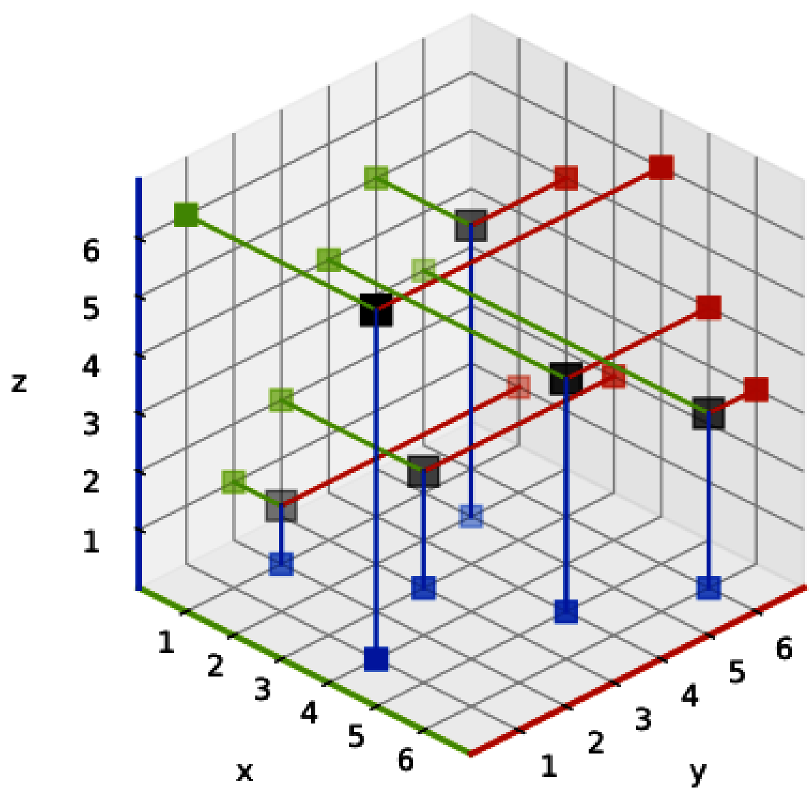
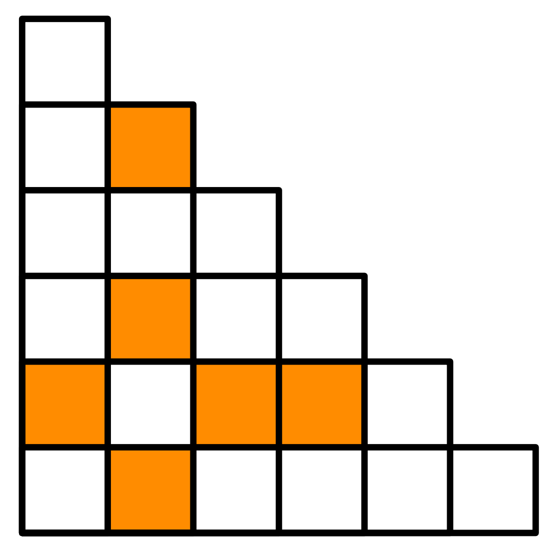
Pattern avoiding 3-permutations and triangle bases

Juliette Schabanel

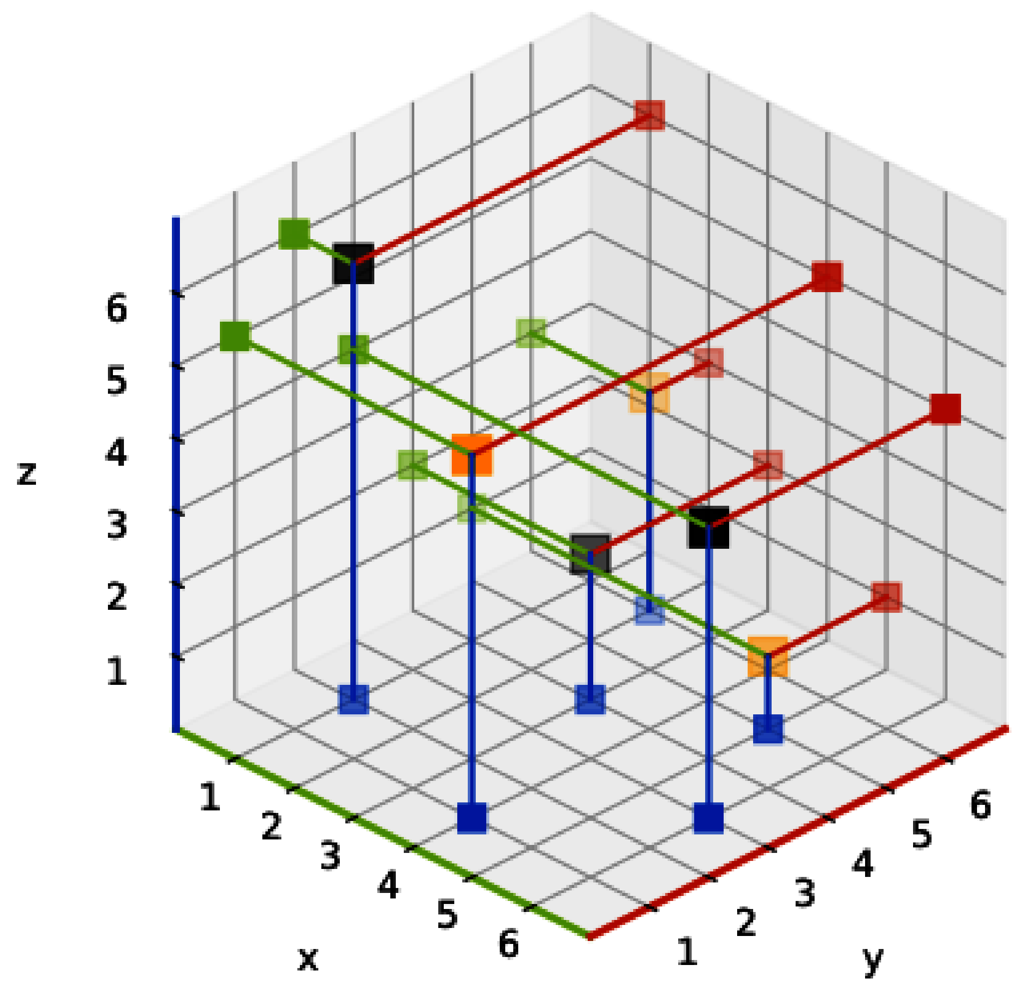
LaBRI





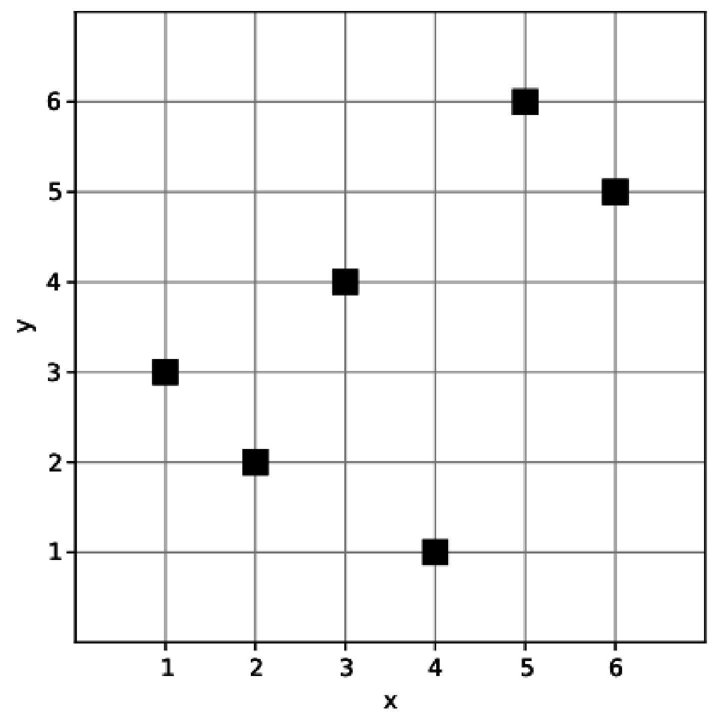
I- The objects

a) Pattern avoiding 3-permutations



Permutations and diagrams

A **permutation** $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$ is a bijection from $[\![1,n]\!] = \{1,2,\ldots,n\}$ to itself.
Its **diagram** is the set of points $P_\sigma = \{(i,\sigma(i)) \mid 1 \leqslant i \leqslant n\}$.

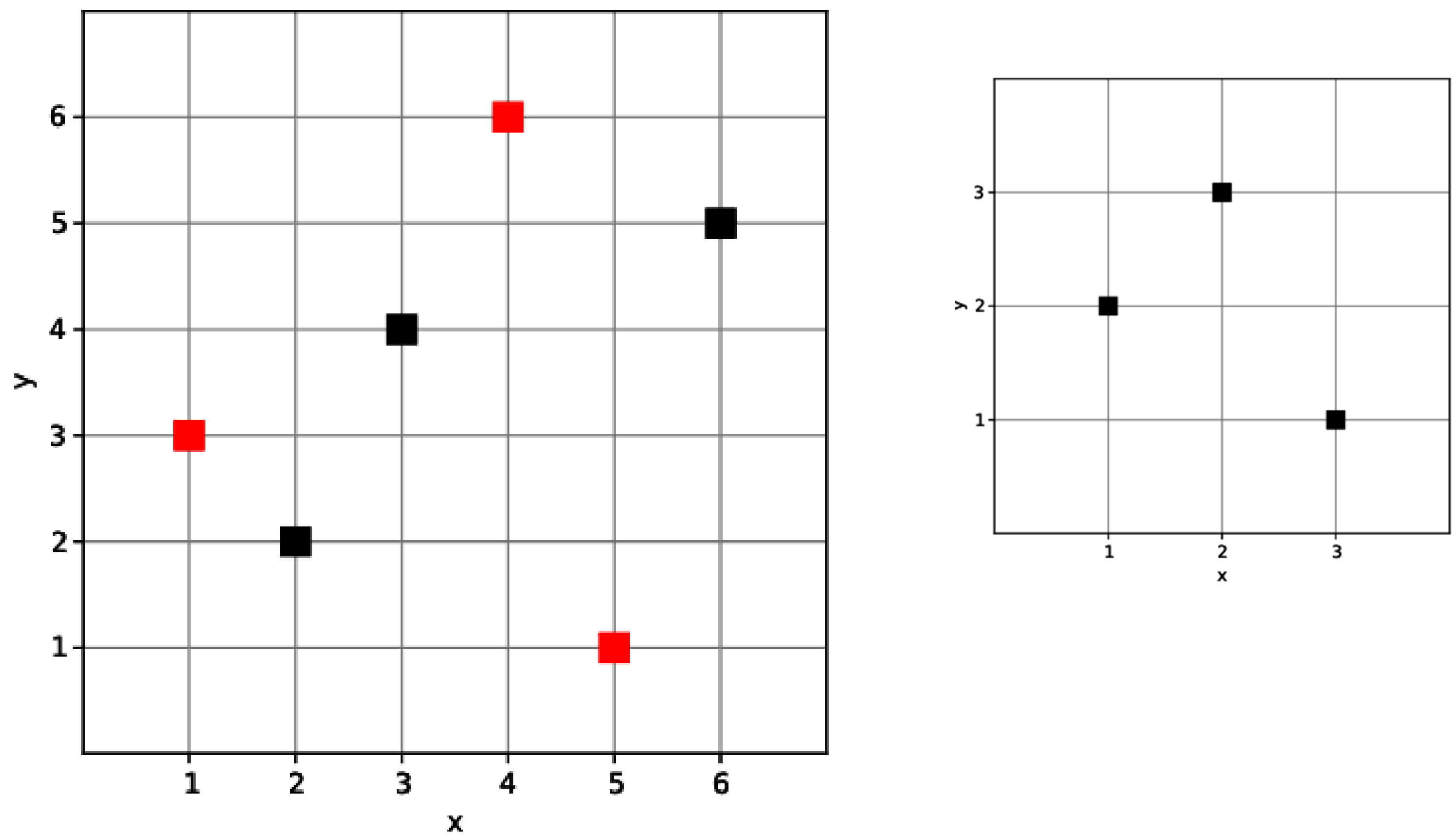


The diagram of $\sigma = 324165$.

A **diagram** is a set of points on $[\![1,n]\!] \times [\![1,n]\!]$ with exactly one point per row and per column.

Pattern avoidance in permutations

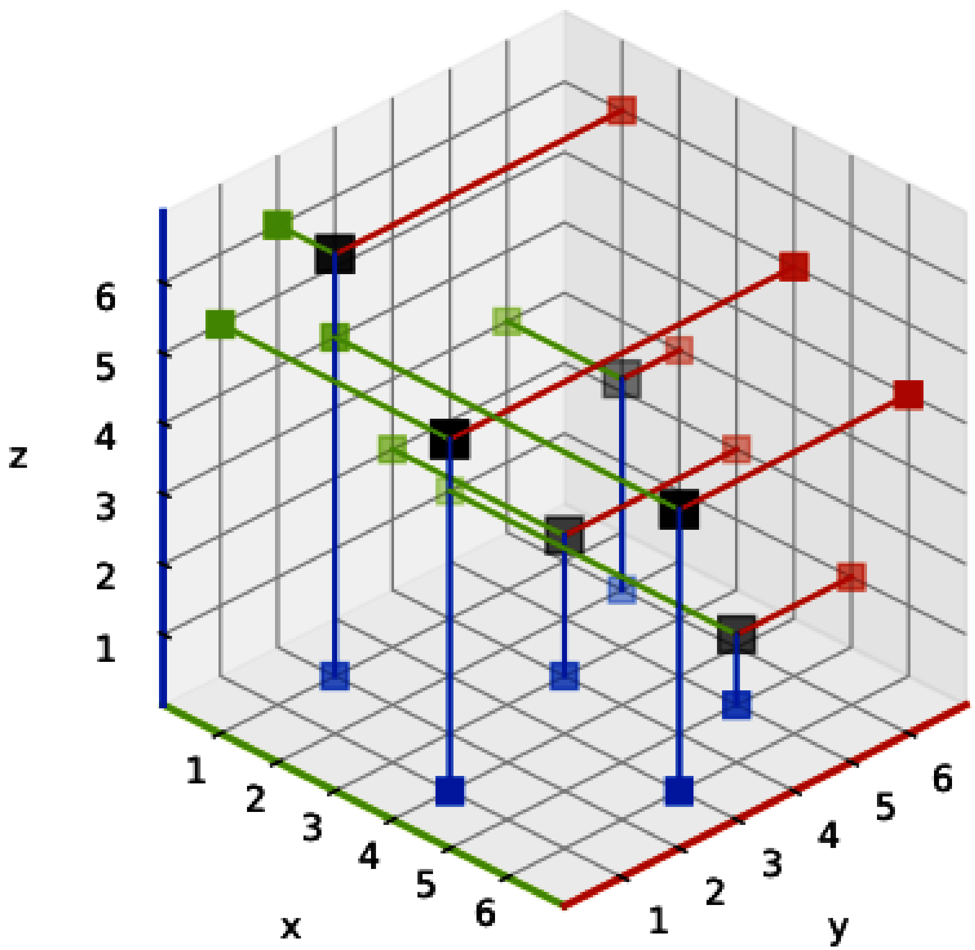
A permutation $\sigma \in \mathfrak{S}_n$ **contains** a pattern $\pi \in \mathfrak{S}_k$ if there is a set of indices I such that $\sigma_{|I} = \pi$. Otherwise, it **avoids** it.



$\sigma = 324615$ contains the pattern $\pi = 231$.

3-Permutations

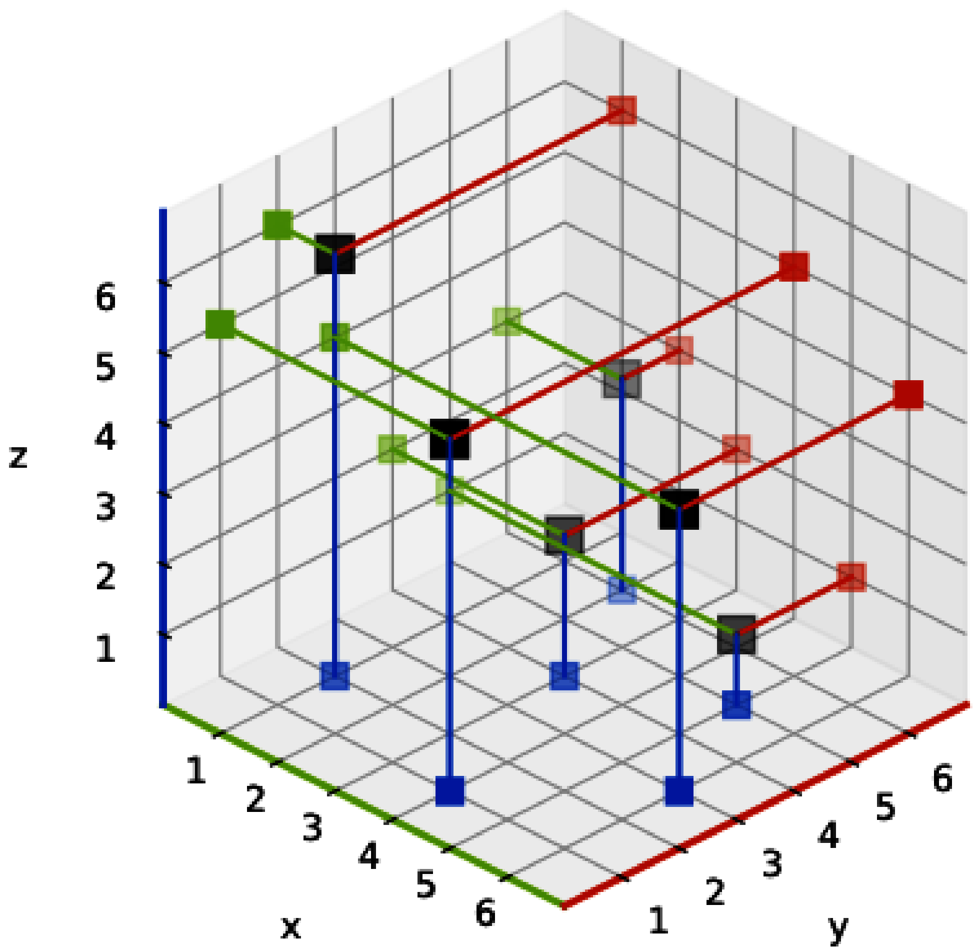
A 3-**diagram** is a set of points in $[\![1, n]\!]^3$ with exactly one point per plane of the grid.



Points :
(1, 2, 6)
(2, 6, 3)
(3, 4, 2)
(4, 1, 5)
(5, 5, 1)
(6, 3, 4)

3-Permutations

A **3-diagram** is a set of points in $[\![1,n]\!]^3$ with exactly one point per plane of the grid.
A **3-permutation** is a couple of permutations $(\sigma,\tau) \in \mathfrak{S}_n^2$.
It is represented by the **diagram** $P_{(\sigma,\tau)} = \{(i,\sigma(i),\tau(i)) \mid 1 \leqslant i \leqslant n\}$.

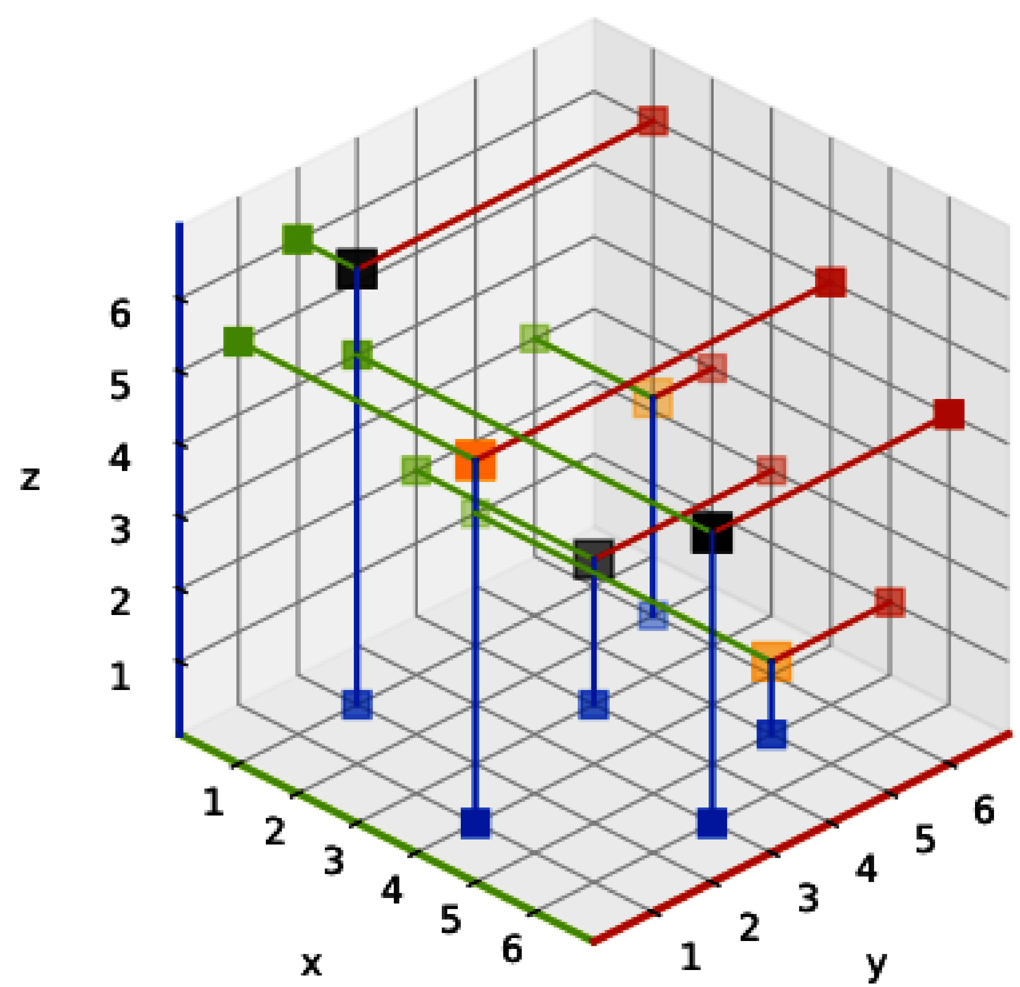


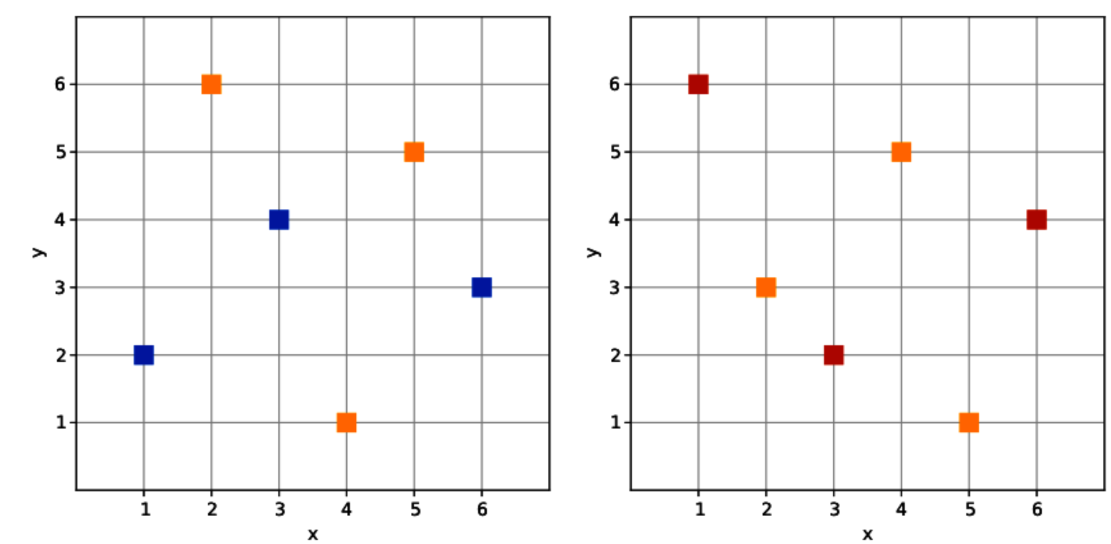
$(264153, 632514)$

Points :
(1, 2, 6)
(2, 6, 3)
(3, 4, 2)
(4, 1, 5)
(5, 5, 1)
(6, 3, 4)

Pattern avoidance in 3-permutations

A 3-permutation $(\sigma, \tau) \in \mathfrak{S}_n^2$ **contains** a pattern $(\pi_1, \pi_2) \in \mathfrak{S}_k^2$ if there is a set of points of (σ, τ) that is equal to (π_1, π_2) (once standardized). Otherwise it **avoids** it.





$(264153, 632514)$ contains the pattern $(312, 231)$.

It avoids the pattern $(12, 12)$ although both 264153 and 632514 contain 12.

Pattern avoidance classes

Patterns	TWE	Sequence	Comment
$(12, 12)$	4	$1, 3, 17, 151, 1899, 31711, \cdots$	weak-Bruhat intervals
$(12, 12), (12, 21)$	6	$n! = 1, 2, 6, 24, 120 \cdots$	$\sigma_1 \Rightarrow \sigma_2$
$(12, 12)$, $(12, 21)$, $(21, 12)$	4	$1, 1, 1, 1, 1, 1, \cdots$	1 diagonal
$(12, 12)$, $(12, 21)$, $(21, 12)$, $(21, 21)$	1	$1, 0, 0, 0, 0, 0, \cdots$	
$(123, 123)$	4	$1, 4, 35, 524, 11774, 366352, \cdots$	*new*
$(123, 132)$	24	$1, 4, 35, 524, 11768, 365558, \cdots$	*new*
$(132, 213)$	8	$1, 4, 35, 524, 11759, 364372, \cdots$	*new*
$(12, 12), (132, 312)$	48	$(n+1)^{n-1} = 1, 3, 16, 125, 1296 \cdots$	**[Atkinson et al. 93,95]**
$(12, 12), (123, 321)$	12	$1, 3, 16, 124, 1262, 15898, \cdots$	distributive lattices inter.
$(12, 12), (231, 312)$	8	$1, 3, 16, 122, 1188, 13844, \cdots$	*A295928?*

[Bonichon & Morel, 22]

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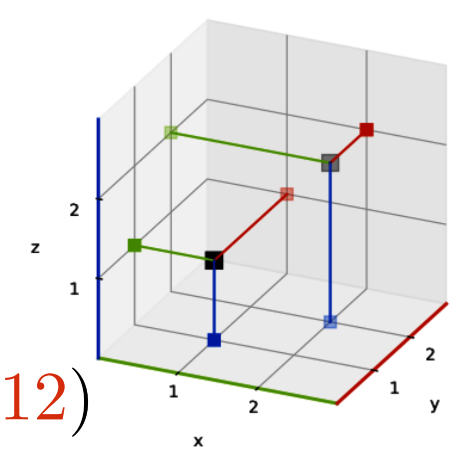
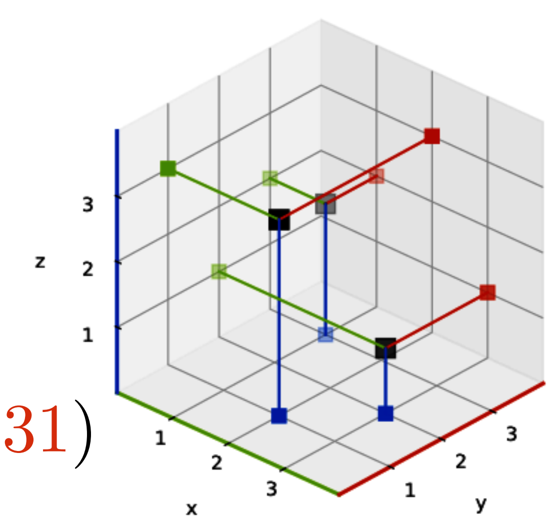
[Bonichon & Morel, 22]

$(12, 12)$ 

$(312, 231)$ 

Pattern avoidance classes

A295928 Number of triangular matrices T(n,i,k), k <= i <= n, with entries "0" or "1" with the property that each triple {T(n,i,k), T(n,i,k+1), T(n,i-1,k)} containing a single "0" can be successively replaced by {1, 1, 1} until finally no "0" entry remains. 1

1, 3, 16, 122, 1188, 13844, 185448, 2781348, 45868268

(list; graph; refs; listen; history; text; internal format)

OFFSET 1,2

COMMENTS A triple {T(n,i,k), T(n,i,k+1), T(n,i-1,k)} will be called a primitive triangle. It is easy to see that b(n) = n(n-1)/2 is the number of such triangles. At each step, exactly one primitive triangle is completed (replaced by {1, 1, 1}). So there are b(n) "0"- and n "1"-terms. Thus the starting matrix has no complete primitive triangle. Furthermore, any triangular submatrix T(m,i,k), k <= i <= m < n cannot have more than m "1"-terms because otherwise it would have less "0"-terms than primitive triangles. The replacement of at least one "0"-term would complete more than one primitive triangle. This has been excluded.

So T(n, i, k) is a special case of U(n, i, k), described in A101481: a(n) < A101481(n+1).

A start matrix may serve as a pattern for a number wall used on worksheets for elementary mathematics, see link "Number walls". That is why I prefer the more descriptive name "fill matrix".

The algorithm for the sequence is rather slow because each start matrix is constructed separately. There exists a faster recursive algorithm which produces the same terms and therefore is likely to be correct, but it is based on a conjecture. For the theory of the recurrence, see "Recursive aspects of fill matrices". Probable extension a(10)-a(14): 821096828, 15804092592, 324709899276, 7081361097108, 163179784397820.

The number of fill matrices with n rows and all "1"- terms concentrated on the last two rows, is A001960(n). See link "Reconstruction of a sequence".

LINKS Table of n, a(n) for n=1..9.

Gerhard Kirchner, Recursive aspects of fill matrices

Gerhard Kirchner, Number walls

Gerhard Kirchner, VB-program

Gerhard Kirchner, Reconstruction of a sequence

Ville Salo, Cutting Corners, arXiv:2002.08730 [math.DS], 2020.

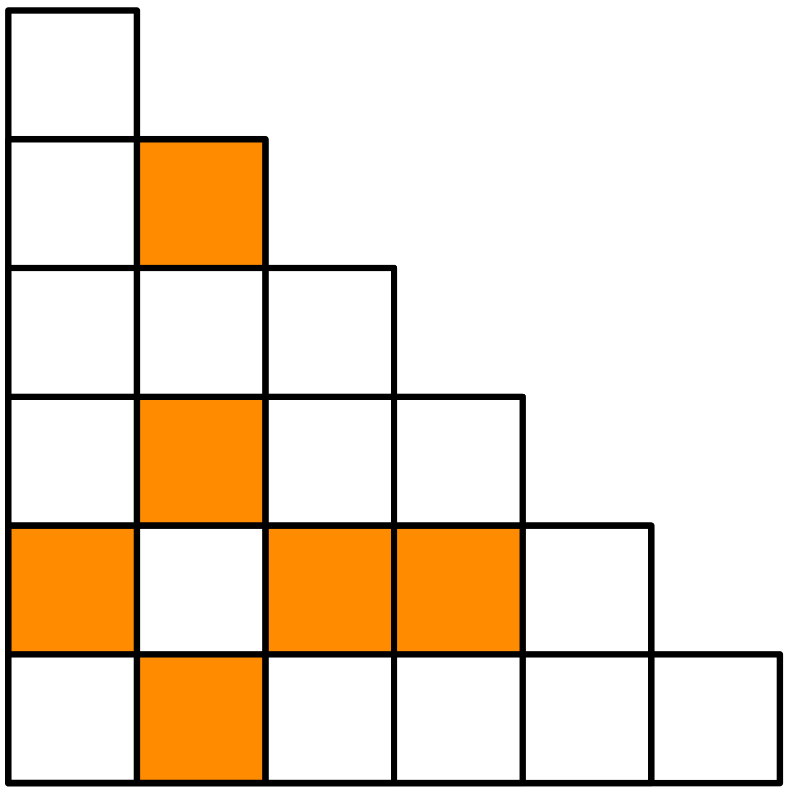
Yuan Yao and Fedir Yudin, Fine Mixed Subdivisions of a Dilated Triangle, arXiv:2402.13342 [math.CO], 2024.

EXAMPLE

```
Example (n=2):    0     1    1
    a(2)=3       1 1   0 1  1 0
Example for completing a 3-matrix (3 bottom terms):
    1        1       1       1
   0 0  ->  1 0 ->  1 1 ->  1 1
  1 1 0    1 1 0   1 1 0   1 1 1
```

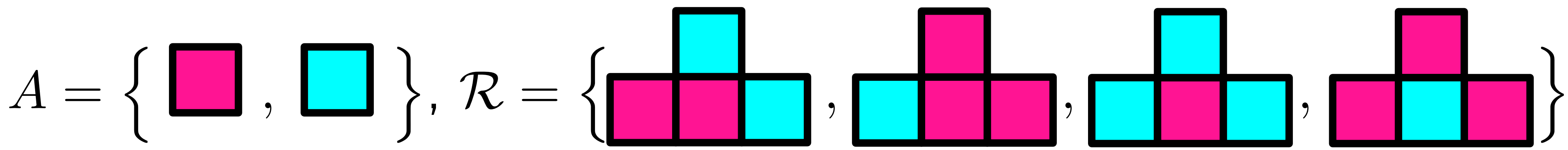
I- The objects

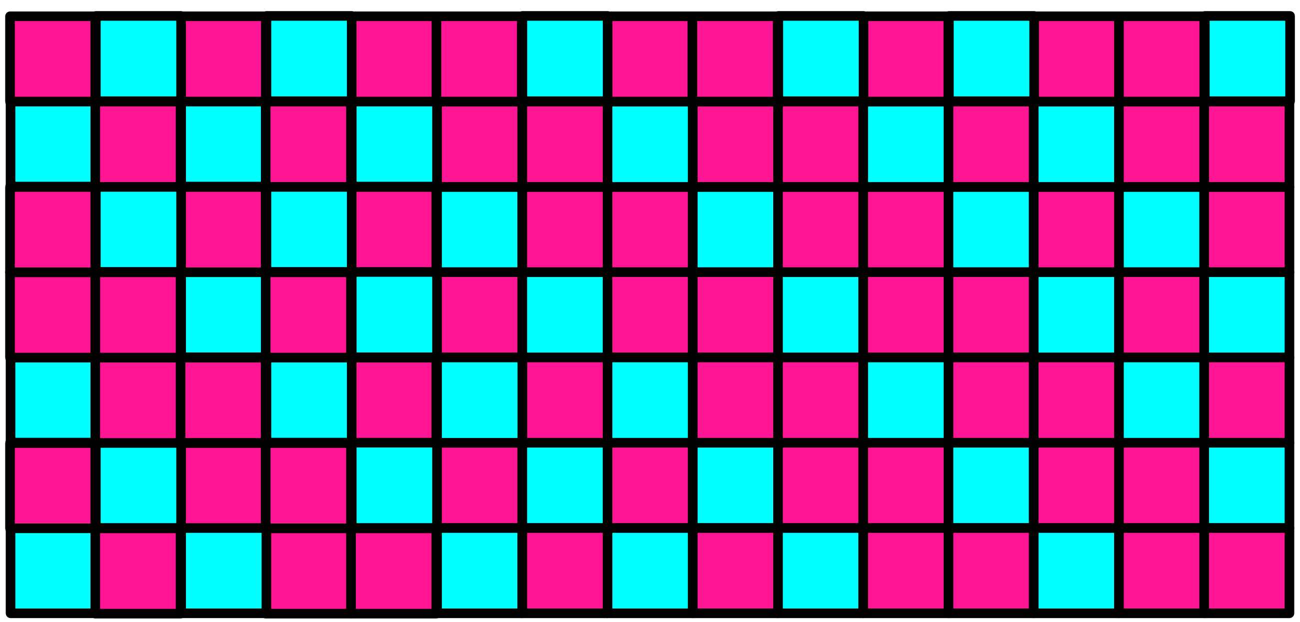
b) Triangle Bases



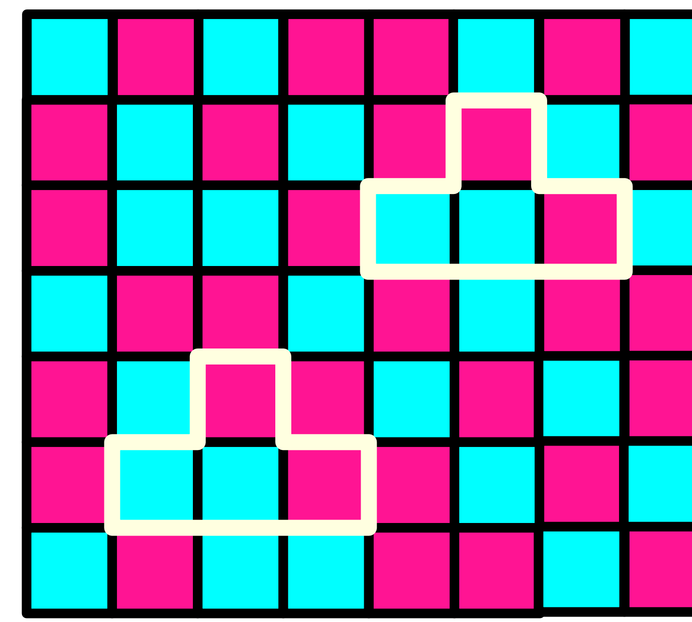
Tilings

A **tiling** with alphabet A and rule set $\mathcal{R} \subset S^A$ with $S \subset \mathbb{Z}^2$ is a coloring of $\mathbb{Z}^2$ with A such that every S-subpattern of it is in $\mathcal{R}$.
The set of tilings for a rule set is called a **subshift**.





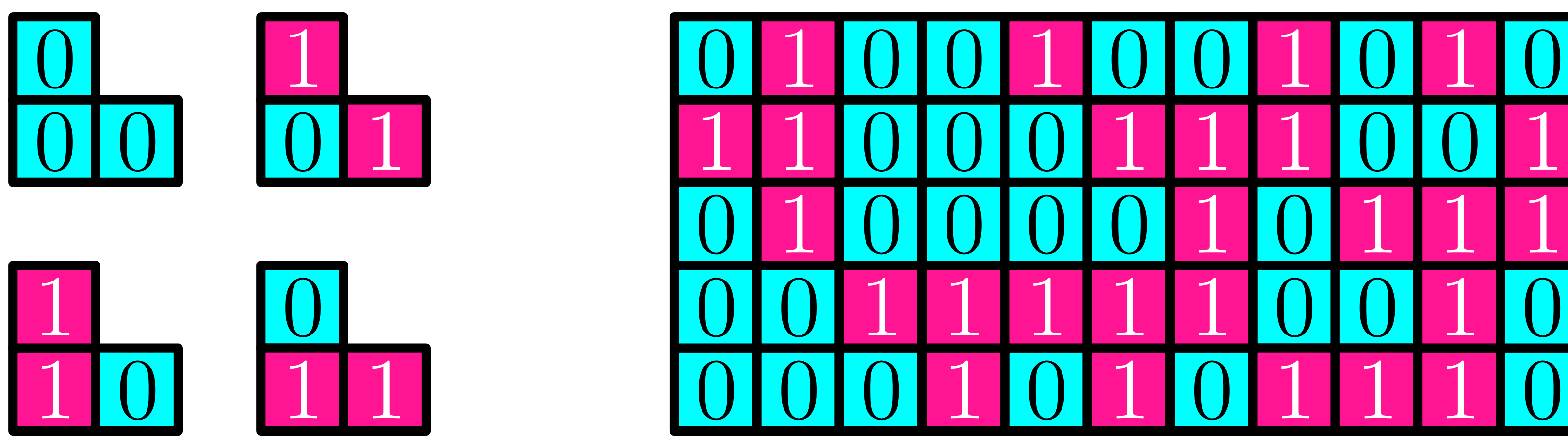
Valid



Not valid

TEP subshifts

A subshift is **TEP** (totally extremally permutive) if for all $x \in S$, for all $c : S \setminus \{x\} \to A$, there is a unique $a \in A$ that extends c into an allowed subpattern.



A TEP subshift

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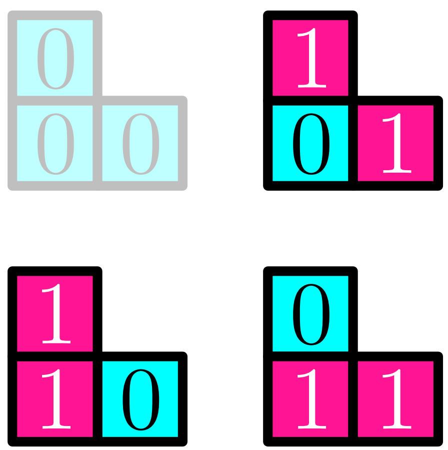

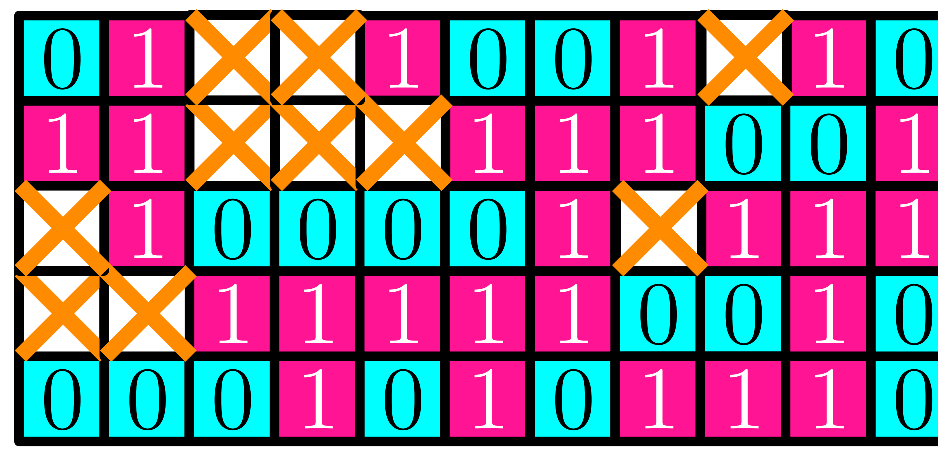

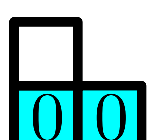 cannot be extended so the rule is not TEP.

TEP subshifts

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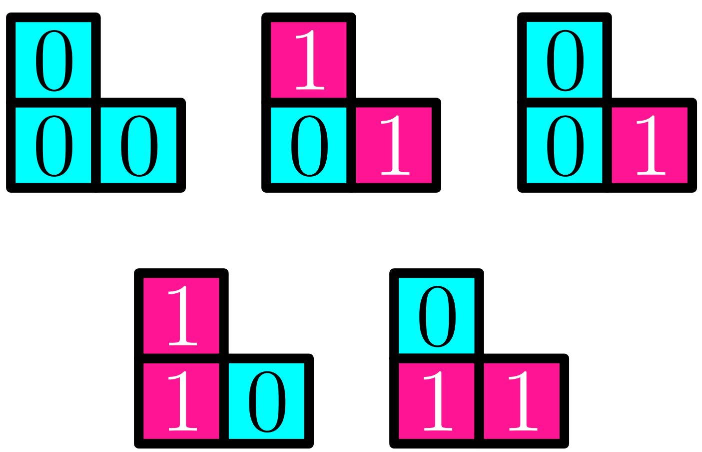
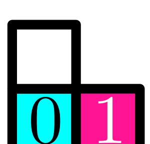


0	1	1	1	1	0	0	1	0	1	0
1	1	0	1	0	1	1	1	0	0	1
0	1	0	1	0	0	1	0	1	1	1
0	0	0	1	0	1	1	0	0	1	0
0	0	0	1	0	1	0	1	1	1	0

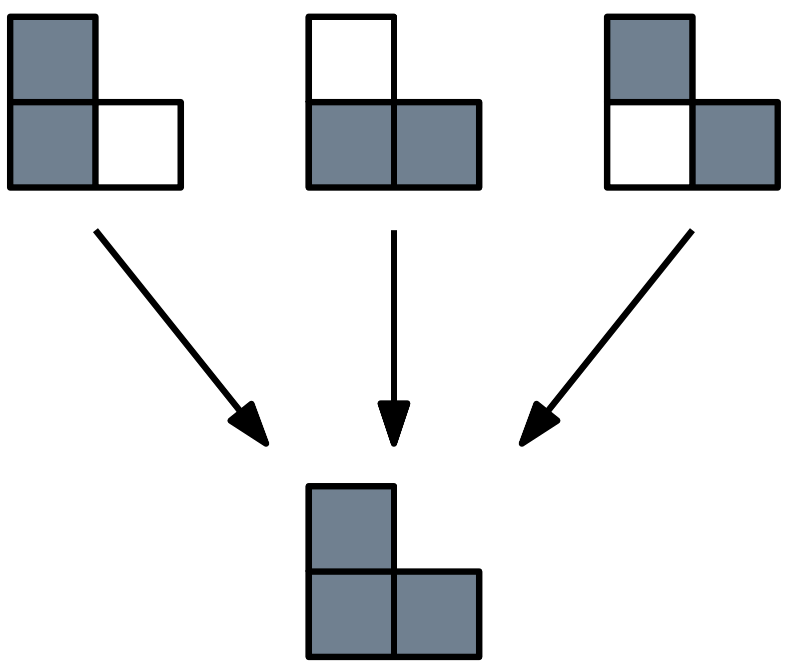


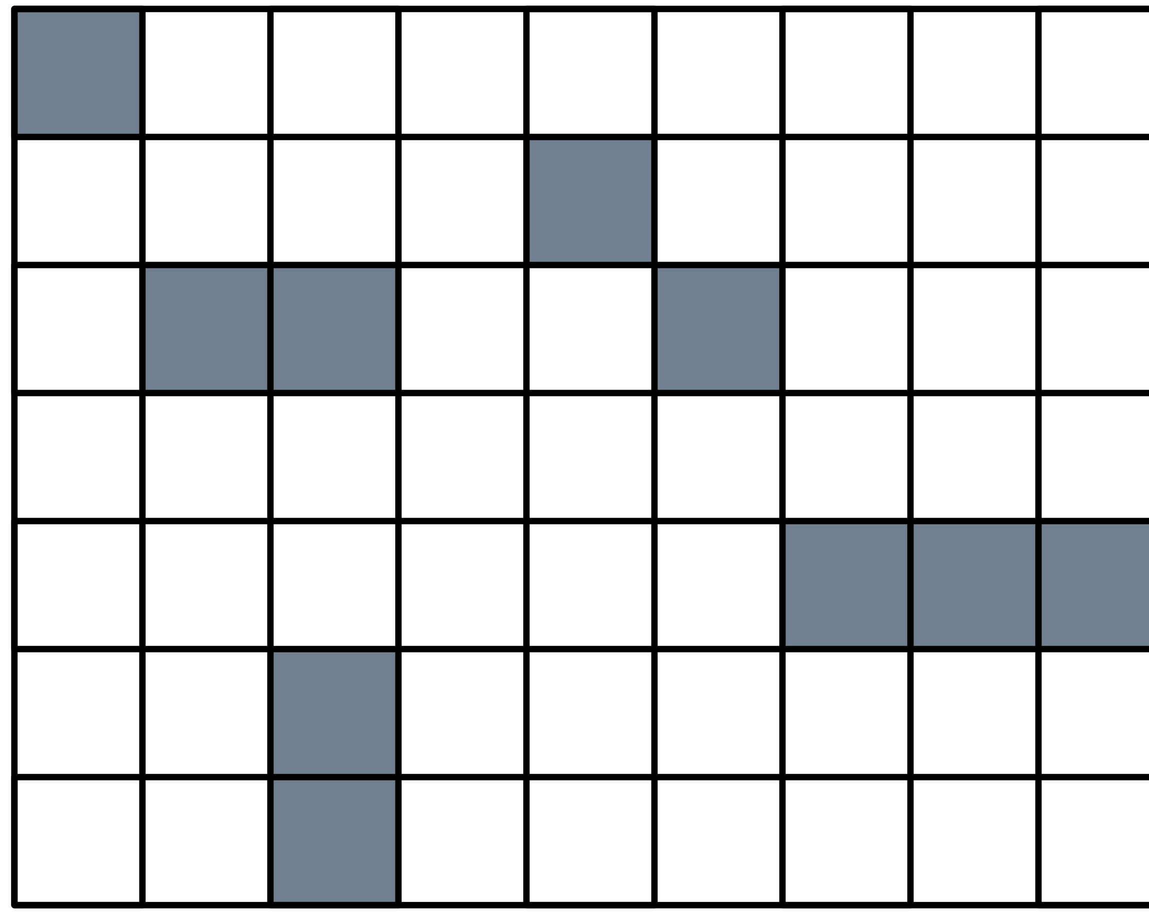
admits two valid extensions so the rule is not TEP.

Filling

Knowing the values in the cells of P, which others can be deduced ?

A **filling step** adds the missing point to a triangle with only 2 points.





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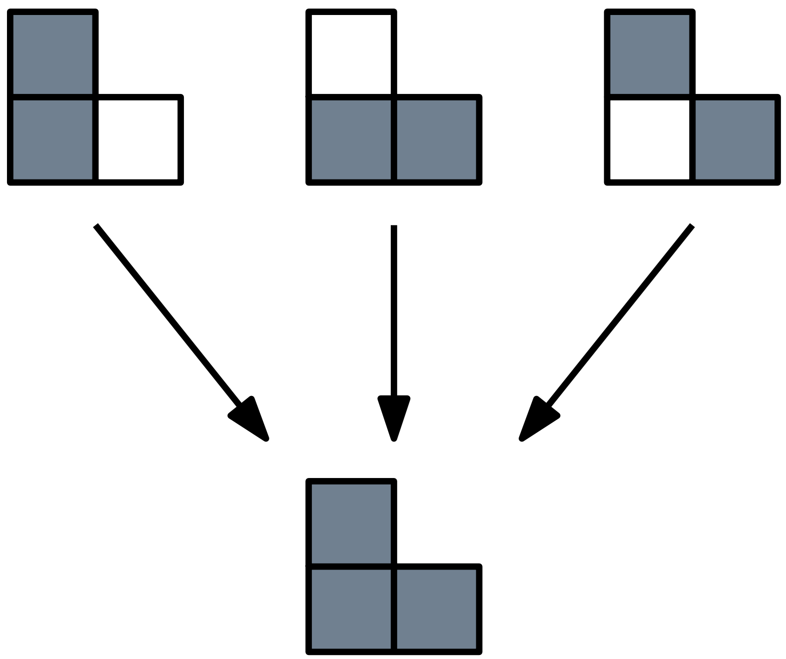

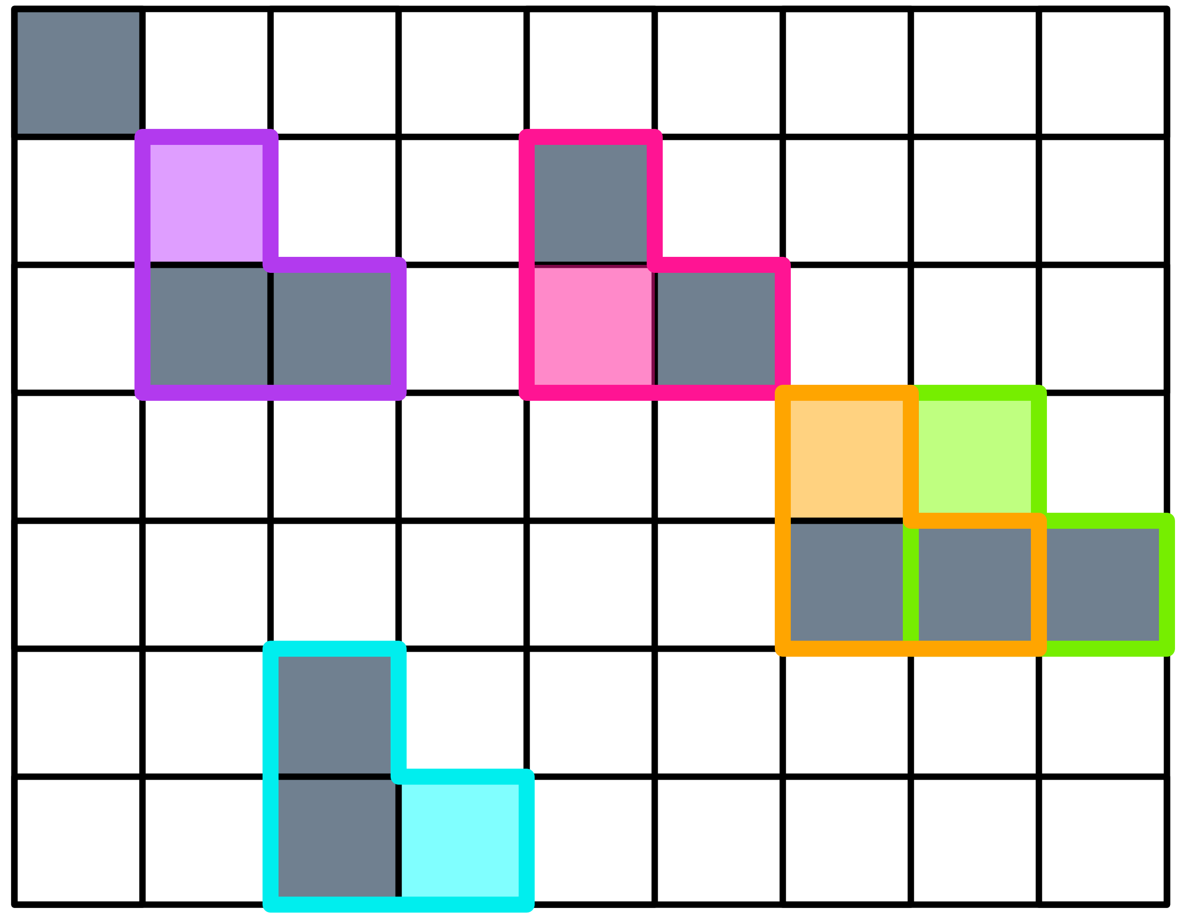

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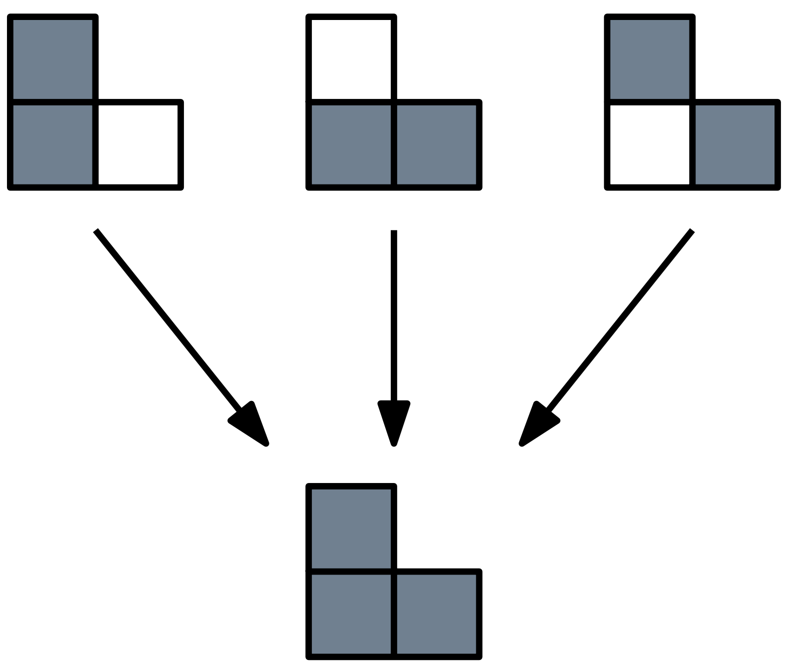

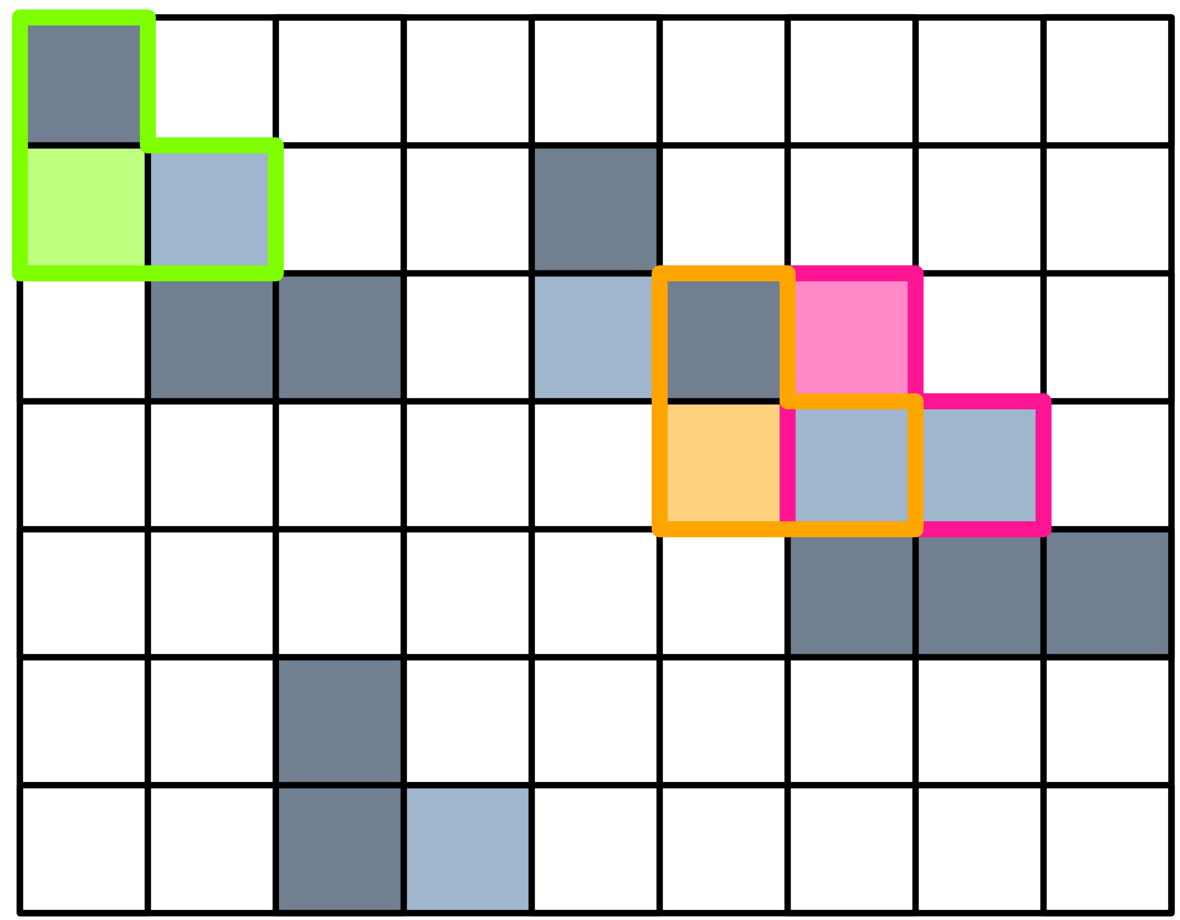

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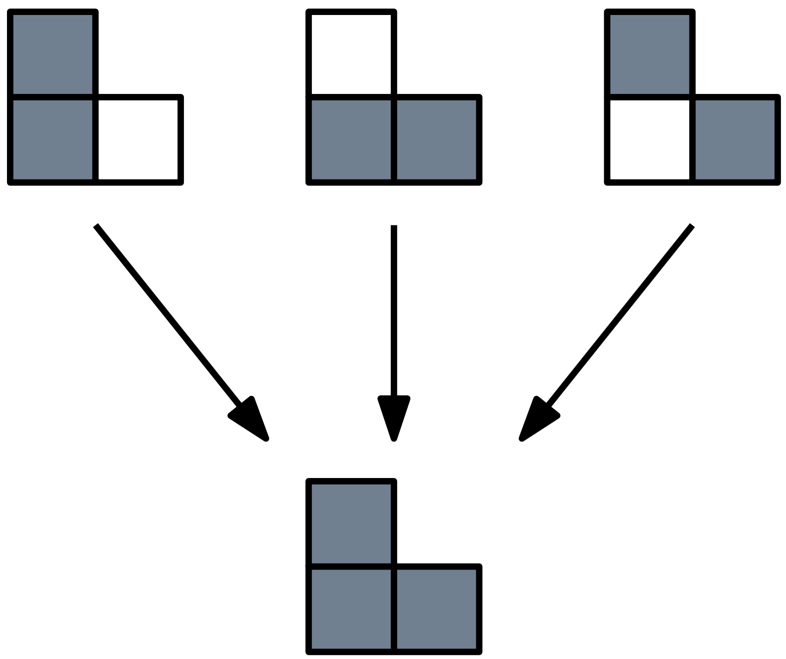

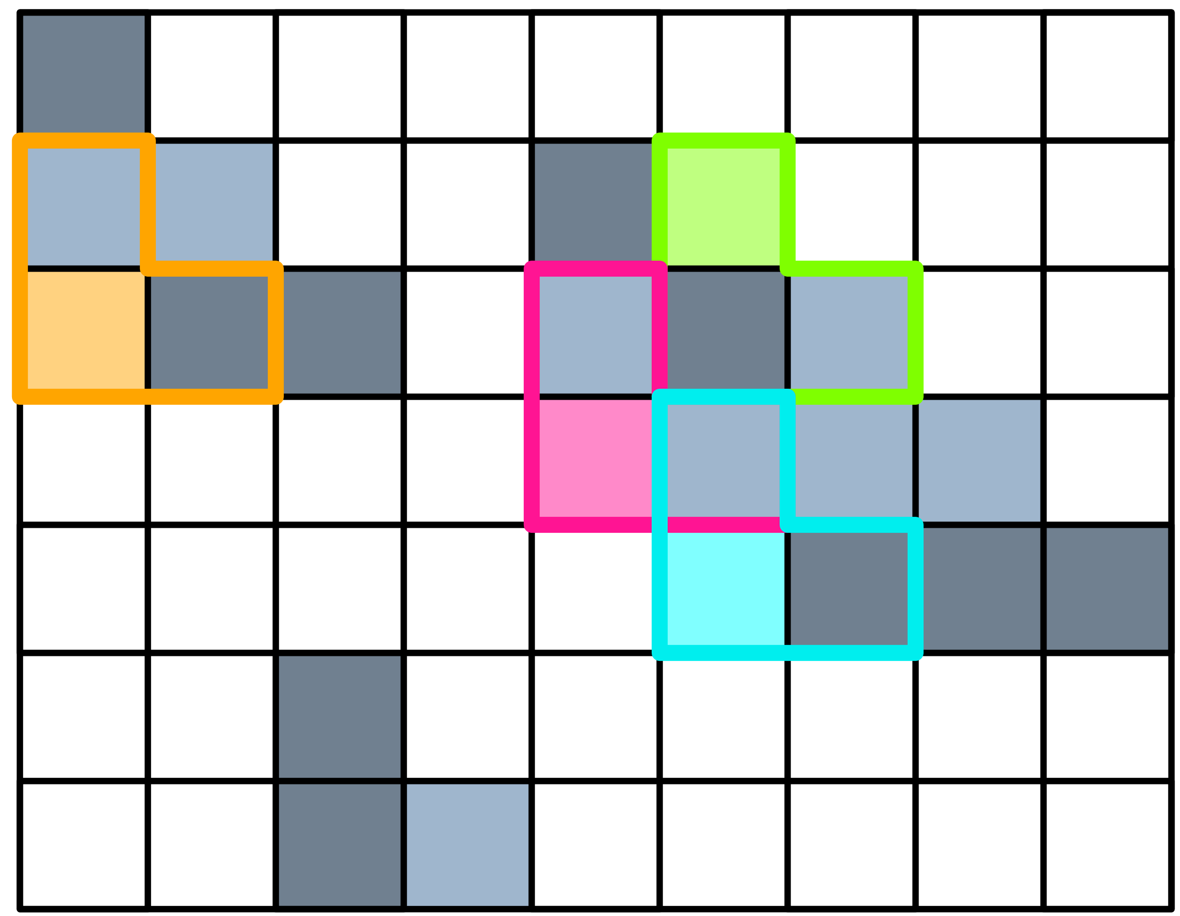

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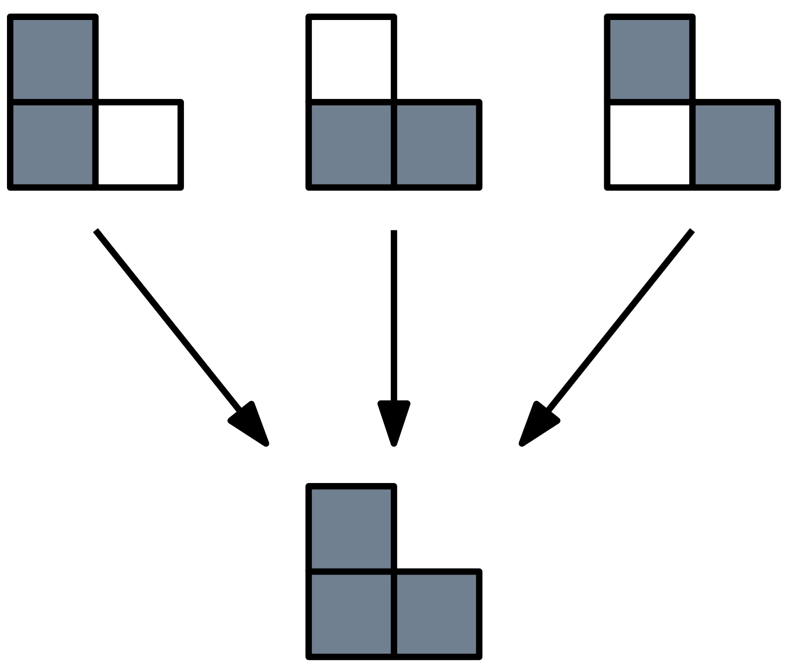

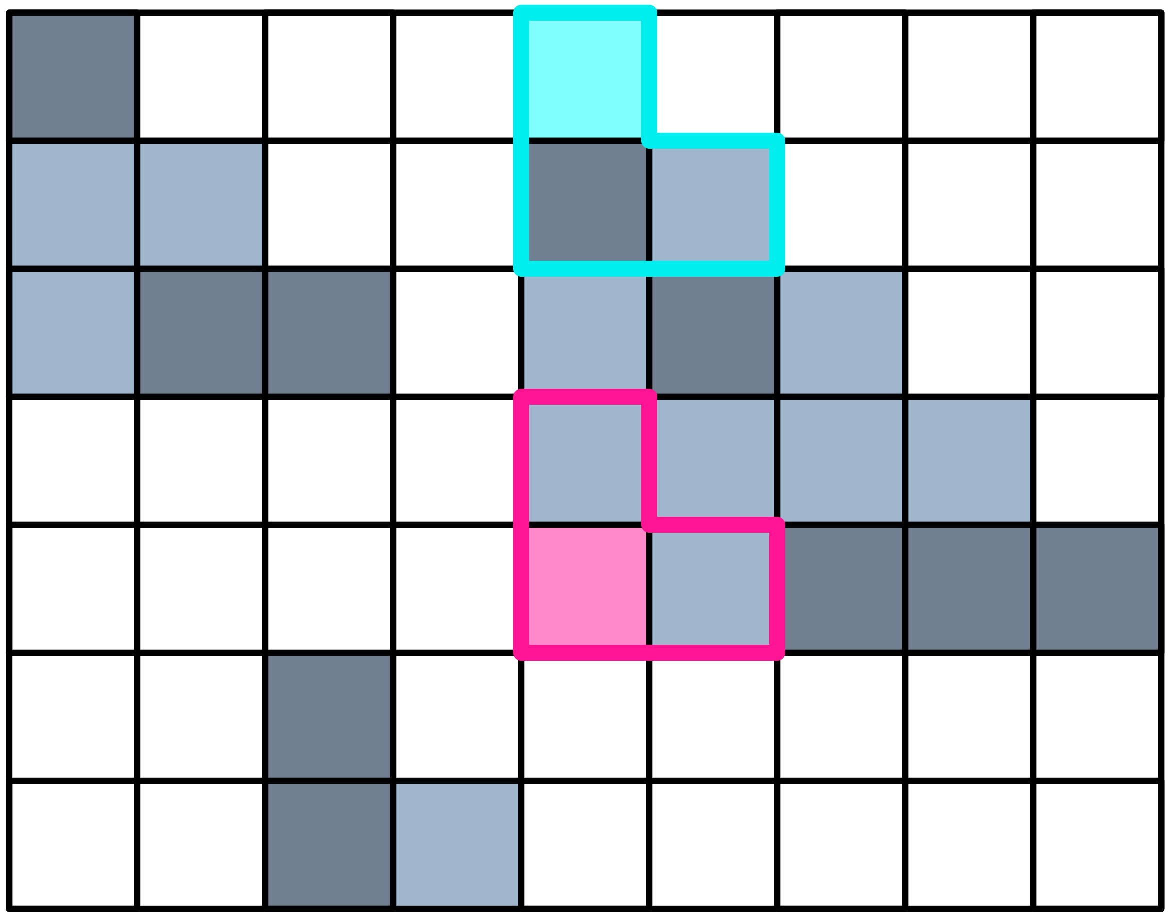

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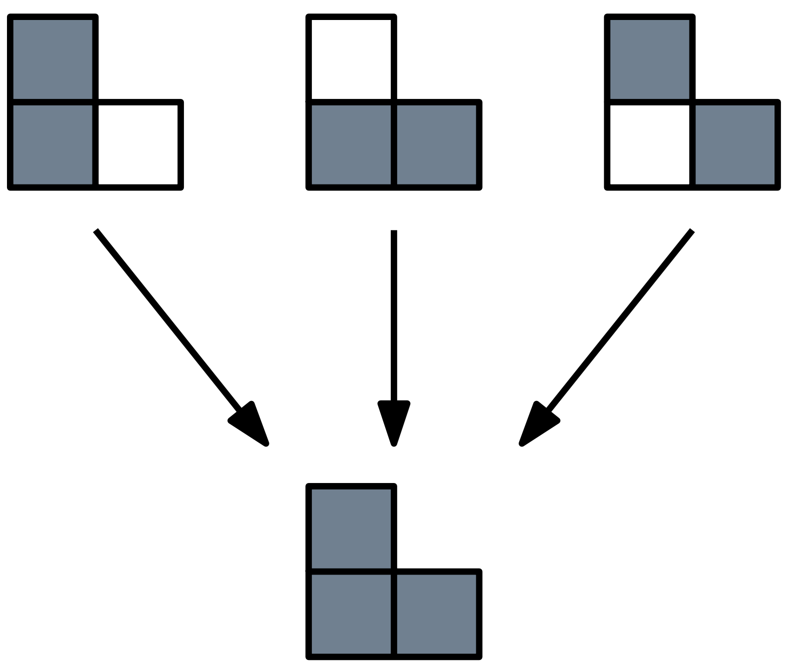
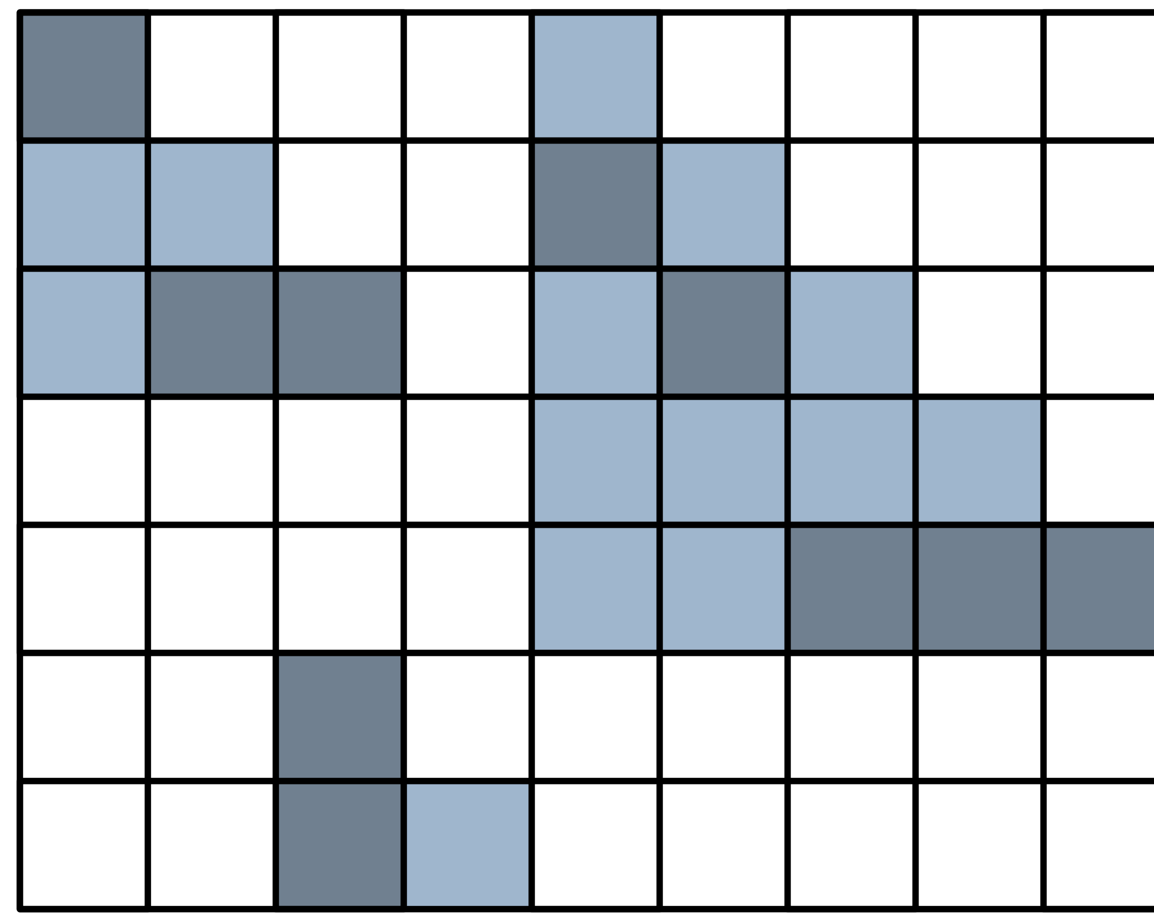
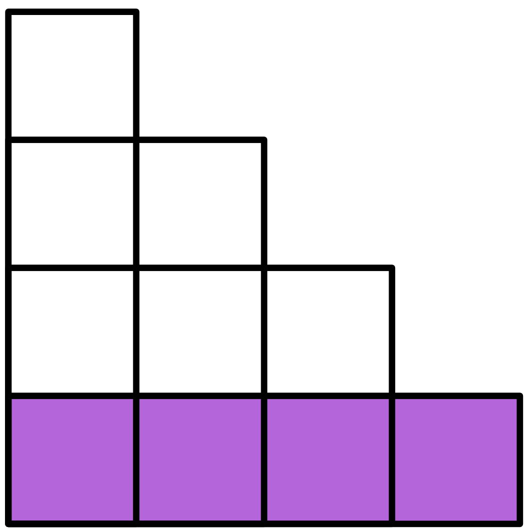




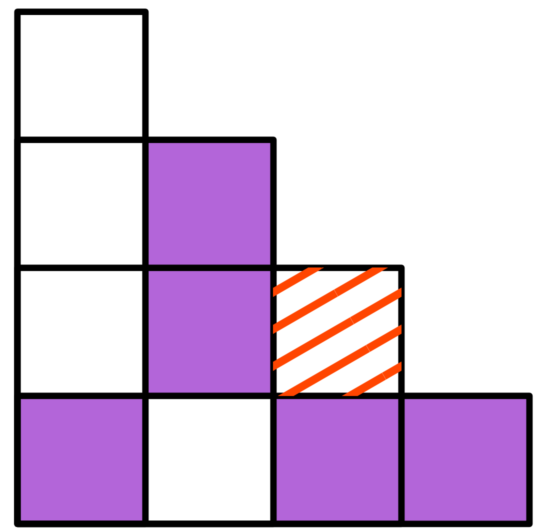
Independence

Given a set of cells P, when can its content be chosen freely?

A set of cells $P \subset \mathbb{Z}^2$ is **independent** if all its colorings admit a valid extension to $\mathbb{Z}^2$.



Independent



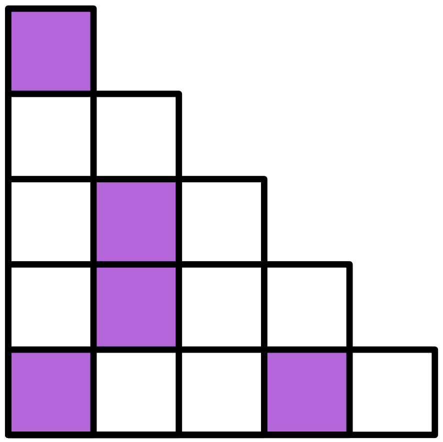
Not independent

Bases

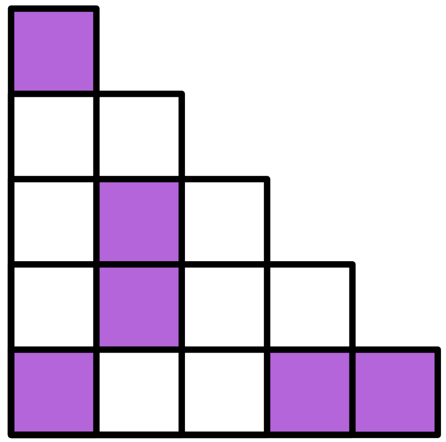
$T_n = \{(a,b) \in \mathbb{N}^2 \mid a+b<n\}$ **triangle** of size n.

A **configuration** of size n is a set of n points $C \subset T_n$.

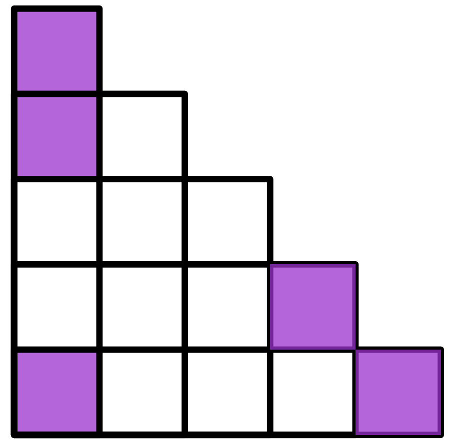
A **triangle basis** of size n is an independent set that fills T_n. Denote $\mathcal{B}_n$ their set.



A basis.



Not independent.



Not filling.

Proposition. [Salo, S. 22] A configuration of size n is a basis if and only if it fills T_n.

Bases

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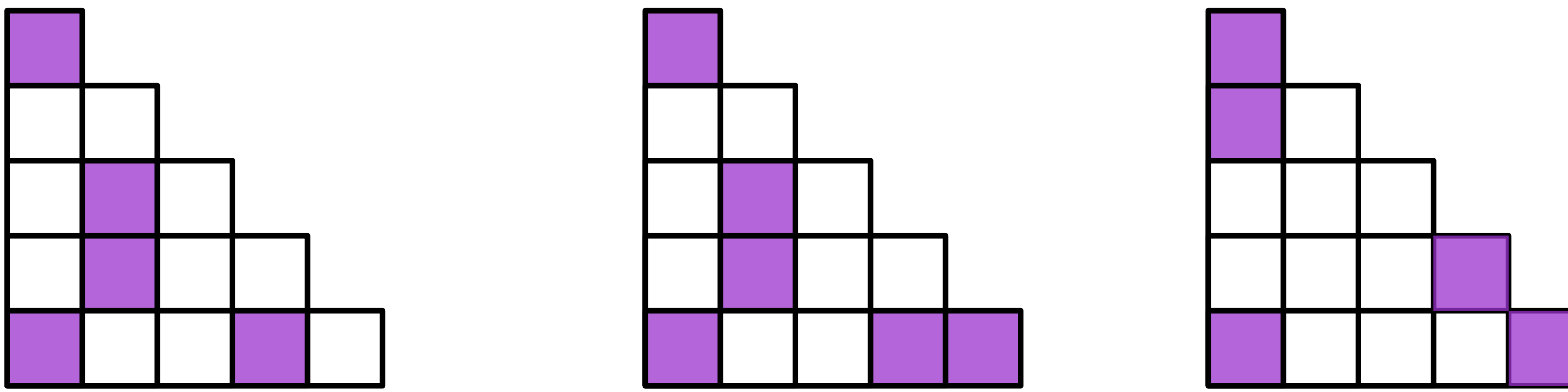
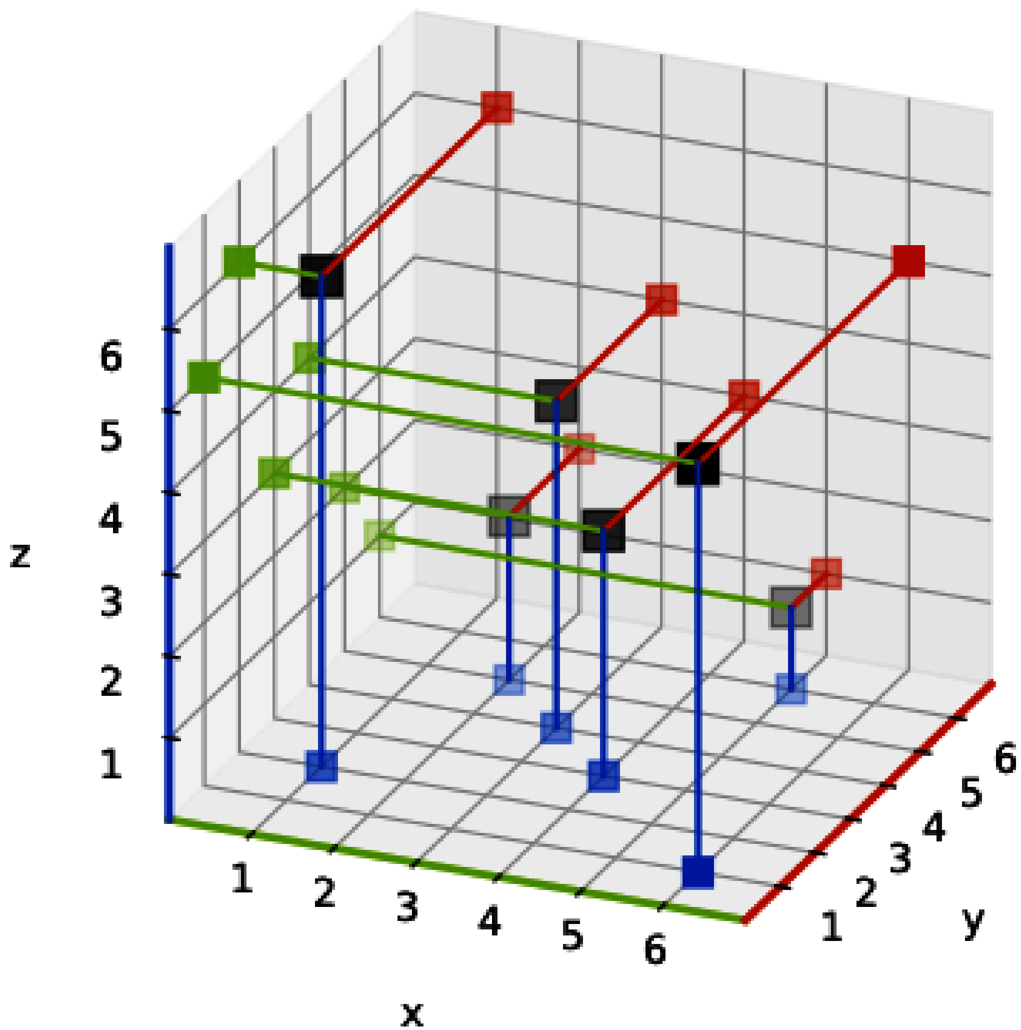


A basis. Not independent. Not filling.

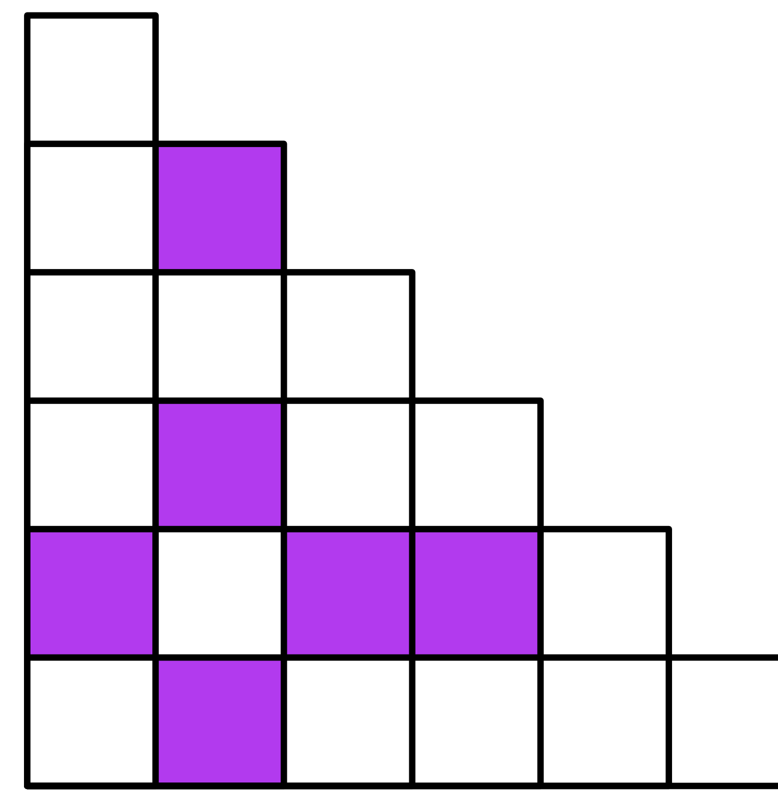
Proposition. [Salo, S. 22] A configuration of size n is a basis if and only if it fills T_n.

Theorem. [S. 24] For all n, the set of bases of size n is in bijection with $Av_n((12, 12), (312, 231))$.

II- A bijection



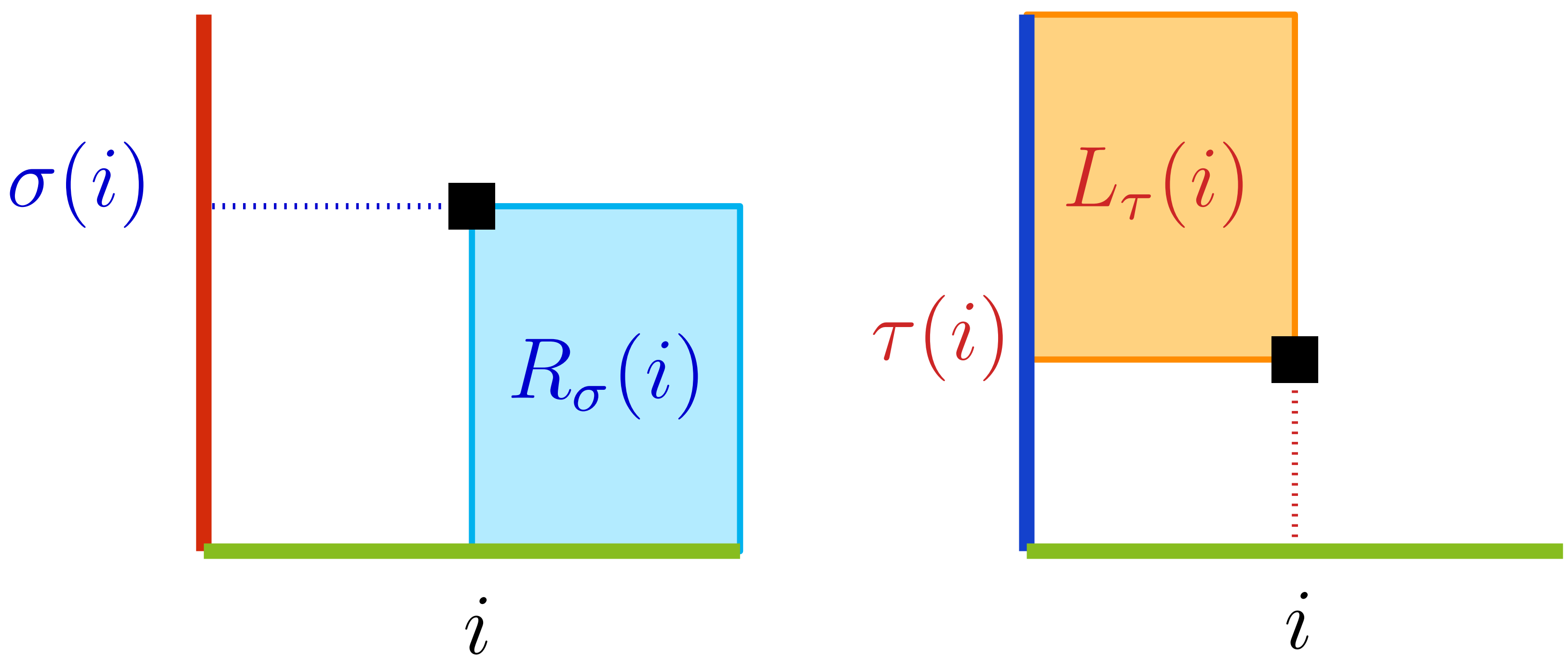
$\xrightarrow{\Gamma}$



Inversions

An **inversion** for $\sigma \in \mathfrak{S}_n$ is $(i,j) \in [\![1,n]\!]$ with $i < j$ and $\sigma(i) > \sigma(j)$.

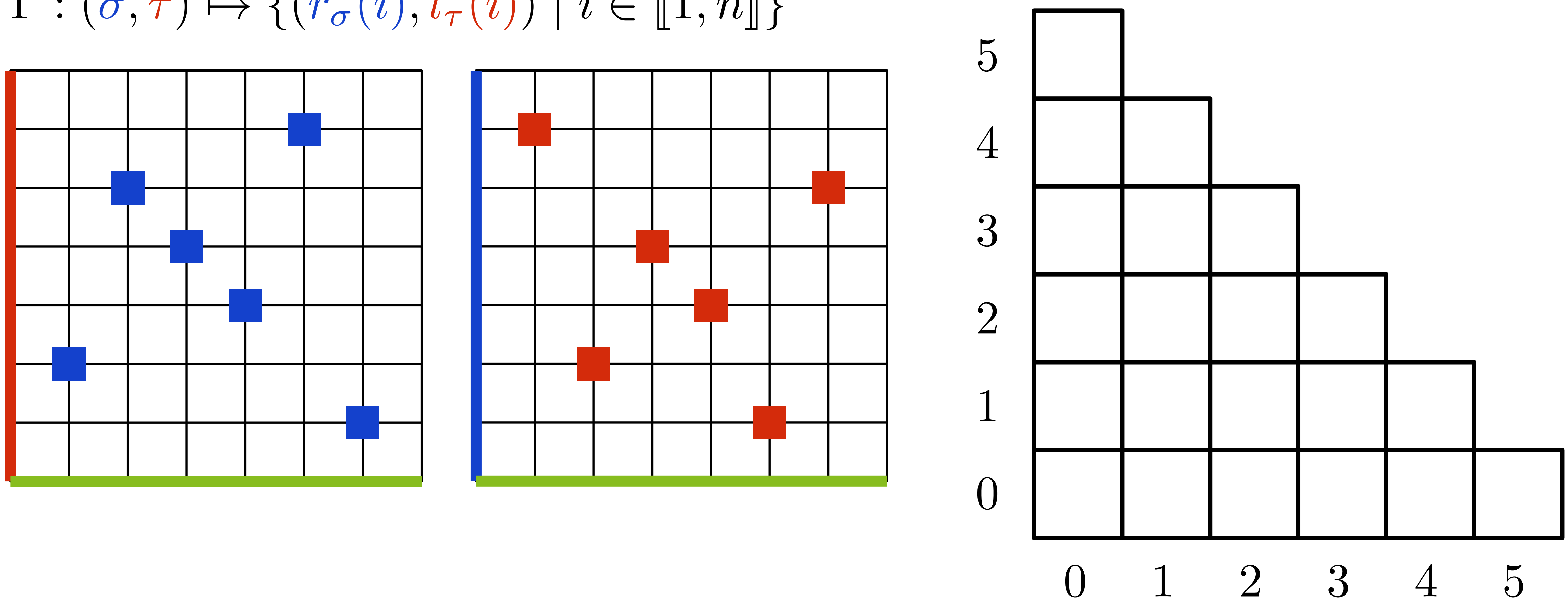
The **inversion sets** are $R_\sigma(i) = \{j > i \mid \sigma(j) < \sigma(i)\}$ (right inversions) and $L_\sigma(i) = \{j < i \mid \sigma(j) > \sigma(i)\}$ (left inversions).
We denote $r_\sigma(i) = |R_\sigma(i)|$ and $l_\sigma(i) = |L_\sigma(i)|$.



$\Gamma : (\sigma, \tau) \mapsto \{(r_\sigma(i), l_\tau(i)) \mid i \in [\![1,n]\!]\}$

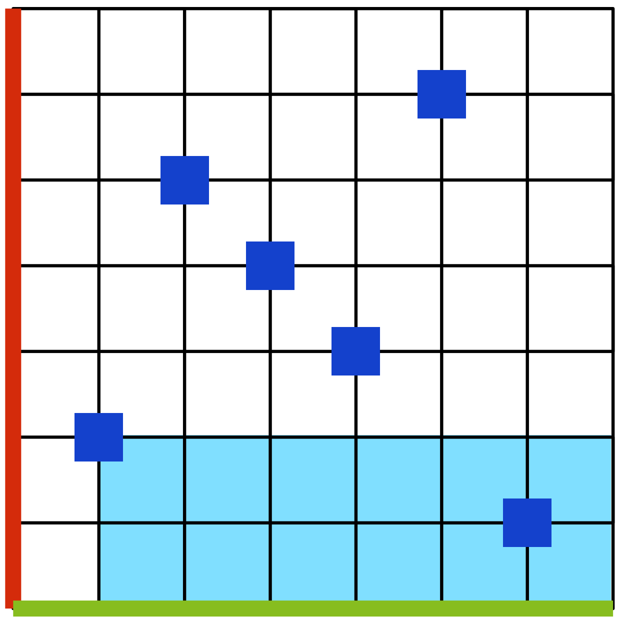
The bijection

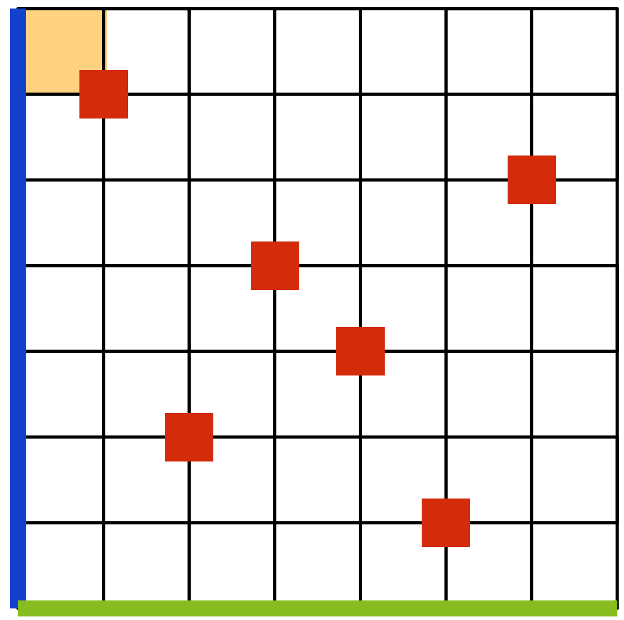
$$\Gamma : (\sigma, \tau) \mapsto \{(r_\sigma(i), l_\tau(i)) \mid i \in [\![1, n]\!]\}$$

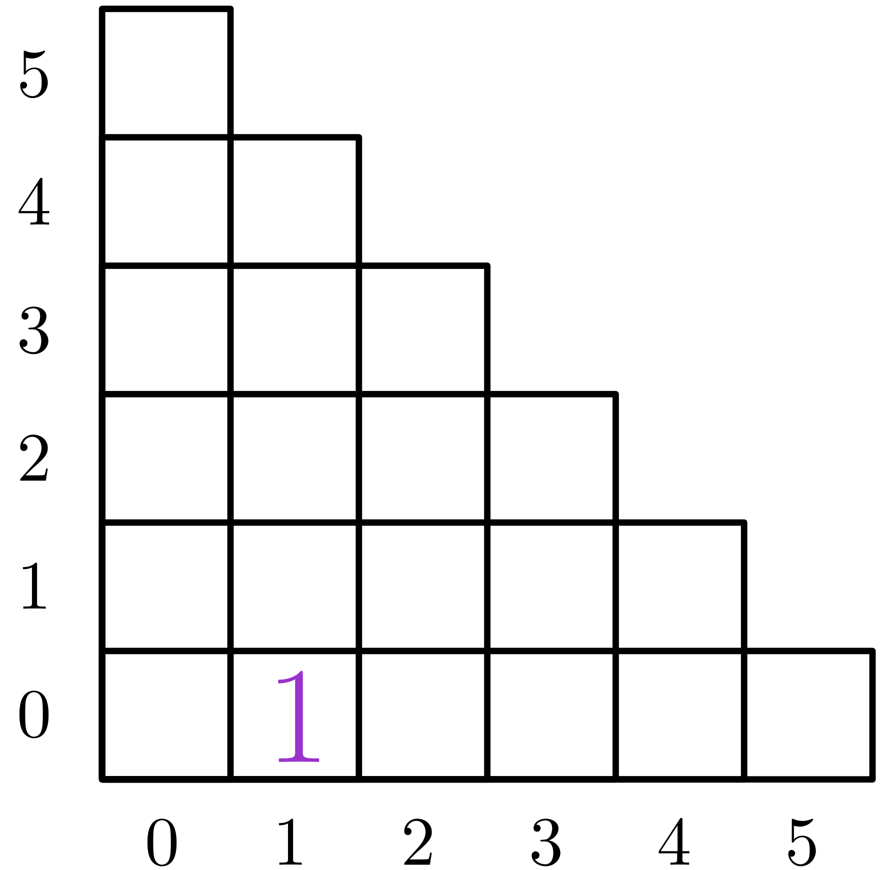


The bijection

$$\Gamma : (\sigma, \tau) \mapsto \{(r_\sigma(i), l_\tau(i)) \mid i \in [\![1, n]\!]\}$$







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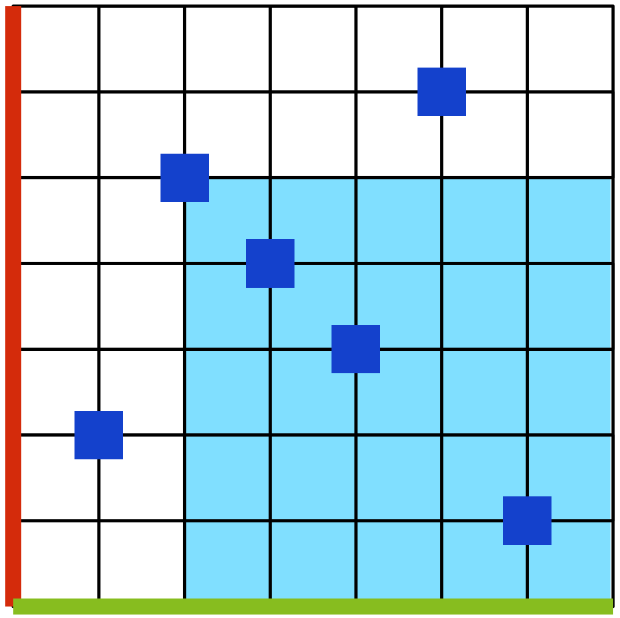

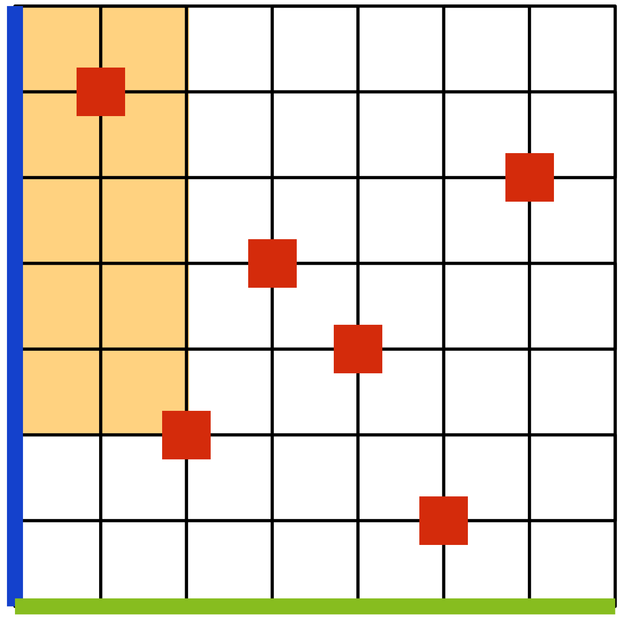

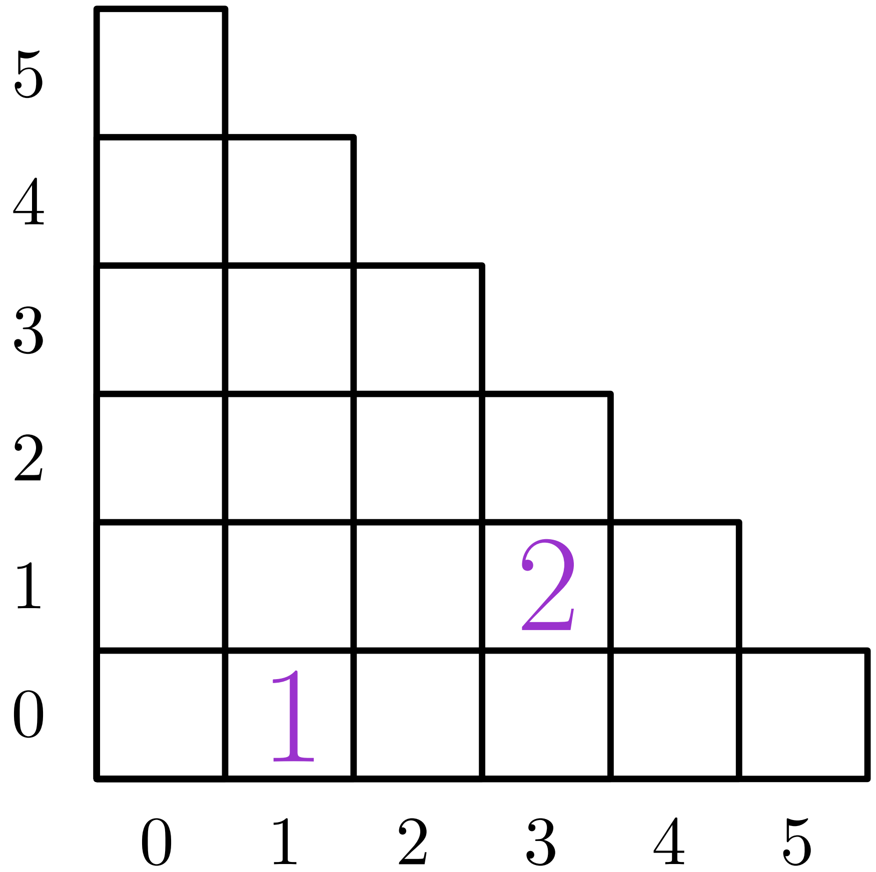

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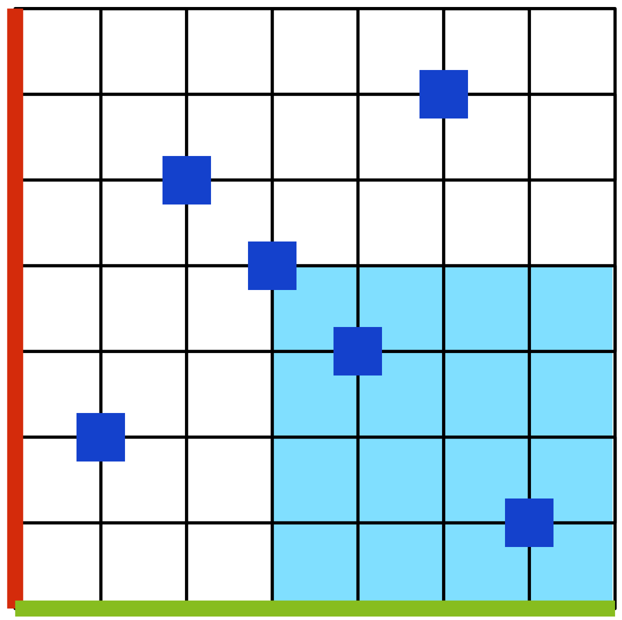

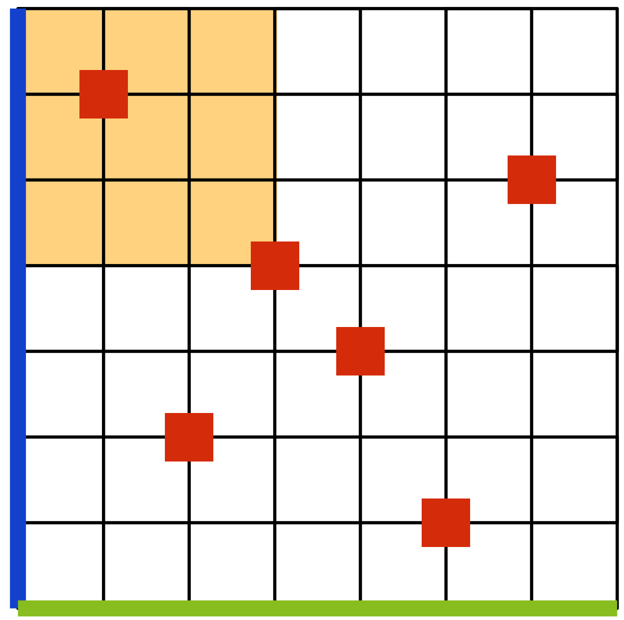

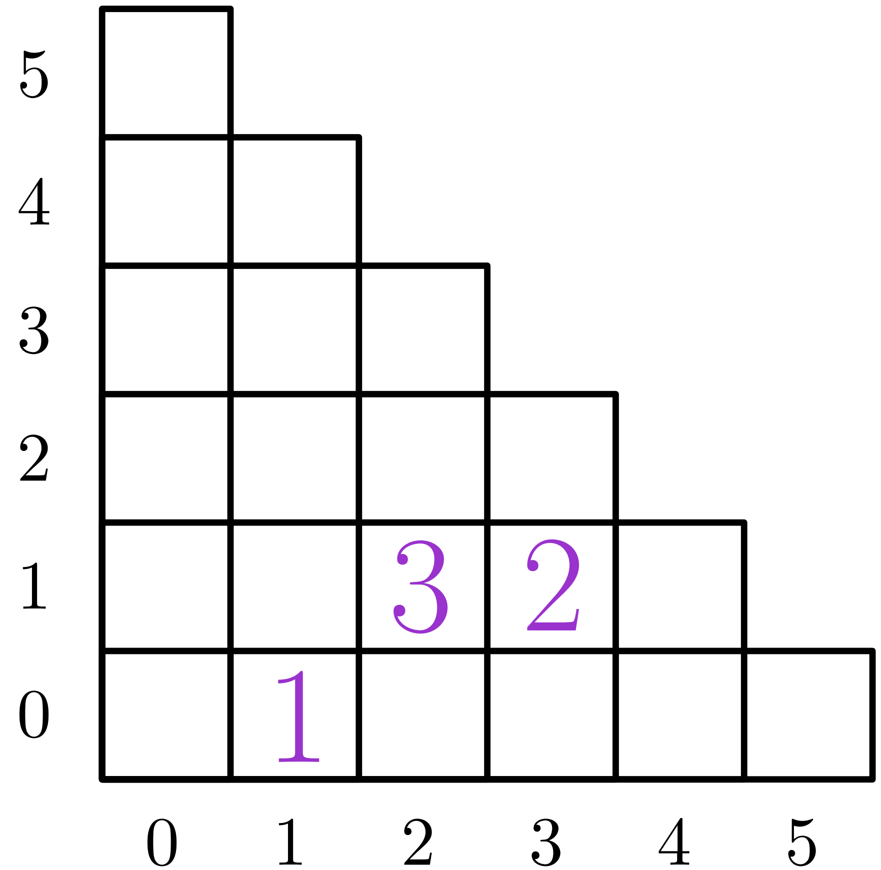

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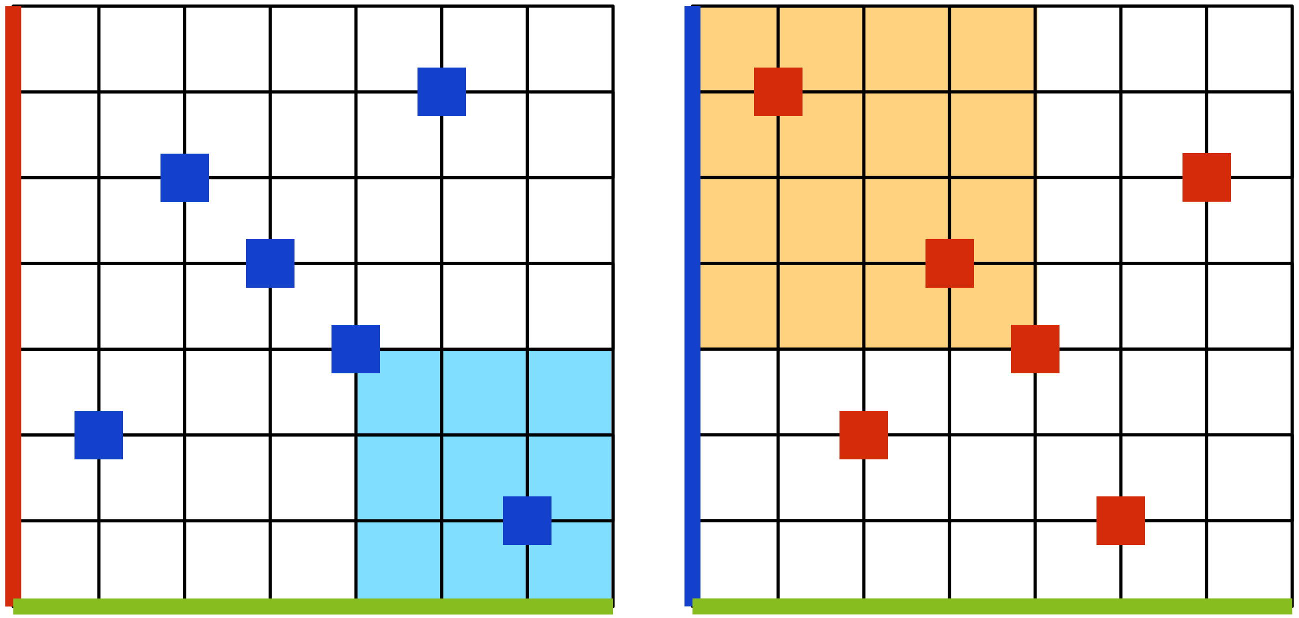

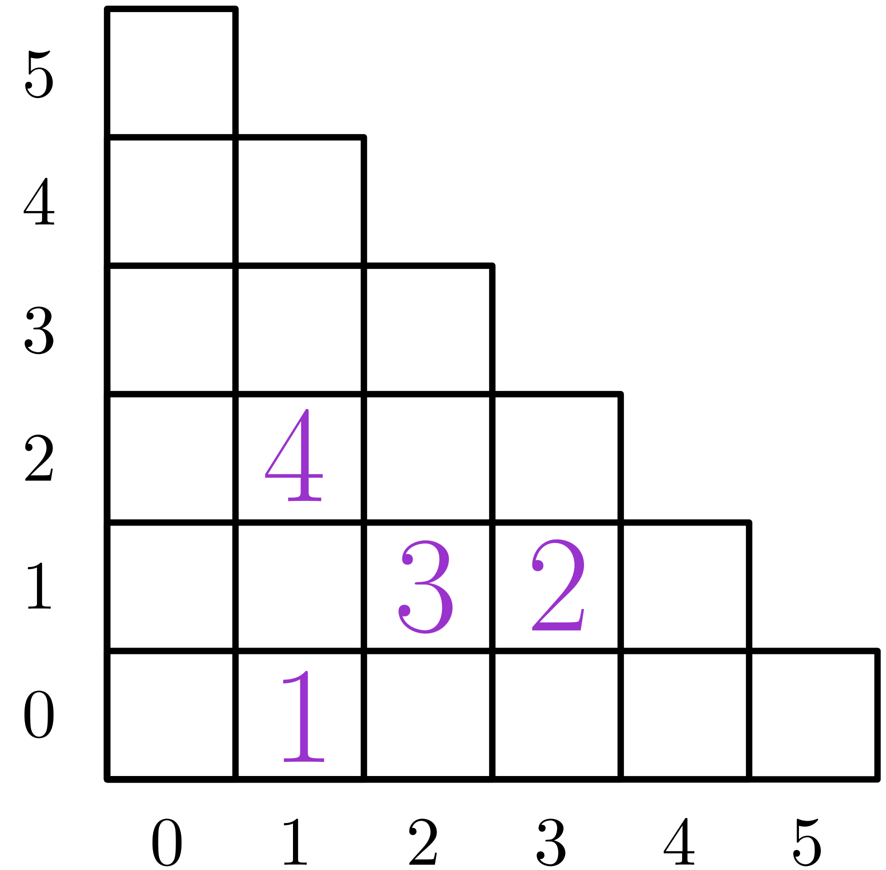

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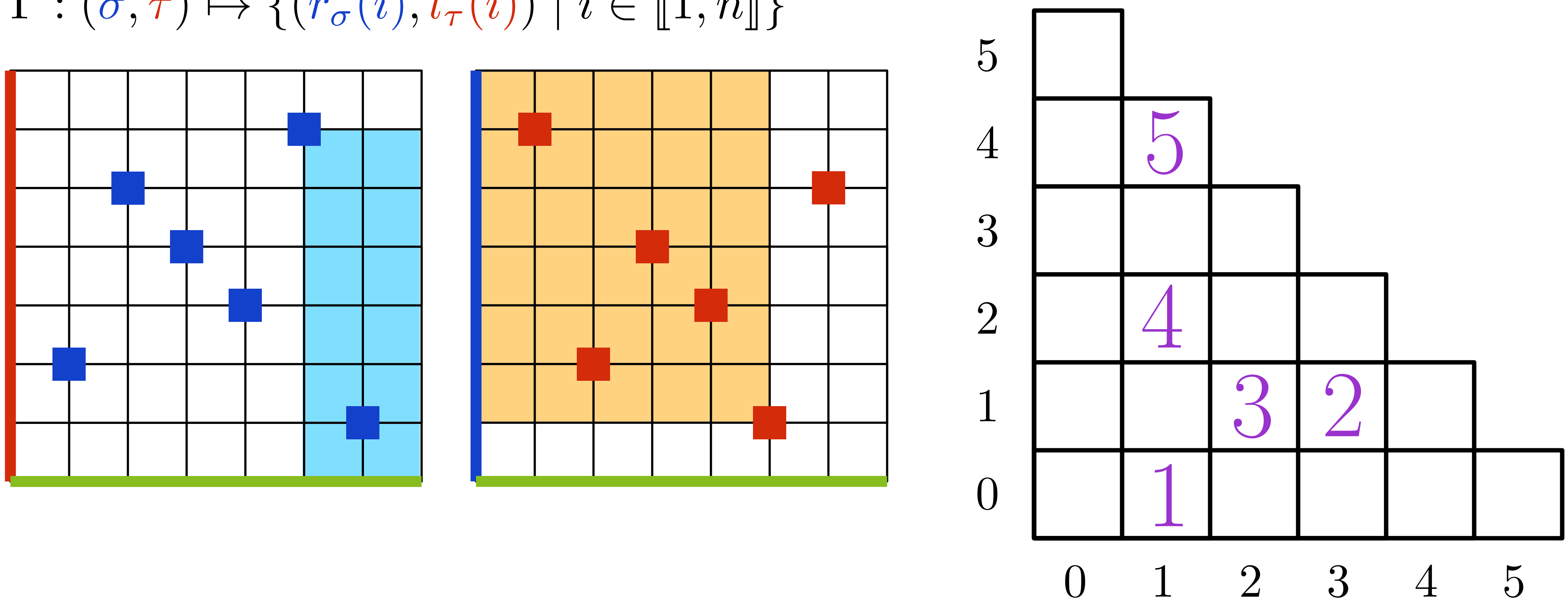

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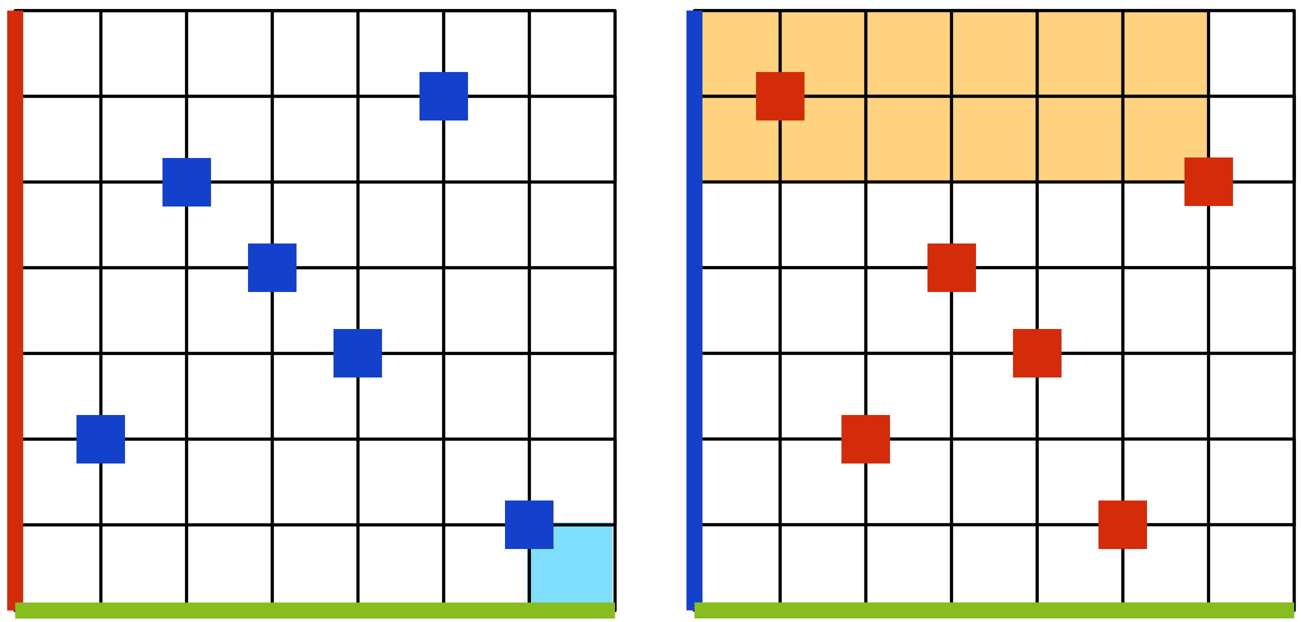

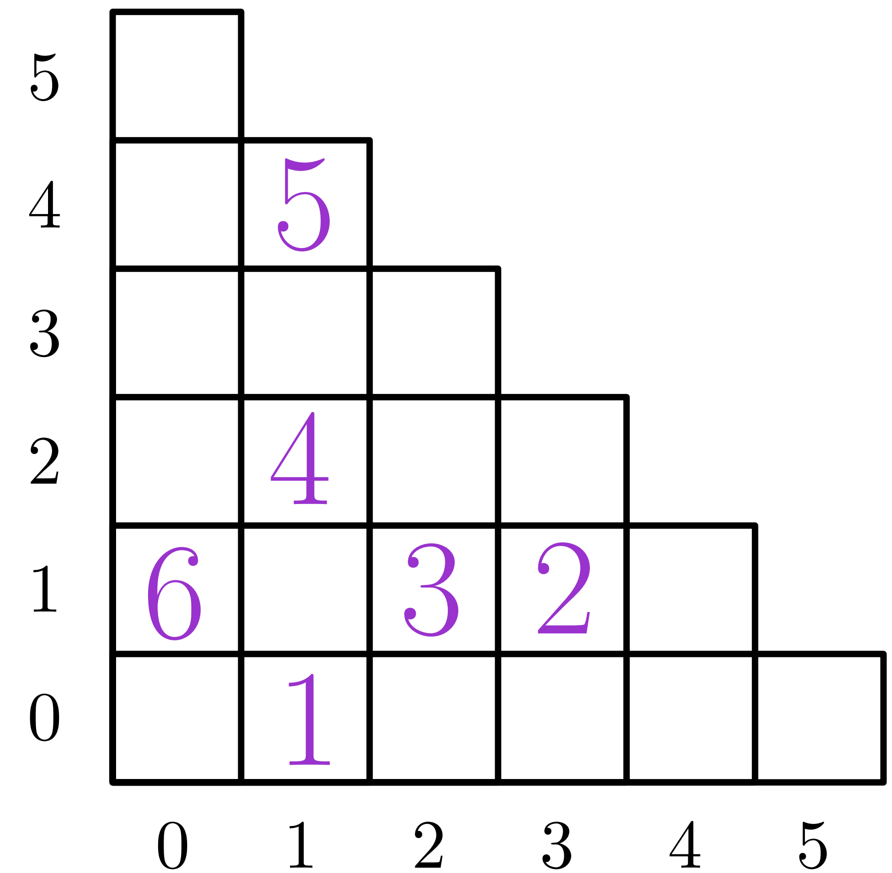

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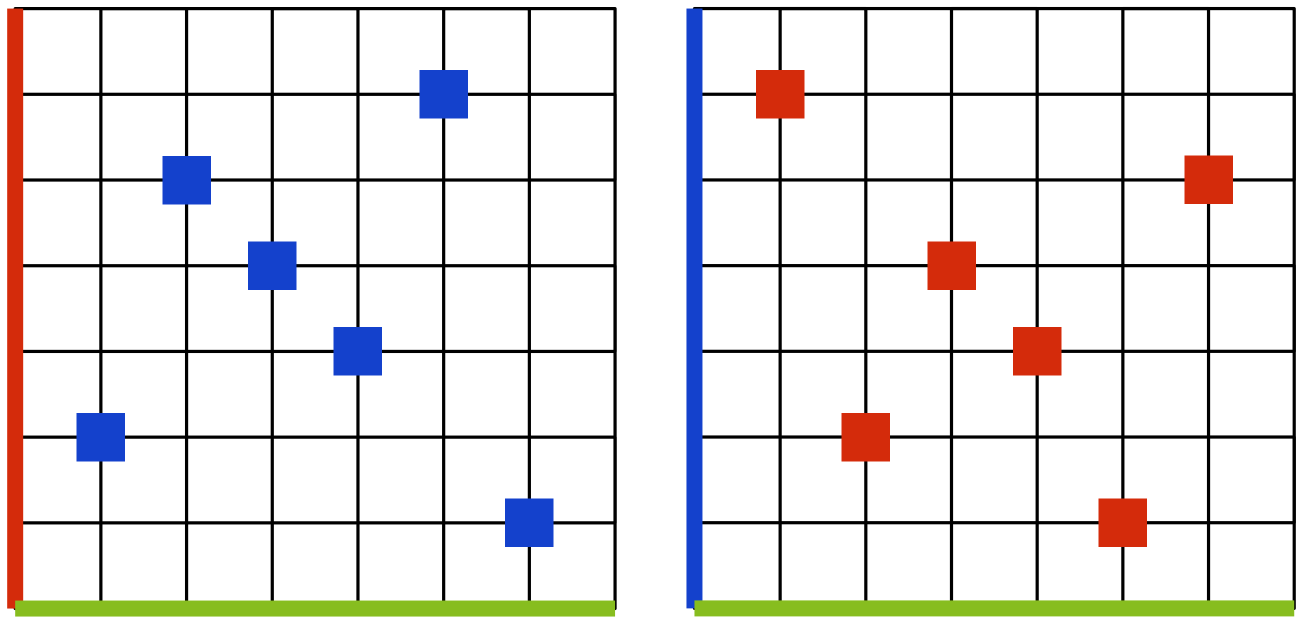
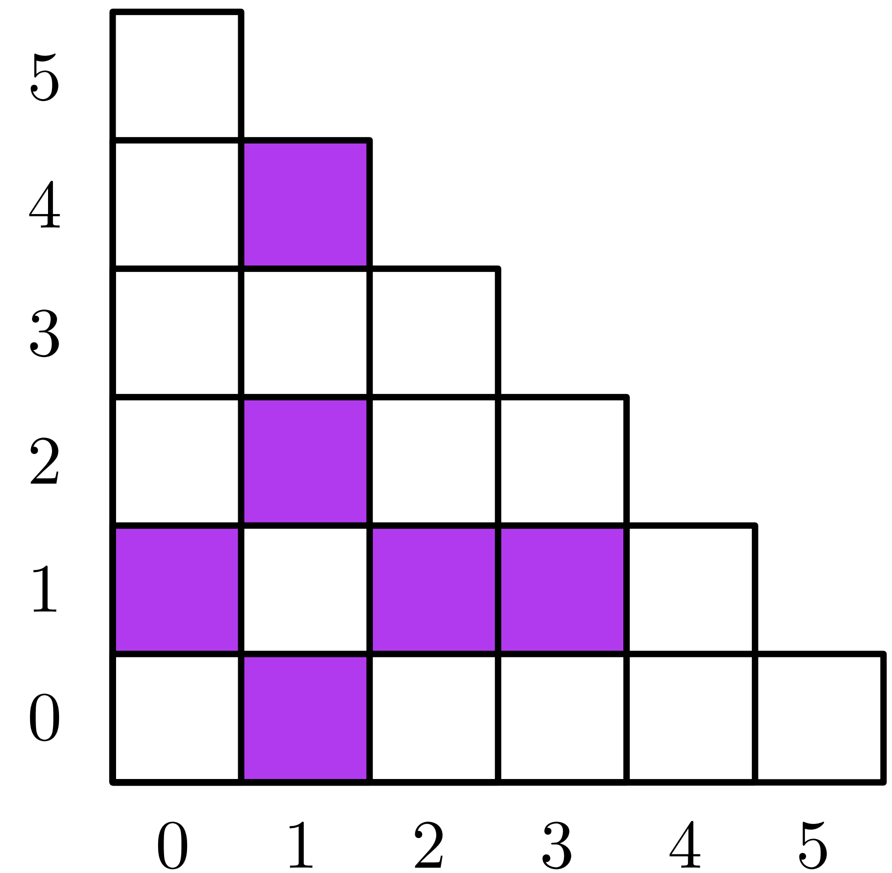


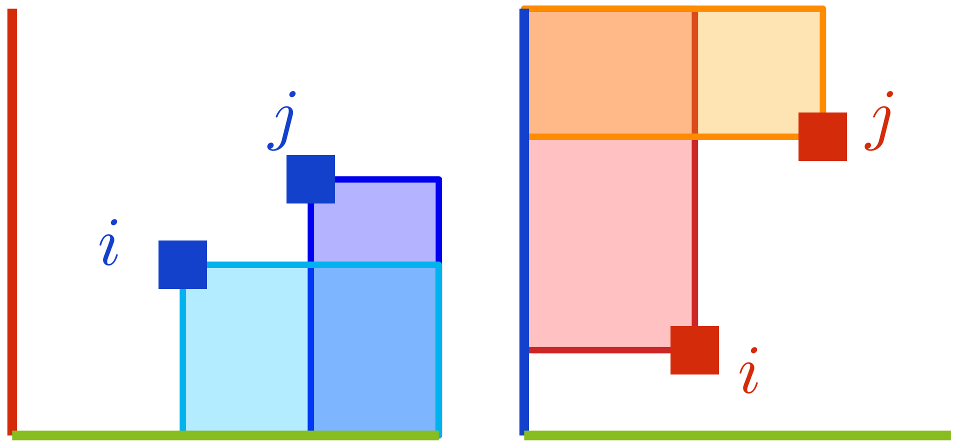


Theorem. [S. 24] For all n, Γ is a bijection between $Av_n((12, 12), (312, 231))$ and the triangle bases of size n.

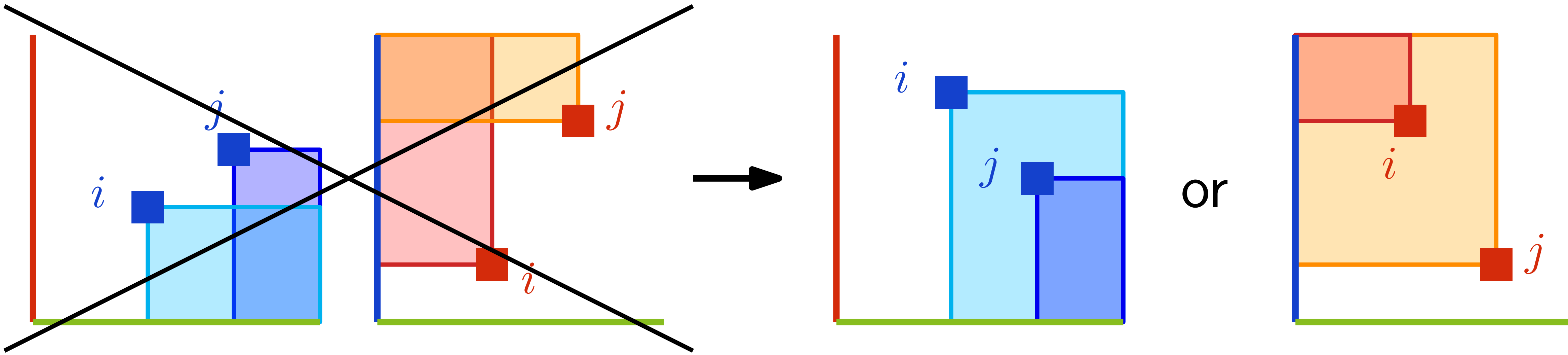
We need to prove that:

- If $(\sigma, \tau) \in Av_n((12, 12), (312, 231))$ then $\Gamma(\sigma, \tau)$ is a basis.
- One can recover the labels.

Avoiding $(12, 12)$: no “points too close”



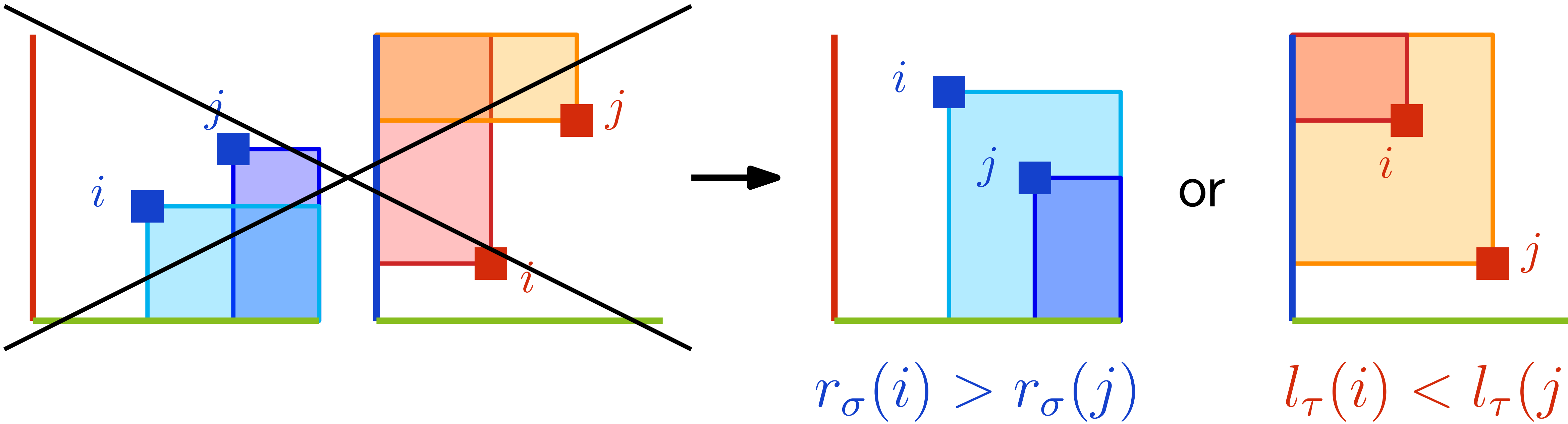
Avoiding $(12, 12)$: no "points too close"



$r_\sigma(i) > r_\sigma(j)$

$l_\tau(i) < l_\tau(j)$

Avoiding $(12, 12)$: no "points too close"

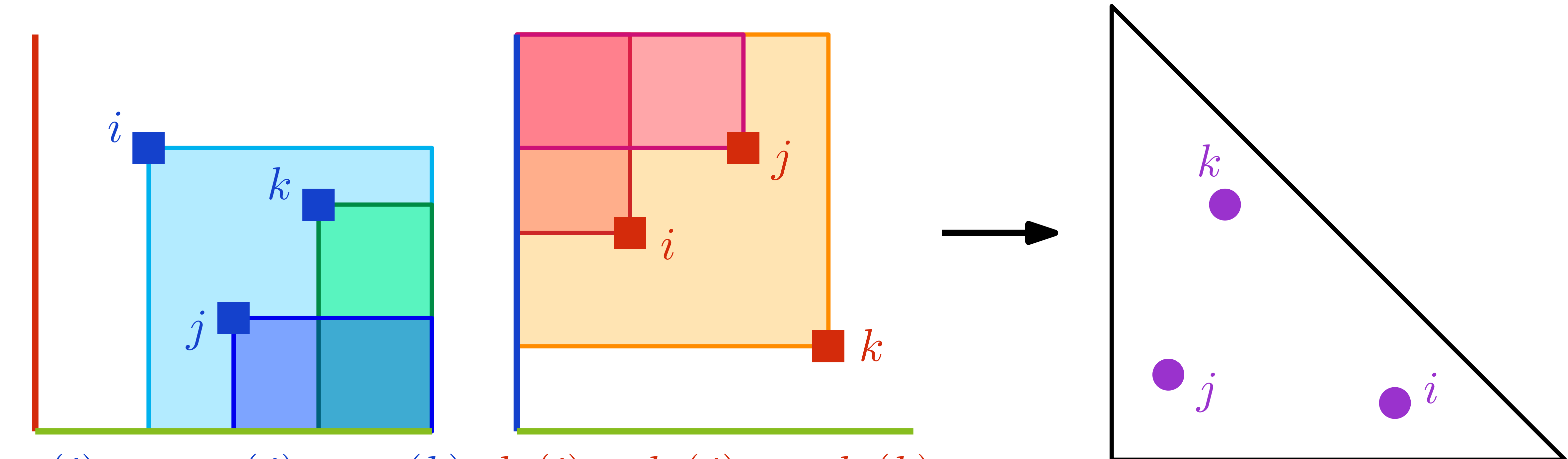


A configuration C is **sparse** if there is no triangle of size k such that $|C \cap T| > k$.

Proposition. [Salo, S. 22] If a configuration C is not sparse then it is not independent (and in particular C is not a basis).

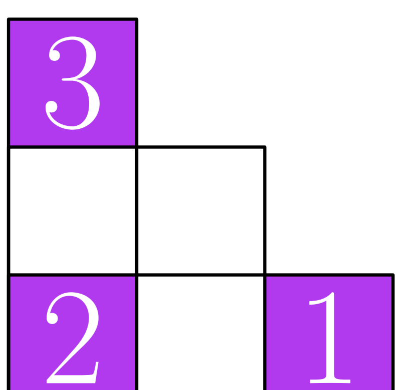
Proposition. If (σ, τ) avoids $(12, 12)$ then $\Gamma(\sigma, \tau)$ is sparse.

Avoiding $(312, 231)$: no "points too far"



$r_\sigma(i) >> r_\sigma(j) \approx r_\sigma(k)$ $\quad l_\tau(i) \approx l_\tau(j) << l_\tau(k)$

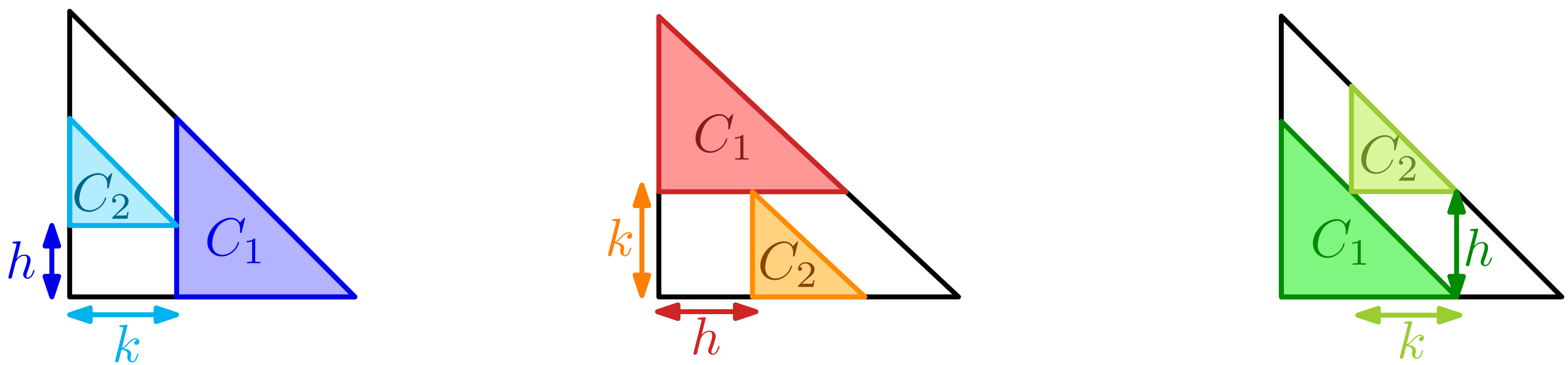
Points too far to fill



$= \Gamma(312, 231)$ is the only sparse configuration of size 3 that does not fill.

Key of the proof: a recursive decomposition

Lemma. [Salo, S. 22] Any basis of size $n \geqslant 2$ can be cut into two smaller bases in one of the 3 following ways.



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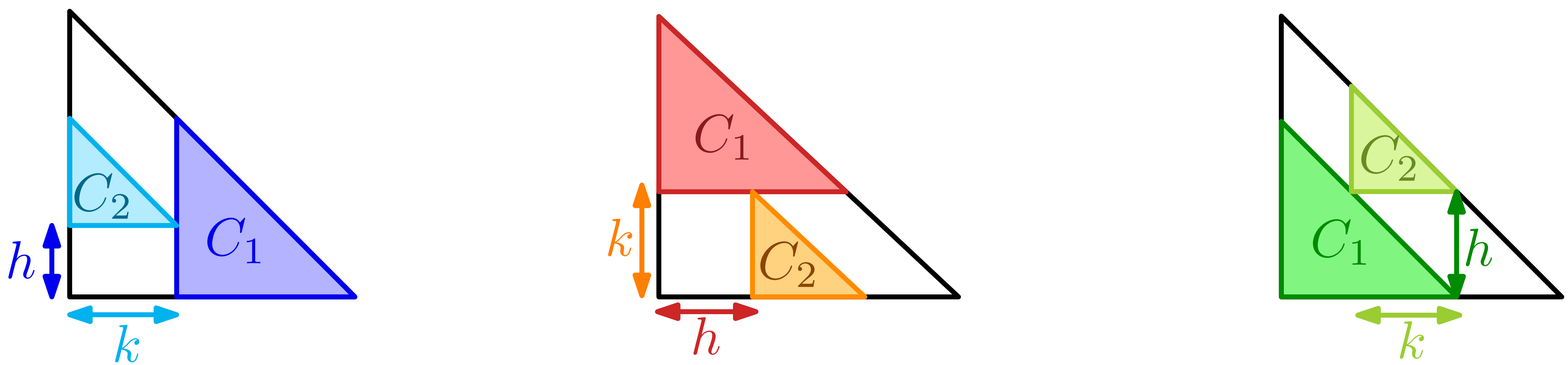
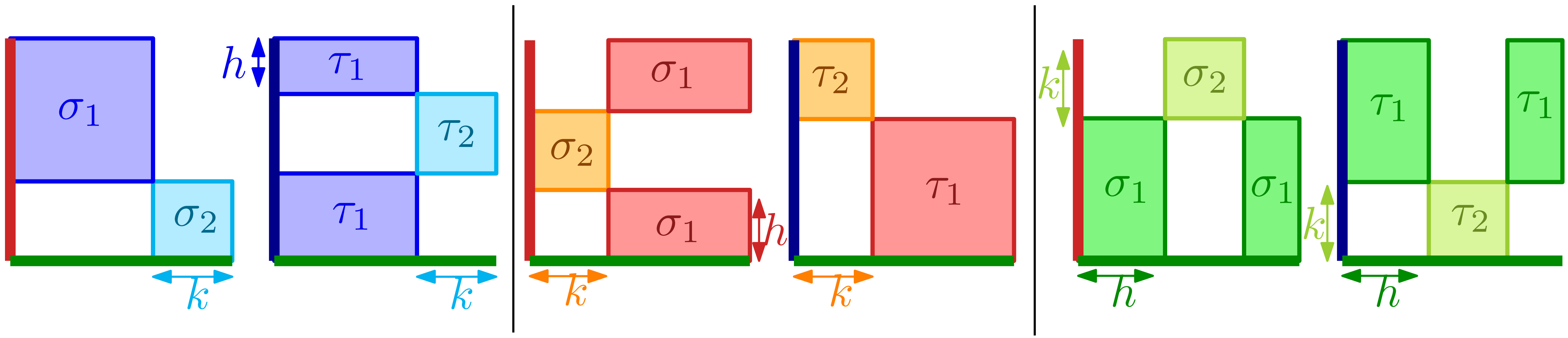


Lemma. Γ transports the cuts : $\Gamma(\sigma, \tau)$ admits a given cut if and only if (σ, τ) does.



Key of the proof: a recursive decomposition

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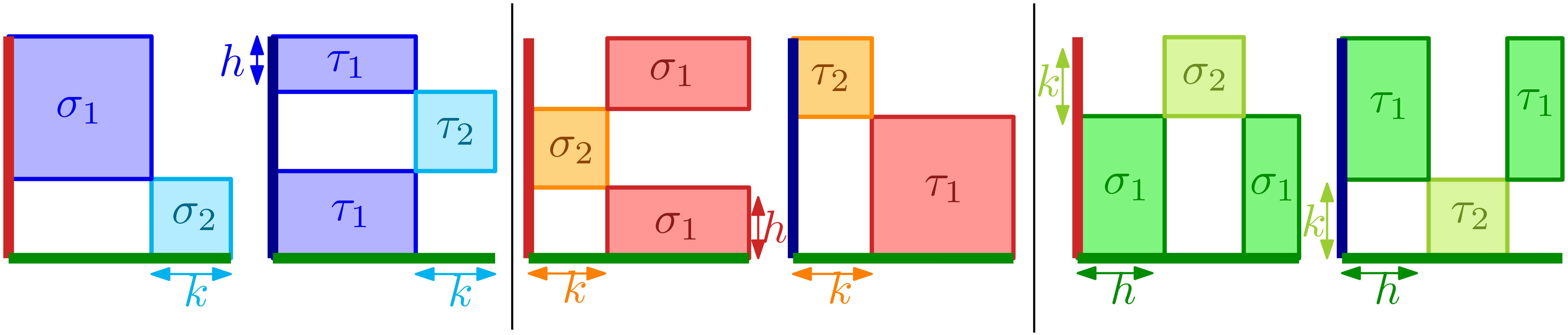
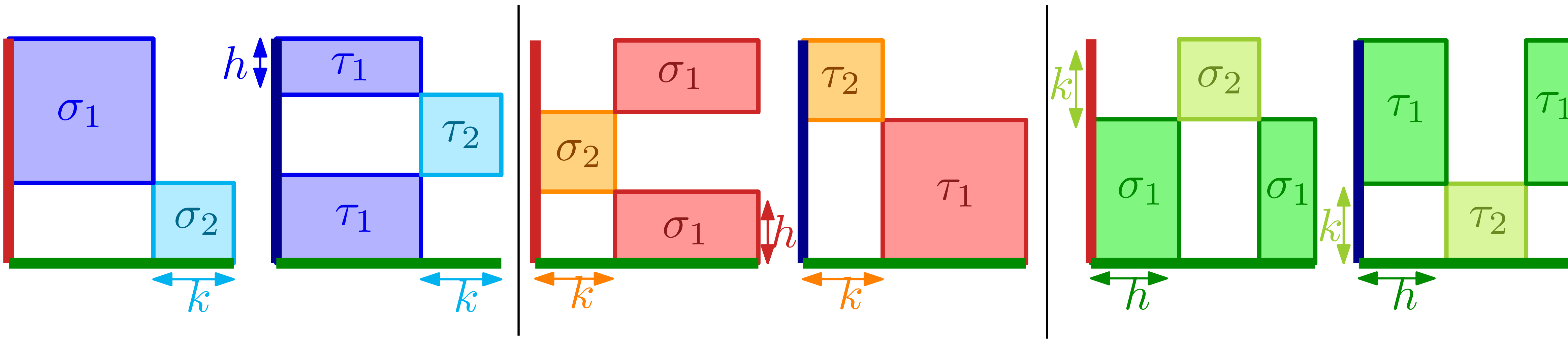


Lemma. All 3-permutations of $Av_n((12, 12), (312, 231))$ admit a cut.

Key of the proof: a recursive decomposition

Lemma. Γ transports the cuts : $\Gamma(\sigma,\tau)$ admits a given cut if and only if (σ,τ) does.

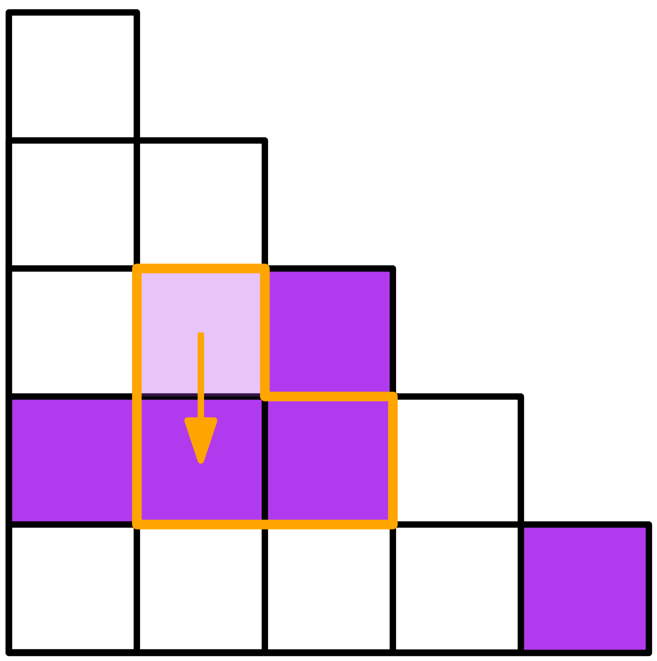


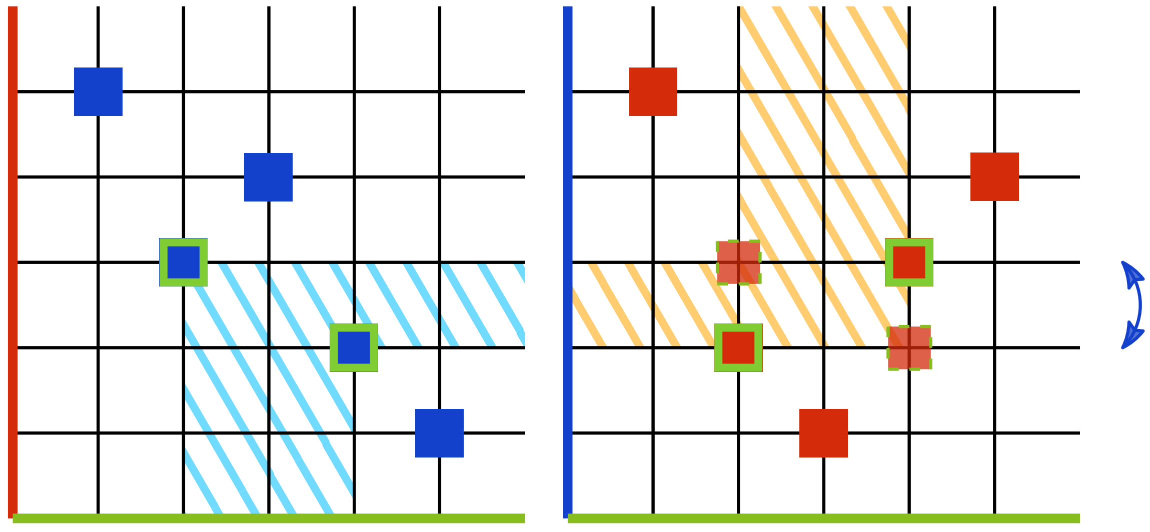
Lemma. All 3-permutations of $Av_n((12,12),(312,231))$ admit a cut.

Corollary. Γ is a bijection between $Av_n((12,12),(312,231))$ and $\mathcal{B}_n$.

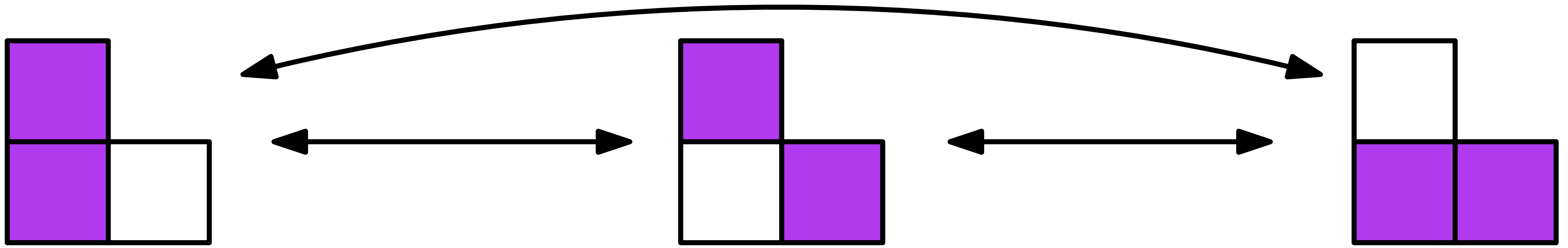
Remark. The cut is not unique, but a canonical one can be chosen.

III- Solitaire game

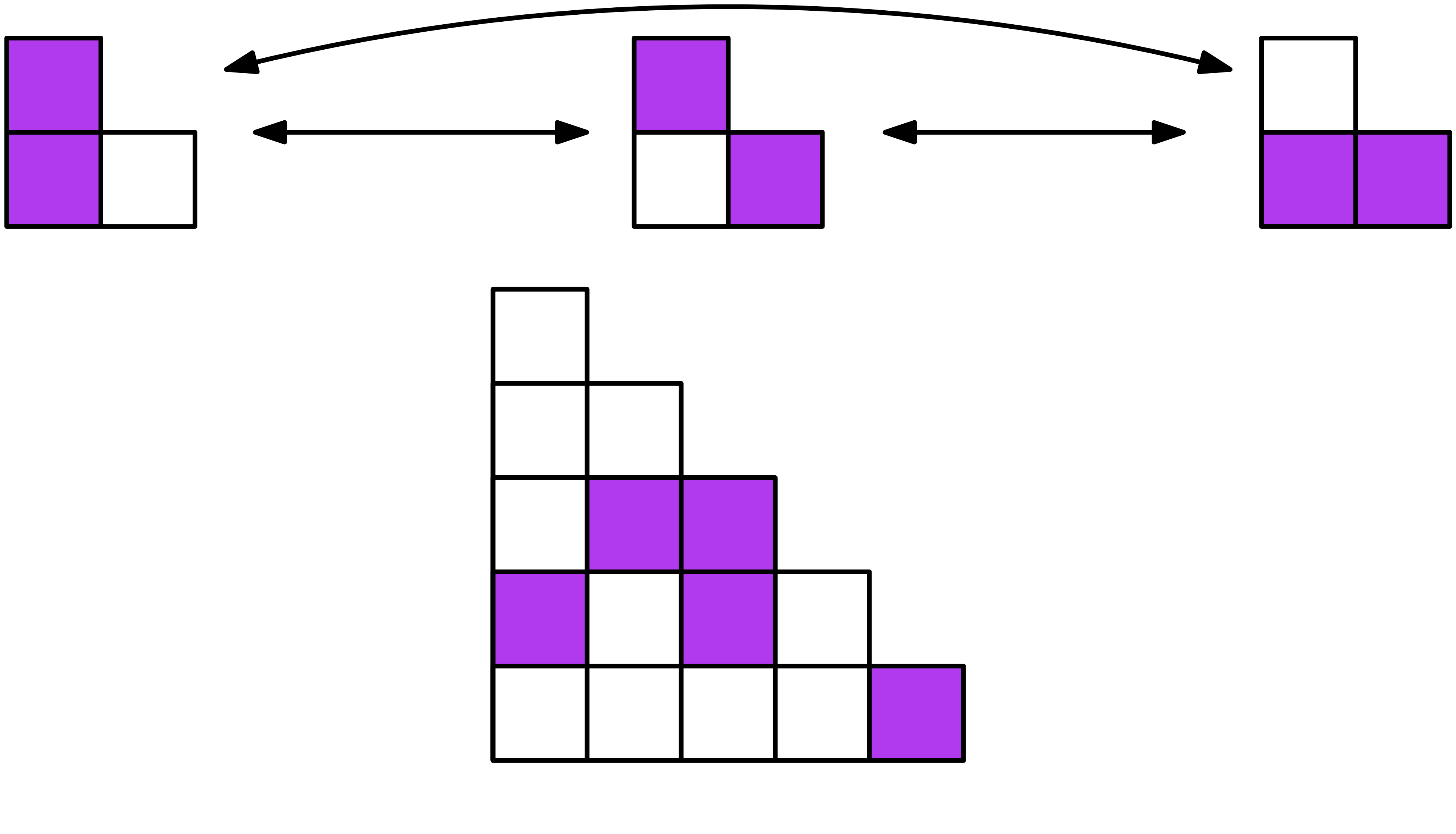




Solitaire game on configurations



Solitaire game on configurations



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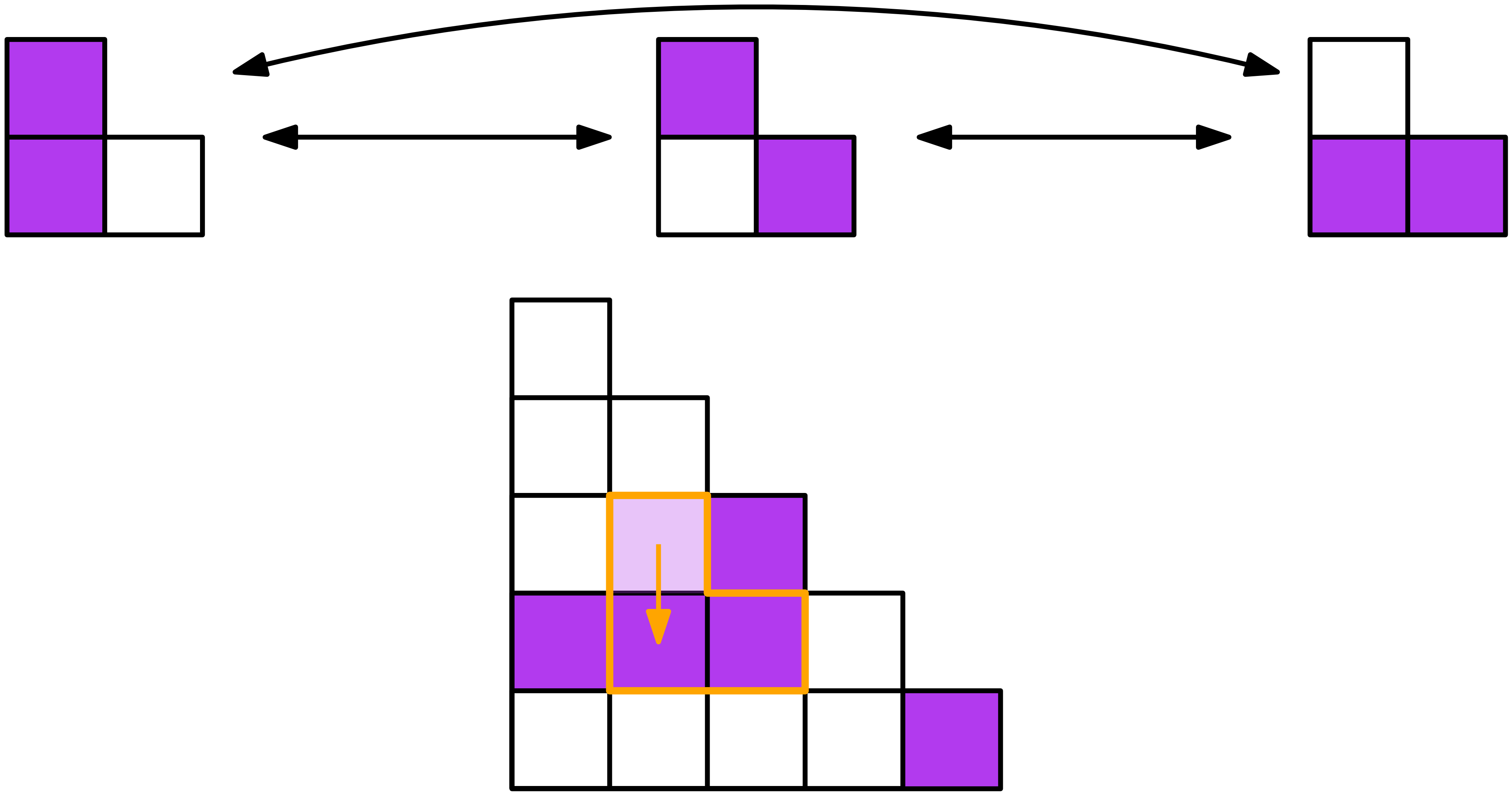

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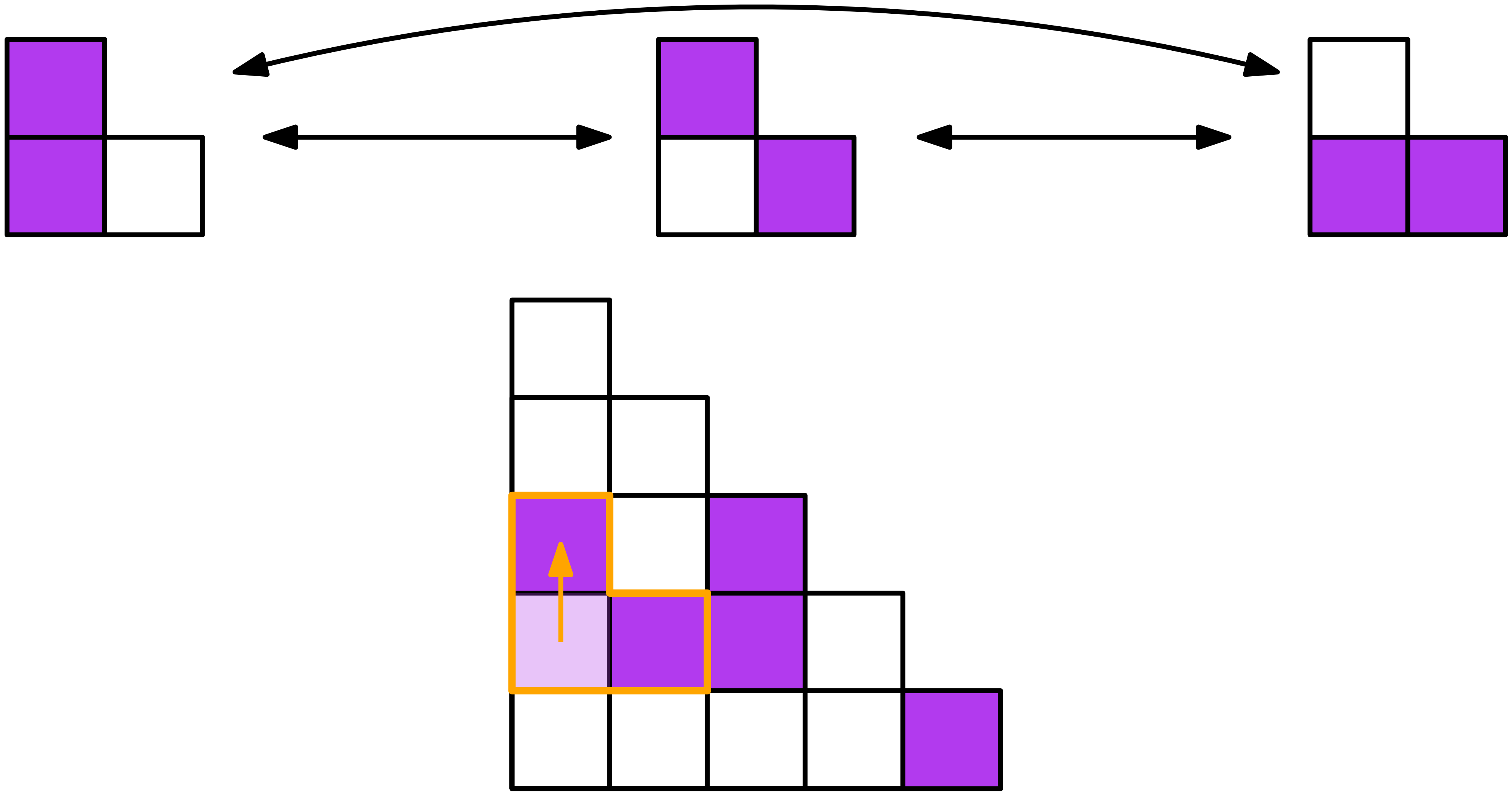

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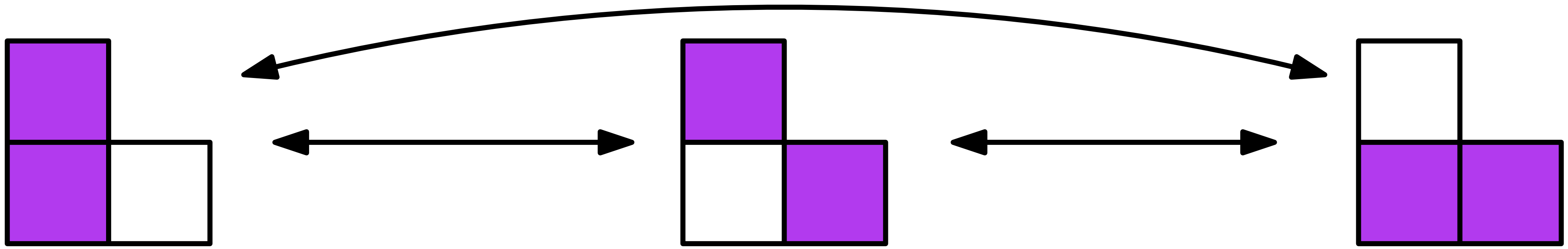

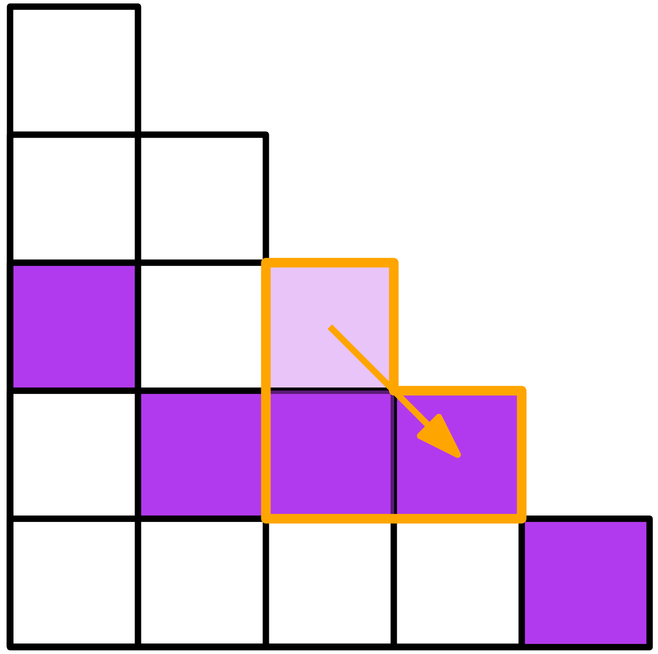

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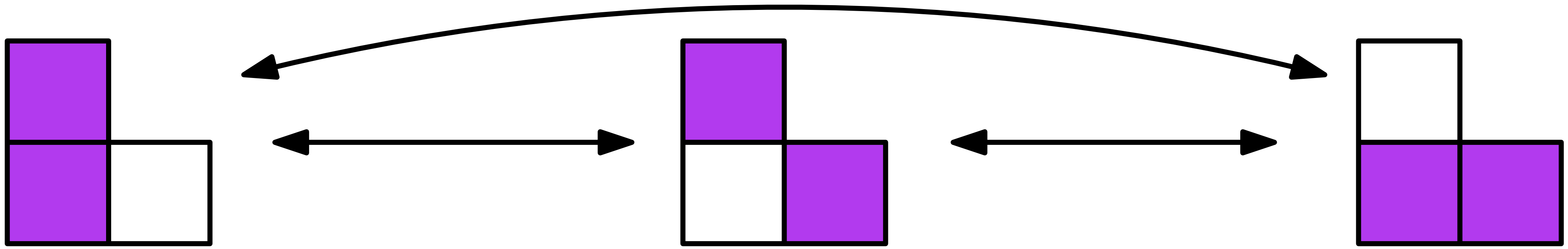

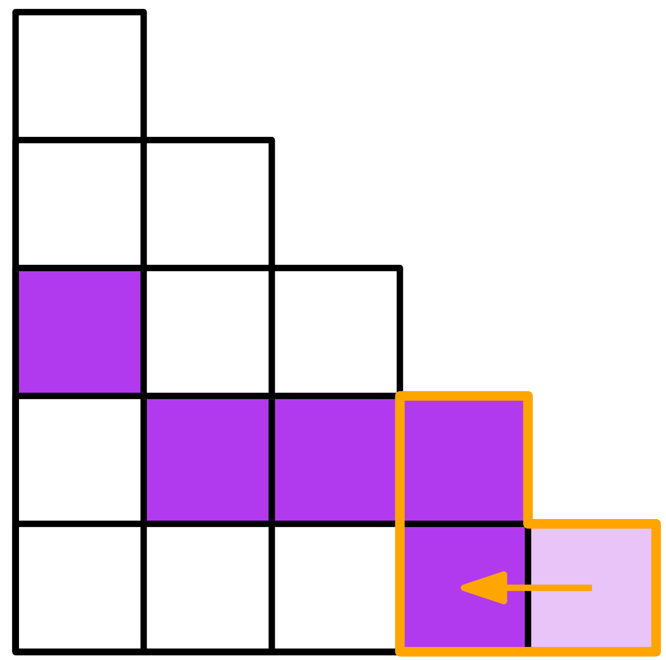

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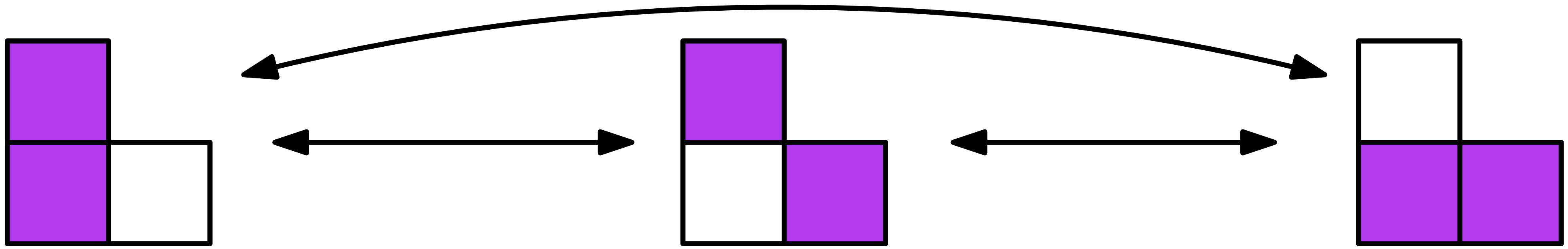
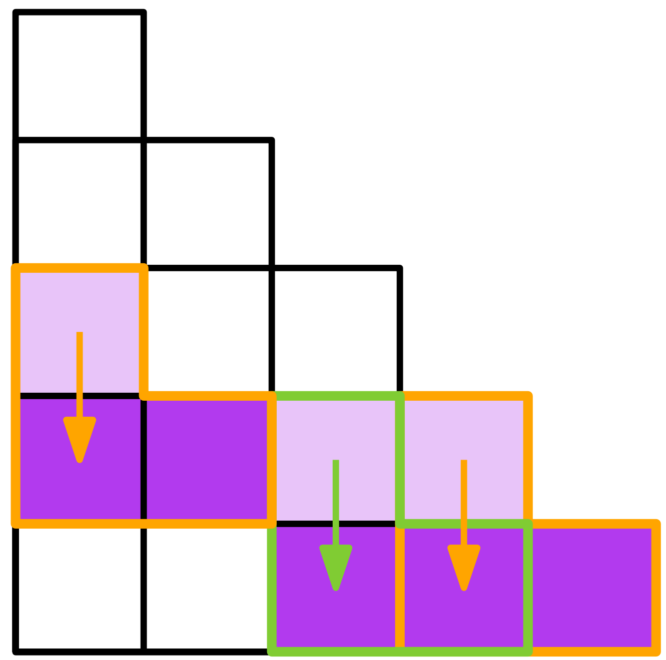
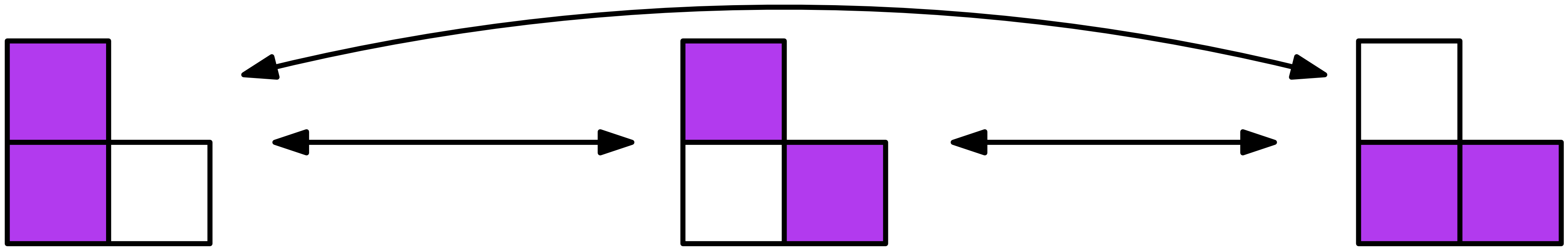
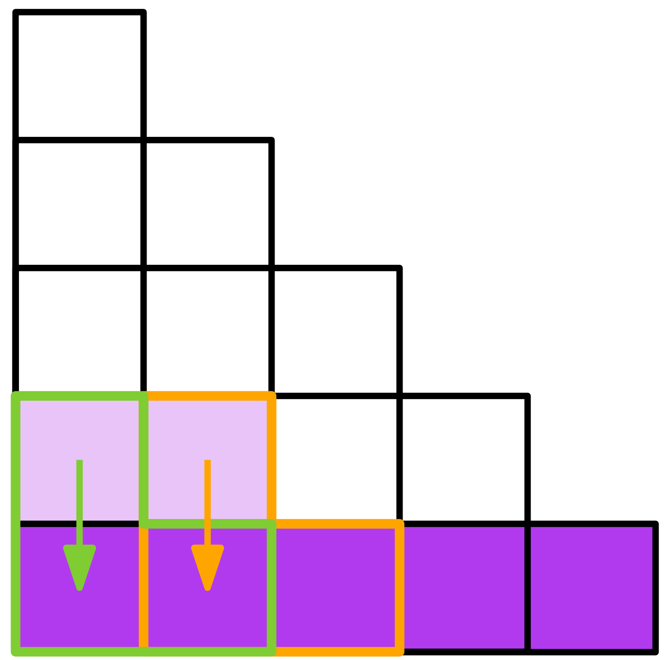




Solitaire game on configurations





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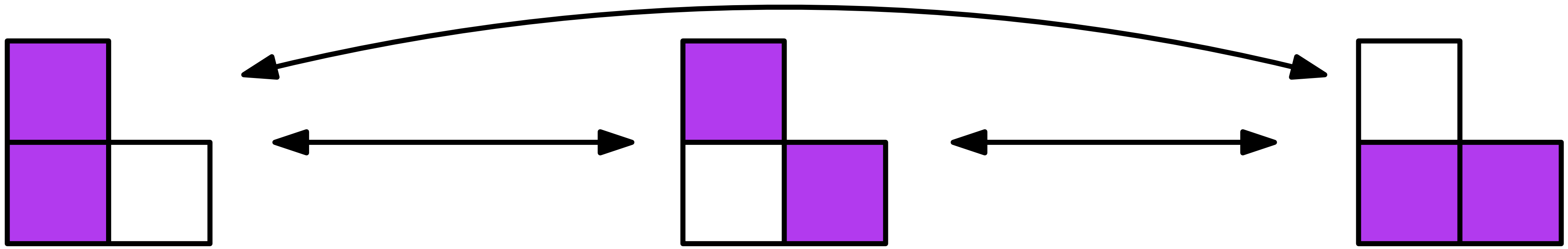
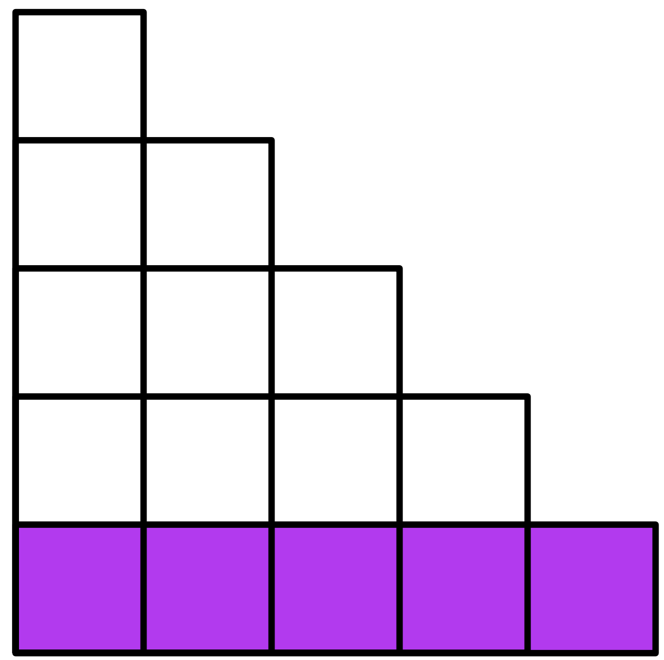
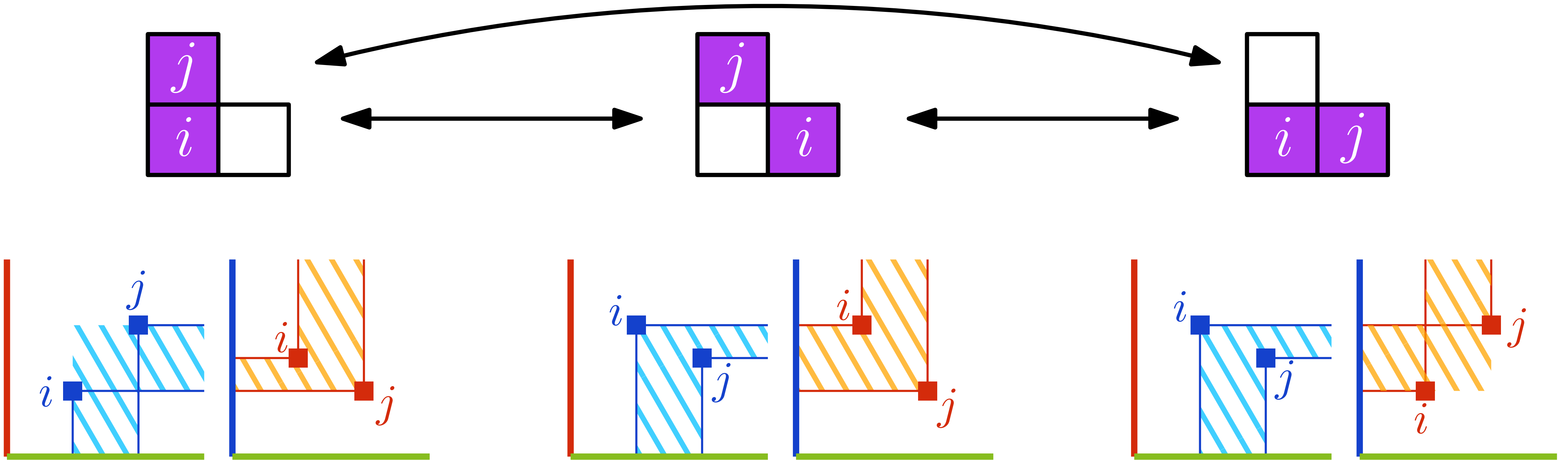


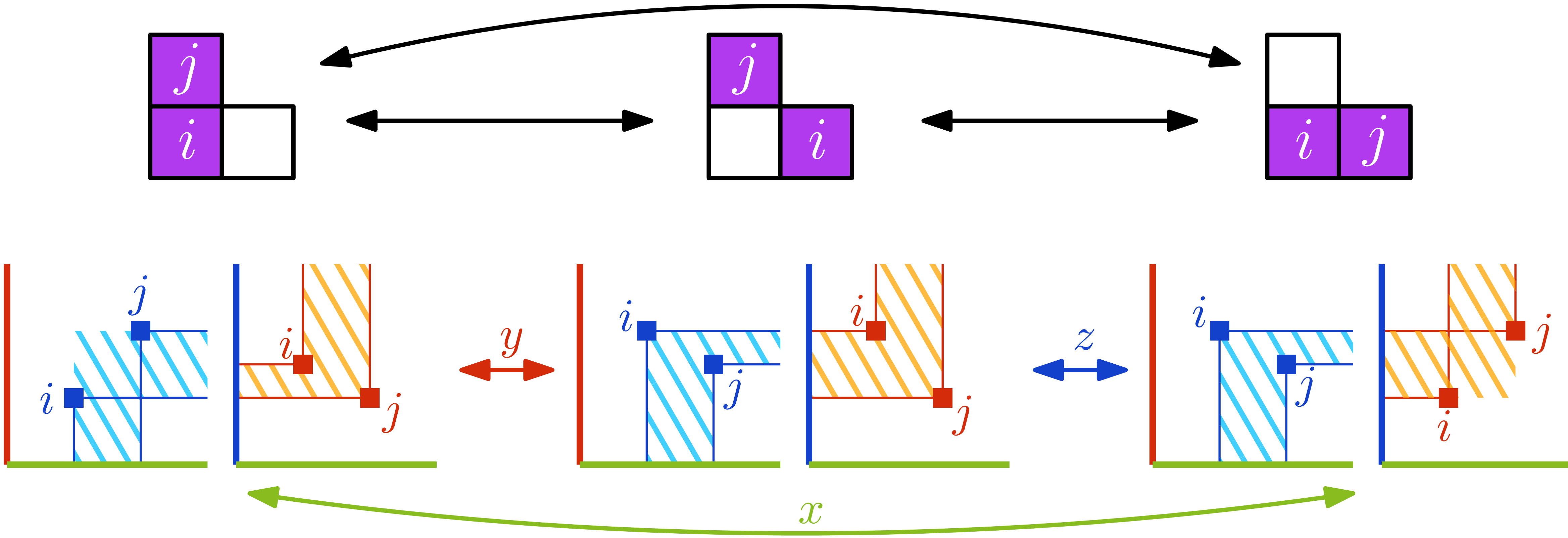


Theorem. [Salo, S. 22] The orbit of the line $[\![0, n-1]\!] \times \{0\}$ consists of the triangle bases of size n.

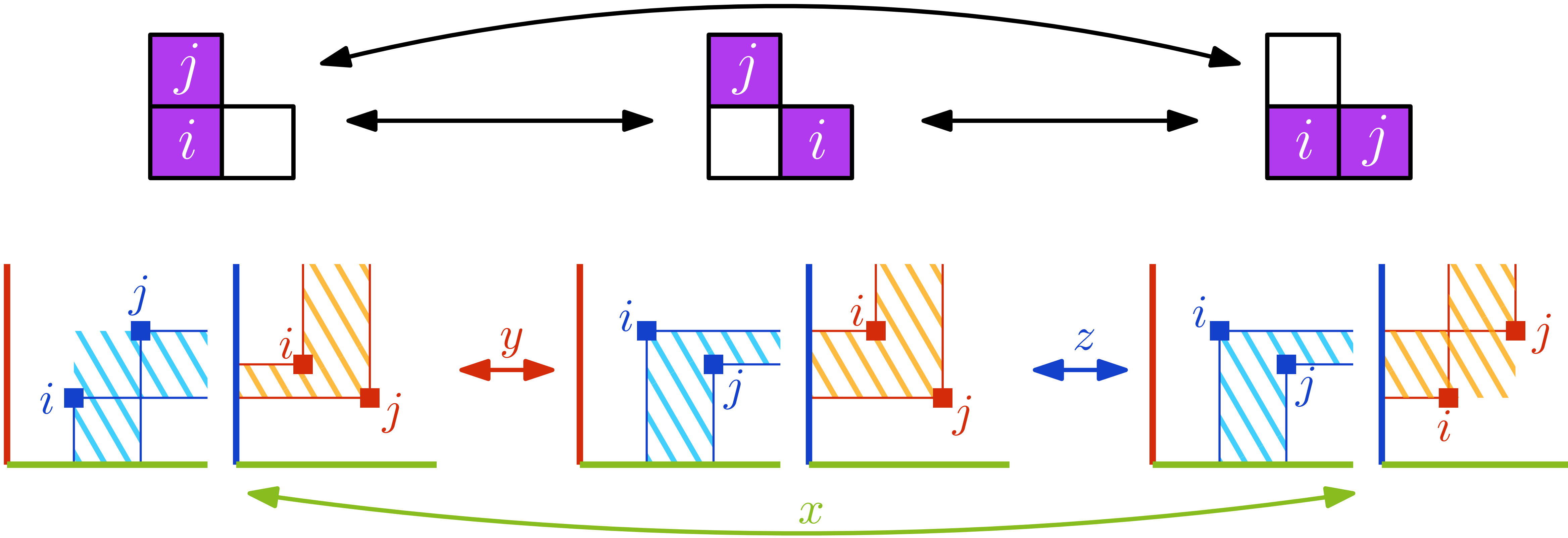
Solitaire game on permutations



Solitaire game on permutations



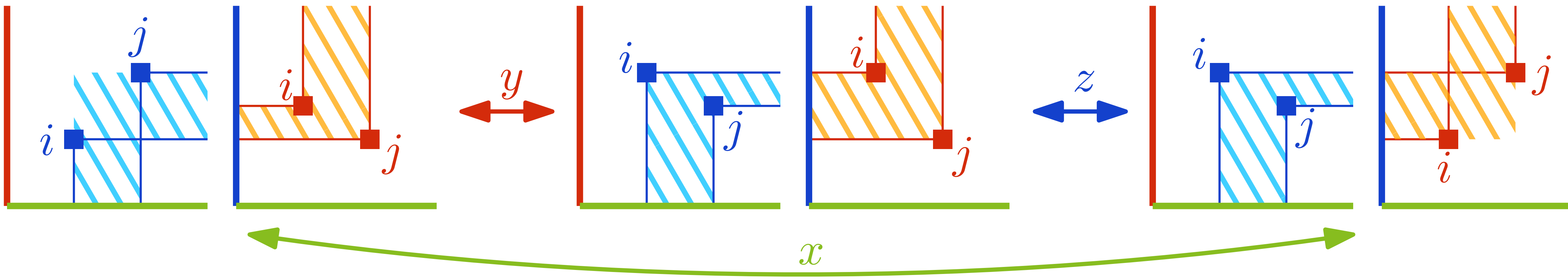
Solitaire game on permutations

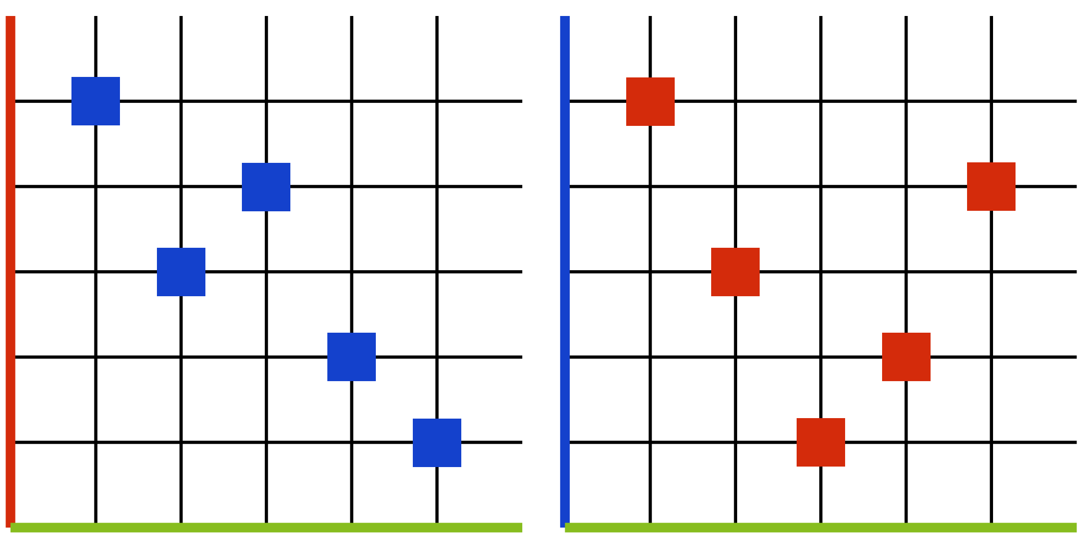


Theorem. [Salo, S. 22] The orbit of the line $[\![0, n-1]\!] \times \{0\}$ is the triangle bases of size n.

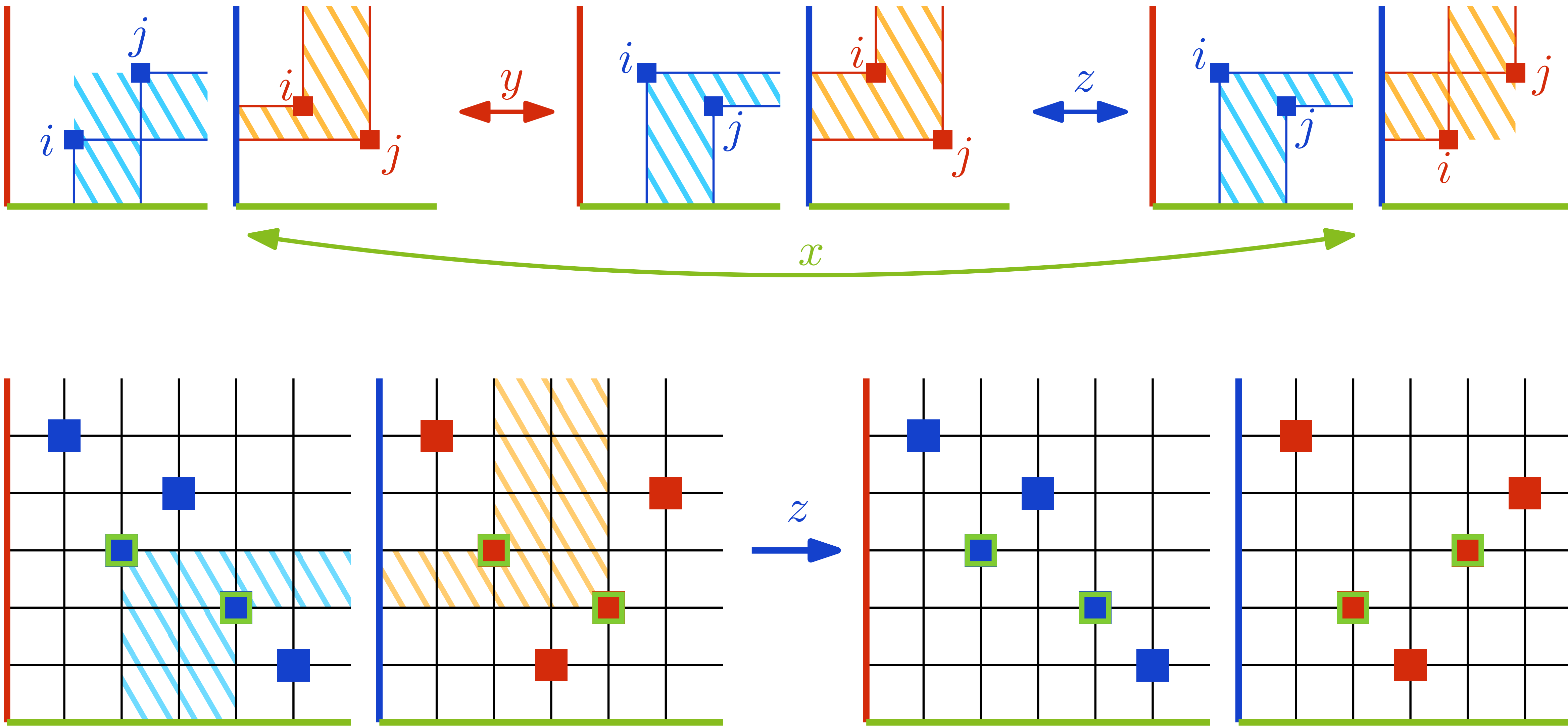
Theorem. The orbit of the line $(\overline{\mathrm{id}}, \mathrm{id})$ is $Av_n((12, 12), (312, 231))$.

Solitaire game on permutations

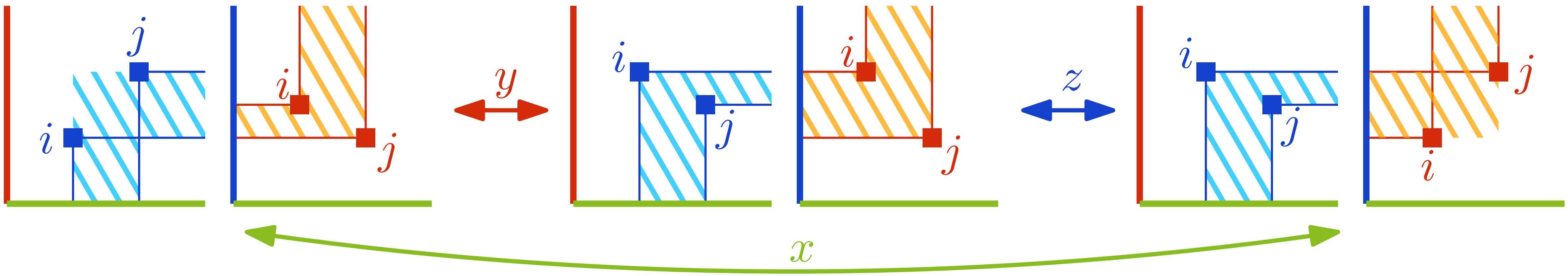


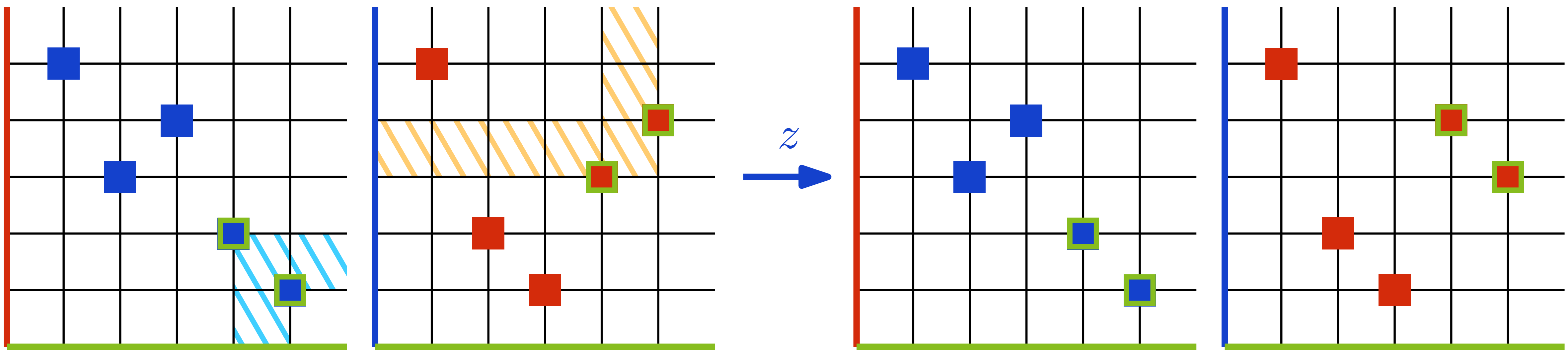


Solitaire game on permutations

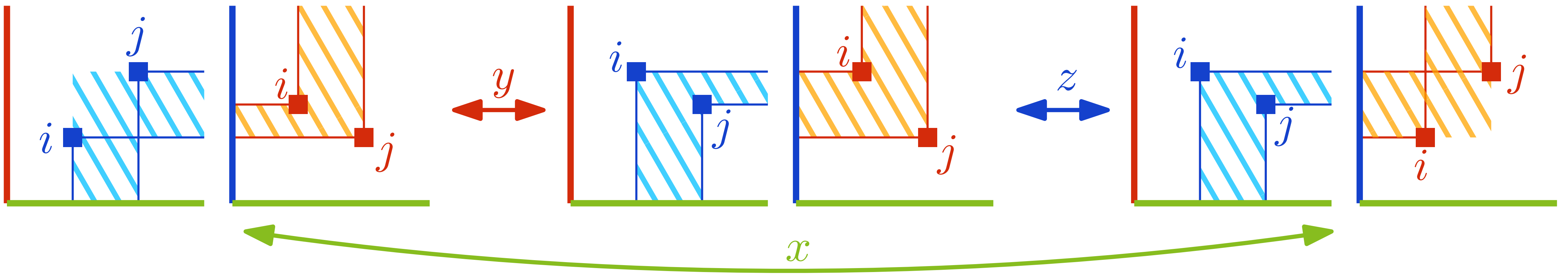


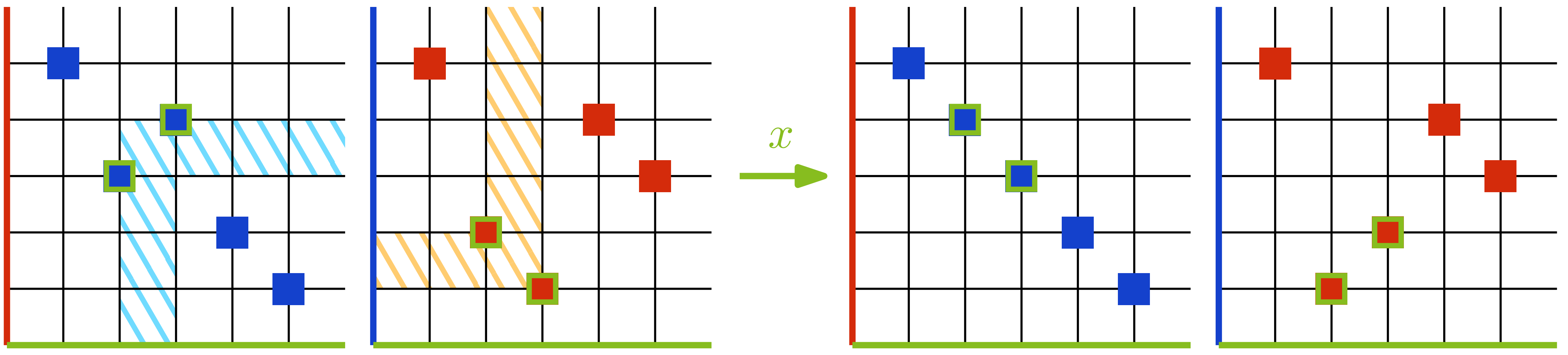
Solitaire game on permutations



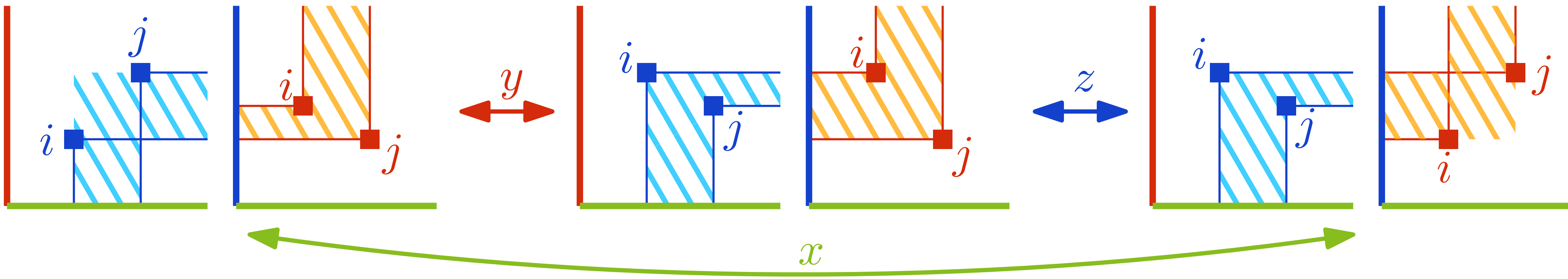


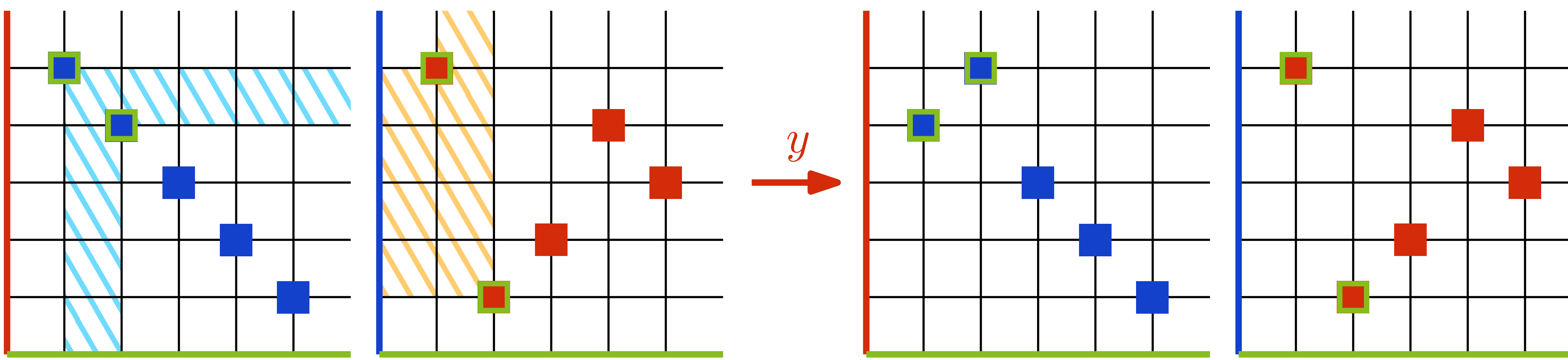
Solitaire game on permutations



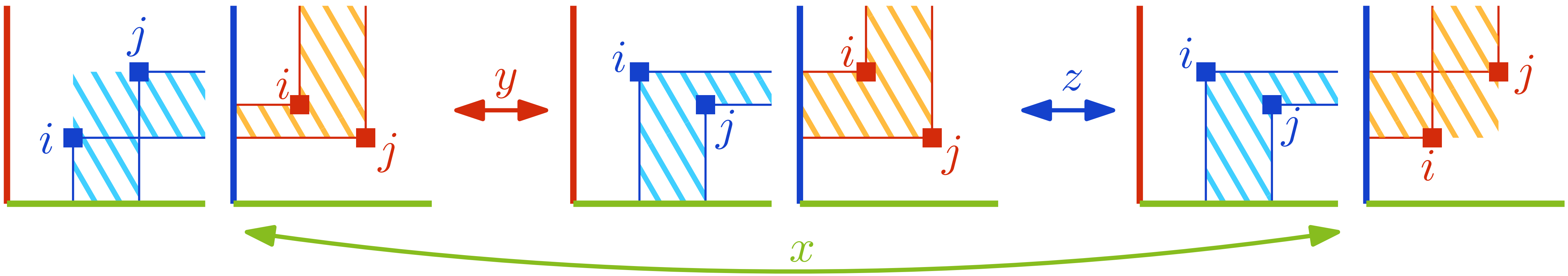


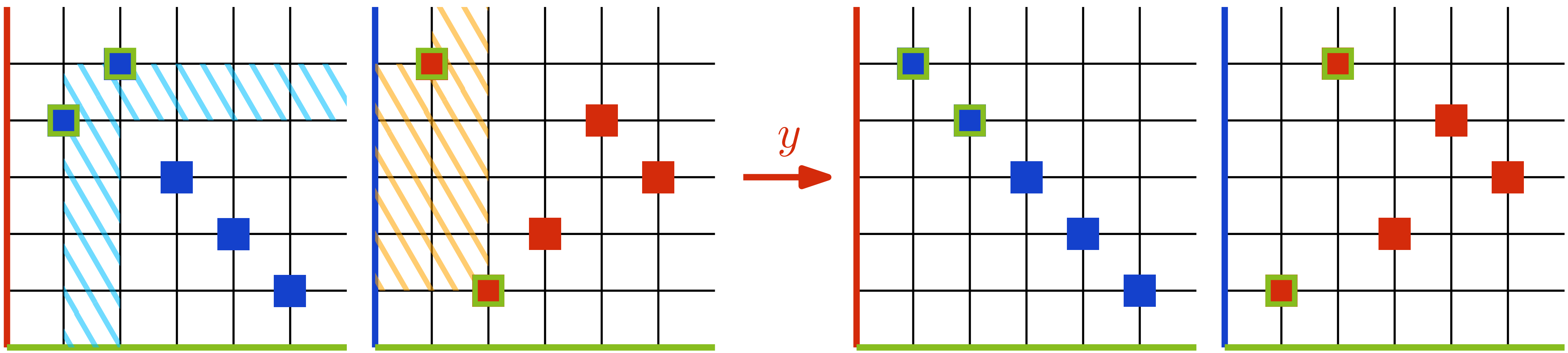
Solitaire game on permutations





Solitaire game on permutations





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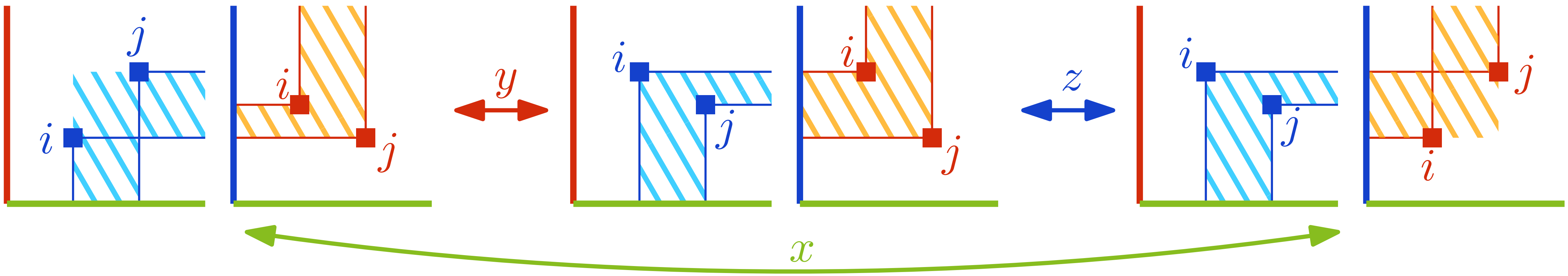
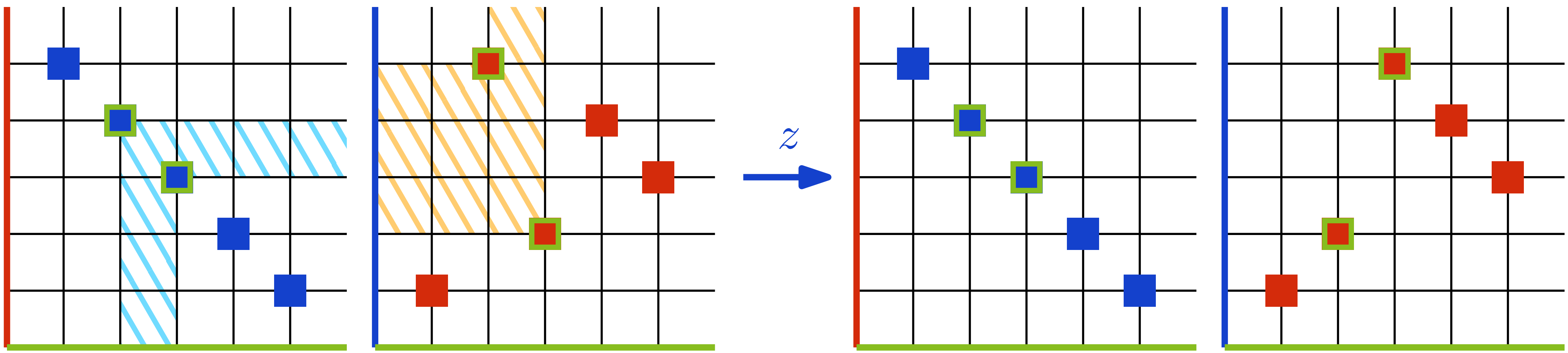
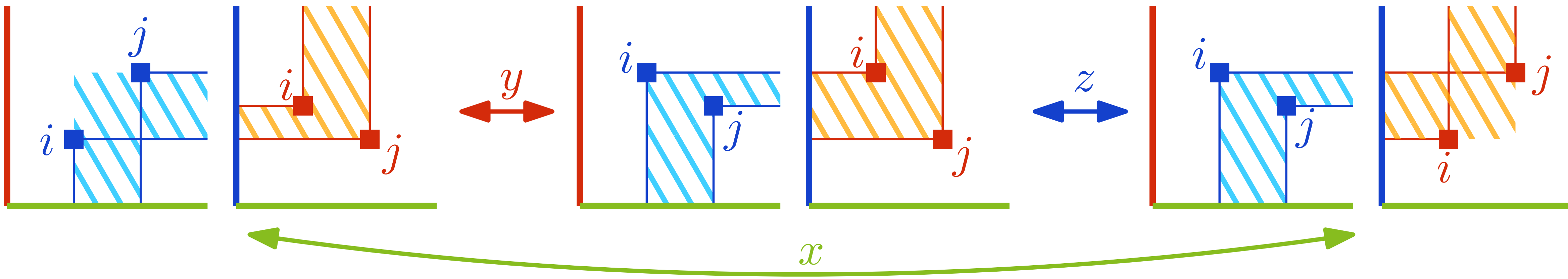
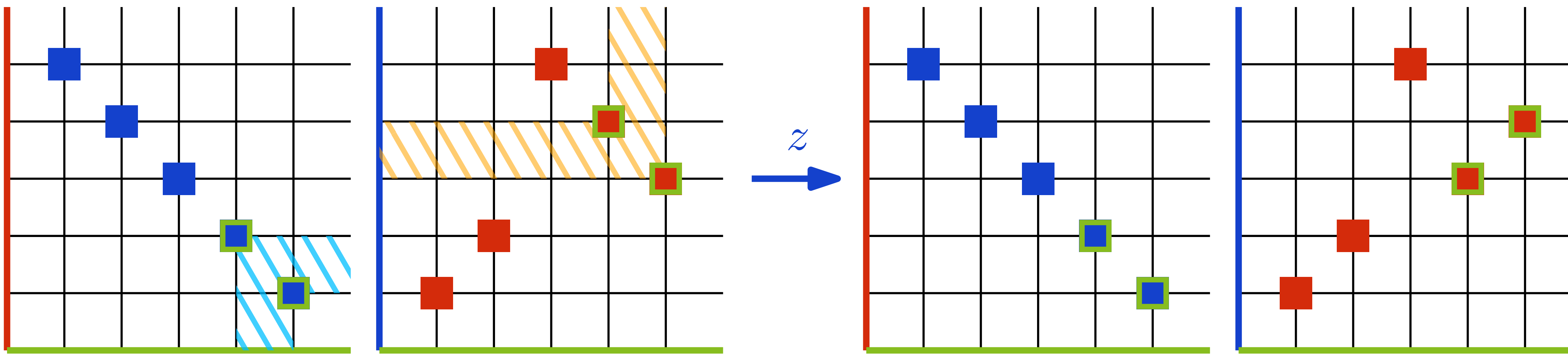


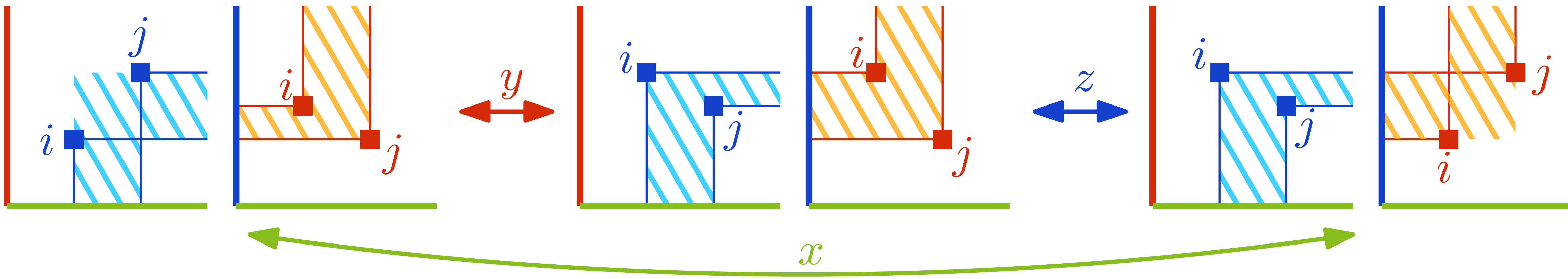


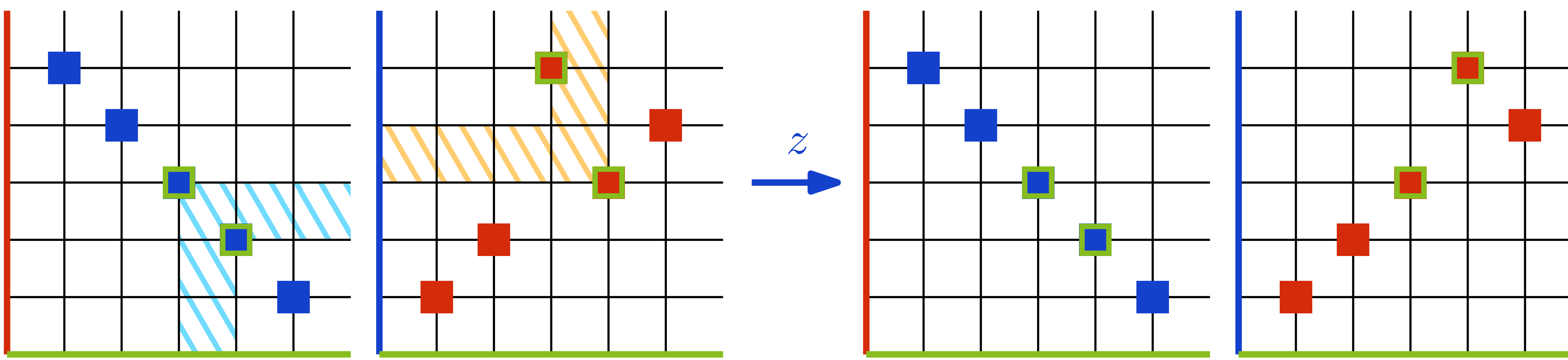
Solitaire game on permutations



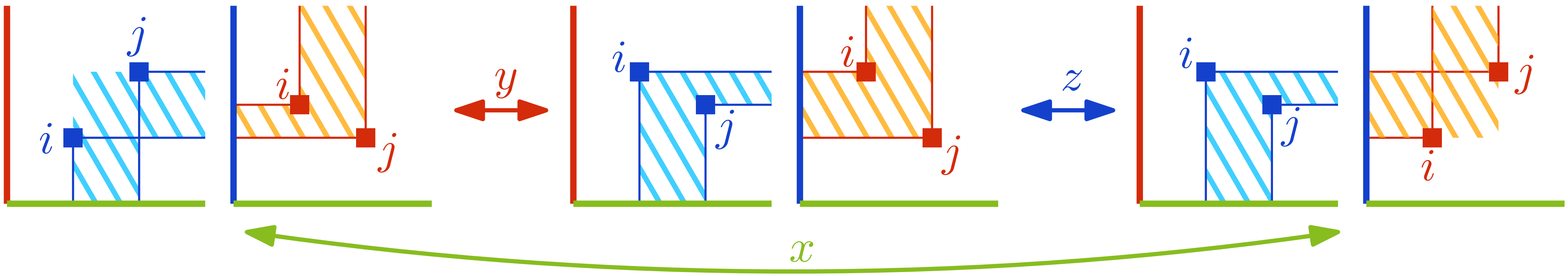


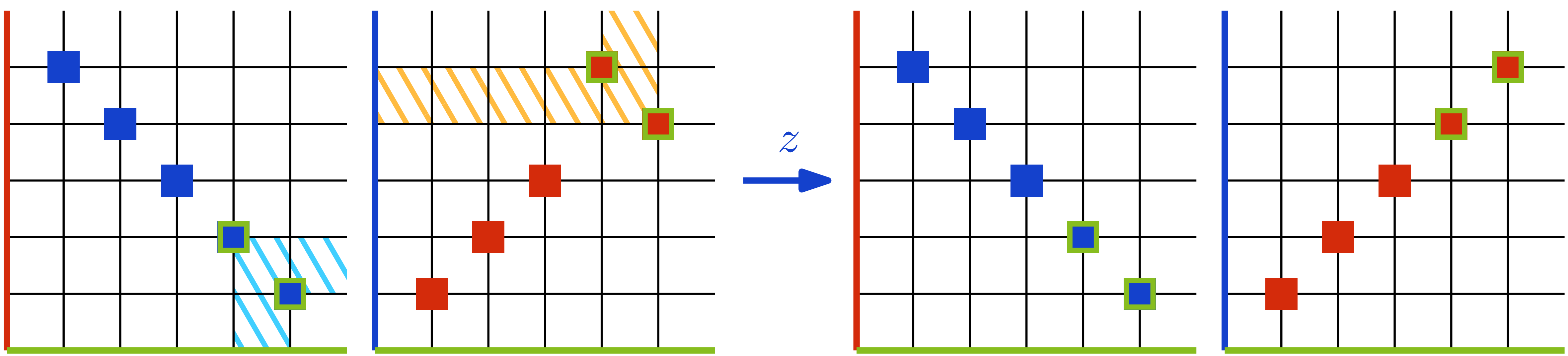
Solitaire game on permutations





Solitaire game on permutations





Uniform sampling ?

Finding one basis is easy but enumerating them is hard $\Rightarrow$ use the solitaire for random sampling !

Question. What is the mixing time of the solitaire ?

Lemma. [Salo, S. 22] The diameter of the line orbit for the solitaire is $\Theta(n^3)$.

What's next ?

- No enumerative result.
  - Best known bounds : $3n! \leqslant |\mathcal{B}_n| \leqslant c\left(\frac{e}{2}\right)^n n^{n-\frac{5}{2}}$ with $c > 0$.

  $|Av_n((12, 12), (312, 231))| \leqslant |Av_n(12, 12)|$ = number of weak Bruhat intervals.

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