Triangle solitaire of independence

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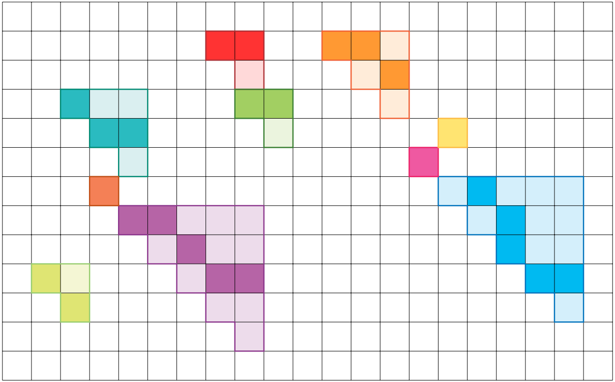
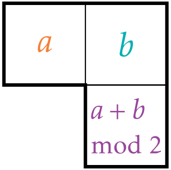


Tillings and independent sets

The triangle solitaire arises form the study of a class of tillings [Cutting Corners, V. Salo, 2020].



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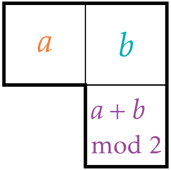

An *independent set* is a set $X \subset \mathbb{Z}^2$ whose content can be chosen freely.

The solitaire moves

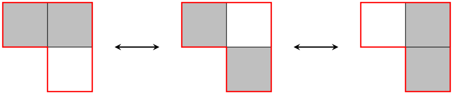

Figure: The action of the triangle shape.

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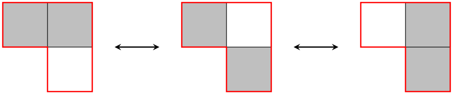

Figure: The action of the triangle shape.

The set of independent sets is stable under the solitaire moves.

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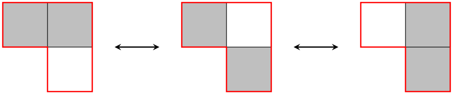

Figure: The action of the triangle shape.

The set of independent sets is stable under the solitaire moves.
The *orbit* of a pattern P, denoted $\gamma(P)$ is the set of patterns reachable from it using the triangle moves.

Questions :

- What are the orbits ? In particular what is the orbit of the line ?
- Can we recognise them easily ?
- What are their sizes ?

The solitaire graph

Consider the graph G_n with vertices the patterns of size n and edges between p and q if there is a solitaire move that changes p into q.

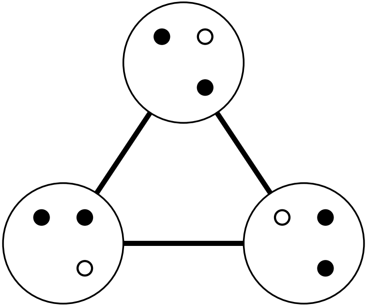

Figure: The Solitaire graph for $n = 2$

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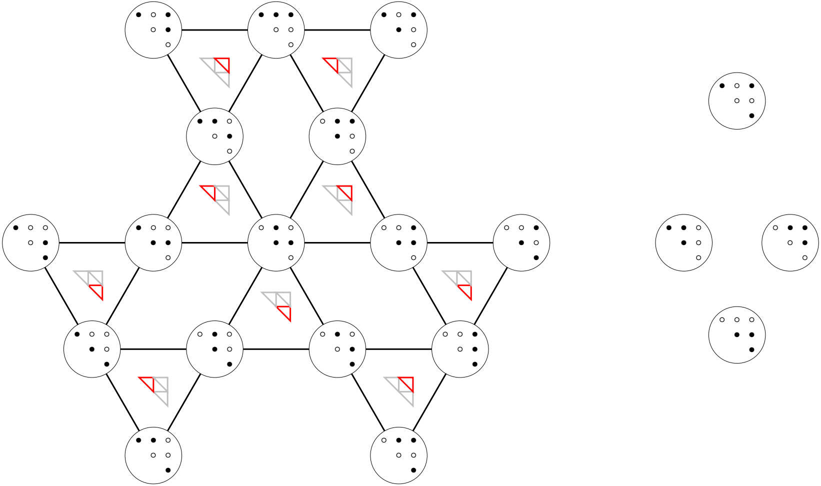


Figure: The Solitaire graph for $n = 3$

The solitaire graph

Consider the graph G_n with vertices the patterns of size n and edges between p and q if there is a solitaire move that changes p into q.

The orbits of a pattern is its connected component in this graph.
Questions :

- What are the connected components of this graph ? In particular what is the connected component of the line ?
- Can we recognise them easily ?
- How are they structured ?

The orbit of the lines

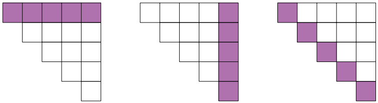
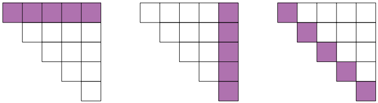


Figure: The lines for $n = 5$

Proposition 1

For every n, the three edges of T_n are in the same orbit.

The orbit of the lines



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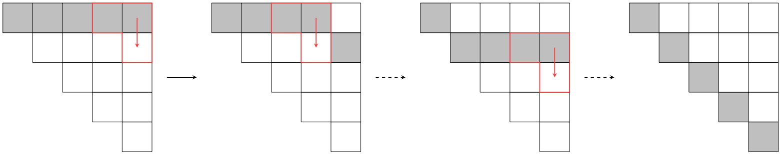

□

The filling process

In a filling step we may complete a triangle that has exactly one point missing.

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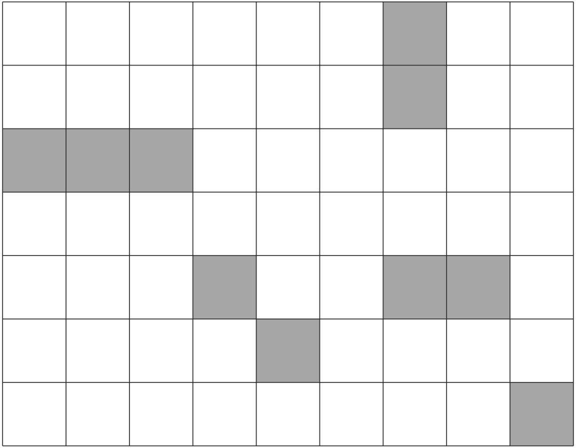

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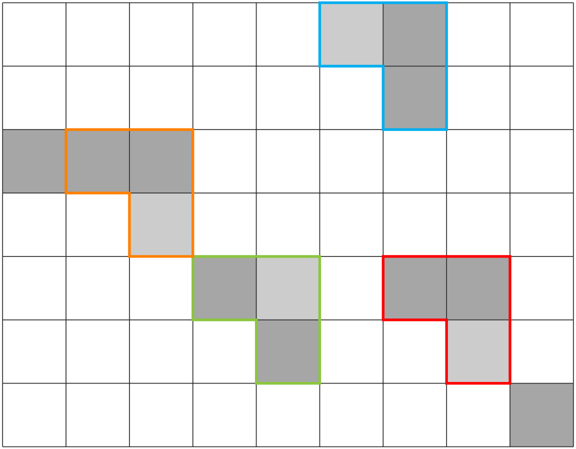

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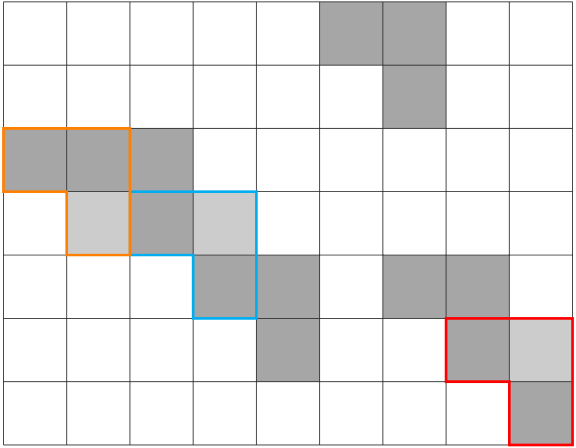

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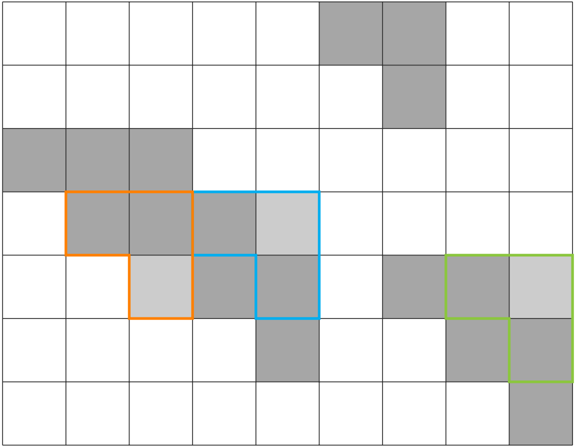

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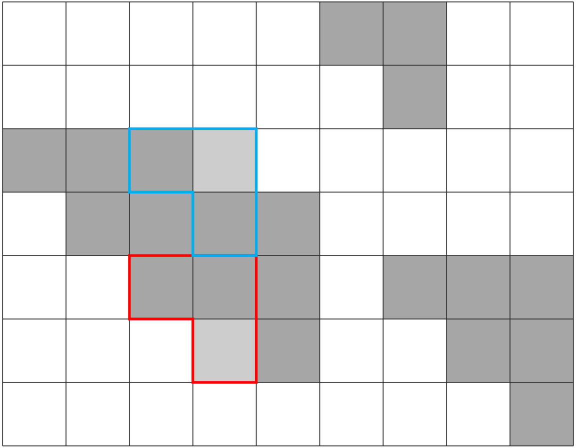

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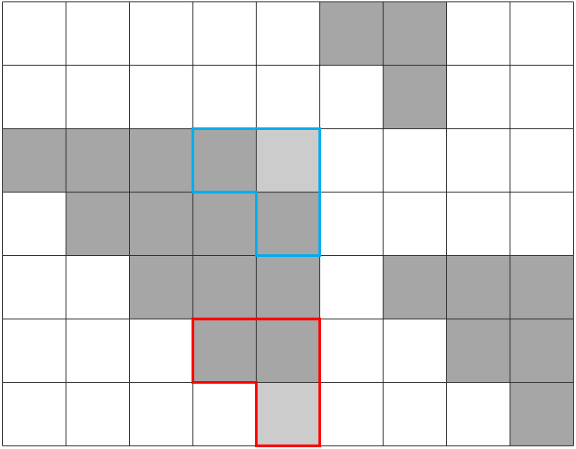

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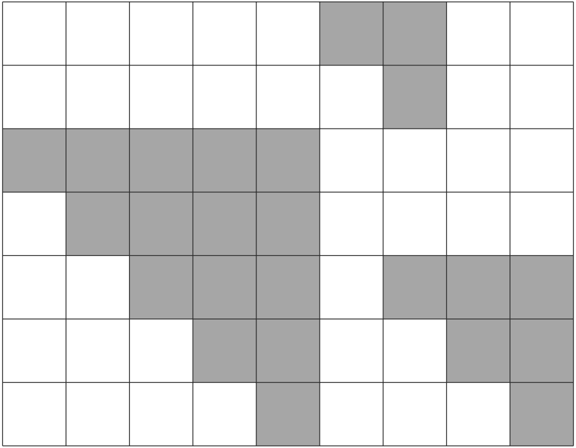


We denote by $\varphi(P)$ its unique the fixed point.

Shape of the filling

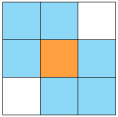


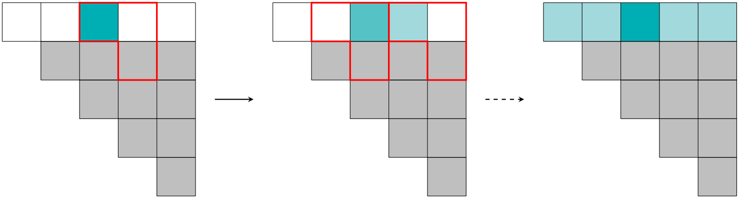
Figure: The orange point touches all the blue points and himself.

Lemma 2

For any pattern P, $\varphi(P)$ is an union of non touching triangle whose sizes sum up to less that $|P|$.

Shape of the filling

Proof.



Shape of the filling

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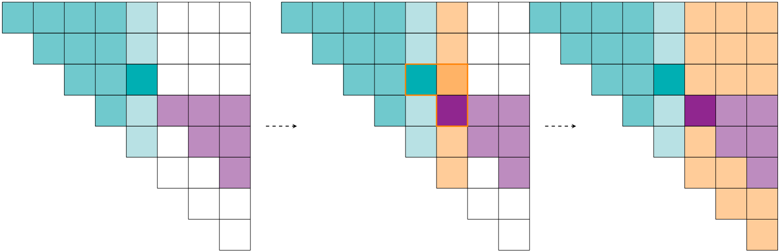

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Invariance property of the filling

Lemma 3

If two patterns P and Q are in the same orbit, then $\varphi(P) = \varphi(Q)$.

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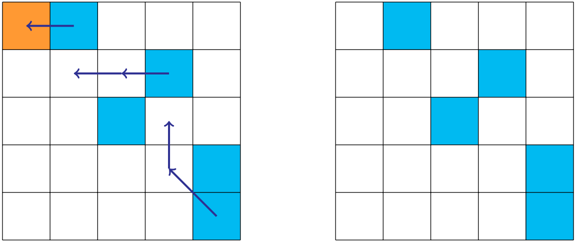

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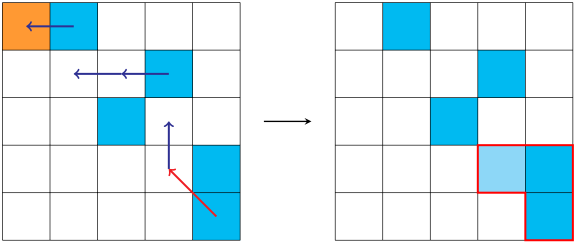

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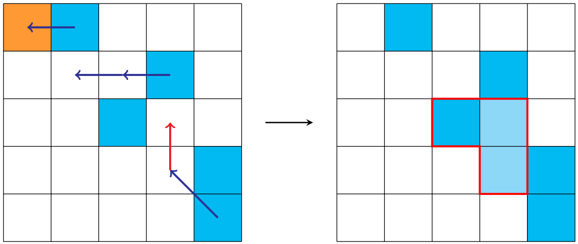

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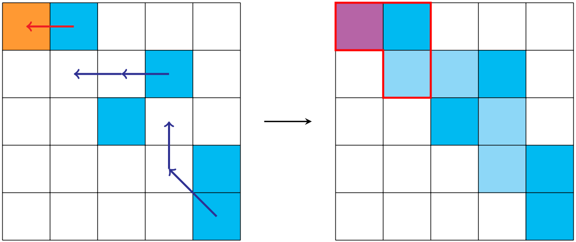

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Excess

The *excess* of P is the difference $e(P) = |P| - \sum_{i=1}^{r} k_i$.

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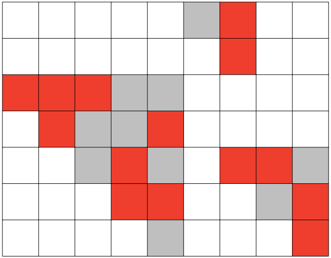

Here, $e(P) = 14 - (5 + 3 + 2) = 4$

Excess

The *excess* of P as the difference $e(P) = |P| - \sum_{i=1}^{r} k_i$.

Lemma 4

If two patterns P and Q are in the same orbit, then $e(P) = e(Q)$.

Excess

The *excess* of P as the difference $e(P) = |P| - \sum_{i=1}^{r} k_i$.

Lemma 4

If two patterns P and Q are in the same orbit, then $e(P) = e(Q)$.

Lemma 5

If Q is a subpattern of P then $e(Q) \leqslant e(P)$.

Characterisation of the line orbit

Theorem 6

A pattern P has no excess if and only if it is in the orbit of the lines that generate the T_{k_i}s.

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Proof.

If $P \in \gamma(L_n)$ then $e(P) = e(L_n) = 0$ according to Lemma 4. □

Characterisation of the other orbits

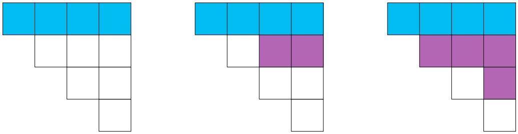

Figure: From left to right: $P_{4,0}$, $P_{4,2}$ and $P_{4,4}$.

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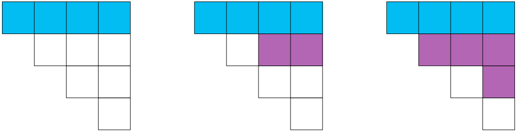

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Theorem 7

If P is a pattern, then $P \in \gamma(P_{n,k})$ if and only if $\varphi(P) = T_n$ and $e(P) = k$.

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A line can still be formed

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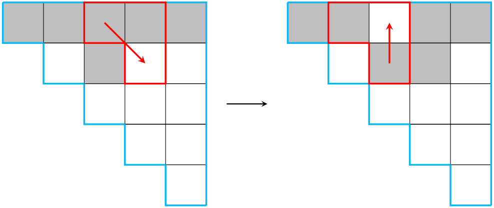

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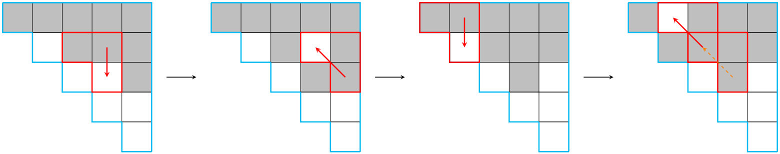

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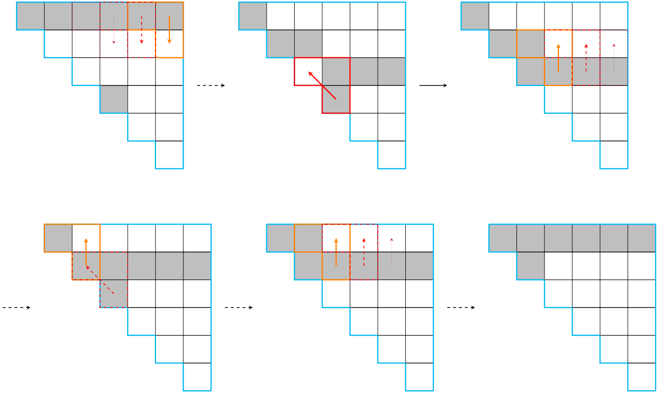

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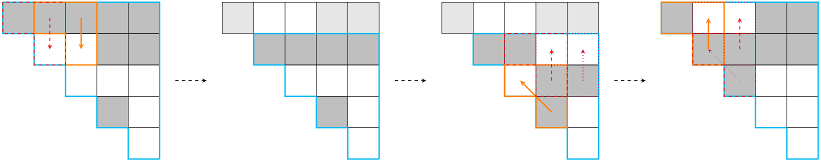

□

Characterisation of the orbits

Theorem 8 (Characterisation of the orbits)

If P is a finite pattern then there are integers $n_1, \dots n_r$ and $k_1, \dots k_r$ and vectors $v_1, \dots v_r$ such that $P \in \gamma(\bigcup_{i=1}^{r} v_i + P_{n_i,k_i})$, the $P_{n_i,k_i} + v_i$ do not touch each other, $\sum_{i=1}^{r} n_i = |P| - e(P)$ and $\sum_{i=1}^{r} k_i = e(P)$.

How to find the canonical form of the pattern

Algorithm 1 (Identify orbit)

Data: pattern P. Result: the canonical representative of the orbit of P.

1. *Fill the pattern.*
2. *Divide the filling into triangles $v_1 + T_{k_1}, \ldots, v_r + T_{k_r}$.*
3. *Count the excess in each triangle, the canonical representative of the orbit of the pattern is $\bigcup_{i=1}^{r} v_i + P_{k_i, e(P \cap (v_i + T_{k_i}))}$.*

How to find the canonical form of the pattern

Algorithm 1 (Identify orbit)

Data: pattern P. Result: the canonical representative of the orbit of P.

1. *Fill the pattern.* $O(|\varphi(P)|) = O(n^2)$
2. *Divide the filling into triangles* $v_1 + T_{k_1}, \ldots, v_r + T_{k_r}$.
   $O(|\varphi(P)|) = O(n^2)$
3. *Count the excess in each triangle, the canonical representative of the orbit of the pattern is* $\bigcup_{i=1}^{r} v_i + P_{k_i, e(P \cap (v_i + T_{k_i}))}$. $O(n)$

The total time complexity of the algorithm is $O(n^2)$.

How to put a pattern in canonical form

Algorithm 2 (Find a path)

1. *Merge the different components and form lines using the process described in Theorem 6.*
2. *Fetch the excess with the process described in Theorem 7.*

How to put a pattern in canonical form

Algorithm 2 (Find a path)

1. *Merge the different components and form lines using the process described in Theorem 6.* $n \cdot O(n^2)$
2. *Fetch the excess with the process described in Theorem 7.* $k \cdot O(n^2)$

The algorithm runs in $O(n^2(n+k))$ time. For $k=0$, this is in fact optimal.

Diameter of the solitaire graph

Theorem 9

The connected component of the line of length n in the graph of the solitaire for the triangle shape has diameter $\Theta(n^3)$.

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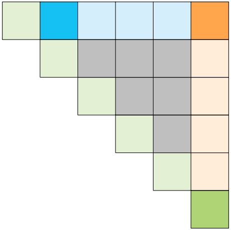

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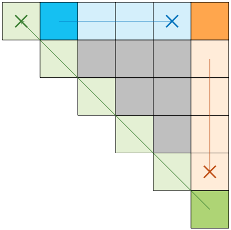


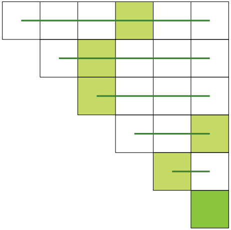
□

Lower bound on the size

The method

1. Choose a corner
2. Choose a point on each line parallel to the edge opposed to the corner

gives an element of the line orbit.



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Lemma 10
$|\gamma(L_n)| \geqslant 3n! - 3.$

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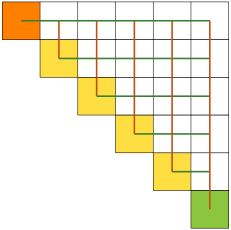

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Upper bound on the size

Proposition 11

If $P \in \gamma(L_n)$ then the number of points in P in the first k columns is at most k.

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If $P \in \gamma(L_n)$ then the number of points in P in the first k columns is at most k.

The number of patterns with this property is equivalent to $c\left(\frac{e}{2}\right)^n (n-1)^{n-\frac{5}{2}}$ with $c \approx 0.086$. [G. Kirchner & V. Kotesovec, OEIS, 2017]

Estimation of the size

Theorem 12

There are constants c_1 and c_2 such that

$$c_1 e^{-n} n^{n+\frac{1}{2}} \leqslant |\gamma(L_n)| \leqslant c_2 \left(\frac{e}{2}\right)^n (n-1)^{n-\frac{5}{2}}.$$

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Conjecture 1

There are two constants $\frac{2}{e} \leqslant c \leqslant e$ and r such that $|\gamma(L_n)| = \Theta\left(\left(\frac{n}{c}\right)^{n+r}\right)$.

Prospects for future work

- What can be said on the structure of the solitaire graph ?
- What can be said about the family of excess sets as a set system ? Can we determine the maximum cardinality of an excess set ? If so, how ?
- Is it easier to decide whether a pattern belongs to the orbit of the line with some additional hypothesis ? (e.g. the pattern is contained in few lines).
- For other convex shapes, similar arguments lead to characterisations of the orbits. But we have no general results yet.
- The solitaire can be played on other groups (F_2, $\mathbb{Z}^3$, ...), do we get similar results there ?

Excess sets

The *excess sets* of P are the subsets $Q \subset P$ such that $\varphi(P \setminus Q) = \varphi(P)$.
Let $E(P)$ be the set of all such sets.

Lemma 9

If $U \in E(P)$ *then* $|U| \leqslant e(P)$.

There is not always a set $U \in E(P)$ such that $|U| = e(P)$.

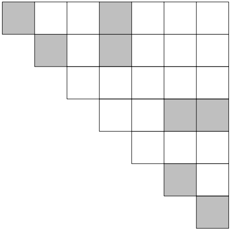

Figure: Here, $e(p) = 1$ but $E(P) = \{\varnothing\}$.

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And maximal excess sets do not all have the same cardinality.