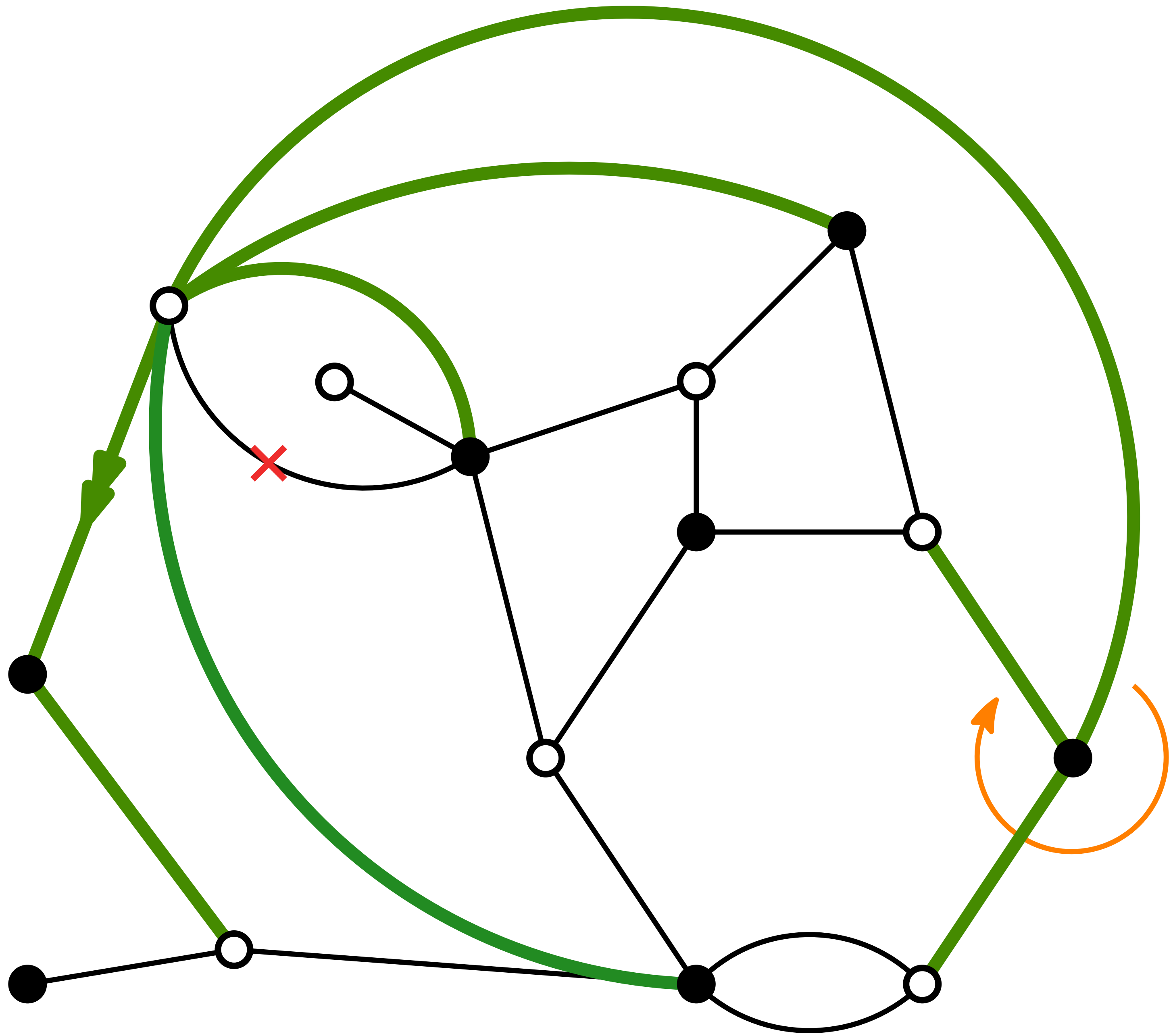
Breadth-first search tree



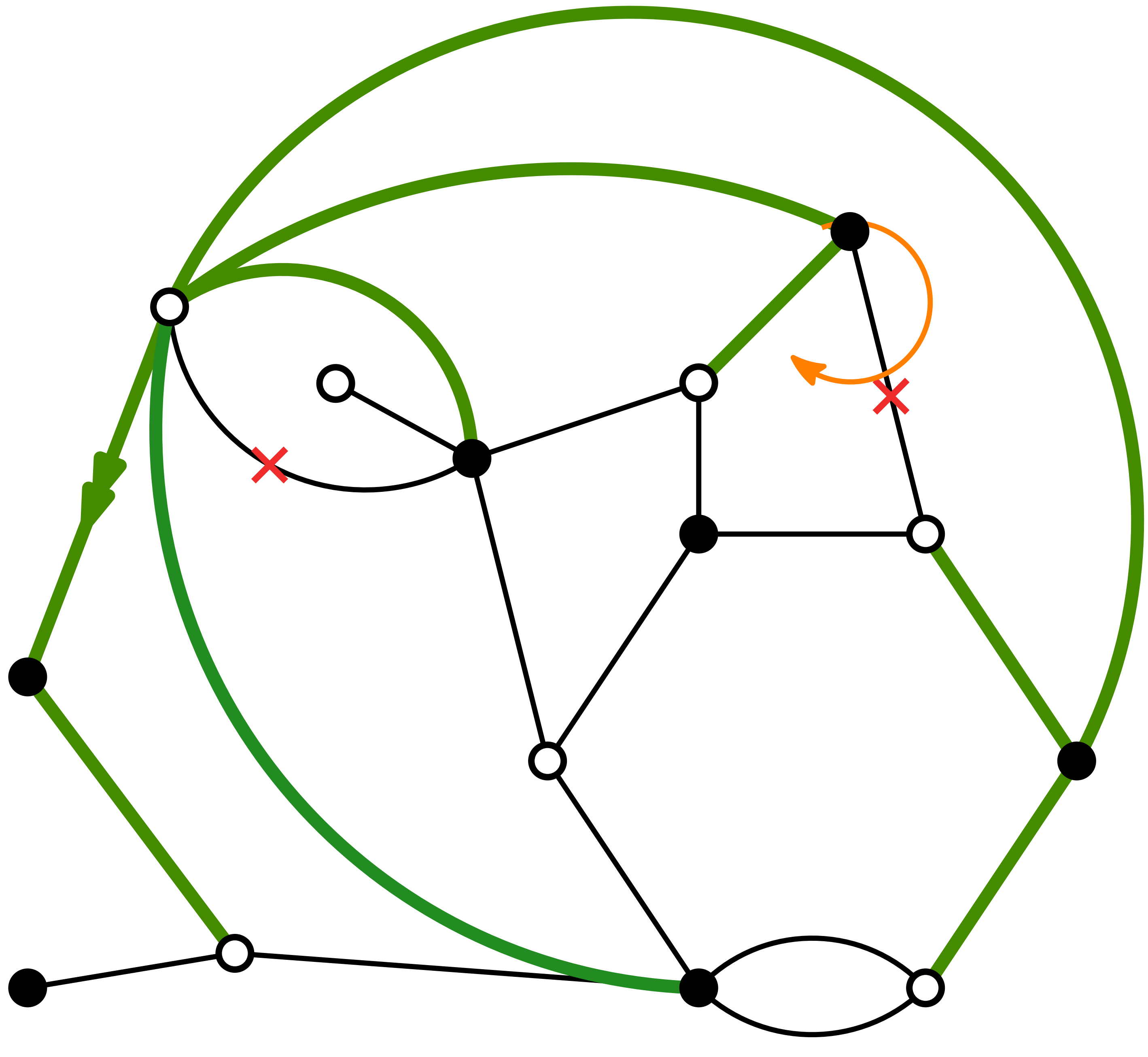
Breadth-first search tree



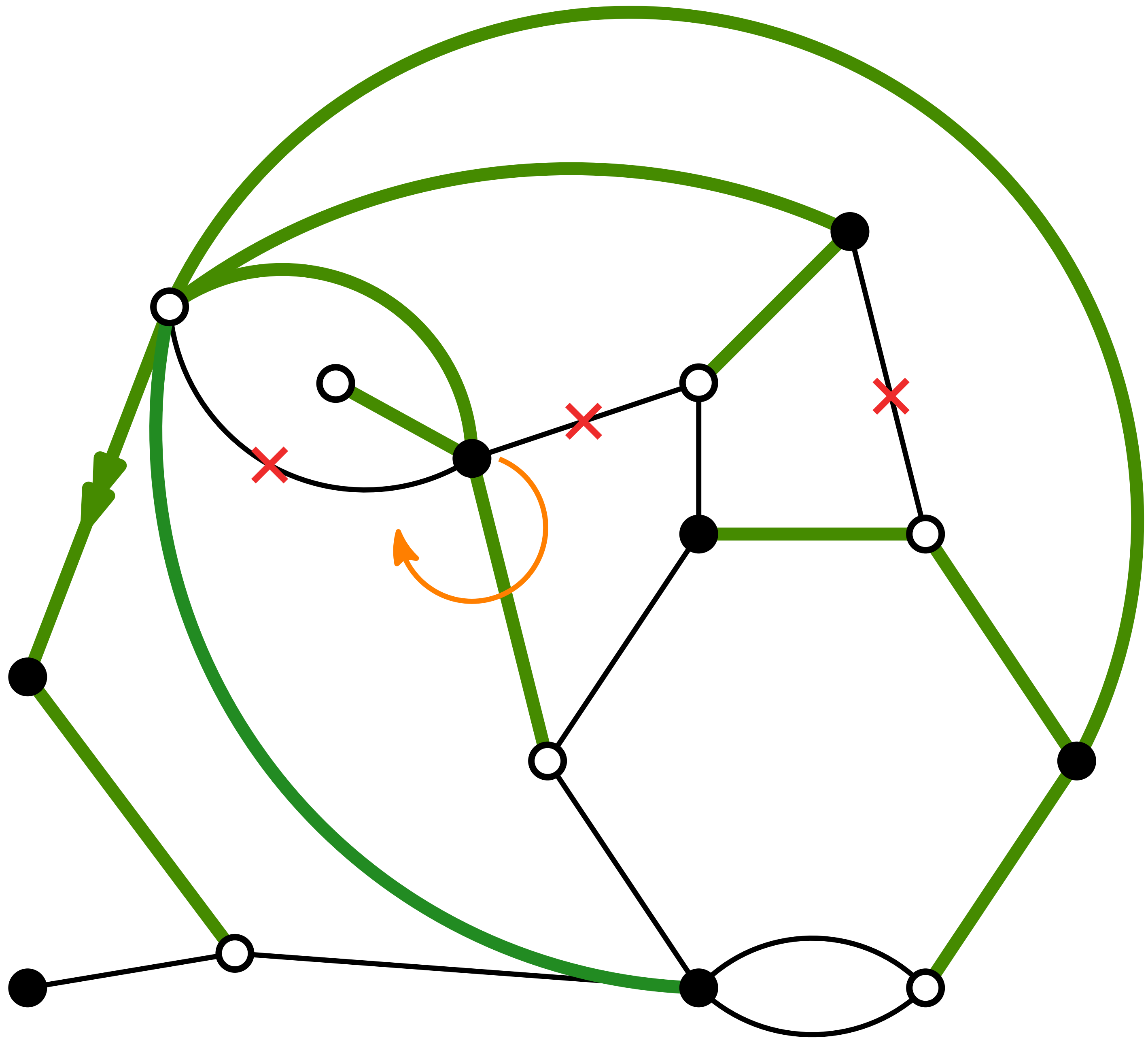
Breadth-first search tree



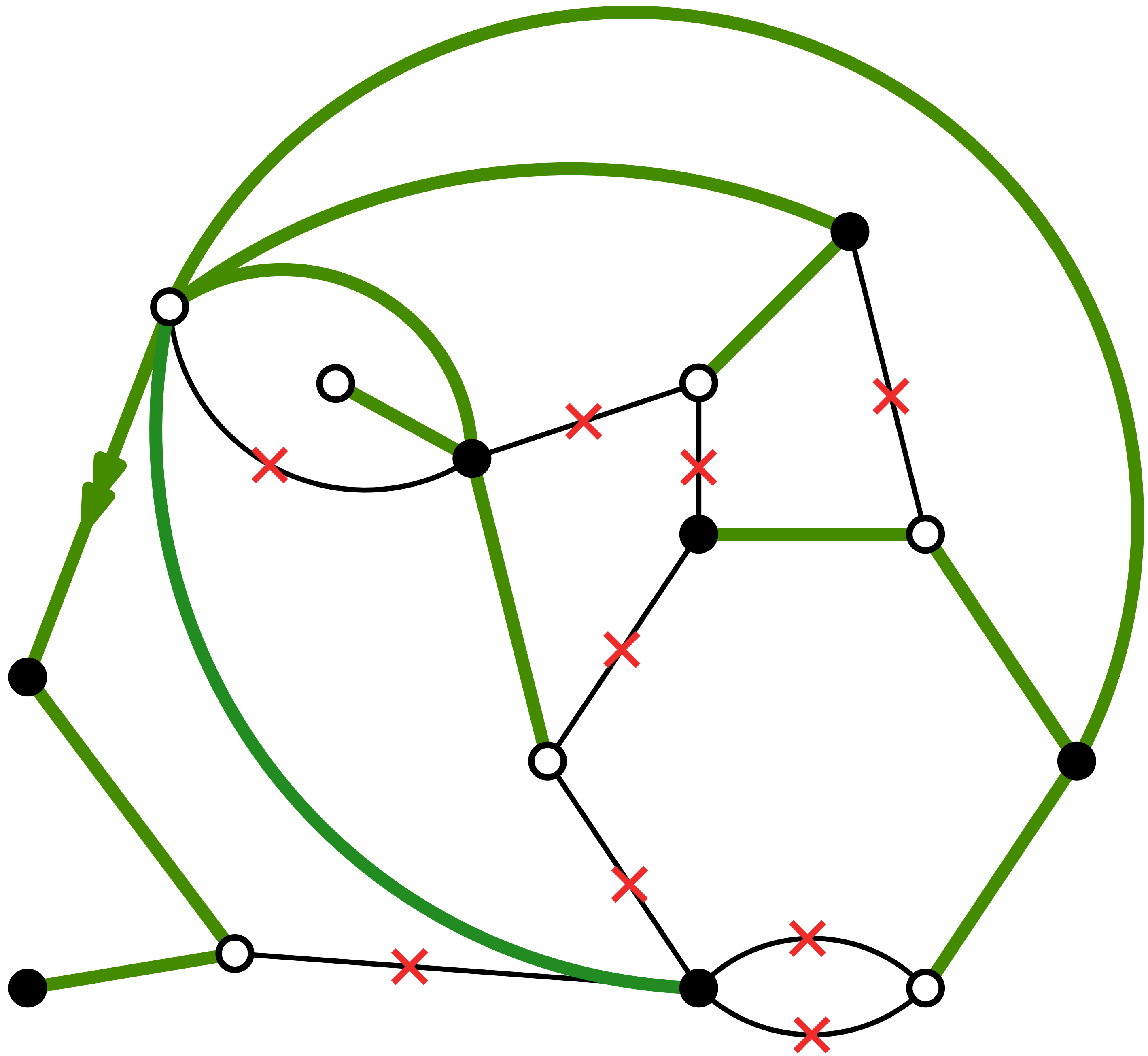
Breadth-first search tree



Breadth-first search tree

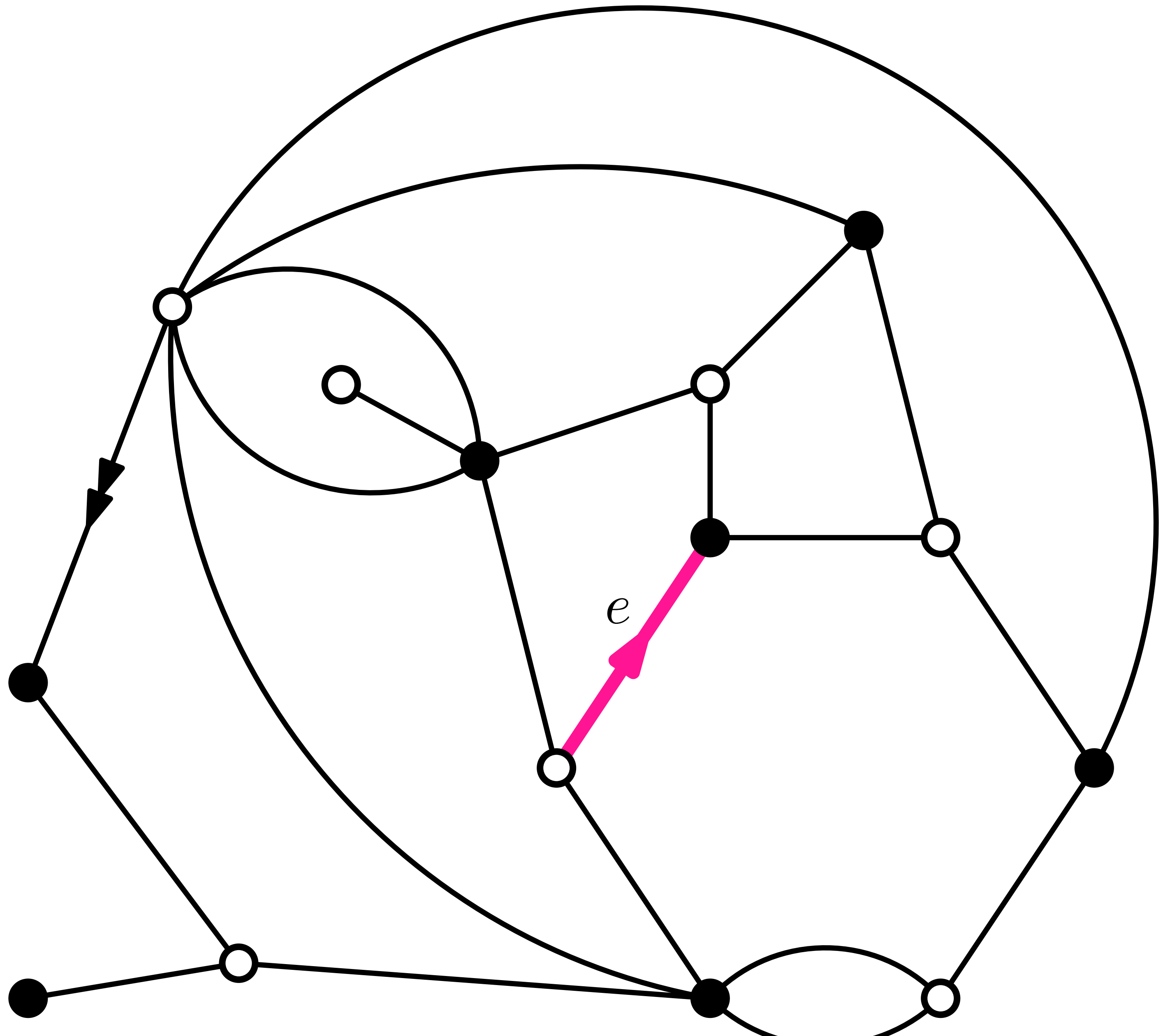


Breadth-first search tree



General case : Cut and Close

$(M, -)$



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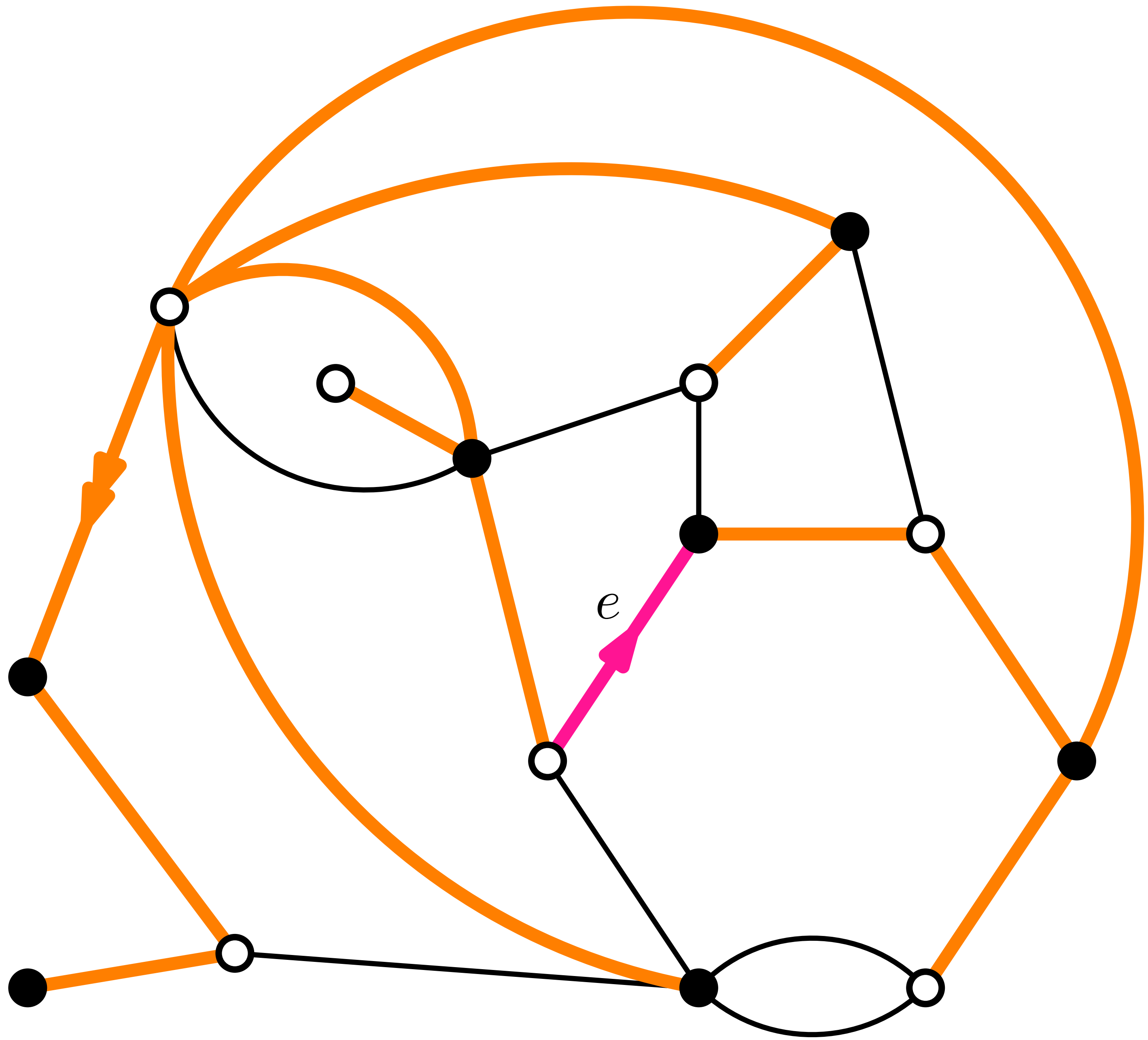

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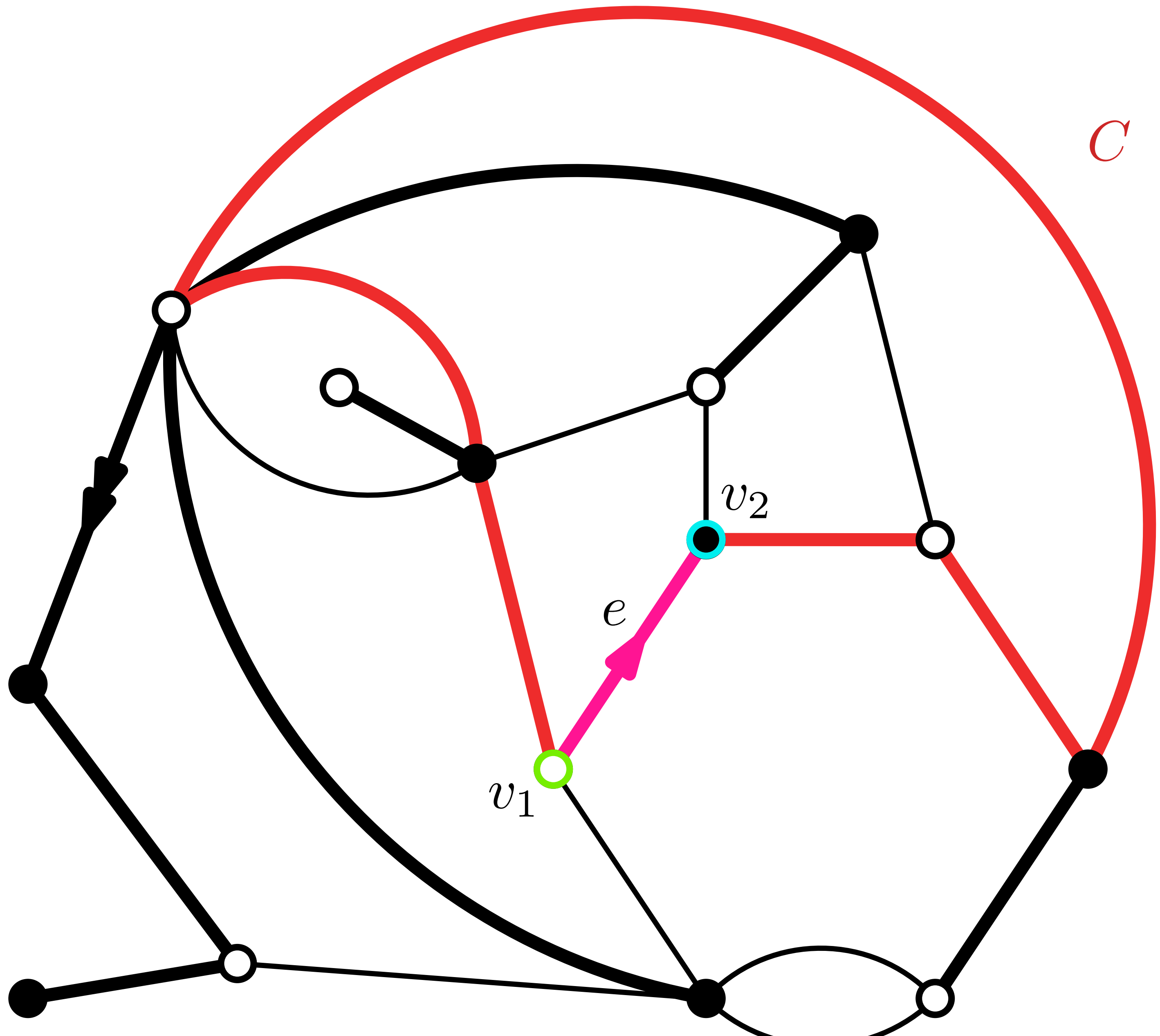

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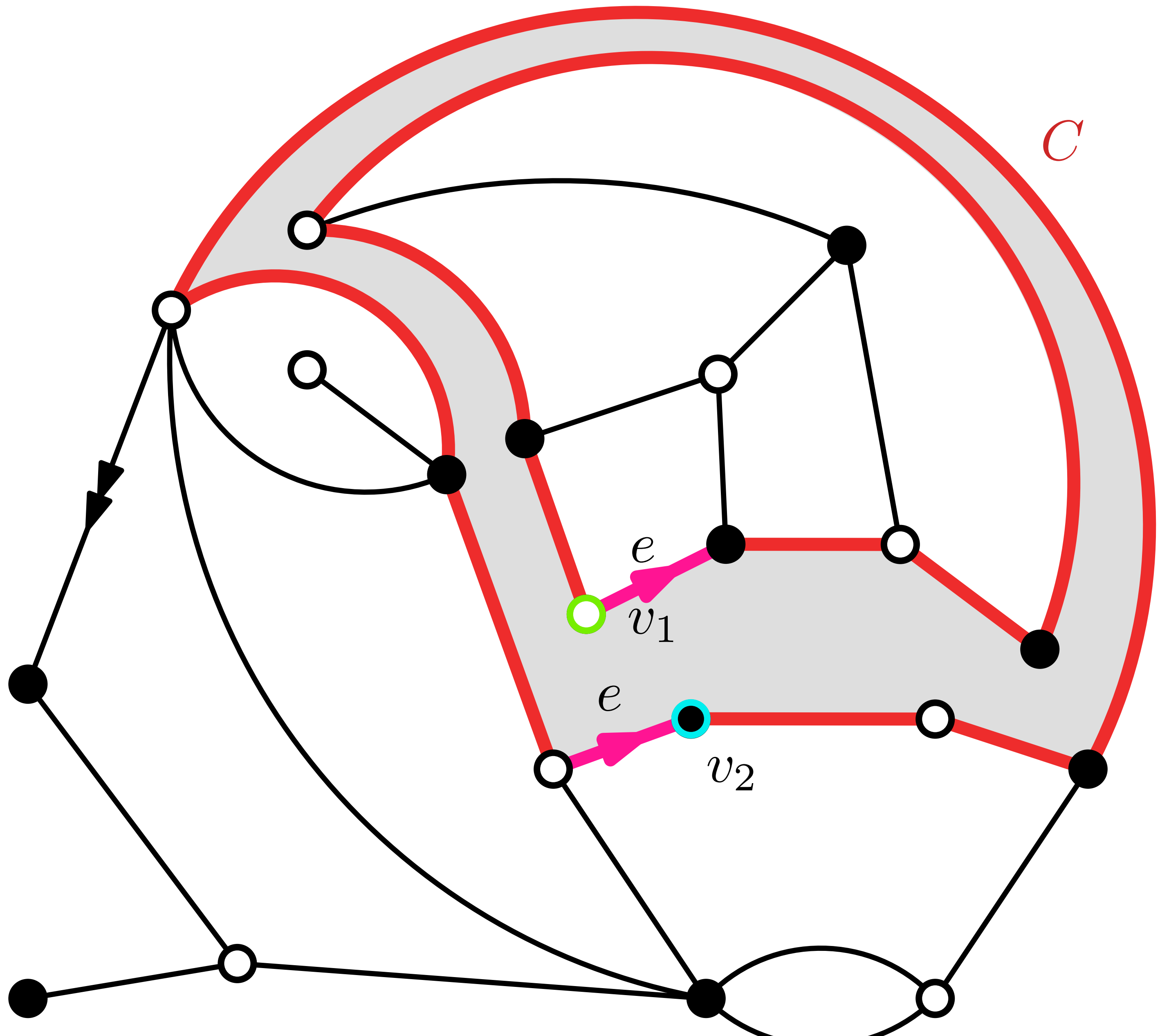

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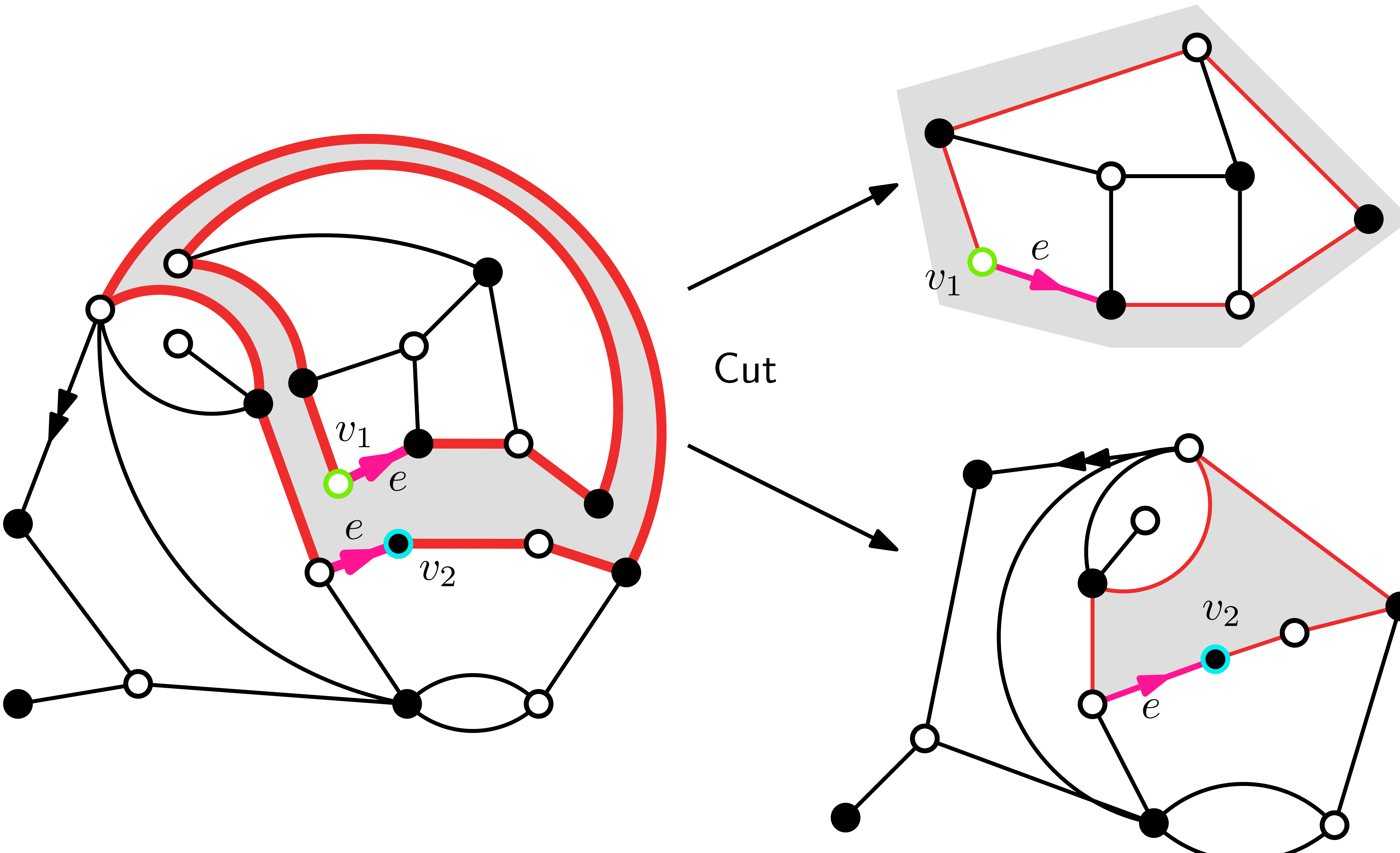

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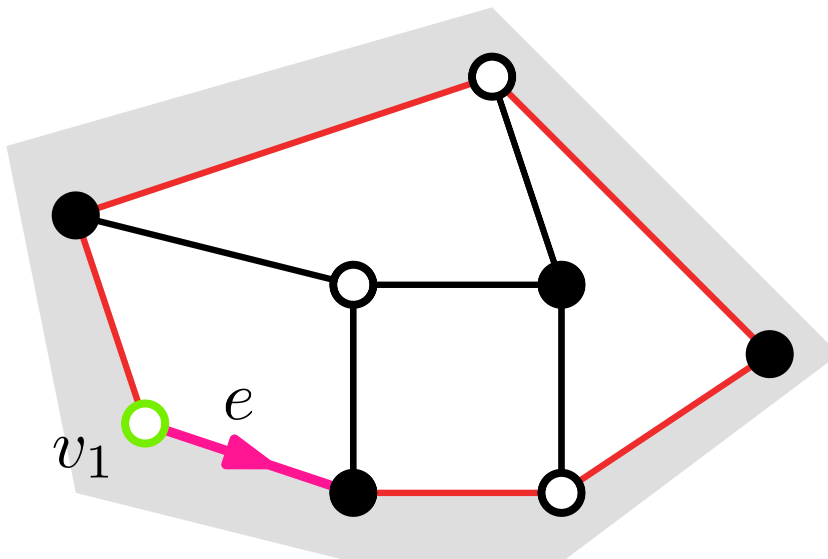
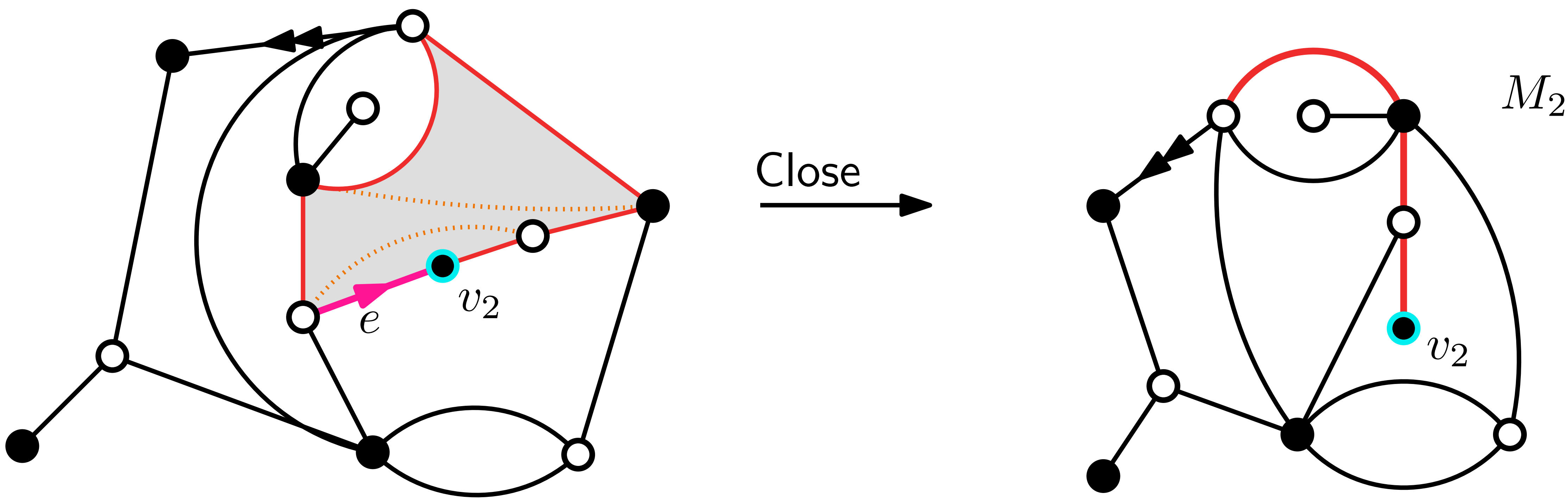
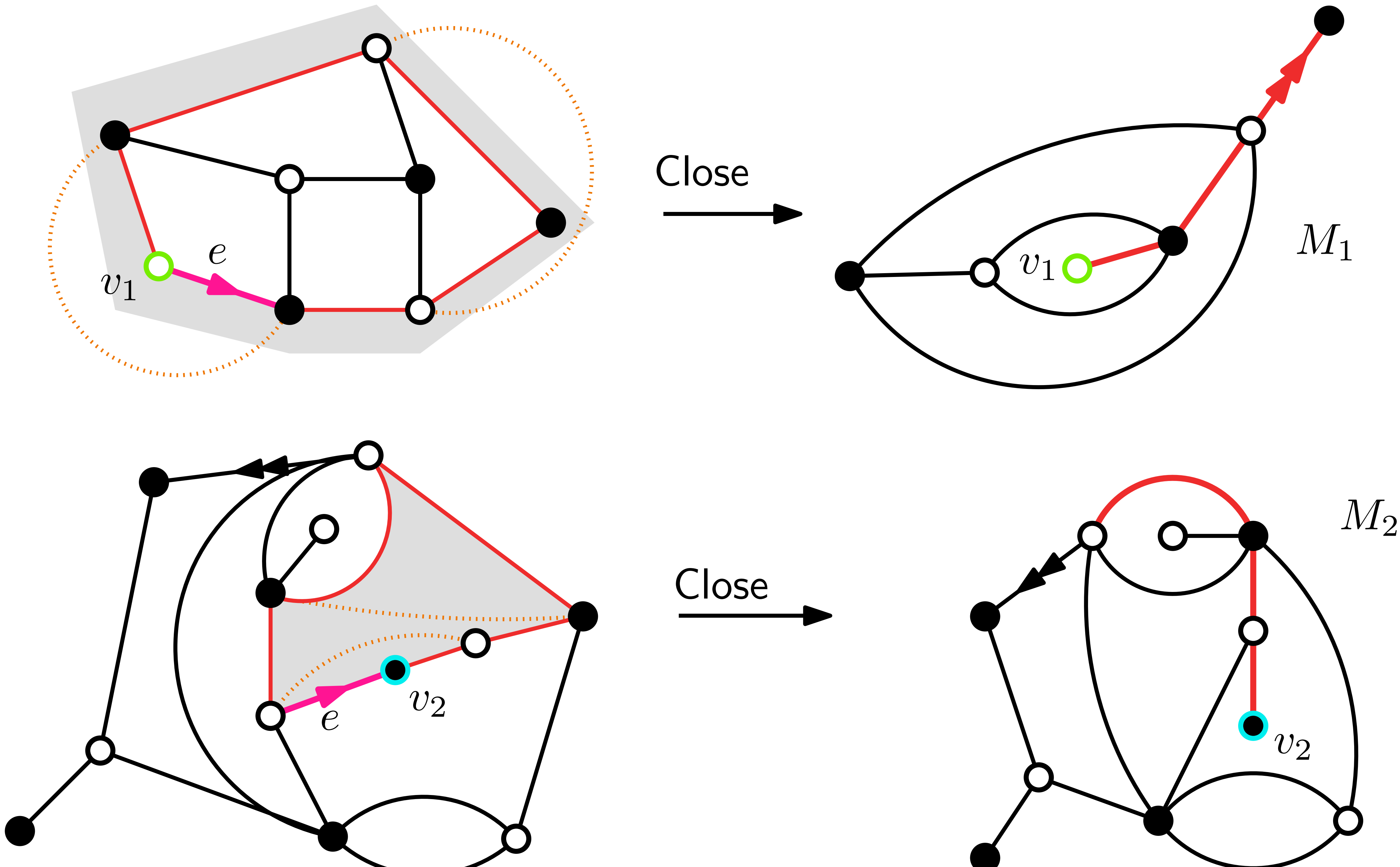
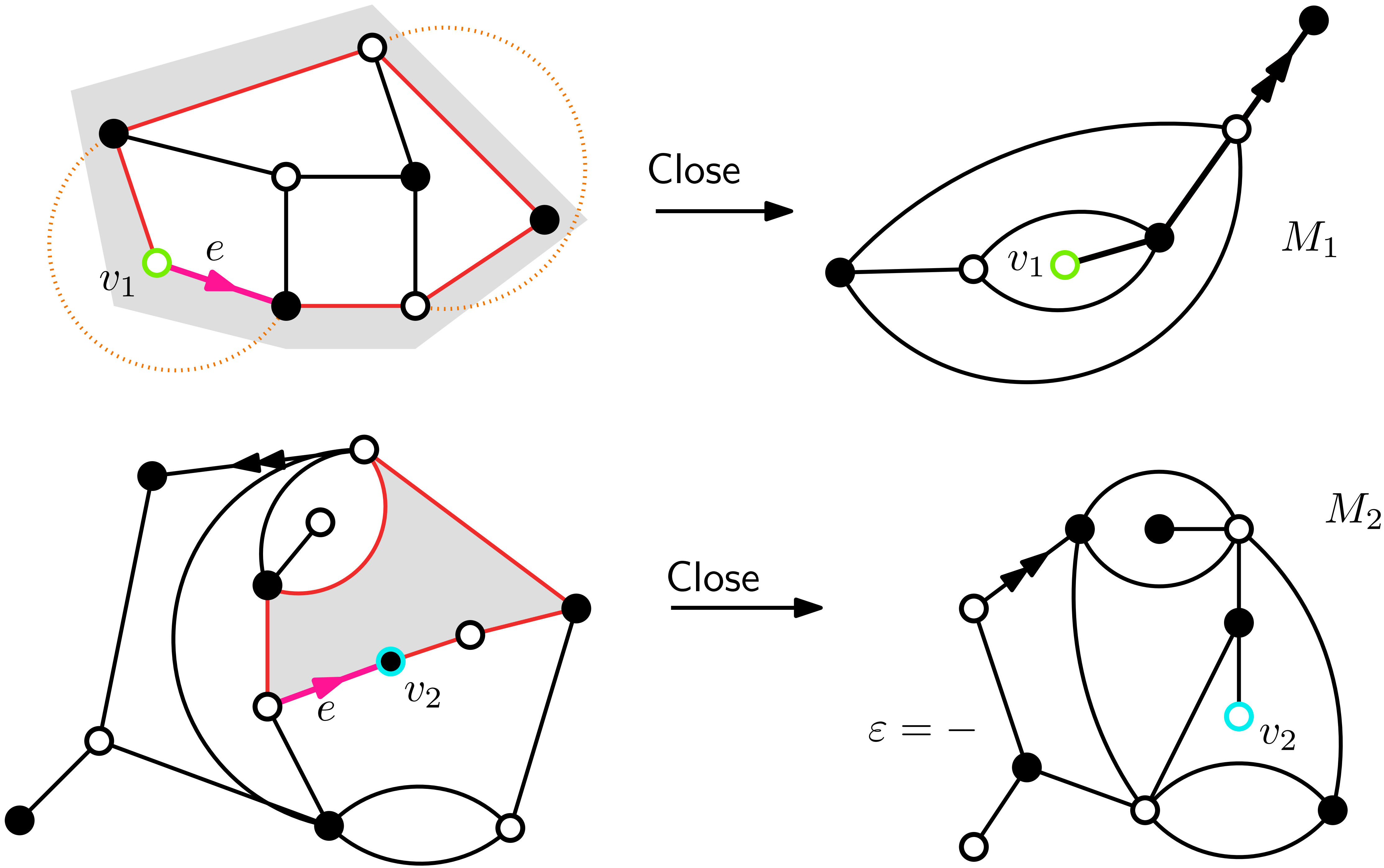




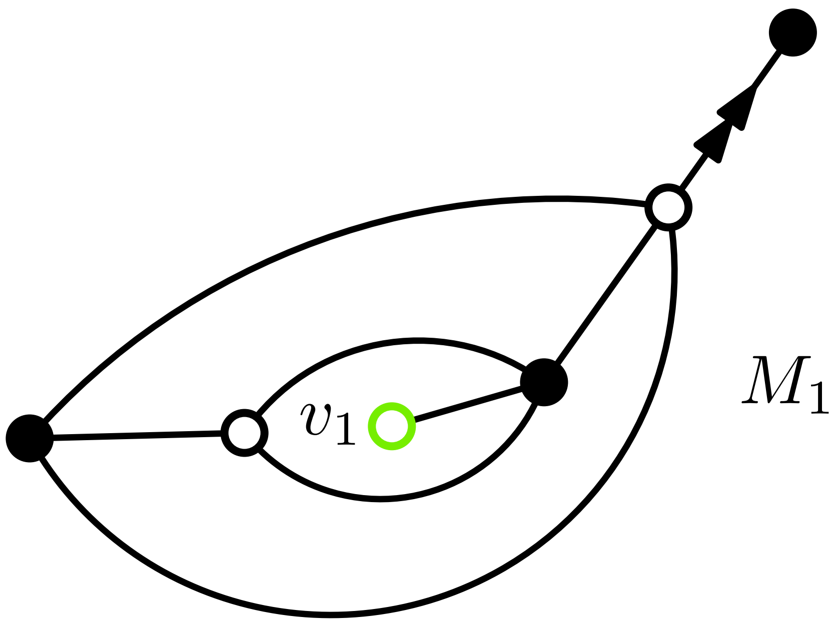
General case : Cut and Close

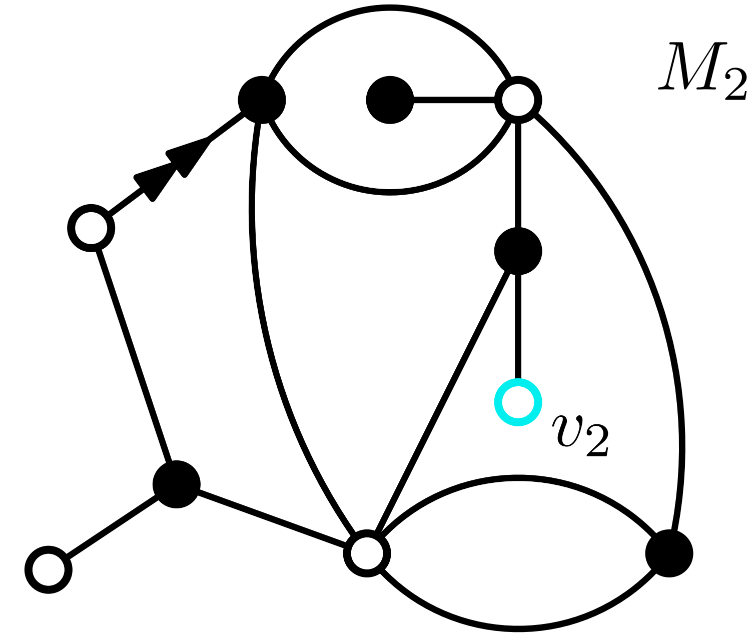


General case : Cut and Close

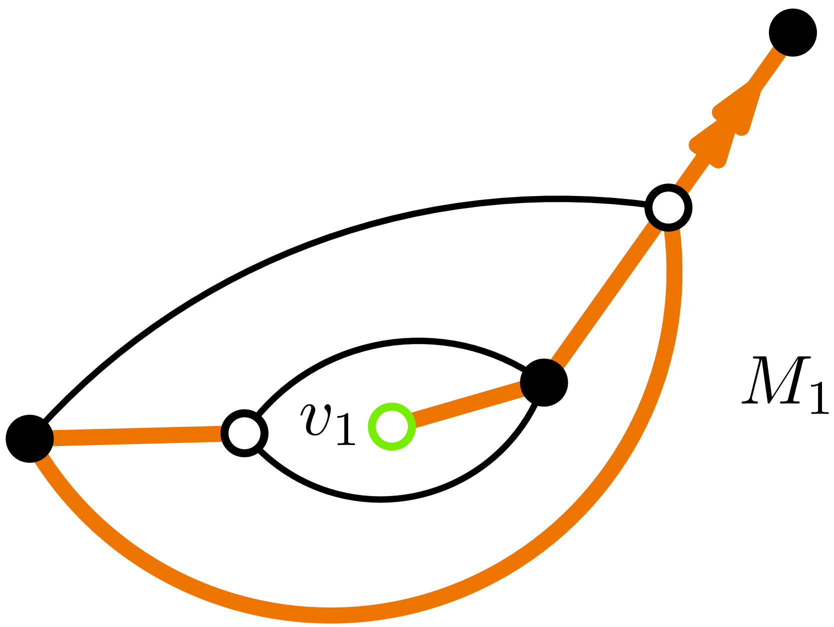


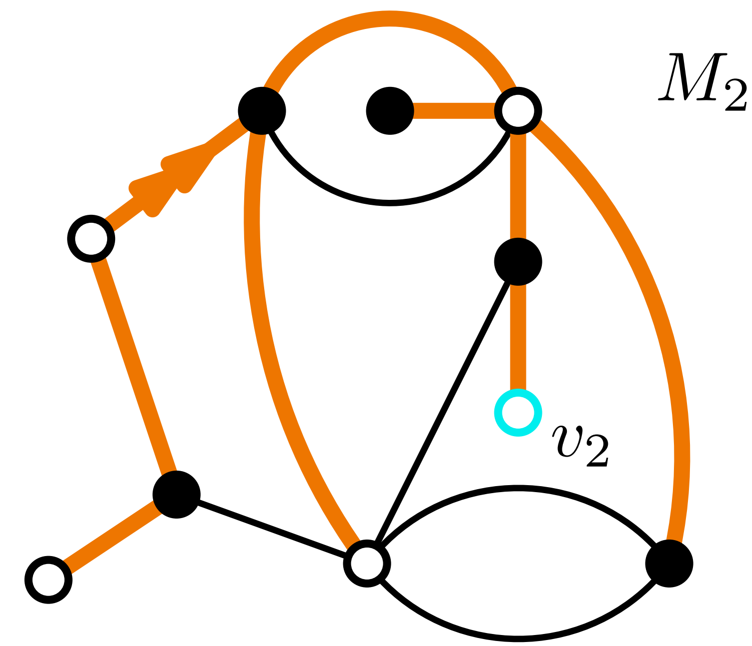
General case : Slit and Sew





General case : Slit and Sew





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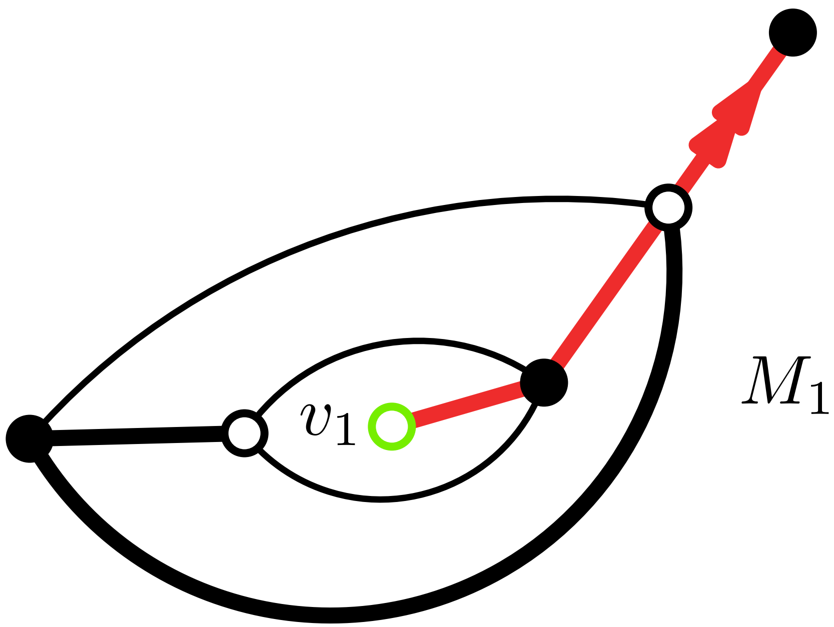

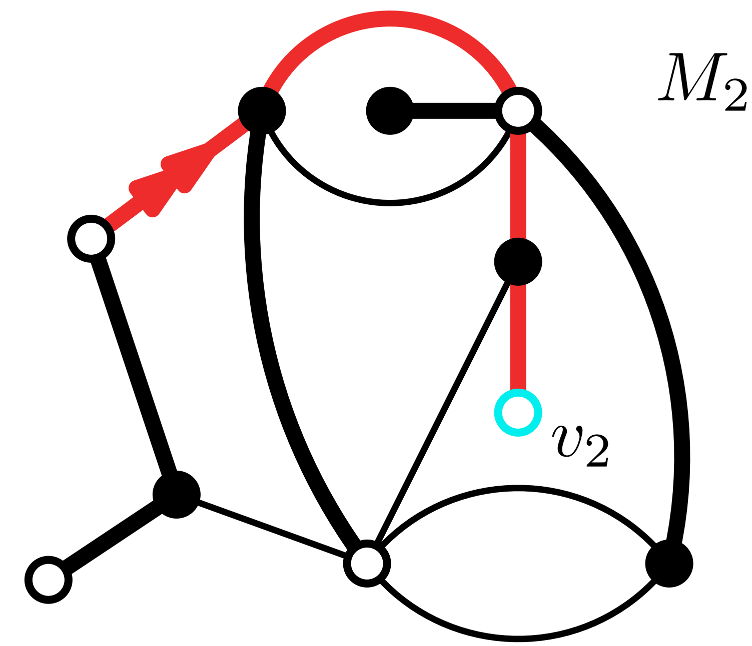

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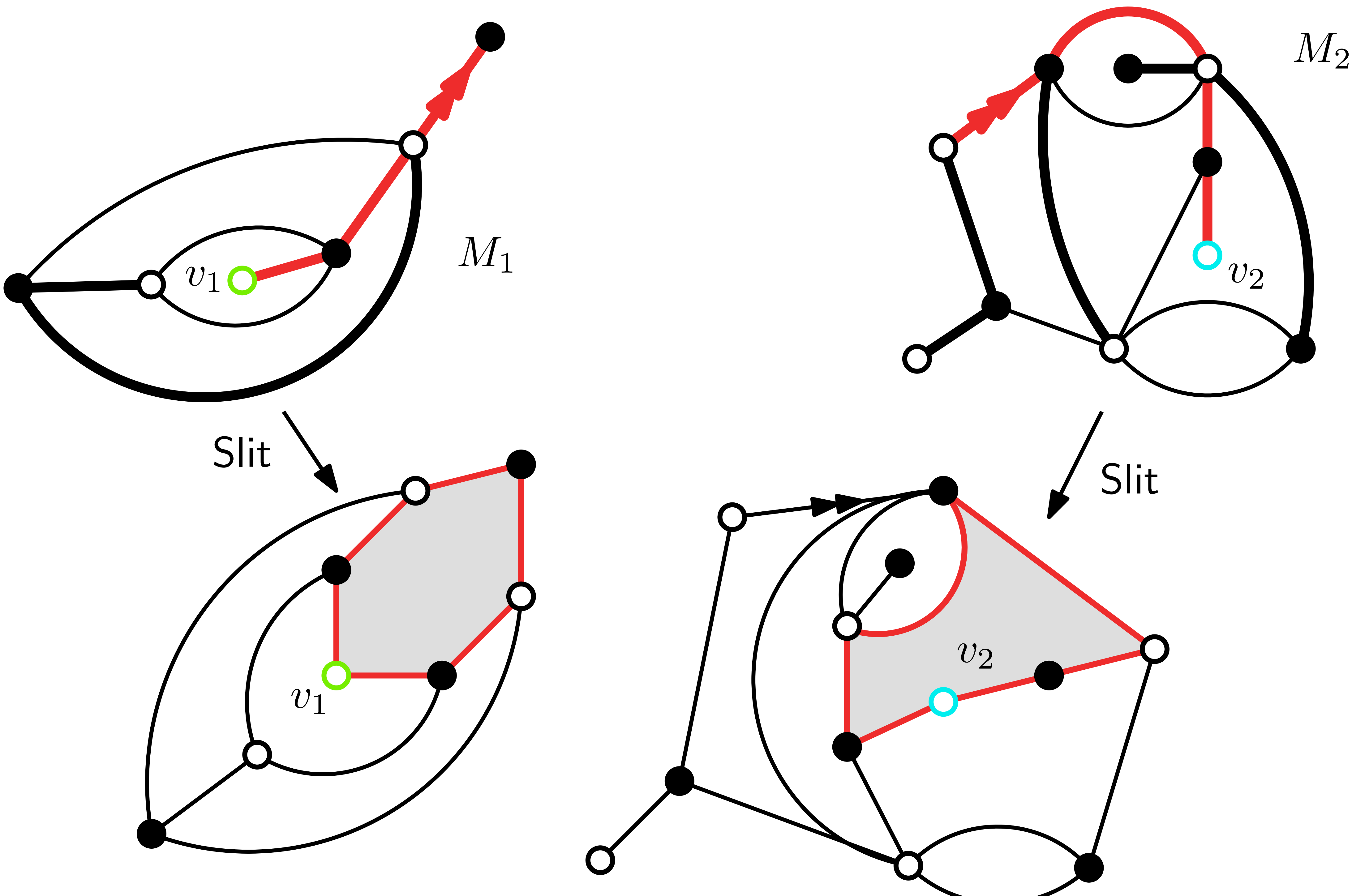
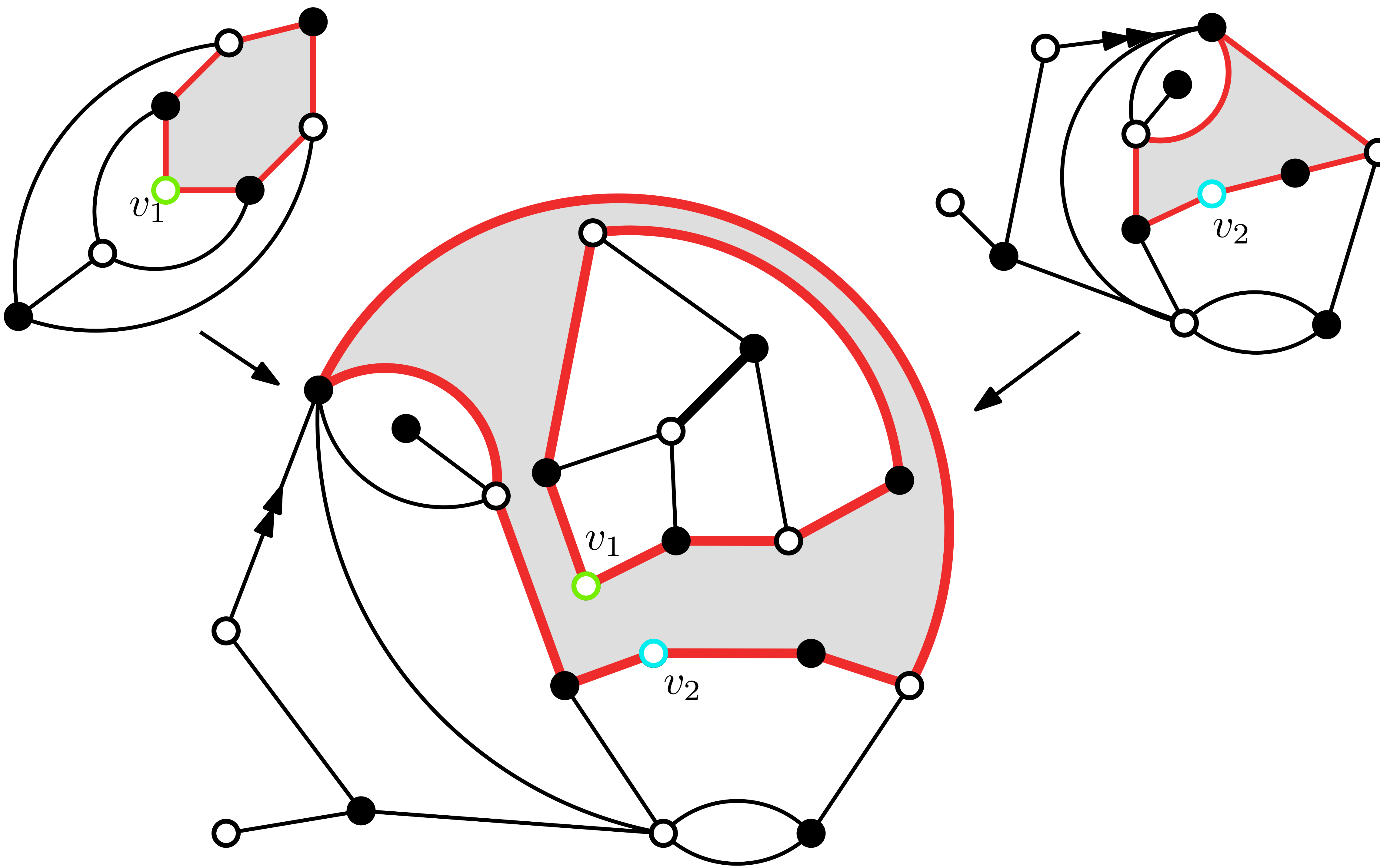
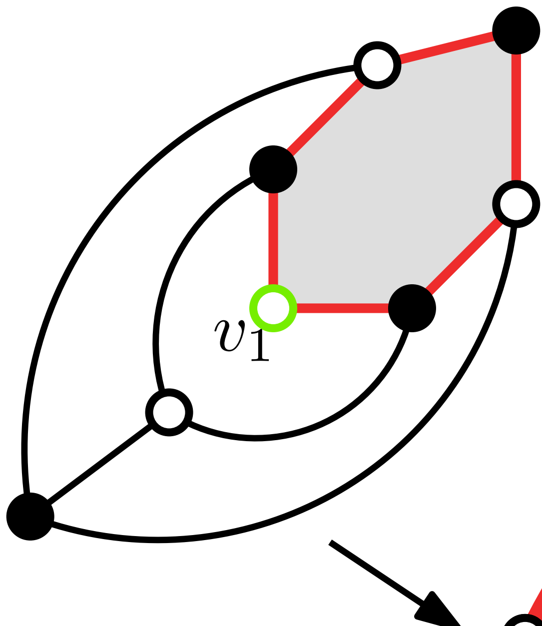
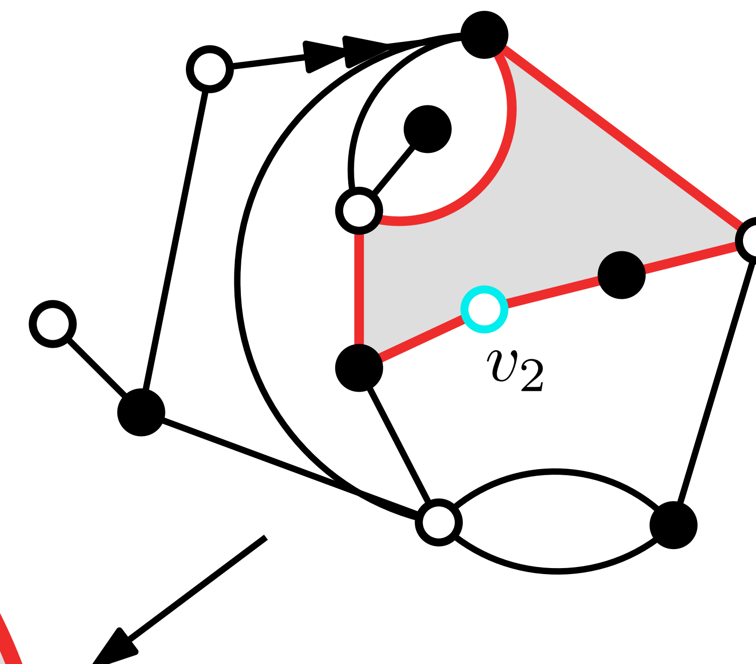


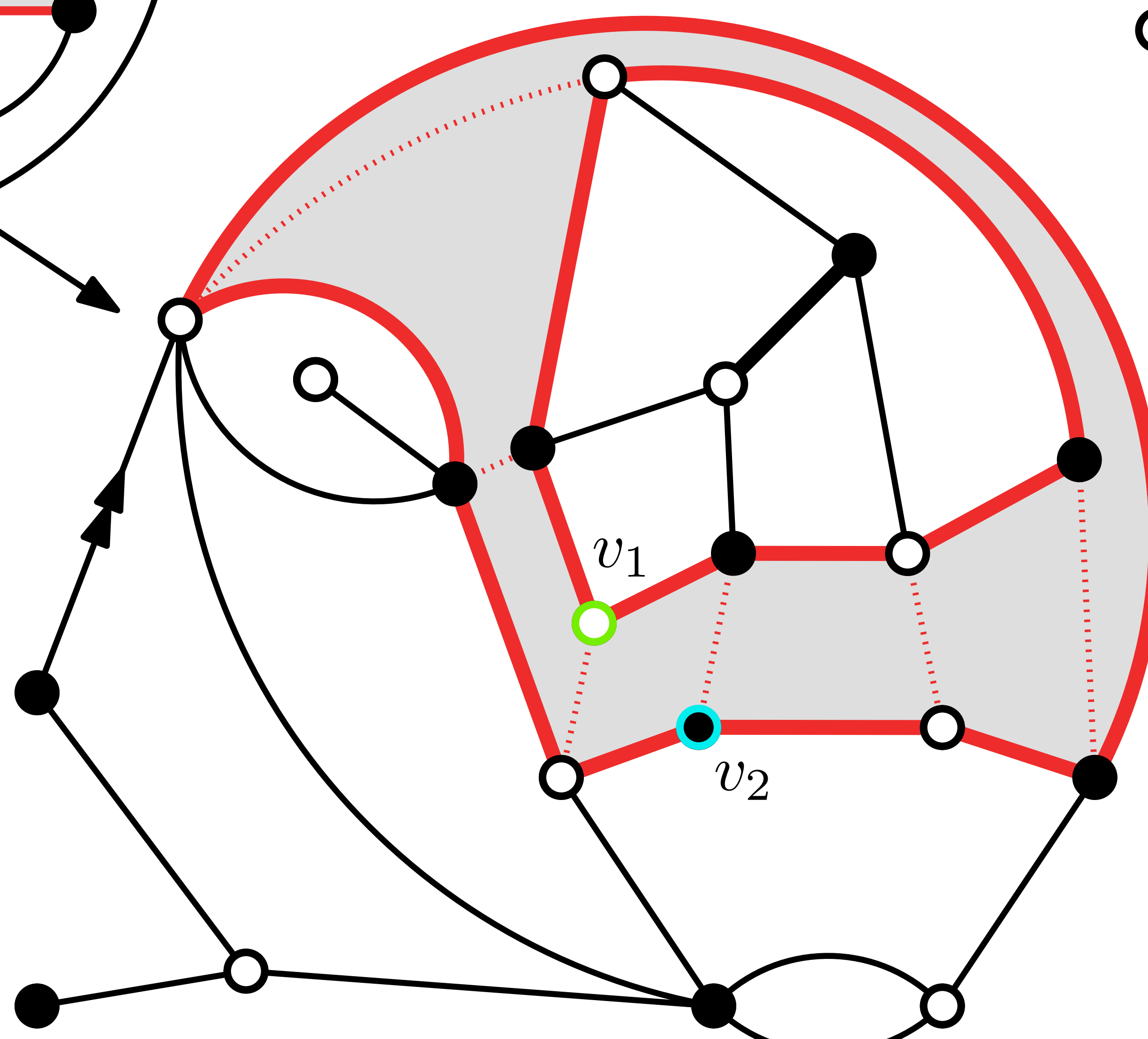
General case : Slit and Sew



General case : Slit and Sew



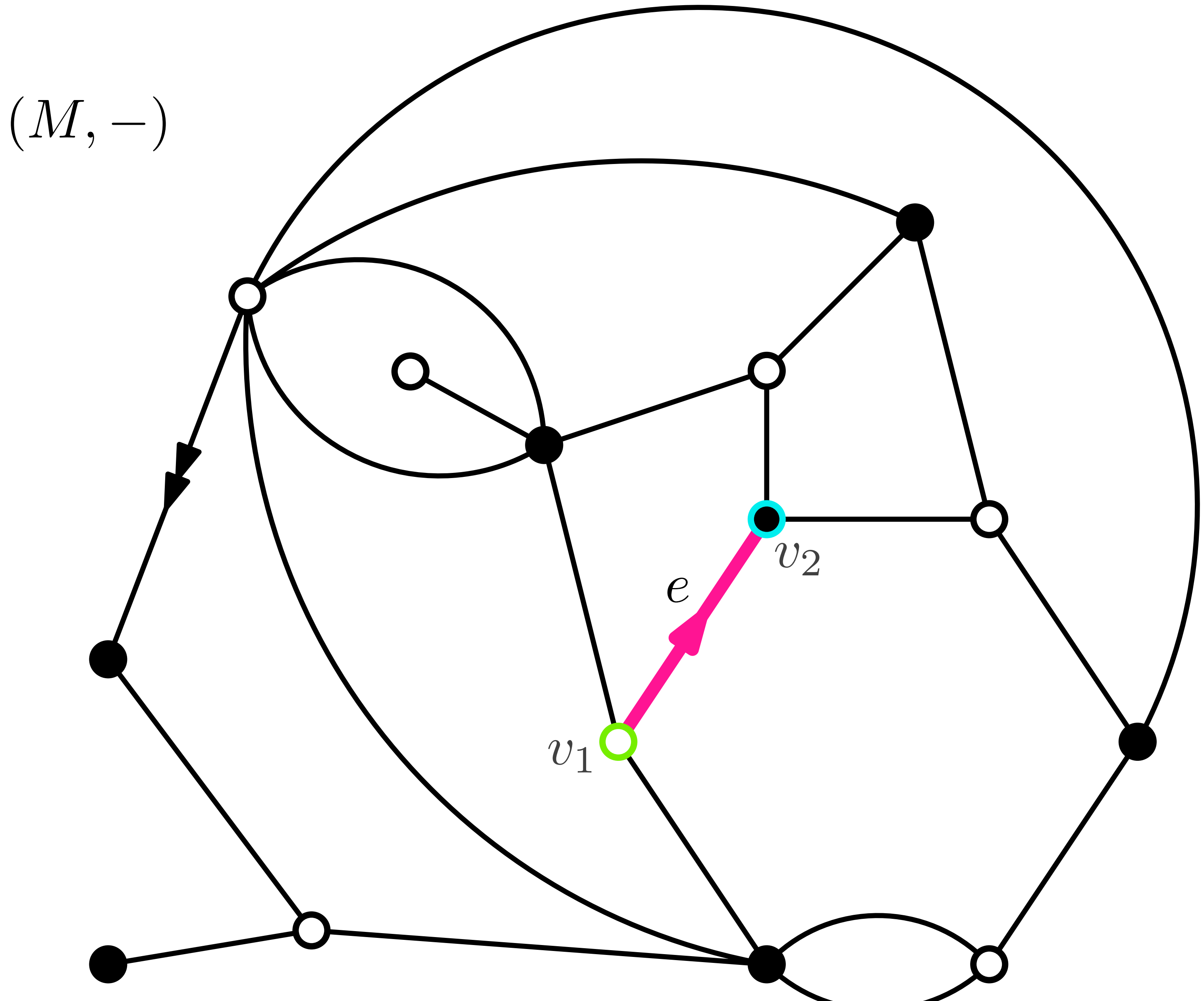




Recolor and Sew

$\varepsilon = -$

General case : Slit and Sew



II- Bijections

The other bijections

The other bijection

$$(f(\mathbf{d})-1)(f(\mathbf{d})-2)B(\mathbf{d}) = \sum_{\mathbf{s}+\mathbf{t}=\mathbf{d}} (f(\mathbf{s})-1)v(\mathbf{t})(v(\mathbf{t})-1)B(\mathbf{s})B(\mathbf{t})$$

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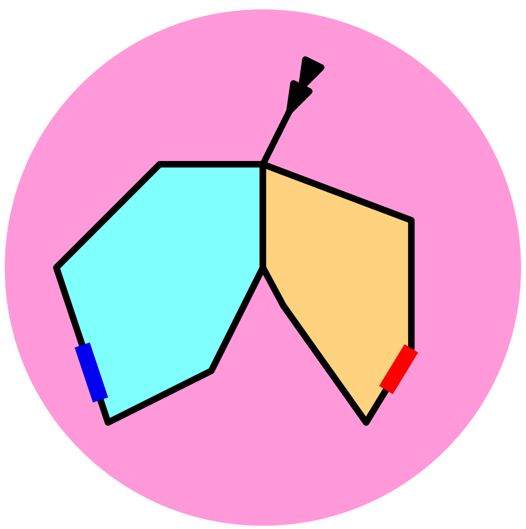
$$(f(\mathbf{d})-1)(f(\mathbf{d})-2)B(\mathbf{d}) = \sum_{\mathbf{s}+\mathbf{t}=\mathbf{d}} (f(\mathbf{s})-1)v(\mathbf{t})(v(\mathbf{t})-1)B(\mathbf{s})B(\mathbf{t})$$



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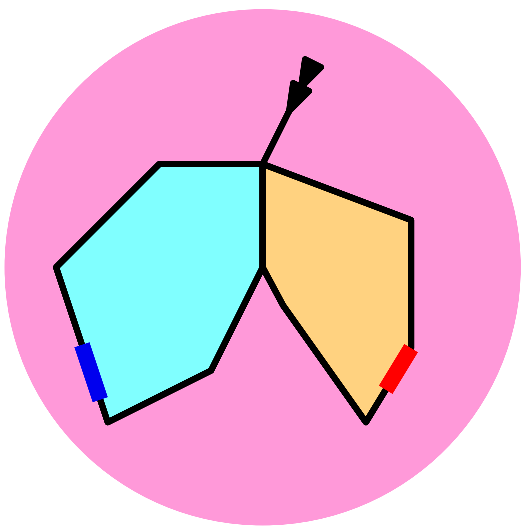

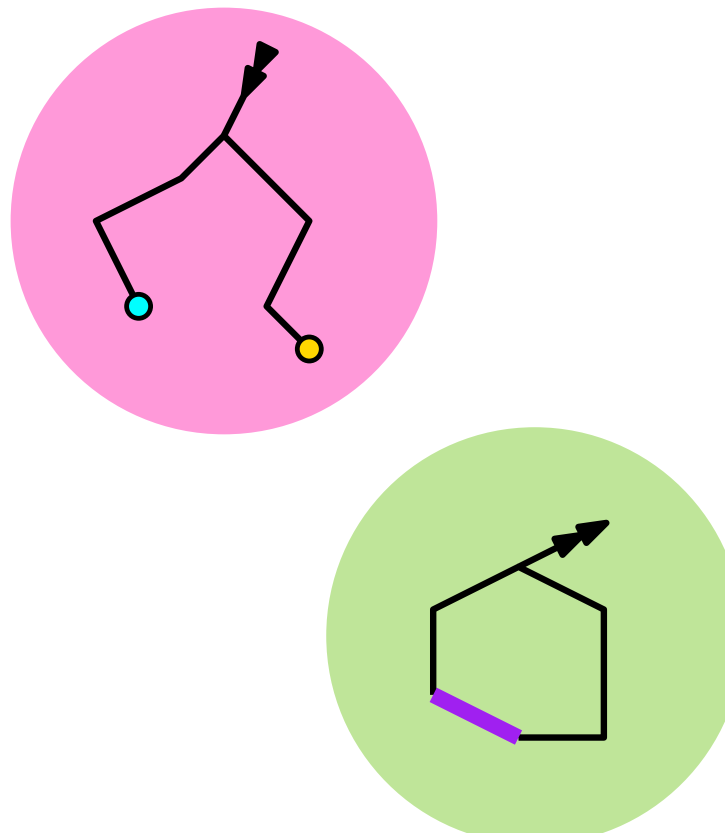

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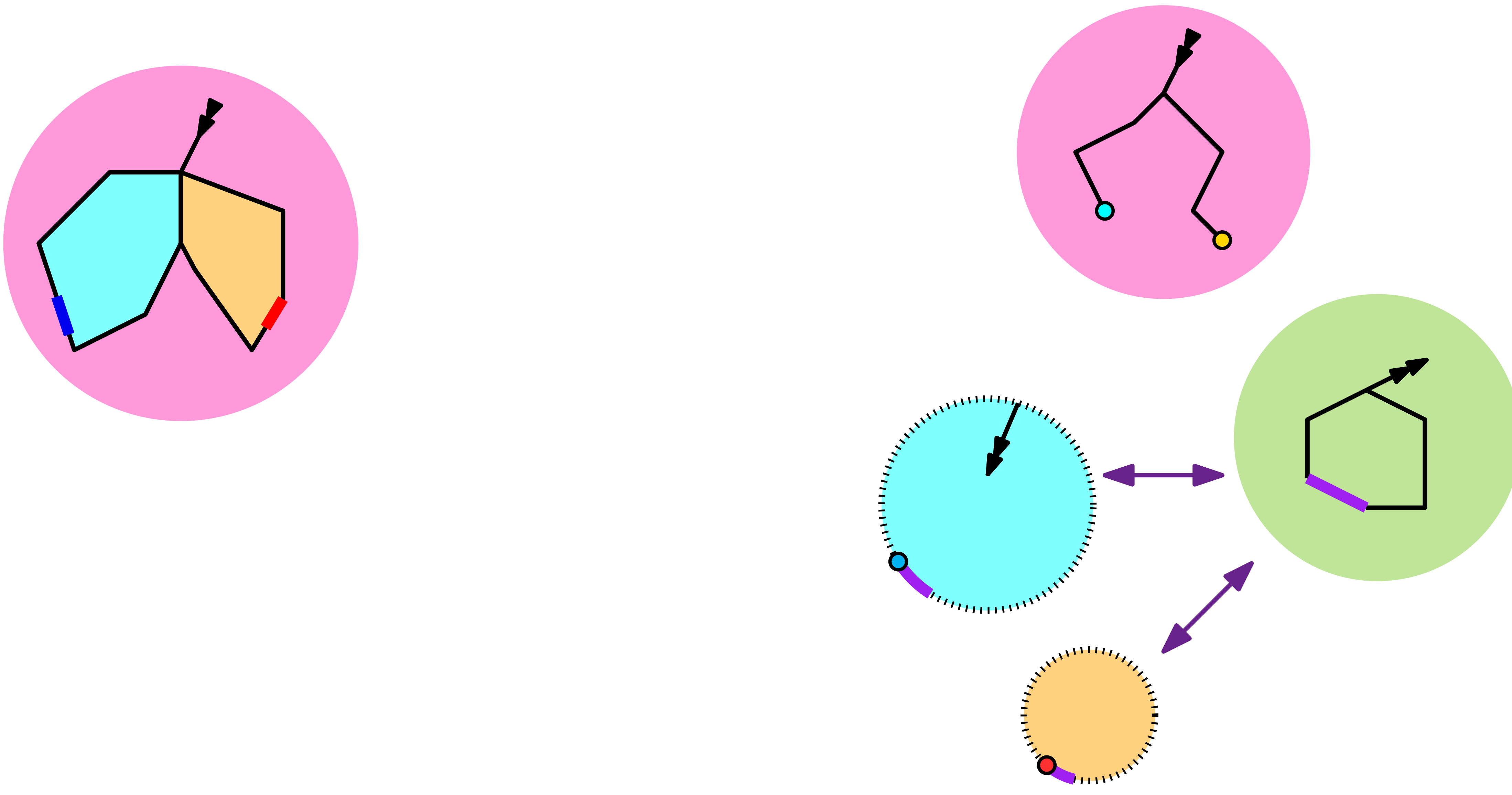

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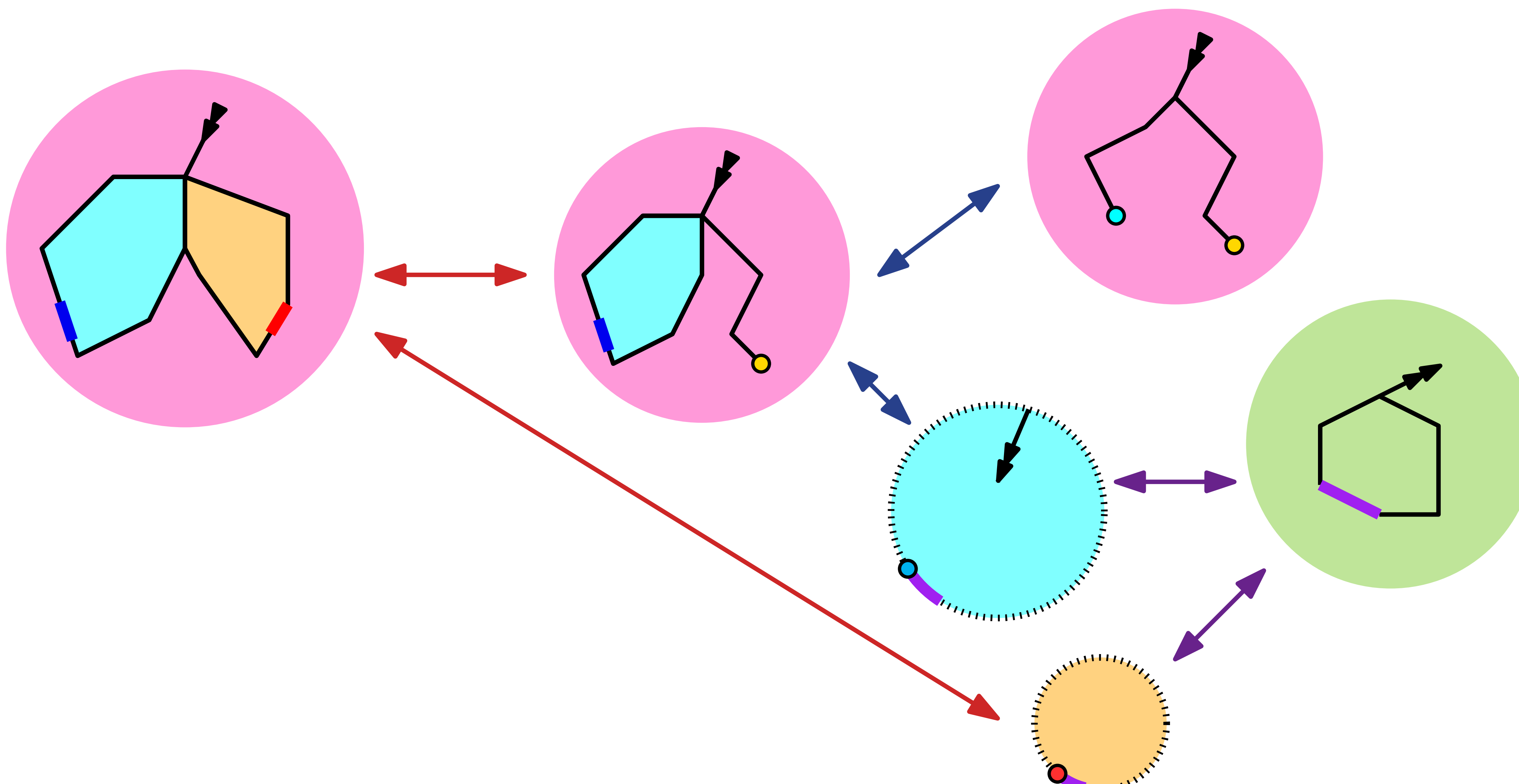

Perspectives

- Unicellular case of Louf's formula:

$$\binom{n+1}{2} B_g(n) = \sum_{g^* \geqslant 0} \binom{v+2g^*}{2g^*+2} B_{g-g^*}(n).$$

Refinement:

$$\binom{n+1}{2} B_{v_\circ, v_\bullet}(n) = \sum_{\substack{w^*, b^* \geqslant 0 \\ 0 < w^* + b^* \leqslant 2g+2 \\ w^* \equiv b^* \mod 2}} \binom{v_\circ - 1 + w^*}{w^*} \binom{v_\bullet - 1 + b^*}{b^*} B_{v_\circ - 1 + w^*, v_\bullet - 1 + b^*}(n).$$

- Tutte's formula for q-colored triangulations:

$$\begin{aligned}(n+3)(n+2)T(n+1) \quad &= \quad 2(q-4)(3n+1)T(n) \\ &\quad + 2\sum_{k=0}^{n} (3k+1)T(k)(n-k+2)(n-k+1)T(n-k).\end{aligned}$$

- General case of formulas from the KP hierarchy (Goulden-Jackson, Carrell-Chapuy).

Perspectives

- Unicellular case of Louf's formula:

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- Tutte's formula for q-colored triangulations:

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- General case of formulas from the KP hierarchy (Goulden-Jackson, Carrell-Chapuy).

Thank you!

A third equation

By refining Louf's formula to count black and white vertices, we get:

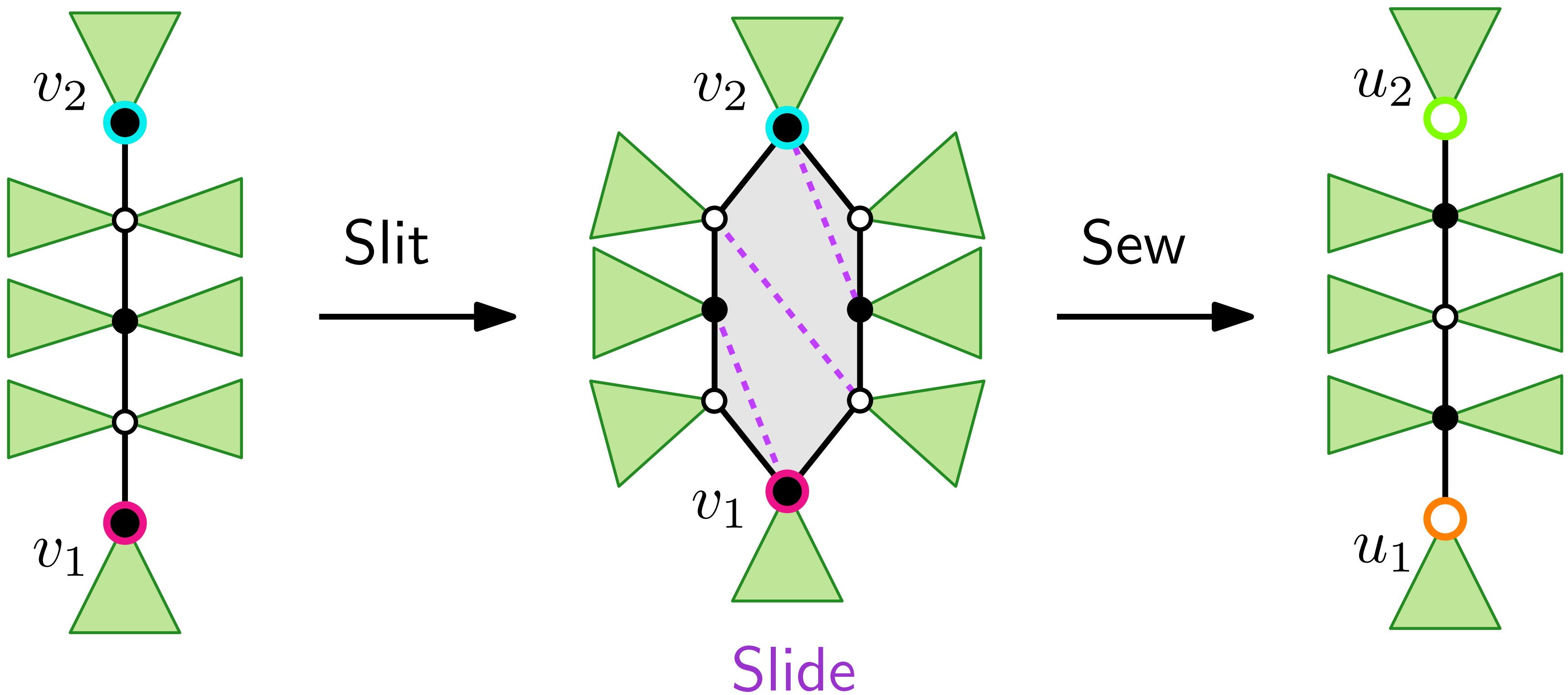
$$\binom{n+1}{2} B_{v_\circ,v_\bullet}(\mathbf{f}) = \sum_{\substack{\mathbf{s}+\mathbf{t}=\mathbf{f}\\ w_1+w_2-w^*=v_\circ\\ b_1+b_2-b^*=v_\bullet\\ w^*+b^*=2}} (1+n_1)\binom{w_2}{w^*}\binom{b_2}{b^*} B_{w_1,b_1}(\mathbf{s}) B_{w_2,b_2}(\mathbf{t}) + \sum_{\substack{w^*,b^*\geqslant 0\\ w^*+b^*=2}} \binom{v_\circ-1+w^*}{w^*}\binom{v_\bullet-1+b^*}{b^*} B_{v_\circ-1+w^*,v_\bullet-1+b^*}(\mathbf{f})$$

So the $\binom{v}{2}B(\mathbf{f}) = \binom{v}{2}B(\mathbf{f})$ becomes

$$\binom{v_\circ+v_\bullet}{2} B_{v_\circ,v_\bullet}(\mathbf{f}) = v_\circ v_\bullet B_{v_\circ,v_\bullet}(\mathbf{f}) + \binom{v_\circ+1}{2} B_{v_\circ+1,v_\bullet-1}(\mathbf{f}) + \binom{v_\bullet+1}{2} B_{v_\circ-1,v_\bullet+1}(\mathbf{f}).$$

Bijection on trees

$$\binom{v_\circ + v_\bullet}{2} T_{v_\circ, v_\bullet} = v_\circ v_\bullet T_{v_\circ, v_\bullet}(\mathbf{f}) + \binom{v_\circ}{2} T_{v_\circ - 1, v_\bullet + 1} + \binom{v_\bullet}{2} T_{v_\circ + 1, v_\bullet - 1}.$$



Bijection on trees

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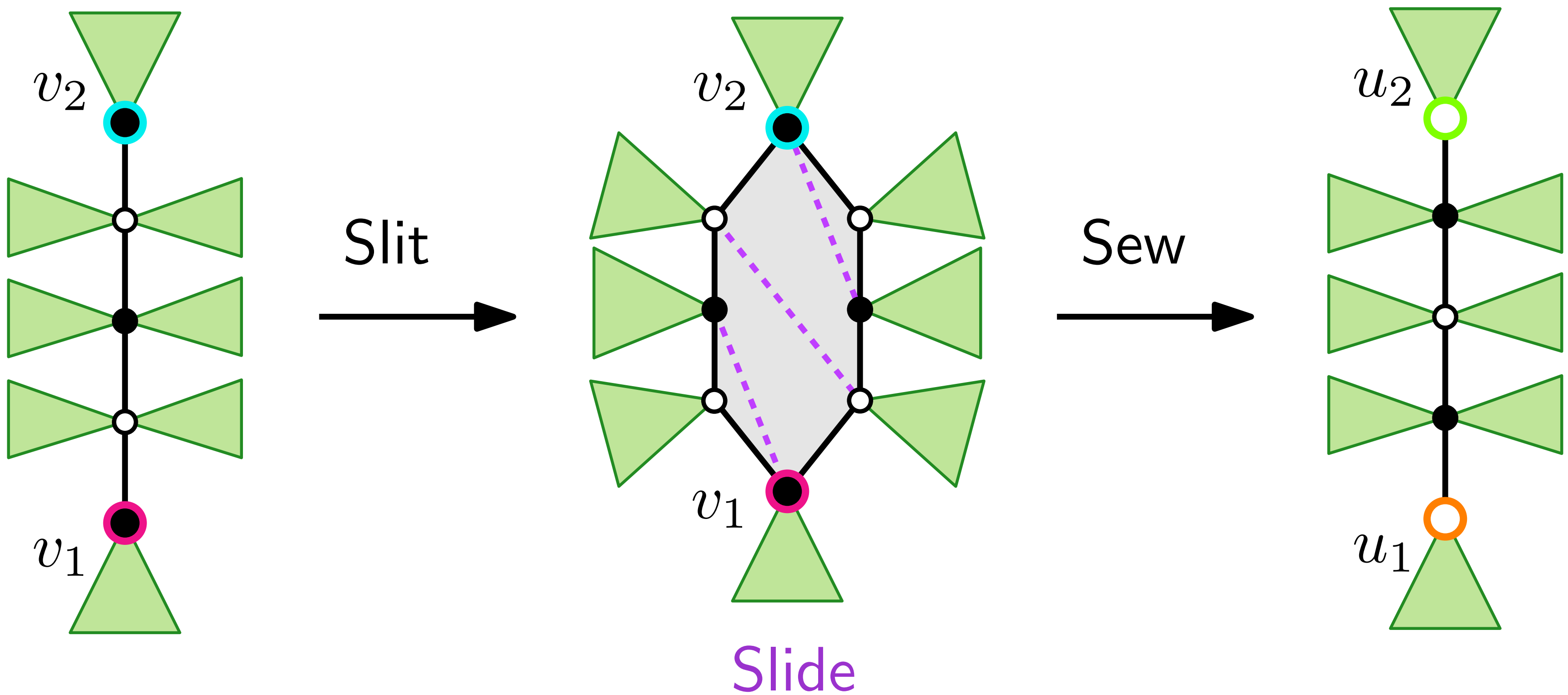

$\hookrightarrow$ Can this be generalized to bipartite maps ?