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The Dehn-Seidel twist, C^0 symplectic geometry and barcodes

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Résumé

Nous étudions la géométrie symplectique C^0 au travers de l'action des homéomorphismes sympectiques sur des sous-variétés lagrangiennes. Plus précisément, nous initions l'étude du mapping class group symplectique C^0, i.e. le groupe des classes d'isotopie des homeomorphismes symplectiques, et nous prouvons les premiers résultats concernant la topologie du groupe des homéomorphismes symplectiques. Pour ce faire, nous développons une méthode provenant de la théorie de Floer et de la théorie des codes-barres.

En appliquant cette stratégie au Dehn-Seidel twist, un symplectomorphisme particulièrement intéressant pour l'étude du mapping class group symplectique, nous généralisons à un contexte C^0 un résultat de Seidel concernant la non-trivialité de la classe de ce morphisme dans le mapping class group symplectique. Nous prouvons que le Dehn-Seidel twist n'est pas dans la composante connexe de l'identité dans le groupe des homéomorphismes symplectiques. Ce faisant, nous prouvons la non-trivialité du mapping class group symplectique C^0 de certains domaines de Liouville.

Notre méthode utilise de très récents résultats comme ceux de Abouzaid-Kragh à propos de la nearby Lagrangian conjecture, ainsi que les dernières avancées en matière de topologie symplectique C^0. En particulier, nous adaptons à notre contexte la continuité locale C^0 des codes-barres, prouvée par Buhovsky-Humilière-Seyfaddini et Kislev-Shelukhin.

Mots-clés

Dehn-Seidel twist, Codes-barres, Topologie symplectique C^0, cohomologie de Floer, sous-variétés Lagrangiennes, mapping class group.

The Dehn-Seidel twist, C^0 symplectic geometry and barcodes

Abstract

We study C^0-symplectic geometry through the action of symplectic homeomorphisms on Lagrangian submanifolds. More precisely, we initiate the study of the C^0 symplectic mapping class group, i.e. the group of isotopy classes of symplectic homeomorphisms, and prove the first results regarding the topology of the group of symplectic homeomorphisms. For that purpose, we develop a method coming from Floer theory and barcodes theory.

Applying this strategy to the Dehn-Seidel twist, a symplectomorphism of particular interest when studying the symplectic mapping class group, we generalize to C^0 settings a result of Seidel concerning the non-triviality of the mapping class of this symplectomorphism. We prove that the generalized Dehn twist is not in the connected component of the identity in the group of symplectic homeomorphisms. Doing so, we prove the non-triviality of the C^0 symplectic mapping class group of some Liouville domains.

Our method uses some very recent results such as those of Abouzaid-Kragh related to the nearby Lagrangian conjecture and the last developments of C^0-symplectic topology. In particular, we adapt and generalize to our context the local C^0-continuity of barcodes proved by Buhovsky-Humilière-Seyfaddini and Kislev-Shelukhin.

Keywords

Dehn-Seidel twist, Barcodes, C^0-symplectic topology, Floer cohomology, Lagrangian submanifolds, mapping class group.

Contents

Introduction

Motivation and main results

Let us start with some basic terminology used in symplectic geometry.

A symplectic manifold (M^{2n}, ω) is an even dimensional smooth manifold equipped with a closed non-degenerate 2-form ω. The diffeomorphisms that preserve the 2-form ω are called symplectomorphisms and their group is denoted $\mathrm{Symp}(M, \omega)$. Among these symplectomorphisms, some of them are called Hamiltonian diffeomorphisms and satisfy additional properties. A *Hamiltonian* on M is a time dependent function

$$H : S^1 \times M \to R.$$

We will denote $H_t(x) = H(t, x)$. This Hamiltonian generates a *Hamiltonian vector field* X_{H_t} defined by

$$dH_t = \omega(\cdot, X_{H_t}).$$

The flow ϕ_H^t of this vector field, when defined, is called the *Hamiltonian isotopy generated by H*. A *Hamiltonian diffeomorphism* is a symplectomorphism that can be written as the time 1 on a Hamiltonian isotopy. We denote by $\mathrm{Ham}(M, \omega)$ the group of Hamiltonian diffeomorphisms on (M, ω). Any Hamiltonian diffeomorphism is isotopic to the identity in $\mathrm{Ham}(M, \omega)$.

The set of symplectomorphisms isotopic to the identity in $\mathrm{Symp}(M, \omega)$ is denoted by $\mathrm{Symp}_0(M, \omega)$. The group $\mathrm{Ham}(M, \omega)$ is a normal subgroup of $\mathrm{Symp}_0(M, \omega)$, which is itself a normal subgroup of $\mathrm{Symp}(M, \omega)$.

In a symplectic manifold, some submanifolds are of particular interest. A *Lagrangian submanifold* of a symplectic manifold (M, ω) is a n-dimensional submanifold where the 2-form vanishes; see Section 1.3 for more details. We call a *Lagrangian sphere* in M an embedding $l : S^n \to M$ such that its image is a Lagrangian submanifold. We will also call the image a Lagrangian sphere.

C^0 symplectic topology

C^0 symplectic topology was born with the famous Gromov-Eliashberg theorem [30] stating that given a symplectic manifold (M,ω), if a sequence of symplectomorphisms C^0-converges to a diffeomorphism, then this diffeomorphism is a symplectomorphism as well. We will precisely define what we mean by "C^0-converges" in Section 1.4.

Considering this theorem, *symplectic homeomorphisms* were naturally defined as the C^0-closure of symplectomorphisms.

Definition 1. *Let (M,ω) be a symplectic manifold. A homeomorphism φ of M is called a* symplectic homeomorphism *if it is the uniform limit of a sequence of symplectic diffeomorphisms.*

The main goal in C^0-symplectic topology is then to understand whether it is possible or not to do symplectic topology with continuous objects.

Laudenbach and Sikorav [55] proved an analogue of the Gromov-Eliashberg theorem, but with Lagrangian submanifolds replacing symplectomorphisms.

More than a decade later, C^0-symplectic topology took a step forward, when Oh and Müller [69] introduced a notion of Hamiltonian homeomorphisms, which they called hameomorphisms. These maps have the property of being generated in some sense by continuous Hamiltonians, hence appearing as good C^0 generalizations of Hamiltonian diffeomorphisms. This notion renewed the interest for C^0 symplectic topology. In particular it was realized by Fathi and Oh that hameomorphisms could be used to tackle the old open question of the simplicity of the group of area preserving and compactly supported homeomorphisms of the 2-disc, recently solved in [26]. Viterbo [97] and Buhovski-Seyfaddini [17] established a uniqueness result for the C^0 Hamiltonians involved in the definition of hameomorphisms. Regarding symplectic homeomorphisms, Opshtein [71] proved that they preserve characteristic foliations on hypersurfaces.

More recently, C^0 symplectic topology took a second step forward. Humilière-Leclercq-Seyfaddini proved a result of coisotropic rigidity in [46] and a reduction result in [47], both papers proving that, on many aspects, symplectic homeomorphisms tend to behave as symplectic diffeomorphisms. At the same time, Buhovsky-Opshtein [16] exhibited, among other rigidity results, the first flexibility behaviour for symplectic homeomorphisms: a symplectic homeomorphism leaving invariant a smooth symplectic submanifold V, and whose restriction to V is smooth but not symplectic. It was shortly followed by the counter-example to the Arnold conjecture by Buhovsky-Humilière-Seyfaddini [14], which is another beautiful example of C^0-symplectic flexibility.

In parallel, much progress has been made regarding the barcodes and action selectors which are the main tools used to study these homeomorphisms. The main results concern the C^0-continuity for action selectors, started by Seyfaddini [86] with his ε-shift trick, and followed by Buhovsky-Humilière-Seyfaddini [15]. Seyfaddini [86], Buhovsky-

Humilière-Seyfaddini [15], Kawamoto [48], Shelukhin [87, 88] proved the C^0-continuity of the action selectors in various settings. Using a result of Kislev-Shelukhin [51], this implies the C^0-continuity of barcodes in the same settings. Le Roux-Seyfaddini-Viterbo [57] proved the continuity of barcodes for Hamiltonians on surfaces, without using Kislev-Shelukhin's result.

Note that there exist other aspects of C^0-symplectic topology, such as the C^0-rigidity for the Poisson bracket, initiated by Cardin-Viterbo [18] and Entov-Polterovich [32]. Nevertheless, these will not be discussed here as they cover an entirely different subject.

In dimension 2 things behave in a different way, for instance Matsumoto [61] proved that the Arnold conjecture holds for Hamiltonian homeomorphisms. The tools involved in dimension 2 are not the same as those in symplectic topology but rather the ones usually used in dynamical systems.

Between rigidity and flexibility, C^0-symplectic geometry raises many open questions. Two examples of central questions of the domain are presented below. Note that these questions appear as problems 19 and 41 in McDuff-Salamon [62].

The first example is called the "C^0-flux conjecture" and was formulated by Banyaga [8]:

Question 1. *Is the group* $\mathrm{Ham}(M,\omega)$ *of Hamiltonian diffeomorphisms closed in* $\mathrm{Symp}_0(M,\omega)$ *with respect to the* C^0*-topology?*

This is well-known in dimension 2 [35]. In higher dimensions the conjecture has been established in some cases by Lalonde-McDuff-Polterovich [54] and Buhovsky ([13]). Note that the C^∞ version of this conjecture was proven by Ono [70] in its full generality.

The second open question is the following:

Question 2. *Does the* 4*-sphere* S^4 *admit a* C^0*-symplectic structure, i.e. an atlas whose transition maps are symplectic homeomorphisms?*

For this question to make sense, let us recall that there is no smooth symplectic structure on S^{2n}, for $n \geq 2$. More generally, an open question is whether there exists a smooth manifold that does not admit a smooth symplectic structure but still admits a C^0 one.

This question is part of the motivation for my work in this thesis on the space of symplectic homeomorphisms, as there is no hope to find (or not) such a structure without a very good understanding of these objects.

Dehn twists and mapping class groups

Dehn twists are diffeomorphisms supported in the neighbourhood of a simple loop in surfaces.

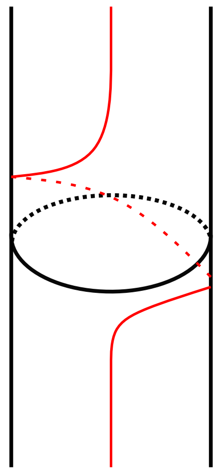

Figure 1 – Dehn Twist in T^*S^1.
The red curve represents the image by the Dehn twist of a fiber of T^*S^1.

Let us first describe the local model. We consider the annulus $S^1 \times [-1, 1] = T_1^* S^1$. We denote $\tau : T_1^* S^1 \to T_1^* S^1$ the map given by

$$\tau(\theta, t) = (\theta + 2\pi f(t), t),$$

where $f : [-1, 1] \to \mathbb{R}^+$ is a smooth function equal to 0 near -1 and and equal to 1 near 1. This map is called a twist map. Now that we have our model, we can describe the Dehn twist for surfaces. It consists of a map which agrees with our local model on the neighbourhood of a given loop l and is equal to the identity away from this loop. It is called the *Dehn twist along* l and it is denoted τ_l. One can prove that the isotopy class of τ_l only depends on the isotopy class of l. If the loop along which the Dehn twist is defined is not contractible, then the Dehn twist is not isotopic to the identity.

The Dehn twists are of particular interest when studying the mapping class group of surfaces. Let us recall that the *mapping class group* is defined, in the case of a smooth oriented manifold M by

$$\mathrm{MCG}(M) = \pi_0(\mathrm{Diff}^+(M)).$$

Let Σ be an oriented smooth surface and denote ω an associated symplectic form on Σ. We denote by $\mathrm{MCG}^\omega(\Sigma)$ the mapping class group for area-preserving diffeomorphisms. This $\mathrm{MCG}^\omega(\Sigma)$ is nothing but $\pi_0(\mathrm{Symp}(\Sigma, \omega))$. Let us also denote by $\mathrm{MCG}(\Sigma, C^0) = \pi_0(\mathrm{Homeo}^+(\Sigma))$ the mapping class group for homeomorphisms and by $\mathrm{MCG}^\omega(\Sigma, C^0) = \pi_0(\mathrm{Homeo}^{+,\omega}(\Sigma))$ the mapping class group for area-preserving homeomorphisms.

One can prove that the mapping class group $\mathrm{MCG}(\Sigma)$ is generated by Dehn twists. We actually have the following isomorphisms:

$$\mathrm{MCG}^\omega(\Sigma) \cong \mathrm{MCG}(\Sigma) \cong \mathrm{MCG}(\Sigma, C^0) \cong \mathrm{MCG}^\omega(\Sigma, C^0). \tag{1}$$

The first isomorphism is a consequence of Moser's trick. The surjectivity of the second one comes from the fact that any homeomorphism is a limit of diffeomorphisms, which is for example proven in [57], together with the fact that the group of homeomorphisms is locally contractible [22]. Its injectivity comes from the local contractibility of the group of diffeomorphisms [56]. Finally, the third isomorphism is due to Fathi [35].

In symplectic geometry, the mapping class group we are interested in is of course related to symplectomorphisms:

$$\mathrm{MCG}^{\omega}(M) = \pi_0(\mathrm{Symp}(M, \omega)).$$

These Dehn twists have been generalized to higher dimensions by Arnold [4] and they have been then intensively studied by Seidel in his PhD thesis [84] and in [80, 81, 82]. We call these higher dimensional maps *generalized Dehn twist*, or *Dehn-Seidel twists*. They are defined in the neighbourhood of a Lagrangian sphere L, and thus will be denoted τ_L. Let us give a brief description of these maps. As in dimension 2, we start by describing a local model in the cotangent bundle of a sphere. We denote

$$T_1^*S^n = \{\xi \in T^*S^n, \quad |\xi| \leq 1\},$$

where $|\cdot|$ denotes the dual of the standard round metric on S^n. In coordinates we have

$$T_1^*S^n = \{(u, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, \ \ |u| \leq 1, |v| = 1, \langle u, v\rangle = 0\},$$

and $\omega_{T_1^*S^n} = \sum_i du_i \wedge dv_i$. We set

$$\sigma_t(u, v) = \left(\cos(2\pi t)u - \sin(2\pi t)v|u|, \cos(2\pi t)v + \sin(2\pi t)\frac{u}{|u|}\right),$$

for $t \in [0, 1]$, and $(u, v) \in T^*S^n \setminus S^n$. When $t = 1/2$, σ corresponds to the antipodal map: $\sigma_{1/2}(u, v) = (-u, -v)$. Note that this antipodal map extends continuously to the zero-section. We choose a cut-off function $\rho : [0, 1] \to \mathbb{R}$ such that ρ is equal to $\frac{1}{2}$ near 0 and equal to 0 near 1. We can now define τ by

$$\tau(\xi) = \sigma_{\rho(|\xi|)}(\xi).$$

This map is a symplectomorphism equal to the antipodal map on the zero-section and equal to the identity near the boundary of $T_1^*S^n$. When $n = 1$, it is isotopic to the model Dehn twist on surfaces described above.

We now want to embed our local model into a symplectic manifold, matching the zero-section with a Lagrangian sphere. Let (M, ω) be a symplectic manifold with boundary, together with a Lagrangian embedding $l : S^n \to M$. Using Weinstein's neighbourhood theorem, we may implant this local model in the neighbourhood of the Lagrangian sphere $l(S^n) = L$. The isotopy class in $\mathrm{Symp}(M, \omega)$ of the resulting map τ_l only depends on l.

This map is called *the generalized Dehn, or Dehn-Seidel twist along l.*

In his PhD thesis [84], Seidel proved that in dimension 4, the square of a Dehn-Seidel twist is isotopic to the identity through smooth diffeomorphisms but is not through symplectomorphisms. He later generalized the last part of this result to higher dimensions using the technology of Lagrangian Floer cohomology in [81].

Using Seidel's notations, we start by describing what an (A_k)-configuration is. Let M be a $2n$-dimensional compact symplectic manifold.

Definition 2. *An (A_k)-configuration in M is a family of Lagrangian spheres $(l_1, ...l_k)$ with images $(L_1, ...L_k)$ such that*

- *they are pairwise transverse*
- *for $2 \leq j \leq k-1$, $|L_i \cap L_j| = 1$ if $i = j \pm 1$ and $|L_i \cap L_j| = \emptyset$ else.*

Before going further, we have to be sure that such configurations really exist. Seidel proved [80] that the affine hypersurface (H, ω) in $\mathbb{C}^{n+1}$ equipped with the standard symplectic form satisfying the equation

$$z_1^2 + z_2^2 + \cdots + z_n^2 = z_{n+1}^{m+1} + \frac{1}{2}$$

contains an (A_m)-configuration of Lagrangian n-spheres. The name comes from the fact that these hypersurfaces are the Milnor fibres of type (A_m)-singularities.

Following Seidel's paper [80], we briefly describe these Lagrangian spheres for $n = 2$. Let us denote $\pi : H \to \mathbb{C}^2$ the projection onto the (z_1, z_2) complex plane and σ the map defined by $\sigma(z_1, z_2, z_3) = (z_1, z_2, e^{2i\pi/(m+1)}z_3)$. The projection is an $(m+1)$-fold covering branched along $C = \{z_1^2 + z_2^2 = \frac{1}{2}, \quad (z_1, z_2) \in \mathbb{C}^2\}$ whose covering group is generated by σ. We now consider the map $f : S^2 \subset \mathbb{R}^3 \to \mathbb{C}^2$ defined by

$$f(x_1, x_2, x_3) = (x_2(1 + ix_1), x_3(1 + ix_1)).$$

For all $x \in S^2$, we have $f(x) \in \mathbb{C}^2 \setminus C$. This map is an immersion with one double point: $f(1, 0, 0) = f(-1, 0, 0)$. Let us denote $\tilde{f} : S^2 \to H$ a lift of f. One can show that $\tilde{f}(S^2)$ and $\sigma \circ \tilde{f}(S^2)$ have only one intersection point, at $\tilde{f}(1, 0, 0)$. In the same way, the family

$$(\tilde{f}(S^2), \sigma\tilde{f}(S^2), ..., \sigma^{m-1}\tilde{f}(S^2))$$

satisfies the intersection conditions of the previous definition. Finally one can choose a 2-form ω_0 on H, diffeomorphic to ω, such that these spheres are Lagrangians, and thus (H, ω_0) admits a (A_m)-configuration. The fact that these two 2-forms are diffeomorphic tells that (H, ω) contains such a configuration as well.

This allows us to have such configurations inside a Liouville domain. These objects were intensively studied by Khovanov-Seidel [50], Seidel-Thomas [85], Seidel [83], Keating [49]...

The theorem of Seidel which interests us in this thesis is the following ([81])

Theorem 3 (Seidel [81])**.** *Let (M^{2n}, ω) be a compact symplectic manifold with contact type boundary, with n even, which satisfies $[\omega] = 0$ and $2c_1(M, \omega) = 0$. Assume that M contains an A_3-configuration (l_∞, l', l) of Lagrangian spheres. Then M contains infinitely many symplectically knotted Lagrangian spheres. More precisely, if one defines $L'^{(k)} = \tau_l^{2k}(L')$ for $k \in \mathbb{Z}$, then all the $L'^{(k)}$ are isotopic as smooth submanifolds of M, but no two of them are isotopic as Lagrangian submanifolds.*

Here, $c_1(M, \omega)$ denotes the first Chern class of the tangent bundle TM. This theorem immediately implies that τ_L^2 is not isotopic to the identity in $\mathrm{Symp}(M, \omega)$. Historically, this is the first higher dimensional result on the symplectic mapping class group.

Remark 4. In Seidel's theorem, it is assumed that n is even. Indeed, for n odd, one can prove that the square of the Dehn-Seidel twist acts non-trivially on homology making the previous result irrelevant. However, in the same paper [81], Seidel also proved an odd-dimensional counterpart of this theorem in which one should consider a composition of non-isotopic Dehn-Seidel twists.

Seidel's result is deeply related to Picard-Lefschetz theory and thus to homological mirror symmetry. Nevertheless, this is an entirely different subject that will not be addressed here. However, many progress have been made on more related topics. For example Evans [33] and Li-Li-Wu [60] showed that the symplectic mapping class group of some specific blow-ups of $\mathbb{CP}^2$ is generated by Dehn-Seidel twists. Khovanov-Seidel [50] and Seidel-Thomas [85] proved that if two Lagrangian spheres intersect transversely at a single point, their associated Dehn twists satisfy a braid relation. This result was generalized in [49] by Keating for more general pairs of Lagrangians. In some specific cases Evans [33] and Wu [99] proved that there is a weak homotopy equivalence between the group of compactly supported symplectomorphisms and a braid group on the disk. Moreover, Dimitroglou-Rizell and Evans [27] constructed from Dehn twists non-contractible families of symplectomorphisms.

As shown by Seidel's result, these questions are closely related to Lagrangian isotopy questions. For instance Coffey [23] showed that under specific conditions, on a 4-dimensional manifold M together with a (very) specific Lagrangian submanifold L, $\mathrm{Symp}(M)$ is homotopy equivalent to the space of Lagrangian embeddings of L.

Dehn-Seidel twist and C^0 symplectic mapping class group

We now turn our attention to the core of this thesis. Inspired by the pioneering work of Seidel on the group of symplectomorphisms, we would like to study the topology of the group $\overline{\mathrm{Symp}}(M,\omega)$ of symplectic homeomorphisms. In particular, we would like to understand the C^0 symplectic mapping class group, i.e. the group $\pi_0(\overline{\mathrm{Symp}}(M,\omega))$.

There is a priori no reason for this group to be non trivial. Indeed, the flexibility results such as the C^0-counter example to the Arnold conjecture ([14]) show that sometimes symplectic homeomorphisms behave very differently than their smooth counter parts. This led Ivan Smith to ask [1] the following question.

Question 3. *Is the square of the Dehn-Seidel twist connected to the identity in* $\overline{\mathrm{Symp}}(M,\omega)$*, where* (M,ω) *is a symplectic manifold as in Seidel's Theorem 3?*

Answering this question would help to understand the relation between the symplectic mapping class group and the C^0 symplectic mapping class group. It would show that the natural map induced by the inclusion

$$\pi_0(\mathrm{Symp}(M,\omega)) \xrightarrow{J} \pi_0(\overline{\mathrm{Symp}}(M,\omega)) \tag{2}$$

is non-trivial. Here, $\overline{\mathrm{Symp}}(M,\omega)$, which denotes the set of symplectic homeomorphisms, is equipped with the C^0-topology, whereas $\mathrm{Symp}(M,\omega)$ is equipped with C^∞-topology.

The main objective of this thesis was to answer Question 3, which we successfully achieved by proving the following theorems.

Theorem A. *Let* (M^{2n},ω) *be a* $2n$*-dimensionnal Liouville domain with* n *even,* $n \geq 4$ *and* $2c_1(M,\omega) = 0$*. Assume that* M *contains an* A_2*-configuration of Lagrangian spheres* (l,l')*.*

Then, τ_l^2 *is not in the connected component of the identity in* $\overline{\mathrm{Symp}}(M,\omega)$*.*

Unlike in Seidel's theorem, we only assume that M contains an A_2-configuration. It was probably known that Seidel's Theorem 3 holds for an A_2-configuration as well, but we were not able to find an appropriate reference. This theorem implies that the group $\pi_0(\overline{\mathrm{Symp}}(M,\omega))$ is not trivial. Of course, an immediate consequence of the previous theorem is the following corollary answering Question 3.

Corollary B. *Under the same assumptions as Theorem A, the map* τ_l^2 *is not isotopic to the identity in* $\overline{\mathrm{Symp}}(M,\omega)$*.*

We have to discuss the relation between Theorem A and its Corollary B. In smooth symplectic geometry, the two results would be equivalent. However, in C^0-symplectic geometry, there is no reason for this equivalence to hold and it is actually related to an important and much harder question that could have been stated along with Question 1 and Question 2. It is the question of the local path-connectedness of $\overline{\mathrm{Ham}}$ or $\overline{\mathrm{Symp}}$, which can be formulated in the following way.

[1] in a private discussion with V. Humilière

Question 4. *Given an arbitrary neighbourhood U of the identity in $\overline{\mathrm{Ham}}(M,\omega)$ or in $\overline{\mathrm{Symp}}(M,\omega)$, is there a neighbourhood V contained in U such that every element in V can be connected to the identity using a path in V?*

Consequently, whether $\overline{\mathrm{Symp}}(M,\omega)$ is locally path-connected in dimension greater or equal to 4 remains an open question. It is unknwon whether the connected component of the identity is equal to the path-connected component of the identity in $\overline{\mathrm{Symp}}(M,\omega)$. It is known that a positive answer to Question 4 would, for example, solve the C^0-flux conjecture of Question 1.

To illustrate the complexity of this question, one could have in mind a counterpart of the nearby Lagrangian conjecture, but for symplectomorphism isotopies instead of Lagrangian isotopies.

The nearby Lagrangian conjecture was proposed by Arnold. It states that given a cotangent bundle T^*L, any closed exact Lagrangian submanifold $L' \subset T^*L$ is Hamiltonian isotopic to the zero-section. This conjecture is exceptionally difficult to prove. However, important progress has been made. It was proved for T^*S^2 by Hind [44] and $T^*\mathbb{T}^2$ by Goodman-Ivrii-Rizell [28]. On general cotangent bundles, a series of works by Fukaya-Seidel-Smith [40], Abouzaid [1], Kragh [52] and Abouzaid-Kragh [2] led to the fact that for any closed exact Lagrangian L', the projection of L' onto the zero section L is a homotopy equivalence. Even if this conjecture is not proven in its full generality, those results have already been used. For instance, in Shelukhin's proof of the Viterbo conjecture [88], it allows him to extend his results to all exact Lagrangian submanifolds.

Moreover, note that Theorem A also implies the following corollary since $\overline{\mathrm{Ham}}(M,\omega)$ is connected (as the closure of a connected space).

Corollary C. *Under the hypothesis of Theorem A, $\tau_{l_0}^2$ does not belong to $\overline{\mathrm{Ham}}(M,\omega)$.*

Denoting $\mathrm{Symp}_c(M,\omega)$ the set of compactly supported symplectomorphisms in (M,ω), the most explicit corollary may be the following one:

Corollary D. *Let L be a n-dimensional manifold with $n > 2$. Then the Dehn-Seidel-twist along L is not in the connected component of the identity in $\overline{Symp_c}(T^*L,\omega)$.*

By Weinstein's neighbourhood theorem (see Section 1.3), the C^∞-counterpart of this corollary was a consequence of Seidel's Theorem 3.

As we will see in Section 6.5, we also have results for $n = 2$ with stronger assumptions on the Lagrangian configuration.

Theorem E. *Let (M^4,ω) be a 4-dimensional Liouville domain, such that $2c_1(M,\omega) = 0$. Assume that M contains an A_3-configuration of Lagrangian spheres (l, l', l_∞).*

Then, τ_l^2 is not isotopic to the identity in $\overline{\mathrm{Symp}}(M,\omega)$.

Let us say a few words on the map J defined by (2). This map is very poorly understood and we have the following open question.

Question 5. *For a general symplectic manifold (M, ω), is the map J injective? Is it surjective?*

Note that Theorem A implies that, at least, this map is non-trivial on some manifolds. Moreover, a positive answer to Question 4 would imply the surjectivity of the map J.

However, in some specific cases, some results exist. As mentioned earlier in (1), we know that, for surfaces, this map is an isomorphism.

The case of the $2n$-ball is also very interesting. Let us denote $\mathrm{Symp}_c(B^{2n}, \omega)$ the group of compactly supported symplectomorphisms of $B^{2n} \subset \mathbb{R}^{2n}$. Using Alexander's trick, i.e. conjugating by $x \mapsto t \cdot x$, one gets that $\overline{\mathrm{Symp}}_c(B^{2n}, \omega)$, the group of compactly supported symplectic homeomorphisms, is contractible. Consequently, we have that $\mathrm{MCG}^\omega(B^{2n}, C^0)$ is trivial and so is the map J. On the other hand, it is not known whether the group $\mathrm{Symp}_c(B^{2n}, \omega)$ is connected, except when $n = 1$ or 2. Indeed, in this case, Gromov showed [43] that this is contractible.

As this example shows, it could well be that the C^0 symplectic mapping class group turns out to be simpler to study in general than the smooth symplectic mapping class group.

Techniques involved

Seidel's proof cannot directly be adapted to symplectic homeomorphisms. Indeed, it is based on Floer homology which only applies to smooth objects. However, we will see that barcodes form a rich enough invariant that can be defined for symplectic homeomorphisms and that offers a good substitute to Floer homology.

Floer homology

Floer Homology was introduced by Floer in [36]. Given a symplectic manifold (M, ω) satisfying good properties and a Hamiltonian H, his idea was to use the action functional $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$ from the Hamiltonian H introduced by Lagrange, to construct a Morse-like complex. Here, $\mathcal{L}M$ denotes the space of contractible loops in M. By Morse-like complex, we mean a complex generated by critical points and the differential given by the trajectories of a pseudo-gradient vector field. This will be explained in Chapter 2.

Floer homology is then defined from a complex whose generators are the critical points of this action functional. The differential is given by counting some perturbed pseudo-holomorphic curves.

The homology obtained this way is called the Floer homology of the Hamiltonian H, and is denoted $HF(H)$. Floer proved the following theorem:

Theorem 5. *For (M,ω) a compact, symplectically aspherical symplectic manifold, and for any Hamiltonian H on M,*

$$HF(H) \cong H_*(M).$$

Part of his motivation was to answer the Arnold conjecture[3]:

Conjecture 6. *A Hamiltonian diffeomorphism of M must have at least as many fixed points as the minimal number of critical points of a smooth function on M.*

Following these ideas, other Floer (co)homologies have been defined. The one we are particularly interested in is the Lagrangian intersection Floer cohomology.

We will be working with exact Lagrangian submanifolds. In an exact symplectic manifold $(M, \omega = d\lambda)$, an *exact Lagrangian submanifold* L is a Lagrangian submanifold such that the restriction $\lambda_{|L}$ of the 1-form λ is exact.

Let L, L' be two closed exact Lagrangian submanifolds in an exact symplectic manifold (M,ω). We assume that their intersections are transverse. The Floer complex is generated by the intersection points $\chi(L, L')$ of the two Lagrangian submanifolds L and L'. One could want to proceed as for Morse homology and find an action functional to compute its differential in order to obtain its gradient vector field and hence the differential. However, even if we can construct this action functional whose critical points correspond to the intersection points, some analytic difficulties make the rest of the construction impossible. To define the differential, we have to count J-holomorphic strips, for a chosen almost complex structure J, between two intersection points, with boundaries on both Lagrangian submanifolds. Of course, for all the objects at stake to be well-defined, some perturbations are required. Once this Floer cohomology is defined, we have an analogue of Theorem 5.

Theorem 7 (Floer [37])**.** *Let (M,ω) be a symplectically aspherical symplectic manifold, together with a closed weakly-exact Lagrangian submanifold L. Then,*

$$HF^*(L, L; \mathbb{Z}/2) \cong H^*(L, \mathbb{Z}/2).$$

Many improvements, for weaker assumptions, have been made since then by Oh [66], Fukaya-Oh-Ohta-Ono [39]...

One of the many interesting properties of this cohomology is its Hamiltonian invariance, i.e. let ϕ be an Hamiltonian diffeomorphism on M, then

$$HF^*(L, L') \cong HF^*(L, \phi(L')).$$

When $L = L'$, we denote $HF(L, H) = HF(L, \varphi_H^1(L))$ and for all $K, H \in \mathrm{Ham}(M,\omega)$, we have $HF(L, H) \cong HF(L, K)$. The invariance property makes this cohomology a great tool to study Hamiltonian diffeomorphisms.

Moreover, the structure of this cohomology is very rich. Indeed, given three closed exact Lagrangian submanifolds L_0, L_1, L_2 in (M, ω), counting pseudo-holomorphic curves between three intersection points, one can define a product structure

$$\mu^2 : HF(L_0, L_1) \otimes HF(L_2, L_0) \to HF(L_2, L_1).$$

This product equips $HF(L, L)$ with a ring structure and the isomorphism of Theorem 7 is a ring isomorphism. Given more Lagrangian submanifolds, we can also define higher products $\mu^k, k \in \mathbb{N}$.

Action selectors and Barcodes

Action selectors were introduced by Viterbo [96] for Lagrangian submanifolds in a cotangent bundle using generating functions theory. After this construction, it was adapted to many contexts by Oh [67], Schwarz [79], Leclercq [58] and others... They contributed to the definition of many useful tools, such as the spectral norm [96], or the study of other ones such as the Hofer norm, defined for Hamiltonian diffeomorphisms [45]. These action selectors are fundamental symplectic invariants and are thus deeply studied. Since these objects will be discussed in much more detail later on, we will give here only a brief overview of their construction and relevant properties.

Given a non-zero homology class $\alpha \in HF(H)$ (respectively a cohomology class in $HF(L, H)$), the associated action selector $l(\alpha, H)$ is the minimal action above (respectively maximal action under) which this class is represented in homology. These action selectors satisfy the following properties

- Finiteness: $l(\alpha, H) < +\infty$,
- Spectrality: $l(\alpha, H)$ is a critical value of the action functional,
- Continuity: $|l(\alpha, H) - l(\alpha, K)| \leq \|H - K\|$, where $\| \cdot \|$ denotes the Hofer norm,
- Triangle inequality: $l(\alpha * \beta, H \sharp K) \leq l(\alpha, H) + l(\beta, K)$, where $*$ denotes the intersection product and $\sharp$ the flow composition (this is for homology, for cohomology, the inequality goes in the other direction and the product is the cup product).

These action selectors have been subject to a lot of works and have been shown to satisfy stronger properties than the above mentioned. For instance, the result relevant for us is that Buhovsky-Humilière-Seyfaddini [15] proved that they are locally C^0-Lipschitz in the Hamiltonian Floer homology case.

Thanks to respectively Theorem 5, one can define the spectral norm γ by

$$\gamma(H) = l([M], H) - l([pt], H).$$

Note that there exists an analogous version of this definition for Lagrangian Floer cohomology denoted by $\gamma(L, H)$.

The above properties easily lead to a certain continuity of this spectral norm.

Barcodes come from a totally different area of mathematics: topological data analysis. A barcode is a collection of intervals (called bars) used to represent certain algebraic structures called persistence modules. They were introduced by Edelsbrunner et al. [29] and, for example, found applications in image recognition with the work of Carlson et al. [20].

The terminology of barcodes was brought into symplectic topology by Polterovich and Shelukhin [75] although germs of this theory were already present in the work of Barannikov [9] and Usher [92, 93]. Indeed, they observed that Floer theories carry natural persistence module structures, coming from the action filtration. See Chapter 4 for details.

The space of barcodes may be equipped with a distance, called the *bottleneck distance*. One can associate a barcode to a Morse function, and this barcode is C^0-continuous with respect to the Morse function. They satisfy many more properties that will be discussed in much more details in Chapters 4 and 5.

Barcodes are of particular interest since they carry the information on the action filtration in Floer (co)homology. Given two exact Lagrangian submanifolds L, L' in a symplectic manifold (M, ω), this filtration is given by the cohomology of the following subcomplexes. For all $\kappa \in \mathbb{R}$, we define

$$CF^{*,\kappa}(L, L') = \mathrm{span}_{\mathbb{Z}/2} \left\{ z \in \chi(L, L'), \quad \mathcal{A}_{L,L'}(z) < \kappa \right\} \subset CF^*(L, L'),$$

where $\chi(L, L')$ denotes the generators of the Floer complex $CF(L, L')$, and $\mathcal{A}_{L,L'}$ the action functional associated to the pair of Lagrangians. When the parameter κ increases, some classes appear while some other ones vanish. The bars of the associated barcode keep track of the levels at which classes appear and disappear.

But maybe the most interesting property of the space of barcodes is that, being equipped with a distance, it has a topology. We will use the existence of this topology on the space of barcodes, together with continuity results to prove our statements.

Some recent continuity results will be extremely useful. The first one comes from a work of Kislev-Shelukhin. In [51], they proved that, in the case of a Lagrangian submanifold together with a Hamiltonian function, the aforementioned barcodes are continuous with respect to the Lagrangian spectral norm $\gamma(L, H)$.

The second result is the one we mentioned before: Buhovsky-Humilière-Seyfaddini [15] proved that action selectors (in the Hamiltonian case) are locally C^0-Lipschitz. This allows to extend these objects and the different spectral invariants to the C^0-closure, i.e. to Hamiltonian homeomorphisms. This provides invariants that will be used to study these objects.

One can put both those results together to get a local C^0-Lipschitz continuity of the barcodes. Adapting these proofs to our context is the point of our following main tool-theorem.

Theorem F. *Let M be a Liouville domain. Let L and L' be two closed exact Lagrangian submanifolds, and assume that $H^1(L', \mathbb{R}) = 0$. The map*

$$\varphi \in \mathrm{Symp}(M, \omega) \mapsto \hat{B}(\varphi(L'), L),$$

where $\hat{B}(L, L')$ denotes the barcodes associated to the exact Lagrangian submanifolds L and L', is continuous and extends continuously to $\overline{\mathrm{Symp}}(M, \omega)$.

Remark 8. This theorem implies that there is a map between the homotopy groups of $\overline{\mathrm{Symp}}(M, \omega)$ and the homotopy groups of the relevant space of barcodes which we strongly hope to use in the future to study the topology of $\overline{\mathrm{Symp}}(M, \omega)$.

Organisation

In the first chapter, we recall basic definitions of symplectic topology and give some notations that will be used in the rest of this thesis.

The second chapter starts with a short presentation of Morse homology, as it will be helpful to understand the following constructions. We then present Lagrangian Floer cohomology of two exact Lagrangian submanifolds.

The third chapter is a presentation of the theory of persistence modules and barcodes as they were introduced in topological data analysis, focusing on the properties we are interested in. We also prove some small topological observations on this set, both completeness and connectivity results.

In the fourth chapter, after having recalled how barcodes are constructed for Morse homology, we propose a definition of barcodes for Lagrangian Floer cohomology. We then prove that the product operations in Floer cohomology respect the filtration. We also present the action selectors for a pair of Lagrangian and define the spectral distance in the case of a pair exact Lagrangians non-necessarily Hamiltonian isotopic, together with some properties. Note that the same definition also appears in Shelukhin's work [88].

We do not claim any original result in Chapters 1-4 but rather produce an adaptation to our context of pre-existing definitions and results.

The fifth chapter is the proof of our main tool-theorem used to get our results on the Dehn twist. We prove a local Lipschitz continuity of the barcodes, with respect to the C^0-norm. The statements presented here are generalizations of similar results obtained by Kislev-Shelukhin in [51] and Buhovsky-Humilière-Seyfaddini in [15]. Using this theorem, we also prove Theorem 5.1.3 and Theorem 5.1.4. The first one associates a continuous path of barcodes to a continuous path in $\overline{\mathrm{Symp}}$ while the second one is a connectivity result.

Finally, in Chapter 6, after a presentation of the Dehn-Seidel twist and Seidel's results, we state and prove our main results, Theorem A and its corollaries, along with their counterparts in dimension 4.

Chapter 1

Preliminaries and notations

All the definitions and proofs of the propositions in this section can be found in McDuff-Salamon's book [62].

1.1 Symplectic geometry

Let M be a smooth oriented manifold endowed with a closed non-degenerate 2-form ω. The pair (M, ω) is said to be a *symplectic manifold.* To be equipped with such ω, M has to be even dimensional. Hence, we will often denote by $2n$ the dimension of M.

The most basic example is the Euclidean space $\mathbb{R}^{2n}$, with coordinates $(x_1, ...x_n, y_1, ..., y_n)$, equipped with the symplectic form

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

A famous theorem of Darboux states that every symplectic form ω on a manifold M is locally diffeomorphic to the standard form ω_0 on $\mathbb{R}^{2n}$.

We are interested in the study of diffeomorphisms preserving the symplectic form, i.e. the diffeomorphisms $\phi \in \mathrm{Diff}(M)$ such that

$$\phi^*\omega = \omega.$$

These diffeomorphisms are called *sympectic diffeomorphisms* or *symplectomorphisms.* The set of symplectomorphisms is a group called the *symplectomorphism group* and is denoted $\mathrm{Symp}(M, \omega)$ or just $\mathrm{Symp}(M)$ when no confusion is possible.

Definition 1.1.1. *We call an* compatible almost complex structure *on M a map $J : TM \to TM$ such that $g_J = \omega(\cdot, J\cdot)$ is a Riemannian metric and $\omega(J\cdot, J\cdot) = \omega$. A triple (ω, J, g) with these properties is called a* Hermitian structure *on M.*

One can prove that the set of compatible almost complex structures on M is non empty and contractible.

As a consequence, the first Chern class of the complex tangent bundle (TM, J) does not depend on the complex structure, and hence will be denoted $c_1(TM)$ or $c_1(M, \omega)$. We will be interested in *symplectically aspherical* symplectic manifolds, i.e. symplectic manifolds such that

$$\langle 2c_1(TM), \pi_2(M)\rangle = 0 \text{ and } \langle \omega, \pi_2(M)\rangle = 0.$$

A symplectic manifold (M, ω) is said to be *exact* if ω is exact, i.e. if there is a 1-form λ such that $\omega = d\lambda$. This 1-form λ is called a *Liouville form*, to which we can associate a vector field X_λ defined by

$$i_{X_\lambda}\omega = \lambda.$$

This vector field X_λ is called the *Liouville vector field* and its flow $\phi^t_{\mathcal{L}}$ satisfies

$$(\phi^t_{\mathcal{L}})^*\omega = e^t\omega.$$

A symplectomorphism φ on $(M, \omega = d\lambda)$ is said to be *exact* (with respect to λ) if $\varphi^*\lambda - \lambda$ is exact.

In this thesis, we will be interested in particular cases of exact symplectic manifolds called *Liouville domains*. Before giving the definition of such manifolds, we have to say a few words on contact manifolds. Contact manifolds can be seen as the odd-dimensional counterparts of symplectic manifolds.

A *contact manifold* is a pair (V, ξ) where V^{2n-1} is an odd dimensional oriented manifold, and ξ is a *contact structure*. A contact structure is a maximally non-integrable hyperplane distribution. For a contact structure ξ, there exists a *contact form* α such that $\ker\alpha = \xi$ and $\alpha \wedge d\alpha^{n-1} > 0$. Note that this contact form is not unique.

A *symplectic manifold with contact type boundary* is a symplectic manifold (M, ω), with $\partial M \neq \emptyset$ which admits a Liouville vector field in a neighbourhood of ∂M which is transverse and pointing outwards along ∂M. The existence of this Liouville vector field implies that ω is exact near the boundary equal, to $d\lambda$. Thus ∂M is a contact manifold for a contact form α obtained as the pull-back of λ to ∂M by the inclusion.

We can now define a Liouville domain.

Definition 1.1.2. *A* Liouville domain *is an exact symplectic manifold* $(M, \omega = d\lambda)$ *with contact type boundary such that the Liouville vector field* X_λ*, defined by* $i_{X_\lambda}d\lambda = \lambda$*, is transverse to the boundary and points outwards.*

The pull-back α of λ on the boundary is a contact form.

For such a manifold, there exists an embedding $\psi : (-\varepsilon, 0] \times \partial M \to M$ such that $\psi(0, \cdot)$ is the identity on ∂M, and $\psi^*\omega = d(e^t\alpha)$. Here, t denotes the $(-\varepsilon, 0]$-coordinate. We can now define the *completion* $(\hat{M}, \hat{\omega})$ of the Liouville domain (M, ω) as

$$\hat{M} = M \cup [0, +\infty) \times \partial M, \tag{1.1}$$

with the symplectic form $d(e^t\alpha)$ as the symplectic form on the cylindrical part. We in fact obtain this completion by simply gluing on the boudary ∂M the cylinder $[0, +\infty) \times \partial M$ equipped with the symplectic form $d(e^t\alpha)$ so that it extends the collar $\mathrm{Im}(\psi)$.

Remark 1.1.3. For the proofs involved in this thesis, we do not need to require that the symplectomorphisms defined on the Liouville domain M are equal to the identity on the boundary ∂M.

1.2 Hamiltonian formalism

Let us recall and detail the preliminaries mentioned in the introduction.

Let (M, ω) be a symplectic manifold. We call a *Hamiltonian function* (or just a *Hamiltonian*), a smooth function

$$H : S^1 \times M \to \mathbb{R}.$$

The S^1 coordinate refers to the time-dependence of the Hamiltonian. We will often denote $H_t(x) = H(t, x)$, where t is the S^1 coordinate and x the M coordinate. A Hamiltonian generates a time-dependent vector field X_{H_t} by the formula

$$dH_t = \omega(\cdot, X_{H_t}).$$

A time-dependent vector field that can be expressed as the vector field generated by a Hamiltonian is called a *Hamiltonian vector field.* When it is defined, the flow ϕ_H^t of this vector field is called the *Hamiltonian isotopy generated by* H. A symplectomorphism φ is called a *Hamiltonian diffeomorphism* if it is the time-1 of a Hamiltonian isotopy ϕ_H^t, i.e. there exist a Hamiltonian H such that $\varphi = \phi_H^1$.

Remark 1.2.1. For this flow to be well defined on a Liouville domain M, we require the Hamiltonian to be constant on the boundary ∂M. This implies that the vector field X_{H_t} is tangent to the boundary and thus ensures that the flow ϕ_H^t is well-defined.

We say that φ^t is a Hamiltonian isotopy supported in $U \subset M$ if it is generated by a Hamiltonian H supported in U, i.e. for all $t \in S^1$, H_t is supported in U.

The set of Hamiltonian diffeomorphisms on (M, ω) is a group called the *Hamiltonian group* and is denoted $\mathrm{Ham}(M, \omega)$ or just $\mathrm{Ham}(M)$ when there is no possible confusion. Let us give the following properties which prove that this is indeed a group. These notations will be useful later.

Let H and K be two Hamiltonian functions on a symplectic manifold (M,ω). The flow $\phi^t_{\bar{H}} = (\phi^t_H)^{-1}$ is generated by the Hamiltonian

$$\bar{H}(t,x) = -H(t,\phi^t_H(x)).$$

The flow $\phi^t_{H\sharp K} = \phi^t_H \circ \phi^t_K$ is generated by the Hamiltonian

$$(H\sharp K)(t,x) = H(t,x) + K(t,(\phi^t_H)^{-1}(x)).$$

The Hamiltonian group can be equipped with a norm, called the *Hofer norm*. We define for a Hamiltonian H its Hofer norm $\|H\|$ by

$$\|H\| = \int_0^1 (\max H_t - \min H_t)dt.$$

This corresponds to the so-called the *Hofer length* of the isotopy $(\phi^t_H)_{t\in[0,1]}$ generated by H. Consequently, given a Hamiltonian diffeomorphism φ, we define the *Hofer norm of* φ as

$$\|\varphi\| = \inf\{\|H\|, \quad \phi^1_H = \varphi\}.$$

It was proven by [45, 74, 53] that the Hofer norm is non-degenerate. It therefore allows us to naturally define the *Hofer distance between two Hamiltonian diffeomorphisms* φ and ψ as

$$d_{\text{Hofer}}(\varphi,\psi) = \|\psi \circ \varphi^{-1}\|.$$

1.3 Lagrangian submanifolds

The notion of Lagrangian submanifold is a fundamental notion in symplectic geometry and its study is a central topic in symplectic topology. To define these objects, we have to start by the linear case.

Let (V,ω) be a symplectic vector space and let W be a linear subspace. We define the *symplectic orthogonal* of W in V as

$$W^\omega = \{u \in V| \quad \omega(u,v) = 0 \quad \forall v \in W\}.$$

The subspace W is called

- isotropic if $W \subset W^\omega$,
- coisotropic if $W^\omega \subset W$,
- symplectic if $W \cap W^\omega = \{0\}$,
- Lagrangian if $W = W^\omega$.

Note than in a $2n$-dimensional vector space, a Lagrangian subspace is n-dimensional. We will denote by $\Lambda(n)$ the *Lagrangian Grassmanian* in $\mathbb{R}^{2n}$, i.e. the manifold consisting of all Lagrangian subspaces.

We can now define *Lagrangian submanifolds* in a symplectic manifold.

Definition 1.3.1. *Let (M^{2n}, ω) be a symplectic manifold, together with $W \subset M$ a smooth submanifold. The submanifold W is called a* Lagrangian *(or respectively isotropic or coisotropic) submanifold if and only if, for every $q \in W$, the subspace T_qW of T_qM is Lagrangian (respectively isotropic or coisotropic).*

A Lagrangian submanifold is n-dimensional.

The most important examples of Lagrangian submanifold for us are the following.

Example 1.3.2. *Let L be a smooth manifold. Its cotangent bundle T^*L admits an exact symplectic 2-form $\omega = d\lambda$ locally given by*

$$\lambda = -pdq \text{ and } \omega = dq \wedge dp,$$

where q is a local coordinate on L and p the dual coordinate on the fibre T_q^*L. Then, the zero section $L \subset T^*L$ is a Lagrangian submanifold.

Moreover, given a closed 1-form β on L, its graph $G_\beta \subset T^*L$ is a Lagrangian submanifold.

The following theorem, called Weinstein's neighbourhood theorem [98] is a perfect illustration of the importance of the previous example.

Theorem 1.3.3. *Let (M, ω) be a symplectic manifold and $L \subset M$ a Lagrangian submanifold. Then there exists a neighbourhood $V \subset T^*L$ of the zero section and a neighbourhood $W(L) \subset M$ of L together with a diffeomorphism $\psi : V \to W(L)$ such that*

$$\psi^*\omega = d\lambda \text{ and } \psi_{|L} = Id,$$

where λ is the canonical 1-form on T^*L.

Some Lagrangian submanifolds satisfy properties that make them easier to work with. This is the case of the so-called *exact Lagrangian submanifolds*, which are of particular interest for our study.

Definition 1.3.4. *Let $(M, \omega = d\lambda)$ be an exact symplectic manifold. A Lagrangian submanifold L is said to be* exact *if the restriction of λ to L is exact, i.e. there exists a function $f : L \to \mathbb{R}$ such that*

$$\lambda_{|L} = df.$$

Remark 1.3.5. Let L be an exact Lagrangian submanifold in $(M, d\lambda)$ together with a function f such that $\lambda_{|L} = df$. Then for all $c \in \mathbb{R}$, the function f_c defined by $f_c = f + c$ also satisfies $\lambda_{|L} = df_c$.

Exact Lagrangian submanifolds are closely related to Hamiltonian transformations. Indeed, let φ be a Hamiltonian diffeomorphism in (T^*L, λ). Then $\varphi(L) \subset T^*L$ is an exact Lagrangian submanifold, where L denotes the zero-section.

More generally, if we are working in an exact symplectic manifold $(M, d\lambda)$ and the path $(L_t)_{t\in[0,1]}$ is a smooth path of exact Lagrangian submanifolds, there is a smooth path $(\phi_t)_{t\in[0,1]}$ in $\mathrm{Ham}(M, d\lambda)$ such that

$$\forall t \in [0,1], \quad \phi_t(L_0) = L_t. \tag{1.2}$$

1.4 Motivations for C^0 symplectic geometry

The interest for C^0-symplectic geometry began with the famous Gromov-Eliashberg theorem. This theorem was in fact proved by Eliashberg [30] building on previous work from Gromov.

Theorem 1.4.1. *Let (M, ω) be a symplectic manifold together with a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of symplectomorphism. Assume that this sequence C^0-converges to a diffeomorphism φ. Then φ is a symplectomorphism.*

Before discussing the consequences of this famous theorem, we have to explain what we mean by "C^0-converges". It is the uniform convergence on compact sets. Let us choose a Riemannian metric on M and denote d its induced distance. For any compact subset K in M, and two homeomorphisms $\varphi, \psi : M \to M$, we denote

$$d_K(\varphi, \psi) = \max \left\{ \sup_{p\in K} d(\varphi(p), \psi(p)), \quad \sup_{p\in K} d(\varphi^{-1}(p), \psi^{-1}(p)) \right\}. \tag{1.3}$$

Consequently, we say that $(\varphi_n)_{n\in\mathbb{N}}$ C^0-converges to φ in M if and only if, for all compact K in M, $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ for the distance d_K. Note that the C^0-convergence is a notion that does not depend on the choice of the Riemannian metric.

One could choose to define d_K only by

$$d_K(\varphi, \psi) = \sup_{p\in K} d(\varphi(p), \psi(p)).$$

However, in that case, it is possible for a sequence of homeomorphisms to converge to a map which is not a homeomorphism, which we would like to avoid. With our choice, the following lemma holds.

Lemma 1.4.2. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of symplectomorphisms which C^0-converges to a map φ. Then φ is a homeomorphism and $(\varphi_n^{-1})_{n\in\mathbb{N}}$ C^0-converges to φ.*

Gromov-Eliashberg's theorem is the first example of C^0 symplectic rigidity. Note that this fact can be surprising as the condition for a diffeomorphism to be symplectic is a C^1 condition, and there is no assumption concerning a convergence of the differential in the theorem. This theorem is the birth of C^0 symplectic topology as the study of objects defined as C^0-limits of their smooth symplectic counterparts. The first consequence of this theorem is the following natural definition.

Definition 1.4.3. *Let (M, ω) be a symplectic manifold. We say that φ is a* symplectic homeomorphism *of M if there is a sequence of symplectomorphisms $(\varphi_n)_{n\in\mathbb{N}}$ of M such that φ is the C^0-limit of $(\varphi_n)_{n\in\mathbb{N}}$.*

The set of symplectic homeomorphisms of (M, ω) is denoted $\overline{\mathrm{Symp}}(M, \omega)$. By Gromov-Eliashberg theorem, we have

$$\mathrm{Diff}(M) \cap \overline{\mathrm{Symp}}(M, \omega) = \mathrm{Symp}(M, \omega).$$

Remark 1.4.4. In the special case of dimension 2, it is proven in [57] that for every compact symplectic surface, the area and orientation preserving homeomorphisms are exactly the symplectic homeomorphisms:

$$\overline{\mathrm{Symp}}(M, \omega) = \mathrm{Homeo}^{+,\omega}(M).$$

The statement in that paper is actually more precise: denoting G_0 the identity component of a topological group G, we have

$$\overline{\mathrm{Symp}}(M, \omega)_0(M, \omega) = \mathrm{Homeo}_0^{+,\omega}(M).$$

In this thesis, we are mainly interested in Lagrangian submanifolds. Hence, a natural question to ask is what happens to these submanifolds when taking their image by a symplectic homeomorphism? A Gromov-Eliashberg -like rigidity theorem was proven by Humilière-Leclercq-Seyfaddini [46] (see also Laudenbach-Sikorav [55]).

Theorem 1.4.5. *Let L be a Langrangian submanifold in a symplectic manifold (M, ω), together with a symplectic homeomorphism φ. If $\varphi(L)$ is a smooth submanifold, then $\varphi(L)$ is a Lagrangian submanifold.*

This theorem is part of the motivation for the upcoming discussions. Indeed, if symplectic homeomorphisms preserve the Lagrangian property of smooth submanifolds, it is legitimate to ask whether they preserve other topological properties or not. In a more general way, the main question is to what extent do symplectic homeomorphisms behave as symplectic diffeomorphisms.

Chapter 2

Different homologies

2.1 Morse homology

In this section, we will briefly sketch the definition of the Morse homology, as many of the objects presented in this thesis can be interpreted as a generalization of Morse theory. It will also help to present and understand these objects. This theory was developed by Morse [65], Thom [91], Smale [90], Milnor [63] . We follow Audin-Damian's work [6] for this section.

For all this section, M will be a n-dimensional compact manifold.

Let us recall that a function $M \to \mathbb{R}$ is said to be *Morse* if and only if all its critical points are non-degenerate, i.e.

$$\forall x \in M, df(x) = 0 \Rightarrow d^2 f_x \text{ non degenerate.}$$

The functions at stake in this section will all assumed to be Morse. The structure of f near the critical points is well understood as shown by the following *Morse lemma.*

Lemma 2.1.1. *Let $a \in M$ be a critical point of Morse function $f : M \to \mathbb{R}$. There exists a neighbourhood U of a, and a local chart, called a Morse chart $\varphi : (U, a) \to (\mathbb{R}^n, 0)$ such that*

$$f \circ \varphi^{-1}(x_1, ..., x_n) = f(a) - \sum_{j=1}^{i} x_j^2 + \sum_{j=i+1}^{n} x_j^2.$$

The $i \in \mathbb{N}$ in the previous sum is actually independent of the choice of the Morse chart, and is called the *Morse index* of a. We will denote $\mathrm{Crit}_k f$ the critical points of f of index k.

Fixing a Morse function f on M, a *pseudo gradient* vector field for f is a vector field X such that for all $x \in M$, $df_x(X_x) \leq 0$ and with equality if and only if x is a critical point of f. Moreover, we ask that in a Morse chart near a critical point, X is equal to the opposite of the gradient vector field of f for the canonical metric on $\mathbb{R}^n$. Given a Morse function, such vector fields always exist.

Let us fix a Morse function f and a pseudo gradient vector field X. By Smale's theorem, one can always find a pseudo gradient vector field Y, C^1-close to X such that, for all critical points a and b of f, the Y-stable manifold of a intersects transversely the Y-unstable manifold of b. The vector field Y is said to satisfy the *Smale condition.*

We can now describe the *Morse chain complex* $CM_*(f)$, with f being a Morse function on M, together with a pseudo-gradient vector field X satisfying the Smale condition. The k^{th} group of the chain complex $CM_k(f)$ is given by

$$CM_k(f) = \left\{ \sum_{c \in \mathrm{Crit}_k(f)} \lambda_c \cdot c, \quad \lambda_c \in \mathbb{Z}/2 \right\}.$$

We now have to define the differential of this complex: $\partial_X : CM_k(f) \to CM_{k-1}(f)$. To do so, we will study the solutions of the following exact differential equation

$$\dot{l}(t) = X(l(t)). \tag{2.1}$$

Let x_- and x_+ be two distinct critical points of f. We set

$$\mathcal{M}(x_-, x_+; f, X) = \left\{ l \text{ solution of 2.1}, \quad \lim_{t \to \pm\infty} l(t) = x_\pm \right\}.$$

Since we have $\mathcal{M}(x_-, x_+; f, X) \cong W^u(x_-) \cap W^s(x_+)$, the transversality condition ensures that $\dim(\mathcal{M}(x_-, x_+; f, X)) = \mathrm{ind}(x_-) - \mathrm{ind}(x_+)$.

Moreover, there is a free and proper action of $\mathbb{R}$ on $\mathcal{M}(x_-, x_+; f, X)$ given for all $c \in \mathbb{R}$ by $l \mapsto l(\cdot + c)$. So we define $\widehat{\mathcal{M}}(x_-, x_+; f, X)$ the quotient of $\mathcal{M}(x_-, x_+; f, X)$ by this action. The manifold $\widehat{\mathcal{M}}(x_-, x_+; f, X)$ is then $(\mathrm{ind}(x_-) - \mathrm{ind}(x_+) - 1)$-dimensional. The *Morse differential* $\partial_X : CM_*(f) \to CM_{*-1}(f)$ is then defined to be the linear map such that for all critical points $x_- \in \mathrm{Crit}_k(f)$,

$$\partial_X(x_-) = \sum_{y \in \mathrm{Crit}_{k-1}(f)} \sharp \widehat{\mathcal{M}}(x_-, y; f, X) \cdot y,$$

where $\sharp \widehat{\mathcal{M}}(x_-, y; f, X)$ denotes the cardinal of $\widehat{\mathcal{M}}(x_-, y; f, X)$. The last point to be checked is whether we have $\partial_X^2 = 0$. For $x \in \mathrm{Crit}_{k+2}(f)$,

$$\partial_X^2 x = \sum_{\substack{z \in \mathrm{Crit}_k(f) \\ y \in \mathrm{Crit} k+1(f)}} \sharp(\widehat{\mathcal{M}}(x, y; f, X) \times \widehat{\mathcal{M}}(y, z; f, X)) \cdot z.$$

Proving that ∂_X^2 is always equal to zero requires to understand the structure of the moduli spaces. This can be achieved through the two following theorems, the first one being a compactness one and the second a gluing one [6].

Let us first define broken gradient trajectories.

Definition 2.1.2. *A* broken gradient trajectory *between two critical points x_- and x_+ is a family $(l_1, ..., l_p)$ of trajectories such that*

1. *for all i, l_i is a solution of 2.1,*
2. *$l_{1,-} = x_-$, $l_{p,+} = x_+$*
3. *$\forall 1 \leq i \leq p-1$, $l_{i,+} = l_{i+1,-}$.*

For two critical points x_- and x_+ we denote $\overline{\widehat{\mathcal{M}}}(x_-, x_+; f, X)$ the space of broken gradient trajectory from x_- to x_+.

Theorem 2.1.3. *The space $\overline{\widehat{\mathcal{M}}}(x_-, x_+; f, X)$ is compact for all critical points x_- and x_+.*

Theorem 2.1.4. *Let $(x_-, z, x_+) \in (Crit_{k+1}(f) \times Crit_k(f) \times Crit_{k-1}(f))$, and $l \in \widehat{\mathcal{M}}(x_-, z; f, X)$, $l' \in \widehat{\mathcal{M}}(z, x_+; f, X)$. There is a continuous, differentiable on the interior of its definition domain, embedding ψ from an interval $[0, \delta)$, $\delta > 0$, to a neighbourhood of (l, l') in $\overline{\widehat{\mathcal{M}}}(x_-, x_+; f, X)$ such that*

1. *$\psi(0) = (l, l') \in \overline{\widehat{\mathcal{M}}}(x_-, x_+; f, X)$,*
2. *$\psi(s) \in \widehat{\mathcal{M}}(x_-, x_+; f, X)$ for all $s \neq 0$.*

Moreover, for any (l_n) sequence in $\widehat{\mathcal{M}}(x_-, x_+; f, X)$ converging to (l, l') then, for n large enough, l_n lies in the image of ψ.

Together with the properties of the index, these theorems lead to

$$\bigcup_{z \in Crit_k(f)} \widehat{\mathcal{M}}(x_-, z; f, X) \times \widehat{\mathcal{M}}(z, x_+; f, X) = \partial \overline{\widehat{\mathcal{M}}}(x_-, x_+; f, X),$$

where (x_-, z, x_+) are defined as in the above theorem. Moreover, $\overline{\widehat{\mathcal{M}}}(x_-, x_+; f, X)$ is a 1-dimensional manifold with boundary. Hence, we get that $\partial_X^2 = 0$.

Remark 2.1.5. This homology does not depend on the choice of the function f, or the vector field X. Indeed, given two Morse functions f and g on M, and associated pseudo-gradient vector fields X and Y, there is a chain homotopy equivalence

$$\psi_* : (CM_*(f), \partial_X) \to (CM_*(g), \partial_Y),$$

which induces an isomorphism in homology.

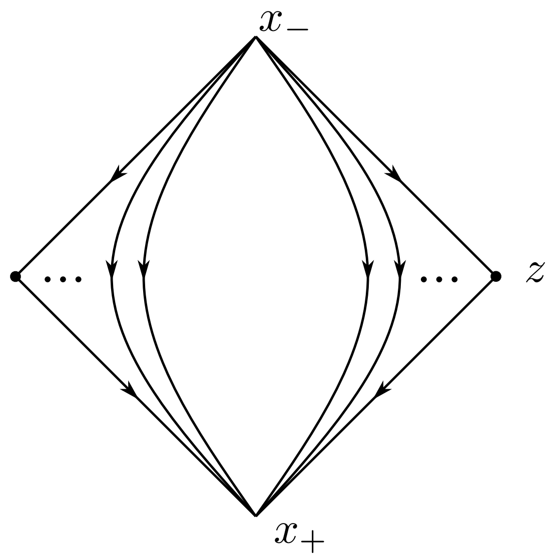

Figure 2.1 – Compactification of $\widehat{\mathcal{M}}(x_-, x_+; f, X)$.

We also make the following remark which will be proved to be useful when discussing persistence modules and barcodes.

Remark 2.1.6. Given a Morse function f and a pseudo-gradient vector field X, the differential ∂_X decreases the value of f, i.e. for all $x \in \mathrm{Crit}(f)$, we have $f(x) \geq f(\partial_X x)$. Indeed, let us consider two critical points x_- and x_+ such that $\mathrm{ind}(x_-) = \mathrm{ind}(x_+) + 1$ and such that there exist a trajectory x from x_- to x_+ satisfying 2.1. Since such a trajectory is a flow line of X, by definition of a pseudo-gradient vector field, we immediately get that $f(x_-) > f(x_+)$. This means that all the points y with non-zero coefficient in the expression of the differential of a point x satisfy $f(x) > f(y)$.

2.2 Floer cohomology for a pair of exact Lagrangian submanifolds

This section will closely follow the work of Denis Auroux [7] together with lectures note from Ghiggini [41] and Oh's book [68].

2.2.1 Motivation and general presentation

One would naively want to define Lagrangian Floer homology in the same way that Morse homology is defined. This means defining an action functional, and setting the generators of the chain complex to be the critical points of this action functional, while the differential would be given by its flow lines.

Let (M, ω) be a Liouville domain, with $d\lambda = \omega$, and let L and L' be two closed connected exact Lagrangian submanifolds in M. We denote $f : L \to \mathbb{R}$ and $f' : L' \to \mathbb{R}$ the functions satisfying $df = \lambda_{|L}$ and $df' = \lambda_{|L'}$. We recall that these functions are well-defined up to a constant by Remark 1.3.5.

Let us consider the path space

$$\mathcal{P}(L, L') = \{\gamma : [0,1] \to M, \gamma \text{ smooth}, \gamma(0) \in L, \gamma(1) \in L'\}.$$

Remark 2.2.1. With this definition, the tangent space of $\mathcal{P}(L, L')$ at γ is:

$$T_\gamma \mathcal{P}(L, L') = \{\xi \in \Gamma(\gamma^* TM), \xi(0) \in T_{\gamma(0)L}, \xi(1) \in T_{\gamma(1)L}\}.$$

Note that this is purely heuristic: we do not describe (or have) a manifold structure on this space of paths.

To do computations, this definition is not the easiest one to consider. As we will see, it is more convenient to consider tangent vectors at γ as the derivative of a 1-parameter family of paths near γ.

We can now define the *action functional* as follows.

Definition 2.2.2. *In our context, the* action functional *on the space of paths $\mathcal{P}(L, L')$ is the map $\mathcal{A}_{L,L'} : \mathcal{P}(L, L') \to \mathbb{R}$ defined by the expression*

$$\mathcal{A}_{L,L'}(\gamma) = \int \gamma^* \lambda + f(\gamma(0)) - f'(\gamma(1)),$$

with $\gamma \in \mathcal{P}(L, L')$.

Remark 2.2.3. This definition of the action presents some unusual properties regarding the classical conventions used in cohomology. Indeed, as we will see in Remark 2.2.16, the differential in cohmology decreases this action. This choice does not fundamentally matter but it makes the definitions of persistence modules and barcodes easier as our setting thus matches with the usual definitions of these objects.

Let us compute the differential to get the critical points. To do so, we compute the differential of $\mathcal{A}_{L,L'}$ at a point γ applied to a tangent vector ξ.

To do so, we extend $\gamma(t)$ to $u_s(t) = u(s,t)$ defined for s in a neighborhood $(-\epsilon, \epsilon)$ of 0, with $\epsilon > 0$, such that

$$\begin{cases} \forall s \in (-\epsilon, \epsilon), u(s,0) \in L, u(s,1) \in L' \\ u(0,t) = \gamma(t) \\ \frac{\partial u}{\partial s}(0,t) = \xi(t). \end{cases}$$

The differential of $\mathcal{A}_{L,L'}$ at γ is then

$$\begin{aligned}
d_\gamma \mathcal{A}_{L,L'}(\xi) &= \frac{\partial}{\partial s}\mathcal{A}_{L,L'}(u(s,t))_{|s=0} \\
&= \lim_{s\to 0}\frac{1}{s}\left(\mathcal{A}_{L,L'}(u(s,t)) - \mathcal{A}_{L,L'}(u(0,t))\right) \\
&= \lim_{s\to 0}\frac{1}{s}\left(\int_{[0,1]} u_s^*\lambda - u_0^*\lambda + f(u(s,0)) - f'(u(s,1)) - f(u(0,0)) + f'(u(0,1))\right) \\
&= \lim_{s\to 0}\frac{1}{s}\int_{\partial([0,s]\times[0,1])} u^*\lambda \\
&= \lim_{s\to 0}\frac{1}{s}\int_{[0,s]\times[0,1]} u^*\omega \\
&= \lim_{s\to 0}\frac{1}{s}\int_0^s\int_0^1 \omega_{u(s,t)}\left(\frac{\partial u}{\partial s},\frac{\partial u}{\partial t}\right)dtds \\
&= \int_0^1 \omega_{\gamma(t)}(\xi(t),\dot\gamma(t))dt.
\end{aligned}$$

Thus, the critical points of $\mathcal{A}_{L,L'}$ are exactly the paths such that for all $t\in[0,1]$, $\dot\gamma(t)=0$, i.e. the constant paths. Since the only constant paths with endpoints on L and L' are the intersection points, we conclude that the critical points of $\mathcal{A}_{L,L'}$ are the intersection points between L and L'. At such a point p, we have

$$\mathcal{A}_{L,L'}(p) = f(p) - f'(p).$$

We denote $\mathrm{Spec}(L,L')$ the set of critical values of $\mathcal{A}_{L,L'}$.

To proceed as done for Morse homology, in order to define the differential, we now have to compute the gradient flow lines. However, the analysis results at work in Morse theory do not work anymore when trying to do infinite-dimensional Morse theory. This is why, even if the generators of our Floer complex will be intersection points, in order to define Floer cohomology, we will have to study moduli spaces of pseudo-holomorphic strips.

Along the definition of Lagrangian Floer cohomology, we will explain how our particular case of an exact symplectic manifold together with two exact Lagrangian submanifolds makes this cohomology easier to define than for the general case.

2.2.2 The complex

In the general case, Floer cohomology is defined with coefficients in a Novikov field. However, thanks to the exactness condition required in our context, we do not need this coefficient field, and we can work with coefficients in $\mathbb{Z}/2$.

Let us take a Liouville domain $(M,\omega=d\lambda)$ and two closed exact Lagrangian submanifolds L and L' that intersect transversely, at a finite set of points. The Floer complex is the free $\mathbb{Z}/2$-module generated by the intersection points $\chi(L, L') = \{p \in L \cap L'\}$ of these two Lagrangian submanifolds:

$$CF(L, L') = \bigoplus_{p \in \chi(L,L')} \mathbb{Z}/2 \cdot p.$$

Before defining the differential and the grading of this complex, we have to choose an ω-compatible almost-complex structure J as in Definition 1.1.1. It will be essential to define a metric on our moduli spaces.

Let us recall [62] that for all (M, ω), the set ω-compatible almost-complex structure is non empty and contractible. So let us fix a time-dependent almost complex structure $(J_t)_{t\in[0.1]} \in \mathcal{J}(M, \omega)$. We are working with time-dependent almost-complex structures because this will allow us to ensure the good definition of the Lagrangian Floer cohomology; this will be discussed in Subsection 2.2.5.

We now define the differential of our chain complex.

2.2.3 The Floer differential

The idea of the Floer differential $\partial : CF(L, L') \to CF(L, L')$ is to count pseudo-holomorphic strips in M with boundary in L and L', between two intersection points.

Let p and q be two points in $\chi(L, L')$. We define the moduli space $\mathcal{M}(p, q; J_t)$ as follows.

Definition 2.2.4. *The moduli space $\mathcal{M}(p, q; J)$ is the set of smooth maps $u : \mathbb{R}\times[0; 1] \to M$ which satisfy the Cauchy-Riemann equation*

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} = 0 \iff \bar{\partial}_{J_t} u = 0, \tag{2.2}$$

together with boundary conditions:

$$\forall s \in \mathbb{R}, u(s, 0) \in L, u(s, 1) \in L',$$

$$\forall t \in \mathbb{R} \lim_{s\to+\infty} u(s, t) = p,$$

$$\forall t \in \mathbb{R} \lim_{s\to-\infty} u(s, t) = q.$$

A map $u \in \mathcal{M}(p, q; J_t)$ is called a Floer strip from p to q, and a map $u \in C^0(\mathbb{R}\times[0, 1], M)$ which satisfies the boundary conditions but which does not satisfy Cauchy-Riemann is called a Floer-like strip from p to q.

Remark 2.2.5. If u is in $\mathcal{M}(p, q; J_t)$, then $s_0 \cdot u$ defined by $s_0 \cdot u(s, t) = u(s_0 + s, t)$ is also in $\mathcal{M}(p, q; J_t)$. This means that there is an $\mathbb{R}$-action on the moduli space $\mathcal{M}(p, q; J_t)$. The fixed points of this action are the constant maps at an intersection point. The only stabilizer of a non-constant map is 0.

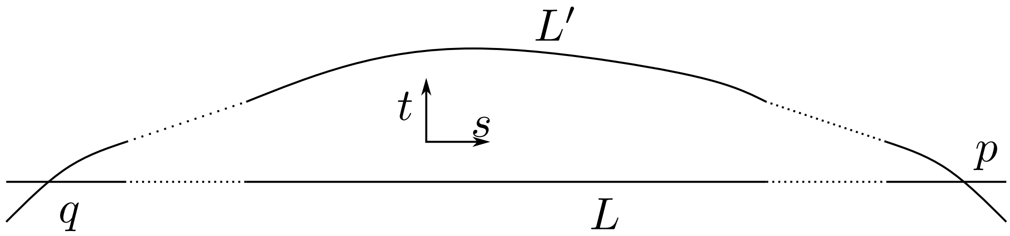

Figure 2.2 – Floer strip between p and q.

Remark 2.2.6. When working with non-exact Lagrangian submanifolds, an other condition is required for the previous definition, which is called the finite energy condition, i.e. the maps u have to satisfy

$$E(u) : \int u^*\omega = \int\int \left|\frac{\partial u}{\partial s}\right|^2 < +\infty.$$

However, a consequence of the following lemma is that this condition is always satisfied, in our exact Lagrangian submanifolds setting.

Lemma 2.2.7. *Let $p, q \in \chi(L, L')$ and let u be a Floer strip from p to q. Then,*

$$E(u) = \mathcal{A}_{L,L'}(p) - \mathcal{A}_{L,L'}(q) = f(p) - f'(p) - f(q) + f'(q) < +\infty.$$

Proof. Applying twice Stoke's theorem to u, we have

$$\begin{aligned} E(u) &= \int u^*\omega \\ &= \int_{\mathbb{R}} u(\cdot, 0)^*\lambda - u(\cdot, 1)^*\lambda \\ &= f(p) - f(q) + f'(q) - f'(p) \\ &= \mathcal{A}_{L,L'}(p) - \mathcal{A}_{L,L'}(q). \end{aligned}$$

□

We now need to study these moduli spaces in order to define our differential. To do so, we will study the different connected components.

Definition 2.2.8. *For an homotopy class $[u] \in \pi_2(M, L \cup L')$, we denote $\mathcal{M}(p, q; [u]; J_t) \subset \mathcal{M}(p, q; J_t)$ the set of maps representing the class $[u]$, and $\widehat{\mathcal{M}}(p, q; [u]; J_t) \subset \widehat{\mathcal{M}}(p, q; J_t)$ its quotient by the $\mathbb{R}$-action.*

This quotient is well-defined by the above Remark 2.2.5.

Remark 2.2.9. Since we are working with a Liouville domain, we need to understand what happens when a pseudo-holomorphic curve goes to the boundary of our domain. Let us consider the completion $(\hat{M}, \hat{\lambda})$ of (M, ω) as defined in 1.1. Let us assume that u is a pseudo-holomorphic curve in $(\hat{M}, \hat{\lambda})$ between two intersection points in (M, ω) and let us denote v its restriction to the end $[-\varepsilon, +\infty) \times \partial M$. One can show that v satisfies a maximum principle in the $[-\varepsilon, +\infty)$ direction. Since u is a strip between two points in M, the component of v in the direction $[-\varepsilon, +\infty)$ is constant. Therefore, a pseudo-holomorphic strip with boundary on Lagrangian submanifolds in M cannot enter this end and we can work in M instead of $\hat{M}$.

The dimension of the connected components is computed thanks to Fredholm operators theory. Given $u \in \mathcal{M}(p, q; J_t)$, we linearize the Cauchy-Riemann operator $\bar{\partial}_{J_t}$ near u in the suitable space of sections that will now be described.

Let us denote $\mathcal{B}^\infty$ the set of smooth Floer-like strips from p to q, with p and q in $\chi(L, L')$. Then, if $u \in \mathcal{B}^\infty$, then $\bar{\partial}_{J_t} u$ is a section of u^*TM and $\bar{\partial}_{J_t}$ is a section of the bundle $\mathcal{E}^\infty \to \mathcal{B}^\infty$ whose fiber at u is $\Gamma(u^*TM)$. Given $p > 2$, we denote $\mathcal{B}$ the set of Floer-like strips from p to q of class $W^{1,p}$ and $\mathcal{E}_u$ the Banach space of sections of u^*TM of class L^p. We have the following proposition.

Proposition 2.2.10. *$\mathcal{B}$ is a Banach manifold, and for every $u \in \mathcal{B}$,*

$$T_u\mathcal{B} = \{\xi \in W^{1,p}(u^*TM) | \xi(s, 0) \in T_{u(s,0)}L, \xi(s, 1) \in T_{u(s,1)}L'\}.$$

In addition, $\mathcal{E} \to \mathcal{B}$ is a bundle of Banach spaces and $\bar{\partial}_{J_t} : \mathcal{B} \to \mathcal{E}$ is a smooth section.

Now, for a given $u \in \mathcal{M}(p, q; J_t)$, we can define the linearized Cauchy-Riemann operator $D_u : T_u\mathcal{B} \to \mathcal{E}_u$. Identifying $\mathcal{B}$ with the zero section of $\mathcal{E}$, there is a canonical isomorphism $T_u\mathcal{E} \cong T_u\mathcal{B} \oplus \mathcal{E}_u$, and a natural projection $\pi_u : T_u\mathcal{E} \to \mathcal{E}_u$. The linearized operator D_u is then defined as $\pi_u \circ d_u\bar{\partial}_{J_t}$. The section $\bar{\partial}_{J_t}$ is transverse to the zero section at u if and only if D_u is surjective.

Proposition 2.2.11. *If L and L' intersect transversely at p and q, then for all $u \in \mathcal{M}(p, q; J_t)$, the linearized operator D_u is a Fredholm operator.*

We define the index of $[u]$ as the Fredholm index of D_u:

$$\mathrm{ind}([u]) = \mathrm{ind}_{\mathbb{R}}(D_u) = \dim \ker(D_u) - \dim \mathrm{coker}(D_u). \tag{2.3}$$

This index can also be computed as an invariant of the class $[u]$. This index is called the Maslov index, and it will be defined in the following section.

Given a class $[u]$, if the linearized operator D_u is surjective at each point of the space of solutions $\widehat{\mathcal{M}}(p, q; [u]; J_t)$, the solutions are said to be regular. Provided that they are all regular, the set $\widehat{\mathcal{M}}(p, q; [u]; J_t)$ is a smooth manifold of dimension $\mathrm{ind}([u])$.

Remark 2.2.12. Both the regularity or transversality question and the compactness of $\widehat{\mathcal{M}}(p, q; [u]; J_t)$ will be briefly discussed later. In more general cases of definition for Floer cohomology, one has to discuss the orientation of the moduli spaces. However, working with $\mathbb{Z}/2$-coefficients, this issue does not appear in our context.

Let us assume for now that we have addressed these questions. We can then define the differential.

Definition 2.2.13. *The Floer differential $\partial : CF(L, L') \to CF(L, L')$ is the $\mathbb{Z}/2$-linear map defined by:*

$$\partial p = \sum_{\substack{q \in \chi(L,L') \\ [u]: ind([u])=1}} (\sharp \widehat{\mathcal{M}}(p, q; [u]; J_t)) \cdot q, \tag{2.4}$$

where $\sharp \widehat{\mathcal{M}}(p, q; [u]; J_t) \in \mathbb{Z}/2$ is the count of points in the moduli space of Floer strips connecting p to q in the class $[u]$.

Remark 2.2.14. In the definition of the differential, we only consider the moduli spaces $\widehat{\mathcal{M}}(p, q; [u]; J_t)$, where $[u]$ is such that $ind([u]) = 1$. Thus, the moduli space $\widehat{\mathcal{M}}(p, q; [u]; J_t)$ is a compact manifold of dimension 0, i.e. a finite set of points. This ensures that the sum 2.4 in the definition of the differential is well defined.

We defined the action of a single point in $\chi(L, L')$. However, as indicated in the previous definition of the differential, we will not be only interested in each generator individually but rather in a formal sum of them. Consequently, we have to define the action for such a sum.

Definition 2.2.15. *Let $p_1, ...p_k$, for $k \in \mathbb{N}$ be points in $\chi(L, L')$. The action of the formal sum of these points is the maximum of the different actions, i.e.*

$$\mathcal{A}_{L,L'}(p_1 + ... + p_k) = \max\{\mathcal{A}_{L,L'}(p_1), ...\mathcal{A}_{L,L'}(p_k)\}.$$

Remark 2.2.16. Since the energy of a Floer strip connecting p to q is always strictly positive by Remark 2.2.6, Lemma 2.2.7 tells that the differential strictly decreases the action, i.e.

$$\mathcal{A}_{L,L'}(p) > \mathcal{A}_{L,L'}(\partial p),$$

for all p in $\chi(L, L')$.

In general, transversality is not automatic. To achieve it, we have to perturb the Cauchy-Riemann equation, and then count perturbed pseudo-holomorphic strips between perturbed intersection points of L and L'. This issues will be discussed later on. Now, we will state Floer's theorem [37].

Theorem 2.2.17. *Assume that $[\omega]\cdot\pi_2(M,L)=[\omega]\cdot\pi_2(M,L')=0$. Then the Floer differential ∂ is well-defined and satisfies $\partial^2=0$. The resulting Floer cohomology $HF(L,L')=H^*(CF(L,L';\mathbb{Z}/2),\partial)$ is independent, up to isomorphism, of the choice of the almost complex-structure J. Moreover, Hamiltonian isotopies of L or L' induces quasi-isomorphisms on the chain complexes.*

The last point of the previous theorem means that the Lagrangian Floer cohomology is invariant under Hamiltonian isotopies of L or L'.

Remark 2.2.18. In our context of a Liouville domain together with exact Lagrangian submanifolds L and L', the condition $[\omega]\cdot\pi_2(M,L)=[\omega]\cdot\pi_2(M,L')=0$ is always verified, by Stoke's theorem.

Remark 2.2.19. As mentioned in the introduction, Lagrangian Floer cohomology has been defined in much more general contexts since Floer's work [66, 39].

2.2.4 Maslov index

Defining a cohomology, one wants a notion of grading. In the case of Lagrangian Floer cohomology, this grading is a relative one which behaves with respect to the index of Floer strip defined earlier by the formula 2.3. This index is called the Maslov index, and first appeared in the work of Viterbo [95].

Let us denote $\Lambda(n)$ the Grassmannian of Lagrangian n-planes in $\mathbb{C}^n$. Let us recall [62] that the group $U(n)$ acts transitively on $\Lambda(n)$, and the stabilizer of $\mathbb{R}^n$ is $O(n)$. This tells that $\Lambda(n)$ is diffeomorphic to $U(n)/O(n)$, and then that $\pi_1(\Lambda(n))\cong\mathbb{Z}$. This diffeomorphism is induced by the map:

$$\begin{array}{rcl} U(n) & \to & \Lambda(n) \\ U & \mapsto & U(\mathbb{R}^n). \end{array}$$

This allows to define a map

$$\rho:\Lambda(n)\to S^1,$$

using the mapping

$$\det{}^2: U(n)\to\Lambda(n)\to S^1,$$

by setting

$$\rho(U(\mathbb{R}^n))=\det{}^2 U.$$

Let l be a loop in $\Lambda(n)$, its Maslov index is then defined to be:

$$\mu_M(l)=\deg(\rho(l))\in\mathbb{Z}.$$

Before adapting this definition to the setting of Lagrangian Floer cohomology, we have to define the canonical short path which will allow us to have a loop.

Definition 2.2.20. *Let l_0 and l_1 in $\Lambda(n)$ be two transverse Lagrangian subspaces. One can find a map $A \in Sp(2n)$ such that $A(l_0) = \mathbb{R}^n$ and $A(l_1) = i\mathbb{R}^n$. The canonical short path is the path $l_t = A^{-1}((e^{-i\pi t/2}\mathbb{R})^n), t \in [0;1]$, which connects l_0 to l_1 in $\Lambda(n)$.*

Remark 2.2.21. The proof of the existence of such A can be found in [62].

Definition 2.2.22. *Let $p, q \in \chi(L, L')$. Let us denote l_p the canonical short path from $T_p(L)$ to $T_p(L')$ and l_q the one from $T_q(L)$ to $T_q(L')$. Let $u : \mathbb{R} \times [0;1] \to M$ be a strip connecting p to q and let us pick a trivialization of u^*TM. We can now see the paths $l = u^*_{\mathbb{R}\times\{0\}}TL$ and $l' = u^*_{\mathbb{R}\times\{1\}}TL'$ as paths in $\Lambda(n)$ (with orientation with s going from $-\infty$ to $+\infty$). The Maslov index $\mu(u)$ of the strip u is then the Maslov index of the closed loop in $\Lambda(n)$ obtained by concatenating $-l$, l_p, l' and $-l_q$.*

We have the fundamental following proposition [37].

Proposition 2.2.23.

$$ind(u) = \mu(u)$$

Remark 2.2.24. Since $\mathbb{R} \times [0;1]$ is contractible, it is immediate that the pullpack u^*TM is a trivial symplectic vector bundle. Moreover, all the trivializations are homotopic.

To ensure that this is well-defined, we have to make sure that this definition does not depend on the choice of the trivialization. Moreover, to be able to use this grading for our Lagrangian Floer homology, we also have to make sure that it does not depend on the choice of the homotopy class of $[u]$.

The first requirement needed is that the first Chern class of M must be 2-torsion: $2c_1(TM) = 0$. Indeed, if we take two Floer strips u and u' between two intersection points p and q, such that they have the same boundary, *i.e;* with the notations of Equation 2.2, for all $s \in \mathbb{R}$, $u(s, 0) = v(s, 0)$ and $u(s, 1) = v(s, 1)$, we have that [95]:

$$\mu(u) - \mu(v) = 2c_1(u\sharp\bar{v}),$$

where $u\sharp\bar{v}$ is the gluing of u and $\bar{v}$, with $\bar{v}$ denoting the strip v with opposite orientation.

The second requirement concerns the Malov classes of L and L'. We require that $\mu_L \in \mathrm{Hom}(\pi_1(L), \mathbb{Z}) = H^1(L, \mathbb{Z})$ and $\mu_{L'} \in \mathrm{Hom}(\pi_1(L'), \mathbb{Z}) = H^1(L', \mathbb{Z})$ vanish. Indeed, according to Viterbo [95], let u and v two Floer strips between two intersection points p and q, with boundary l_u and l_v in L and l'_u and l'_v in L', if $2c_1(TM) = 0$, then

$$\mu(u) - \mu(v) = \mu_L(l_u\sharp l_v^{-1}) - \mu_{L'}(l_u\sharp l_v^{-1}),$$

where $l\sharp l'$ denotes the concatenation of the two path l and l'.

By doing all this, we have defined a relative Maslov index for our intersection points. Indeed, if we fix the degree for a given intersection point, it fixes the degree for all the other intersection points. We just have to set that

$$ind(u) = deg(q) - deg(p).$$

There is a way to define an absolute Maslov index that is described in Appendix A.

Remark 2.2.25. A reassuring fact with this definition is that the Floer differential we defined earlier has degree 1.

2.2.5 Transversality

The two following sections will be short, following Auroux' work [7].

Two types of transversalities are required to define Lagrangian Floer cohomology. The first one is the transversality of the intersections between our two Lagrangian submanifolds L and L', the second one is the transversality of the moduli spaces.

It often appears that the two Lagrangian submanifolds L and L' do not intersect transversely. The main example is simply when $L = L'$. To define the Floer cohomology in this case, the trick is to introduce a Hamiltonian perturbation. Indeed, we require in the construction that this cohomology is independent under Hamiltonian isotopy. Consequently, we will add a Hamiltonian perturbation term to the Cauchy-Riemann equation. Let us fix a generic Hamiltonian $H \in C^\infty([0;1] \times M, \mathbb{R})$, Equation 2.2 then becomes

$$\frac{\partial u}{\partial s} + J_t(u)\left(\frac{\partial u}{\partial t} - X_H(t,u)\right) = 0,$$

with the same boundary conditions on t: $u(s,0) \in L$ and $u(s,1) \in L'$ for all $s \in \mathbb{R}$. If we were not working with exact Lagrangian submanifolds, we would still need the finite energy condition. For the boundary conditions on s, i.e. where u converges as s goes to $\pm\infty$, the intersection points p and q do no longer make sense. The strip converges to trajectories of X_H from L to L'. These are flow lines $\gamma : [0;1] \to M$ such that

$$\dot{\gamma}(t) = X_H(t, \gamma(t)),$$

$$\gamma(0) \in L, \gamma(1) \in L'.$$

They are the actual generators of the Floer complex $CF(L, L')$. An alternative solution is to set $\chi(L, L') = L \cap (\phi_H^1)^{-1}(L')$, where $\phi_H^1 \in \mathrm{Ham}(M, \omega)$ is the time-1 of the flow generated by the Hamiltonian H, and then to proceed as before. With this point of view, we still have intersection points as generators, but they are "the perturbed by H" intersection points between L and L'.

When we define the action in this context, we have to take into account the Hamiltonian perturbation. The Hamiltonian action of a path γ from L to $\phi_H(L)$ is then defined as

$$\mathcal{A}^H_{L,L'}(\gamma) = \int_0^1 \gamma^*\lambda - H(\gamma)dt + f(\gamma(0)) - f'(\gamma(1)). \tag{2.5}$$

We denote by $\mathrm{Spec}(L, L'; H)$ the set of critical values of this action functional. The critical points are the above mentioned generators of the Floer complex.

To be sure that the previous arguments at stake in the definition of Floer cohomology hold for the perturbed Cauchy-Riemann equation, one can consider $\tilde{u}(s,t) = (\phi^t_H)^{-1}(u(s,t))$. We then have

$$\frac{\partial \tilde{u}}{\partial t} = (\phi^t_H)^{-1}_* \left(\frac{\partial u}{\partial t} - X_H \right).$$

The perturbed Cauchy-Riemann equation then becomes

$$\frac{\partial \tilde{u}}{\partial s} + \tilde{J}_t(\tilde{u}) \frac{\partial \tilde{u}}{\partial t} = 0,$$

with $\tilde{J}_t = (\phi^t_H)^{-1}_*(J_t)$. This means that the solutions of the perturbed equation are classical $\tilde{J}$-holomorphic curves with boundaries in L and $(\phi^1_H)^{-1}(L')$, between two points in $\chi(L, L') = L \cap (\phi^1_H)^{-1}(L')$. We can now apply all the arguments involved above to define the Floer complex

$$CF(L, L'; H; \tilde{J}_t),$$

and then the Floer cohomology $HF(L, L'; H; \tilde{J}_t)$.

The natural question then is whether this definition depends on the choice of the Hamiltonian perturbation. As stated in Theorem 2.2.17, Lagrangian Floer cohomology is invariant under Hamiltonian perturbation and consequently independent of the choice of the perturbation. This will be discussed in Subsection 2.2.7.

Remark 2.2.26. If the Lagrangian submanifolds L and L' were transverse, we could of course choose $H = 0$. Moreover, for two given Lagrangian submanifolds, the Hamiltonian perturbation to achieve transversality can be chosen as small as desired.

The transversality of the moduli spaces is actually the question of the surjectivity of the linearized $\bar{\partial}_{J_t}$ at all solutions, which is critical to ensure that the moduli spaces $\mathcal{M}(p, q; [u]; J_t)$ are indeed smooth and of the expected dimension. For that purpose, we will consider a path J_t of almost-complex structure instead of with a autonomous one. Note that we still assume here that L and L' intersect transversely.

Definition 2.2.27. *A time-dependent almost-complex structure J_t is* regular *if D_u is surjective for all Floer strips u.*

This notion of regularity replaces the Morse-Smale condition at work in the definition of Morse homology. For Lagrangian Floer cohomology, we make use of the following theorem from Floer [37].

Theorem 2.2.28. *Regular time-dependent almost complex structures are generic.*

As a consequence, one can always find a regular almost complex structure which will ensures the transversality of the moduli spaces and the correct definition of the Floer cohomology.

2.2.6 ∂ is a differential

As in Morse homology, to prove that the square of the differential is equal to zero, one has to count broken Floer strip. To do so, the idea is to consider these moduli spaces of broken strips as the boundary of index two moduli spaces and easy dimensional arguments will allow us to conclude. This is summarized by the following theorem [43], which is both a compactness and a gluing theorem.

Theorem 2.2.29. *Let (M, λ) be a Liouville domain, and L and L' be two closed exact Lagrangian submanifolds which intersect transversely. Then, for any regular time-dependent compatible almost complex structure J_t, we have*

- *by transversality, if u is a Floer strip with Maslov index $\mu(u) \leq 0$, then u is a constant strip, i.e. for all s and t, $u(s,t) = p \in \chi(L, L')$. Thus $\mu(u) = 0$.*
- *if $p, q \in L \cap L'$, then $\bigcup_{ind([u])=1} \widehat{\mathcal{M}}(p, q; [u]; J_t)$ is a finite set.*
- *if $p, q \in L \cap L'$, and $u \in \widehat{\mathcal{M}}(p, q; J_t)$ such that $ind([u]) = 2$, then $\widehat{\mathcal{M}}(p, q; [u]; J_t)$ is a 1-dimensional manifold and it admits a natural compactification which is homeomorphic to a 1-dimensional manifold with boudaries. The boundary of the compactification is*

$$\partial\overline{\widehat{\mathcal{M}}(p, q; [u]; J_t)} = \bigcup_{\substack{r \in \chi(L,L') \\ [v]+[v']=[u] \\ ind([v])=ind([v'])=1}} \widehat{\mathcal{M}}(p, r; [v]; J_t) \times \widehat{\mathcal{M}}(r, q; [v']; J_t).$$

The compactness of the moduli space is an essential requirement to define Lagrangian Floer cohomology. It will ensure that the sum (2.4) in the definition of the differential is well-defined, which corresponds to the second point of the previous theorem. It also ensures the possibility to compactify the moduli space of index 2 Floer strips, thus giving sense to the last point of the previous theorem. These results are due to Gromov's compactness theorem [43] which addresses sequences of J-holomorphic curves with uniformly bounded energy. This can be straightforwardly applied to our context of exact Lagrangian submanifolds. Indeed Lemma 2.2.7 shows that all J-holomorphic curves between two given intersection points have the same finite energy since this energy only depends on the two intersection points.

The last point of the theorem is larger than a compactness result, it is also a gluing result. Indeed it means that every broken strip (with index 1 and 1) is locally the limit of an unique family of index 2 strips representing the total class. Moreover, the union of these broken strips is the boundary of the compactification of index 2 strips. In this case, broken configurations with more than two components cannot occur. Noting that the index is additive, $ind(u) = 2$ thus implies that the only possible decomposition is $[u] = [v] + [v']$, with $ind([v]) = ind([v']) = 1$ since any non-constant strip has positive index by transversality.

With all these results being stated, we can sketch the proof that $\partial^2 = 0$, before discussing the compactness argument.

Let us recall that (M, λ) is a Liouville domain, L and L' are two exact Lagrangian submanifolds, which are assumed to intersect transversely. If not, we just have to pick an adequate Hamiltonian perturbation. Let J_t be a regular time-dependent almost complex structure, and p be a generator of the Floer complex. Composing ∂ with itself and applying it to the generator p, we obtain

$$\partial^2 p = \sum_{q \in \chi(L,L')} \left(\sum_{\substack{r \in \chi(L,L') \\ [v]+[v']=[u] \\ ind([v])=ind([v'])=1}} (\sharp \widehat{\mathcal{M}}(p, r; [v]; J_t))(\sharp \widehat{\mathcal{M}}(p, r; [v]; J_t)) \right) \cdot q.$$

The coefficient of a generator q in the expression of $\partial^2 p$ is actually the number of points in

$$\bigcup_{\substack{r \in \chi(L,L') \\ [v]+[v']=[u] \\ ind([v])=ind([v'])=1}} \widehat{\mathcal{M}}(p, r; [v]; J_t) \times \widehat{\mathcal{M}}(r, q; [v']; J_t).$$

But we know that this the boundary of

$$\bigcup_{ind([u])=2} \overline{\widehat{\mathcal{M}}(p, q; [u]; J_t)},$$

which is a finite collection of compact connected 1-dimensional submanifolds. Consequently, since we are counting the number of points in the boundary and since we are working with $\mathbb{Z}/2$-coefficients, the coefficient of q appears to be equal to zero. Hence, since it holds for all p,

$$\partial^2 = 0.$$

Let us go back now to the compactness argument. As mentioned earlier it is based on Gromov's work [43], studying J-holomorphic curves.

We denote $(u_n)_{n\in\mathbb{N}}$ a sequence of J-holomorphic strips between L and L' from p to q with uniformly bounded energy. As for Morse homology, we are interested in a phenomena called *strip breaking*, which, with respect to his name, will provide the broken strips at stake in the gluing-compactness theorem. However, when working with non-exact Lagrangian submanifolds, the situation is more complex than in Morse homology. Indeed, two issues may appear as limiting configurations and thus resulting in obstructions to the definition of Floer cohomology. These issues are called bubbling, and they are of two types:

- the sequence (u_n) converges (in some sense that will not be discussed here) to a J-holomorphic curve with a J-holomorphic disc at its boundary. The boundary of this disc is in one of the two Lagrangian submanifolds. It is called *disc bubbling* and appears if $|du_n| \to \infty$ at a boundary point.

- the sequence (u_n) converges to a J-holomorphic curve with a J-holomorphic sphere. It is called *sphere bubbling* and appears if $|du_n| \to \infty$ at an interior point.

As seen earlier, the energy of u is defined by

$$E(u) : \int u^*\omega = \int\int \left|\frac{\partial u}{\partial s}\right|^2$$

These two bubbling situations can thus be interpreted in term of energy. In the case of disc bubbling, the energy concentrates on a boundary point, i.e. t equal 0 or 1, whereas in the case of sphere bubbling, the energy concentrates on an interior point.

In our context, the fact that we are working with exact Lagrangian submanifolds leads to $[\omega] \cdot \pi_2(M, L) = [\omega] \cdot \pi_2(M, L') = 0$. This ensures the absence of disc bubbling and sphere bubbling since their symplectic area would be equal to zero. This is one of the main reason that makes it easy to work with exact Lagrangian submanifolds: $\partial^2 = 0$ does not hold with disc bubbling. The other issue with bubbling is the transversality results discussed in the previous section. The trick using perturbation to achieve transversality does not work anymore when it comes to the limit curves with disc or sphere bubbles.

According to Gromov's compactness theorem, once these bubblings excluded, to achieve compactness of the moduli space, we have to add an other phenomena as limiting configuration, which is the above mentioned *strip breaking.*

In terms of energy, strip breaking happens when the energy escapes toward an end of the strip. This means that the reparametrization $u_n(\cdot + a_n, \cdot)$, with $a_n \to \pm\infty$, leads to different limits. In this case, the limit of (u_n) is a sequence of J-holomorphic strips.

Aside from compactness, strip-breaking is closely related to the gluing aspect of Theorem 2.2.29, and thus plays a major role in the proof of $\partial^2 = 0$.

These issues having been clarified, putting all this together leads to the structure Theorem 2.2.29 for Floer strips moduli space $\mathcal{M}(p, q; J_t)$ in our context of a Liouville domain together with closed exact Lagrangian submanifolds.

Remark 2.2.30. Note that one can actually work with weaker conditions than the exactness of the Lagrangian submanifolds [67, 39].

2.2.7 Hamiltonian invariance

Let us choose (H, J_t) and (H', J'_t) two almost complex structures and Hamiltonian perturbations. Since both the set of Hamiltonians and compatible almost-complex structures are contractible, there is a smooth family $(H(\theta), J_t(\theta))_{\theta\in[0;1]}$ such that $(H(0), J_t(0)) = (H, J_t)$ and $(H(1), J_t(1)) = (H', J'_t)$. To go from this homotopy to the correct moduli spaces, let us pick a smooth function of s, $\theta(s)$ such that

$$s \ll 0, \theta(s) = 1,$$

$$s \gg 0, \theta(s) = 0.$$

The idea of this proof is quite close to the proof of $\partial^2 = 0$. This is fairly natural since both constructions are much alike. The point is to construct a so-called continuation map between $CF(L, L'; H; J_t)$ and $CF(L, L'; H'; J'_t)$ by counting the strips u between $p \in \chi(L, L'; H)$ and $p' \in \chi(L, L'; H')$ of index 0 and satisfying the following equation:

$$\frac{\partial u}{\partial s} + J_t(\theta(s), u)\left(\frac{\partial u}{\partial t} - X_H(\theta(s), t, u)\right) = 0,$$

with

$$u(s, 0) \in L, u(s, 1) \in L'.$$

Denoting Ψ the continuation morphism, the coefficient of $p' \in \chi(L, L'; H')$ in $\Psi(p)$ is the number in $\mathbb{Z}/2$ of such strips satisfying $u \underset{s\to+\infty}{\longrightarrow} p$ and $u \underset{s\to-\infty}{\longrightarrow} p'$.

In the absence of bubbling, it has been shown [37] that Ψ is a chain map, i.e. $\Psi \circ \partial = \partial' \circ \Psi$. The proof requires to study the index 1 moduli space resulting from the previous equation. It is similar to what we saw for the differential. In our context, no bubbling can occur. There are then two possibilities for broken strips. If it occurs at $s \to -\infty$, we obtain a J'-holomorphic strip contributing to ∂', with perturbation H'. Such strips are actually the strips counted by $\partial' \circ \Psi$. If it occurs at $s \to +\infty$, we obtain a J-holomorphic strip contributing to ∂, with perturbation H. Such strips are the strips counted by $\Psi \circ \partial$. The moduli spaces of degree 1 strips are consequently 1-dimensionnal manifolds, whose end points are broken strips, consisting of an index 0 solution of the previous equation, together with an index 1 perturbed Cauchy-Riemann solution. Once again, the fact that there is an even number of end points and that we are working with $\mathbb{Z}/2$ leads to conclusion:

either both types of broken strips come together as the boundary of the same connected component, or one type counts two ends, which goes to zero in $\mathbb{Z}/2$.

To conclude, we just have to note that we can follow the homotopy in the other direction. This gives another chain map Ψ' from $CF(L, L'; J'_t; H')$ to $CF(L, L'; J_t; H)$. The composition of the chain maps Ψ and Ψ' is homotopic to identity. We will not compute the homotopy here, but it could be done by counting index -1 solutions of a one-parameter family of equations similar to the previous one, with different θ.

Remark 2.2.31. In general, even if $H = H'$, the continuation map Ψ from $CF(L, L'; J_t; H)$ to $CF(L, L'; J'_t; H)$ does not induce the identity on the level of cochain complexes. However if the continuation data $(H(\theta), J(\theta))$ does not depend on s, it induces the identity. By continuity, this result still holds for a small perturbation of this s-independent continuation data, i.e. a continuation data C^1-close to a continuation data independent of s. Consequently, for $CF(L, L'; J_t; H)$ and $CF(L, L'; J'_t; H')$, if H and H' are close enough, a continuation data C^1-close to a continuation data independent of s will induce a $1:1$ correspondence on the level of cochain complexes.

2.2.8 General properties and remarks

Let (M, ω) and (M', ω') be two Liouville domains, together with two pairs of closed exact Lagrangian submanifolds $(L_0, L_1) \subset M$ and $(L'_0, L'_1) \subset M'$. Let us recall that we are working with $\mathbb{Z}/2$-coefficients. Then, there is a Künneth-type formula

$$HF(L_0, L_1; H, J) \otimes HF(L'_0, L'_1; H', J') \cong HF(L_0 \times L'_0, L_1 \times L'_1; H \oplus H', J \oplus J'). \quad (2.6)$$

This isomorphism is natural, resulting from the fact that a pseudo holomorphic curve v in $(M \times M', J \oplus J')$ can be written as $v = (u, u')$, where u is pseudo-holomorphic curve in M and u' in M'. At the chain level, for $(p, p') \in \chi(L_0, L_1) \times \chi(L'_0, L'_1)$, the isomorphism is simply defined by

$$(p, p') \mapsto (p, p') \in \chi(L_0 \times L'_0, L_1 \times L'_1).$$

In the following chapters, we will not be interested in the Hamiltonian or almost-complex structure perturbation, we will just want these Hamiltonian perturbations to be ε-small, for a given $\varepsilon > 0$. Thus, using Kislev-Shelukhin's notations [51], we will denote the Floer cohomology of L and L'

$$CF^*(L, L'; \mathcal{D}),$$

where $\mathcal{D}$ denotes the data perturbation, i.e. the pair (H, J). The perturbation data is said to be ε-small if the Hamiltonian is ε-small. When not needed, we will just write $CF^*(L, L')$, and assume that there is a suitable perturbation data implied.

We firstly point out here a staightforward consequence of the previous section. Let (M,ω), L and L' be as before, together with a perturbation data $\mathcal{D}$ and a Hamiltonian H. Then, the Hamiltonian invariance results in

$$HF(L,L';\mathcal{D}) \cong HF(L,\phi_H(L');\mathcal{D}'),$$

where $\mathcal{D}'$ is any suitable data perturbation and ϕ_H is the time-1 flow of the Hamiltonian H. Together with 1.2 this implies that, for a path of exact Lagrangian submanifolds in M denoted $(L'_t)_{t\in[0,1]}$ and a suitable choice of perturbations data $(\mathcal{D}_t)_{t\in[0,1]}$, we have

$$\forall t \in [0,1], \quad HF(L,L'_0;\mathcal{D}_0) \cong HF(L,L'_t;\mathcal{D}_t).$$

This invariance does not hold when considering (non-Hamiltonian) symplectomorphisms. However given a symplectomorphism ψ of (M,ω), taking the images by ψ of all the objects involved in the construction of the Floer cohomology of L and L' leads to

$$CF(L,L';\mathcal{D}) \cong CF(\psi(L),\psi(L');\mathcal{D}') \tag{2.7}$$

for $\mathcal{D}$ and $\mathcal{D}'$ two suitable perturbation datas.

One can naturally ask about the relation between $CF(L,L';\mathcal{D})$ and $CF(L',L;\mathcal{D})$. Indeed, a Floer strip from $p \in \chi(L,L')$ to $q \in \chi(L',L)$ in the Floer complex $CF(L,L';\mathcal{D})$ corresponds to a Floer strip from q to p in $CF(L',L;\mathcal{D}')$ for suitably chosen data perturbations $\mathcal{D}$ and $\mathcal{D}'$. In fact the two complexes are dual to each other: there exists $i \in \mathbb{Z}$ (coming from the fact that the index in $\chi(L,L')$ is defined up to a constant) such that

$$CF^*(L,L';\mathcal{D}) \cong CF^{n+i-*}(L',L;\mathcal{D}').$$

Consequently, the two cohomologies $HF(L,L';\mathcal{D})$ and $HF(L',L;\mathcal{D}')$ are dual to each other for two suitably chosen perturbation datas $\mathcal{D}$ and $\mathcal{D}'$.

In the following chapters, we will either consider the Lagrangian submanifolds L and L' as Lagrangian submanifolds in M or as Lagrangian submanifolds in T^*L. We will denote $HF(L,L';\mathcal{D},M)$ when the Floer cohomology is computed in M and $HF(L,L';\mathcal{D},T^*L)$ when the Floer cohomology is computed in T^*L.

Lastly we point out that the case when L and L' are Hamiltonian isotopic is particularly interesting in the perspective of the Arnold conjecture. Let us assume that $L' = L$. This choice is not restrictive thanks to the Hamiltonian invariance of the Floer cohomology.

In this case, it is indeed easier to work with more general conditions on the Lagrangian submanifolds considered. Due to Weinstein's neighbourhood theorem and energy estimates, choosing to work in the cotangent bundle T^*L of the Lagrangian L will not be restrictive. A longer and more detailed discussion on this subject will be held in Section 5.2.

Let $\varepsilon > 0$ and choose a ε-small Morse function $f : L \to \mathbb{R}$. We extend this function to T^*L by setting

$$H = f \circ \pi : T^*L \to \mathbb{R}, \tag{2.8}$$

where $\pi : T^*L \to L$ is the natural projection. The exact Lagrangian submanifold $\phi_H(L)$ is the graph of df and intersects L transversely. Note that if we work in a symplectic manifold M instead of T^*L, the cotangent bundle of L, we have to multiply H by a cut-off function equal to 1 near L.

With this perturbation, a critical point p of f is exactly an intersection point between L and $\phi_H(L)$. We then obtain

$$\mathcal{A}^H_{L,\phi_H(L)}(p) = -H(p). \tag{2.9}$$

For a good choice of almost-complex structure J and of shift in the definition of the degree of the intersection points, the matching associates a generator of the Floer cochain complex $CF(L, L; H, J)$ of degree i to a critical point of Morse index $n - i$, i.e a generator of the Morse cochain complex $CM(L, H)$ of index i [38].

This identification is associated to a correspondence between the moduli spaces. The Floer cochain complex $CF(L, L; H, J)$ is then identified with the Morse cochain complex $CM(L, H)$. Together with the Hamiltonian invariance of Floer cohomology, it implies the following proposition.

Proposition 2.2.32. *Let L and L' be two Lagrangian submanifolds which are Hamiltonian isotopic to each other, such that $[\omega] \cdot \pi_2(M, L) = [\omega] \cdot \pi_2(M, L') = 0$, then*

$$HF^*(L, L') \cong HF(L, L) \cong H^*(L; \mathbb{Z}/2).$$

We assume in this statement that the choice of shift in the definition of the degree for the generators of the Floer complexes make the degree equal to the Morse index.

Remark 2.2.33. Both the Floer cochain complex and the Morse cochain complex carry a natural filtration that will be discussed in details in Chapter 4. The filtration for the Floer complex is given by the action functional. The filtration for the Morse complex is given by the Morse function f.

However, with our choice of action for Floer cohomology, the identification between these two complexes does not respect these natural filtrations. Indeed the differential decreases the action functional in Floer cohomology while the differential increases the action in Morse cohomology. Consequently we have to consider the filtration given by $-f$. We denote $CF(L, L; H, J; \mathcal{A}^H_{L,L})$ the Floer cochain complex with the filtration given by $\mathcal{A}^H_{L,L}$ and $CM(L, f; -f)$ the Morse cochain complex with the filtration given by $-f$.

Together with the formula 2.9, this leads, for the ε-small Hamiltonian defined in the formula 2.8, to

$$CF(L, L; H, J; \mathcal{A}^H_{L,L}) \cong CM(L, H; -H).$$

Remark 2.2.34. We can choose the Morse function f to have a unique maximum and a unique minimum on L. This implies that there is a unique generator of $CM^0(L, H)$ and a unique generator of $CM^n(L, H)$. With the previously mentioned good choice of grading, this implies that there is also a unique generator of $CF^0(L, L; H, J)$ and a unique generator of $CF^n(L, L; H, J)$.

To finish this section, we make some remarks concerning the relation between action and energy, when there is a data perturbation $\mathcal{D}$. As we will later only be concerned about C^2-small perturbations, we will only describe this situation here. However, if one is interested in a particular Hamiltonian H, this Hamiltonian term has to be taken into account when defining the action of the generators of the Floer complex as mentioned in Subsection 2.2.5. We can choose a perturbation data to achieve transversality everywhere and conduct the same argument as the following.

Let p, q be two perturbed intersection points in $\chi(L, L')$ together with u, a J-holomorphic strip from p to q. When computing the energy $E(u)$, one has to take into account the perturbation data. So the energy writes down as

$$E(u) = \mathcal{A}_{L,L'}(p) - \mathcal{A}_{L,L'}(q) + f_{\mathcal{D}}(p, q),$$

where $f_{\mathcal{D}}$ is a function depending smoothly on $\mathcal{D}$ and such that $f_{\mathcal{D}}$ converges to zero when the Hamiltonian part of the perturbation data $\mathcal{D}$ goes to zero. According to Remark 2.2.26, this perturbation data can be chosen as small as wished, so that, for all $\varepsilon > 0$, we can find $\mathcal{D}$ such that

$$E(u) \leq \mathcal{A}_{L,L'}(p) - \mathcal{A}_{L,L'}(q) + \varepsilon, \tag{2.10}$$

and thus

$$\mathcal{A}_{L,L'}(q) \leq \mathcal{A}_{L,L'}(p) + \varepsilon.$$

This last remark will one of the key arguments in Section 4.2 to define persistence modules and barcodes associated to Lagrangian Floer cohomology.

Chapter 3

Persistence modules and barcodes

These notions come from Topological data analysis and the work of Edelsbrunner et al. [29] and Carlson et al. [20]. They have been introduced later in symplectic geometry by Polterovich and Shelukhin [75] although some germs of these notion were already present in the works from Barannikov [9] and Usher [92] [93]. We will here present the abstract framework, and we will discuss its relevance for symplectic geometry in the following chapter. The proof of the lemmas unproven in this chapter can be found in Chazal, De Silva, Glisse and Oudot's book [21] along with much more discussions. One can also refer to [42, 24, 19].

3.1 Persistence modules

Definition 3.1.1. *A persistence module over a field $\mathbb{K}$ is a family $(V^t)_{t\in\mathbb{R}}$ of finite dimensional vector spaces over $\mathbb{K}$ equipped with a doubly-indexed family of linear maps, called structure maps, $i_t^s : V^s \to V^t$, for all $s \leq t \in \mathbb{R}$ satisfying:*

1. *$V^t = 0$ for $t \ll 0$,*
2. *for all $s, t, r \in \mathbb{R}$, such that $r \leq s \leq t$, we have $i_t^s \circ i_s^r = i_t^r$ and $i_s^s = \mathrm{Id}_{V^s}$,*
3. *for all $r \in \mathbb{R}$, there is $\varepsilon > 0$ such that i_t^s are isomorphisms for all $r - \varepsilon < s \leq t \leq r$,*
4. *there is a set of points $S(V) \subset \mathbb{R}$ such that for all $r \in \mathbb{R} \setminus S(V)$, there exists $\varepsilon > 0$ such that i_t^s are isomorphisms for all $r - \varepsilon < s \leq t < r + \varepsilon$.*

We will denote the persistence module V or (V, i).

We will denote by V^∞ the direct limit

$$V^\infty = \varinjlim_{t\to+\infty} V^t,$$

together with $i^s : V^s \to V^\infty$ the natural map.

Definition 3.1.2. *$S(V)$ is called the spectrum of V.*

Remark 3.1.3. If $S(V)$ is finite, then for s, t large enough we may assume $V^s = V^t$ and i_t^s is equal to the identity. These persistence modules are said to be *tame* or of *finite type*. For s large enough, we then have $V^s = V^\infty$, and $i_t^s = i^s$.

Lemma 3.1.4. *If $a < b$ are two consecutive points in $S(V)$, then for all $a < s \leq t \leq b$, i_t^s is an isomorphism.*

We can define a selector for persistence modules, which takes its values in $S(V)$. This definition is to be related to the notion of action selectors which is one of the main tools to study C^0-symplectic geometry, as we will see in the next section.

Definition 3.1.5. *Let (V, i) be a persistence module and $\alpha \in V^\infty \setminus \{0\}$. We set*

$$c(\alpha, V) = \inf\{s \in \mathbb{R}, \quad \alpha \in \operatorname{Im} i^s\}.$$

Persistence modules being defined, we can now define morphisms of such objects. This is essential because it is the key to equip the space of persistence modules with a distance.

Definition 3.1.6. *Let (V, i) and (V', i') be two persistence modules. A morphism of persistence modules $h : (V, i) \to (V', i')$ is a family of morphisms $h^t : V^t \to V'^t, t \in \mathbb{R}$ compatible with the structure maps, i.e., for all $s \leq t \in \mathbb{R}$, the following diagram is commutative*

$$\begin{tikzcd} V^t \arrow[r, "h^t"] & V'^t \\ V^s \arrow[u, "i_t^s"] \arrow[r, "h^s"'] & V'^s. \arrow[u, "i'^s_t"'] \end{tikzcd}$$

The kernel and image of a morphism of persistence modules $h : (V, i) \to (V', i')$ are the families of vector spaces $(\ker h_t \subset V^t)_{t \in \mathbb{R}}$ and $(\operatorname{Im} h_t \subset V'^t)_{t \in \mathbb{R}}$. Since the structure maps i and i' restrict to each family, $(\ker h, i)$ and $(\operatorname{Im} h, i')$ are also persistence modules.

Let us define a natural operation on persistence modules: the shift.

Definition 3.1.7. *Let (V, i) be a persistence module, and $\delta \geq 0$. The δ-shifted persistence module $(V[\delta], i[\delta])$ is the persistence module with vector spaces $V[\delta]^t = V^{t+\delta}$ and maps $i[\delta]_t^s = i_{t+\delta}^{s+\delta}$. We will denote $sh(\delta)_V : V \to V[\delta]$ the natural shift morphism of persistence modules given by*

$$sh(\delta)_V^t = i_{t+\delta}^t : V^t \to V^{t+\delta}.$$

A morphism of persistence modules $h : V \to V'$ naturally induces a shifted morphism of shifted persistence modules $h[\delta] : V[\delta] \to V'[\delta]$.

For $\delta \leq 0$, we denote $V[\delta]$ the persistence module such that $V[\delta][-\delta] \cong V$.

We have now defined all we need to introduce a distance on the space of persistence modules.

Definition 3.1.8. *Let V and V' be two persistence modules and let $\delta, \varepsilon \geq 0$. They are* (δ, ε)-interleaved *if there exist two morphisms of persistence modules $f : V \to V'[\delta]$ and $g : V' \to V'[\varepsilon]$ such that*

$$g[\delta] \circ f = sh(\delta+\varepsilon)_V \text{ and } f[\varepsilon] \circ g = sh(\delta+\varepsilon)_{V'},$$

i.e. the two following diagrams commute for all $s \leq t$

$$\begin{tikzcd}
V^t \arrow[r, "f^t"] \arrow[rr, bend left, "i^t_{t+\delta+\varepsilon}"] & V'^{t+\delta} \arrow[r, "g^{t+\delta}"] & V^{t+\delta+\varepsilon} \\
V^s \arrow[u, "i^s_t"] \arrow[r, "f^s"] \arrow[rr, bend right, "i^s_{s+\delta+\varepsilon}"'] & V'^{s+\delta} \arrow[u, "i'^{s+\delta}_{t+\delta}"] \arrow[r, "g^{s+\delta}"] & V^{s+\delta+\varepsilon} \arrow[u, "i'^{s+\delta+\varepsilon}_{t+\delta+\varepsilon}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
V'^t \arrow[r, "g^t"] \arrow[rr, bend left, "i'^t_{t+\delta+\varepsilon}"] & V^{t+\varepsilon} \arrow[r, "f^{t+\varepsilon}"] & V'^{t+\delta+\varepsilon} \\
V'^s \arrow[u, "i'^s_t"] \arrow[r, "g^s"] \arrow[rr, bend right, "i'^s_{s+\delta+\varepsilon}"'] & V^{s+\varepsilon} \arrow[u, "i^{s+\varepsilon}_{t+\varepsilon}"] \arrow[r, "f^{s+\varepsilon}"] & V'^{s+\delta+\varepsilon} \arrow[u, "i'^{s+\delta+\varepsilon}_{t+\delta+\varepsilon}"]
\end{tikzcd}.$$

The pair (f, g) is called a (δ, ε)-interleaving. If $\varepsilon = \delta$, it is a δ-interleaving, *and V, V' are δ-interleaved.*

The distance on persistence modules is then naturally defined from this notion of interleaving:

Definition 3.1.9. *Let V and V' be two persistence modules. The interleaving distance between V and V' is*

$$d_{inter}(V, V') = \inf\{\delta \mid V, V' \text{ are } \delta\text{-interleaved}\}.$$

If there is no interleaving between V and V', then $d_{inter}(V, V') = +\infty$.

This distance is non degenerate in the following sense: two persistence modules are 0-interleaved if and only if they are isomorphic.

Proposition 3.1.10. *The interleaving distance satisfies the triangle inequality. Let U, V and W be three persistence modules, then*

$$d_{inter}(U, W) \leq d_{inter}(U, V) + d_{inter}(V, W).$$

Proof. Let (f_1, g_1) be a δ_1-interleaving between U and V and let (f_2, g_2) be a δ_2-interleaving between V and W.

$$U \xrightarrow{f_1} V \xrightarrow{f_2} W$$

$$U \xleftarrow{g_1} V \xleftarrow{g_2} W$$

Set $f = f_2 \circ f_1$ and $g = g_1 \circ g_2$. These maps are a $\delta = \delta_1 + \delta_2$-interleaving between U and W. Indeed we have

$$\begin{aligned}
g[\delta] \circ f &= g_1[\delta + \delta_2] \circ g_2[\delta] \circ f_2[\delta_1] \circ f_1 = g_1[\delta + \delta_2] \circ sh(2\delta_2)_{V[\delta_1]} \circ f_1 \\
&= g_1[2\delta_2 + \delta_1] \circ f_1[2\delta_2] \circ sh(2\delta_2)_U = sh(2\delta_1)_{U[2\delta_2]} \circ sh(2\delta_2)_U \\
&= sh(2\delta)_U, \\
f[\delta] \circ g &= f_2[\delta + \delta_1] \circ f_1[\delta] \circ g_1[\delta_2] \circ g_2 = f_2[\delta + \delta_1] \circ sh(2\delta_1)_{V[\delta_2]} \circ g_2 \\
&= f_2[\delta + \delta_1] \circ g_2[2\delta_1] \circ sh(2\delta_1)_W = sh(2\delta_2)_{W[2\delta_1]} \circ sh(2\delta_1)_W \\
&= sh(2\delta)_W.
\end{aligned}$$

Taking the infimum over δ_1 and δ_2 concludes the proof. □

Before moving on, we present a classical example of persistence module.

Example 3.1.11. *Let X be a compact topological space, together with a continuous function $f : X \to \mathbb{R}$. For $t \in \mathbb{R}$, we denote $f^{\leq t}$ the set of points x in X such that $f(x) \leq t$. This induces a filtration on the singular chain complex and hence a persistence module V. The vector spaces are $V^t = H_*(f^{\leq t}, \mathbb{R})$ and the structure maps are induced by the inclusions $f^{\leq s} \subset f^{\leq t}$ for all $s \leq t$. If the function f is Morse, the spectrum of V is given by the critical values and consequently V is of finite type. Suppose that g is another function on the same topological space X, and set $\delta = \max(g - f)$ and $\varepsilon = \max(f - g)$. We have the following diagrams whose arrows are induced by inclusions, hence obviously commute.*

$$\begin{tikzcd}
f^{\leq t} \arrow[r] \arrow[rr, bend left] & g^{\leq t+\delta} \arrow[r] & f^{\leq t+\delta+\varepsilon} \\
f^{\leq s} \arrow[u] \arrow[r] \arrow[rr, bend right] & g^{\leq s+\delta} \arrow[u] \arrow[r] & f^{\leq s+\delta+\varepsilon} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
g^{\leq t} \arrow[r] \arrow[rr, bend left] & f^{\leq t+\varepsilon} \arrow[r] & g^{\leq t+\delta+\varepsilon} \\
g^{\leq s} \arrow[u] \arrow[r] \arrow[rr, bend right] & f^{\leq s+\varepsilon} \arrow[u] \arrow[r] & f^{\leq s+\delta+\varepsilon} \arrow[u]
\end{tikzcd}$$

This implies that the (δ, ε)-interleaving diagrams of $H_*(f^{\leq t}, \mathbb{R})$ and $H_*(g^{\leq t}, \mathbb{R})$ commute. As a consequence these two persistence modules are (δ, ε)-interleaved.

Finally the following property will also be used.

Proposition 3.1.12. *Let (V, i) and (V', i') be two persistence modules. Then the direct sum*

$$(V, i) \oplus (V', i') = (V \oplus V', i \oplus i')$$

is a persistence module as well. The structure maps are given for all $s \leq t$ by

$$(i \oplus i')_t^s = i_t^s \oplus i_t'^s.$$

Remark 3.1.13. The persistence modules actually form a category usually denoted **pmod**. This is in fact an abelian category, with respect to the sum given by the previous proposition.

Remark 3.1.14. Let V and V' be two (δ, ε)-interleaved persistence modules such that V^∞ and V'^∞ are isomorphic vector spaces. Taking the limit when t goes to $+\infty$, in the interleaving distance diagrams, we obtain that for all $\alpha \in V^\infty$,

$$-\delta \leq c(\alpha, V) - c(\psi(\alpha), V') \leq \varepsilon,$$

where ψ is the isomorphism induced by the (δ, ε)-interleaving.

3.2 Barcodes

A more intuitive way to “draw" these persistence modules is based on a “structure theorem" for persistence modules. Before stating this theorem and presenting barcodes, we introduce some definitions.

Definition 3.2.1. *Let J be a non-empty interval in $\mathbb{R}$ of the form $(a, b]$ or $(a, +\infty)$, with a and b in $\mathbb{R}$. The* interval module $I = \mathbb{K}^J$ *is the persistence module with vector spaces*

$$I^t = \begin{cases} \mathbb{K}, & \text{if } t \in J \\ 0, & \text{otherwise,} \end{cases}$$

and structure maps

$$i_t^s = \begin{cases} \mathrm{Id}, & \text{if } s, t \in J, \\ 0, & \text{otherwise.} \end{cases}$$

We can now state the structure theorem which is proven in [25].

Theorem 3.2.2. *For any persistence module V, there is a unique collection of pairwise distinct intervals $(J_i)_{i \in \mathcal{I}}$ of the form $(a_i, b_i]$ or $(a_i, +\infty)$, with $a_i, b_i \in S(V)$, and multiplicity $m_i \in \mathbb{N}$ such that*

$$V \cong \bigoplus_{i \in \mathcal{I}} (\mathbb{K}^{J_i})^{m_i}.$$

From this theorem, we can define the barcode associated to V. Let us firsttly define a *multiset*.

Definition 3.2.3. *A* multiset *is a pair $B = (S, m)$ where S is a set and $m : S \to \mathbb{N} \cup \{+\infty\}$ is the multiplicity function. This function tells how many times each $s \in S$ occurs in B.*

Definition 3.2.4. *We denote by $\mathcal{B}(V)$ the multiset containing m_J copies of each interval J appearing in the structure theorem, and $\mathcal{I}(\mathcal{B}(V))$ the set of intervals J_i without multiplicity. $\mathcal{B}(V)$ is called the barcode associated to V, and the intervals J_i are called bars. We will denote*

$$\mathcal{B}(V) = \bigoplus_{J \in \mathcal{I}(\mathcal{B}(V))} J^{m_J}.$$

Remark 3.2.5. The starting points of the semi-infinite bars are exactly the values of the selectors for V introduced in Definition 3.1.5. Each possible value of these selectors is the starting point of a semi-infinite bar.

We can equip the set of barcodes with a distance, which is called the bottleneck distance.

Definition 3.2.6. *Let I be a non-empty interval of the form $(a, b]$ or $(a, +\infty)$, and $\delta \in \mathbb{R}$ such that $2\delta < b - a$. We denote $I^{-\delta}$ the interval $(a - \delta, b + \delta]$ or $(a - \delta, +\infty)$. Let B and B' be two barcodes, and $\delta \geq 0$. They admit a δ-matching if we can forget in both of them some bars of length smaller than 2δ to get two barcodes $\bar{B}$ and $\bar{B}'$ and find a bijection $\phi : \bar{B} \to \bar{B}'$ such that if $\phi(I) = J$, then*

$$I \subset J^{-\delta} \text{ and } J \subset I^{-\delta}.$$

As it was the case for persistence modules, the definition of the distance follows:

Definition 3.2.7. *Let B and B'. The bottleneck-distance between them is*

$$d_{bottle}(B, \mathcal{B}') = \inf\{\delta | \ B \text{ and } B' \text{ admit a } \delta\text{-matching}\}.$$

The bottleneck distance is non-degenerate: if B and B' are two barcodes such that $d_{bottle}(B, B') = 0$, then $B = B'$.

The two notions of interleaving and bottleneck distance are closely related as shown in the following *isometry theorem* [10].

Theorem 3.2.8. *Let V, V' be two persistence modules. Then*

$$d_{inter}(V, V') = d_{bottle}(\mathcal{B}(V), \mathcal{B}(V')).$$

Remark 3.2.9. As for persistence modules, given a barcode B and $\delta \in \mathbb{R}$, we will denote $B[\delta]$ the barcode obtained from B by an overall shift of δ. If B is a barcode associated to a persistence module V, then $B[\delta]$ is the barcode associated with the persistence module $V[\delta]$.

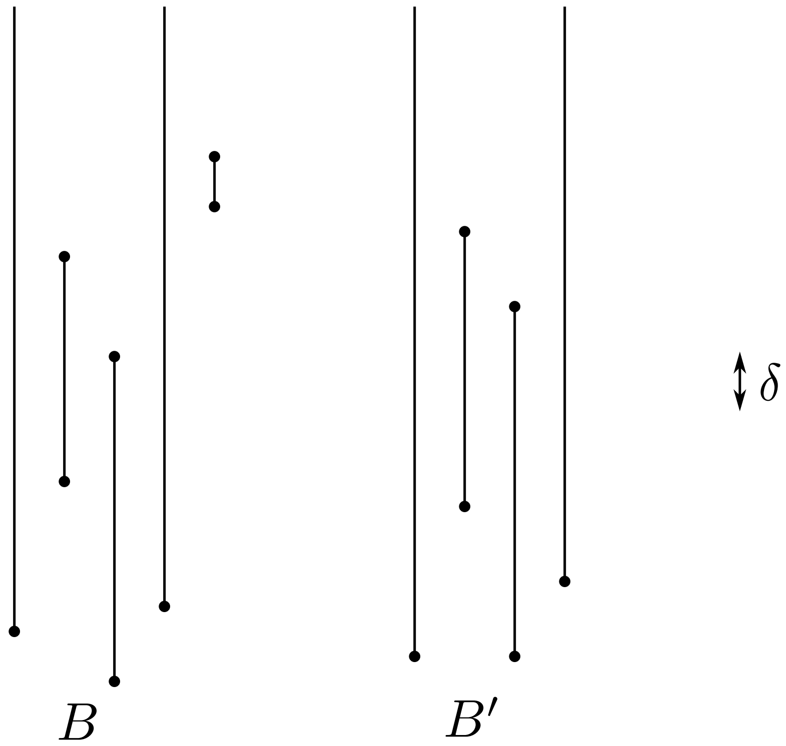

Figure 3.1 – Two barcodes B and B' such that $d_{bottle}(B, B') \leq \delta$

3.3 A bit of topology

One of the main objectives of this work is to obtain new information concerning C^0-symplectic topology using the technology of barcodes applied to Floer homology. We will be working on cases where the number of generators of the chain complex is finite. Consequently, most of our persistence modules will be of finite type, and this will naturally be translated into barcodes.

Definition 3.3.1. *A barcode is said to be* finite *if it contains finitely many intervalls counted with multiplicity. We will denote $\mathcal{B}_f$ the set of finite barcodes.*

This finiteness condition is closely related to the one related to persistence modules, as stated by the following lemma.

Lemma 3.3.2. *Let V be a persistence module and $\mathcal{B}(V)$ its associated barcode. Then V is of finite type if and only if $\mathcal{B}(V)$ is finite.*

Since we study C^0 objects in a world of smoothness, we need, at some point, to take limits, and hence limits of finite barcodes. Thus, the question of closedness and completeness naturally arise. That will be achieved through the set-up given in the following definition.

Definition 3.3.3. *We denote $\mathcal{B}$ the set of barcodes satisfying the following condition: for all $\varepsilon > 0$, the number of bars of length greater or equal to ε is finite.*

Remark 3.3.4. In litterature [21, 12] such barcodes are often referred as "q-tame barcodes".

The following proposition, proved by Bubenik and Vergili [12], justifies the introduction of the set $\mathcal{B}$.

Proposition 3.3.5. *The space $\mathcal{B}$ is complete.*

Aside from completeness, the following proposition explain why the set $\mathcal{B}$ is of particular interest for us, as we will be working with limits of sequences of finite barcodes.

Proposition 3.3.6. *The set $\mathcal{B}_f$ is dense in $\mathcal{B}$ for the topology induced by the bottleneck distance.*

Proof. Let us pick $B \in \mathcal{B}$. We set $(B_n)_{n\in\mathbb{N}}$ a sequence of barcodes defined by

$$B_n = \bigoplus_{\substack{I\in\mathcal{I}(B)\\ l(I)\geq\frac{1}{n}}} I^{m_I},$$

where $l(I)$ is the length of the interval I, and m_I its multiplicity in B. By definition of $\mathcal{B}$, for all $n \in \mathbb{N}$, B_n is a finite barcode, and for all $n \in \mathbb{N}$, B_n satisfies

$$d_{bottle}(B_n, B) = \frac{1}{2n}.$$

This implies that $(B_n)_{n\in\mathbb{N}} \subset \mathcal{B}_f$ converges to B for the bottleneck distance. □

When studying homology or cohomology, the presence of a $\mathbb{Z}$-grading is important. We can easily incorporate this notion to obtain those of persistence modules of $\mathbb{Z}$-graded vector spaces such that the structure maps respect the grading. For instance, if we have a family of persistence modules V_r indexed by the integers, the persistence module

$$(\oplus_{r\in\mathbb{Z}}V_r, \oplus_{r\in\mathbb{Z}}i_r)$$

has such a structure. We can then define an interleaving distance as

$$d_{inter}(V, V') = \max_{r\in\mathbb{Z}}\{d_{inter}(V_r, V'_r)\},$$

where $V = \oplus_r V_r$ and $V' = \oplus_r V'_r$ are two $\mathbb{Z}$-graded persistence modules.

We can incorporate this notion in the same way for barcodes. A $\mathbb{Z}$-graded barcode is a family of barcodes $(B_r)_{r\in\mathbb{Z}}$. We denote

$$B = \bigoplus_{r\in\mathbb{Z}} B_r,$$

the $\mathbb{Z}$-graded barcode B associated to the family $(B_r)_{r\in\mathbb{Z}}$. Then, as for persistence modules, the bottleneck distance for graded barcodes is defined by

$$d_{bottle}(B,B') = \max_{r\in\mathbb{Z}}\{d_{bottle}(B_r,B'_r)\},$$

where $B = \bigoplus_{r\in\mathbb{Z}} B_r$ and $B' = \bigoplus_{r\in\mathbb{Z}} B'_r$.

A finite graded barcode $B = (B_r)_{r\in\mathbb{Z}}$ is a graded barcode such that there is finitely many bars in the whole family $(B_r)_{r\in\mathbb{Z}}$. Since we will always consider graded barcodes, we also denote $\mathcal{B}_f$ the set of finite graded barcodes. By abuse of notations, we also denote $\mathcal{B}$ the set of graded barcodes $B = (B_r)_{r\in\mathbb{Z}}$ such that for all $\varepsilon > 0$ there is a finite number of bars of length greater than ε in the whole family $(B_r)_{r\in\mathbb{Z}}$. From now on, when using the notation $\mathcal{B}$ or $\mathcal{B}_f$, we will always refer to their graded version.

Remark 3.3.7. Let $B = (B_r)_{r\in\mathbb{Z}}$, if B is finite, then B_r has more than 0 bars for only finitely many $r \in \mathbb{Z}$. In the same way, if $B \in \mathcal{B}$, then for all $\varepsilon > 0$, B_r has more than 0 bars of length greater than 0 for only finitely many $r \in \mathbb{Z}$.

With these graded barcodes, we still have

$$\overline{\mathcal{B}_f} = \mathcal{B},$$

and for the same reason as in the non-graded case, $\mathcal{B}$ is complete.

Before moving on and defining barcodes for objects of real interest, we have to make some observations regarding the connectedness of $\mathcal{B}$.

First of all, let us introduce the map that counts the number of semi-infinite bars in each degree.

Definition 3.3.8. *We define $\sigma^\infty : \mathcal{B} \to \mathbb{N}^{\mathbb{Z}}$ by*

$$\sigma^\infty(B) = (\sigma^n)_{n\in\mathbb{Z}} \textit{ with } \forall n \in \mathbb{Z}, \quad \sigma^n = \sum_{\substack{I\in\mathcal{I}(B)\\ l(I)=+\infty \\ \mathrm{Ind}(I)=n}} m_I.$$

This map will be very useful. Indeed the following property shows that its relation with the bottleneck distance is quite straightforward.

Proposition 3.3.9. *For all $B, B' \in \mathcal{B}$,*

$$d_{bottle}(B,B') < +\infty \iff \sigma^\infty(B) = \sigma^\infty(B').$$

Proof. We will first prove that $d_{bottle}(B,B') < +\infty \Rightarrow \sigma^\infty(B) = \sigma^\infty(B')$. We will actually prove the converse: assume that $B, B' \in \mathcal{B}$ have different images by σ^∞. This means that there is a degree r for which $\sigma^r(B) \neq \sigma^r(B')$. By definition of a δ-matching, since $\delta > 0$ is finite, a semi-infinite bar of B has to be associated to a semi-infinite bar of B' of same degree in a δ-matching. Then, since there is a degree for which B and B' do not have the same number of semi-infinite bars, such a δ-matching is impossible, and this holds for all $\delta > 0$. Consequently $d_{bottle}(B, B') = +\infty$, which concludes this part of the proof.

Now we prove: $\sigma^\infty(B) = \sigma^\infty(B') \Rightarrow d_{bottle}(B, B') < +\infty$. Let $B, B' \in \mathcal{B}$ such that $\sigma^\infty(B) = \sigma^\infty(B')$. We denote

$$B^\infty = \bigoplus_{\substack{I \in \mathcal{I}(B) \\ l(I) = +\infty}} I^{m_I} \text{ and } B'^\infty = \bigoplus_{\substack{J \in \mathcal{I}(B') \\ l(J) = +\infty}} J^{m_J}.$$

Let us recall that each barcode B^∞ and B'^∞ has the same finite number of bars with multiplicity in each degree. So we can define $A \in \mathbb{R}$ to be the diameter of $(S(B) \cup S(B'))$, and then B^∞ and B'^∞ admit an A-matching. Now, denote

$$B^f = \bigoplus_{\substack{I \in \mathcal{I}(B) \\ l(I) < +\infty}} I^{m_I} \text{ and } B'^f = \bigoplus_{\substack{J \in \mathcal{I}(B') \\ l(J) < +\infty}} J^{m_J}.$$

Since for all $\varepsilon > 0$, each barcode has only a finite number of bars of length greater than ε, we can define $2C \in \mathbb{R}$ to be the maximal length of a bar in $B^f \cup B'^f$. Then, B^f and B'^f admit a C-matching. So finally we have that B and B' admit a $\max(A, C)$-matching and then $d_{bottle}(B, B') < +\infty$. □

Remark 3.3.10. The barcode B^∞ introduced in the proof strongly relates to what we defined as V^∞ after Definition 3.1.1. The number of bars in each degree $r \in \mathbb{Z}$ is equal to the dimension of the degree r component of V^∞.

This proposition immediately implies the following corollary, which is topologically really useful.

Corollary 3.3.11. *σ^∞ is locally constant.*

Thanks to this corollary and Definition 3.3.8 of σ^∞, we can now state the following proposition.

Proposition 3.3.12. *The connected components of $\mathcal{B}$ are indexed by the graded number of semi-infinite bars, i.e. two barcodes belong to the same connected component of $\mathcal{B}$ if and only if they have the same number of semi-infinite bars in each degree. Moreover the connected components are path-connected.*

Proof. Since the map σ^∞ is locally constant, it is constant on the connected components of $\mathcal{B}$. This means that if two barcodes $B, C \in \mathcal{B}$ are in the same connected component,

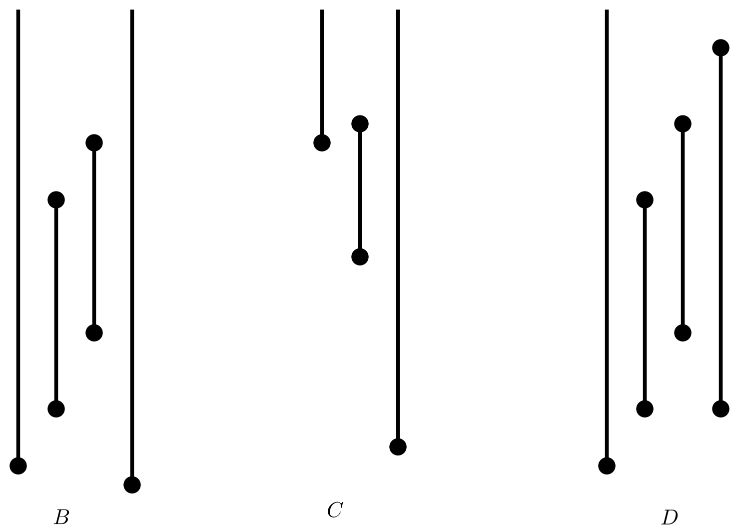

Figure 3.2 – The barcodes B and C are in the same connected component, while D is not.

then $\sigma^\infty(B) = \sigma^\infty(C)$, i.e. B and C have the same number of semi-infinite bars in each degree.

Conversely, let B be a barcode in $\mathcal{B}$. With $r \in \mathbb{Z}$ denoting the degree, we write $B = \bigoplus_{r\in\mathbb{Z}} B^r$ and denote

$$B^r = \bigoplus_{i\in\mathcal{I}_B^r}(a_i, +\infty) \oplus \bigoplus_{i\in\mathcal{J}_r}(a_j, b_j].$$

We define for all $t \in [0,1]$

$$B_t^r = \bigoplus_{i\in\mathcal{I}_B^r}((1-t)a_i, +\infty) \oplus \bigoplus_{i\in\mathcal{J}_B^r}((1-t)a_j, (1-t)b_j],$$

and $B_t = \bigoplus_{r\in\mathbb{Z}} B_t^r$. The path $(B_t)_{t\in[0,1]}$ is a continuous path of barcodes from B to

$$B_0(B) = \bigoplus_{r\in\mathbb{Z}} \bigoplus_{i\in\mathcal{I}_B^r}(0, +\infty).$$

Let B and C be two barcodes in $\mathcal{B}$ such that they have the same number of semi-infinite bars in each degree. Then for all $r \in \mathbb{Z}$, $\mathcal{I}_B^r = \mathcal{I}_C^r$ so $B_0(B) = B_0(C)$.

This implies that the two barcodes B and C are isotopic and thus in the same connected component of $\mathcal{B}$ which concludes the proof of this proposition. □

The following corollary is a direct and obvious consequence of the previous Proposition 3.3.12, but its formulation will be useful later.

Corollary 3.3.13. *Let $(B^t)_{t\in[0;1]}$ be a continuous path of graded barcodes. Then for all $t \in [0;1]$ and for all k, the number of semi-infinite bars of B_k^t is constant with respect to* *the parameter t.*

Let us now introduce another space of barcodes which will allow us to get our desired results.

Definition 3.3.14. *We define $\hat{\mathcal{B}}$ as the set of barcodes $\mathcal{B}$ quotiented by the action by overall shift of $\mathbb{R}$ on $\mathcal{B}$, i.e. B and B' represent the same class in $\hat{\mathcal{B}}$ if and only if there is $c \in \mathbb{R}$ such that $B = B'[c]$.*

Since the action of $\mathbb{R}$ by an overall shift on $\mathcal{B}$ is free and proper, all the above mentioned topological properties also hold for $\hat{\mathcal{B}}$.

The only remaining question is the completeness of $\hat{\mathcal{B}}$. The distance on $\hat{\mathcal{B}}$ is given by the Hausdorff distance between the equivalence classes which will be denoted d_H.

Lemma 3.3.15. *The set $\hat{\mathcal{B}}$ is complete for the distance d_H.*

Proof. Let $(\hat{b}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\hat{\mathcal{B}}$. There is a strictly increasing sequence $(N_p)_{p\in N}$ such that
$$\forall k \in \mathbb{N}, \quad d_H(\hat{b}_{N_p} - \hat{b}_{N_p+k}) \leq \frac{1}{2^p}.$$
Let us choose $b_0 \in \mathcal{B}$ a representative of $\hat{b}_{N_0}$ and $b'_1 \in \mathcal{B}$ a representative of $\hat{b}_{N_1}$. Then, by definition of the equivalence classes, there exists $c_1 \in \mathbb{R}$ such that
$$d_{bottle}(b_0, b'_1[c_1]) \leq \frac{1}{2}.$$
Indeed, for all $c \in \mathbb{R}$ and all $b, b' \in \mathcal{B}$, we have $d_{bottle}(b, b') = d_{bottle}(b[c], b'[c])$. Now set $b_1 = b'_1[c_1]$. We will inductively construct a sequence $(b_p)_{p\in\mathbb{N}}$ such that for all p, the barcode b_p is a representative of $\hat{b}_{N_p}$ and $d_{bottle}(b_p, b_{p+1}) \leq \frac{1}{2^{p+1}}$. Let $p_0 \in \mathbb{N}$ and assume that for all $p \in \{0, ..., p_0\}$, the barcode b_p is constructed.

The barcode b_{p_0} represents the class of $\hat{b}_{N_{p_0}}$. Let us fix b'_{p_0+1} representing the class of $\hat{b}_{N_{p_0}+1}$. Since $d_H(\hat{b}_{N_{p_0}}, \hat{b}_{N_{p_0}+1}) \leq \frac{1}{2^{p_0+1}}$, there exists c_{p_0+1} such that
$$d_{bottle}(b_{p_0}, b'_{p_0+1}[c_{p_0+1}]) \leq \frac{1}{2^{p_0+1}}.$$
We define $b_{p_0+1} = b'_{p_0+1}[c_{p_0+1}]$. And thus we obtain our sequence $(b_p)_{p\in\mathbb{N}}$ inductively.

By the triangle inequality of Proposition 3.1.10 and a classical high school result, we obtain for all $p, k \in \mathbb{N}$
$$d_{bottle}(b_p, b_{p+k}) \leq \frac{1}{2^p}.$$
Consequently $(b_p)_{p\in\mathbb{N}}$ is a Cauchy sequence which converges to a barcode $b \in \mathcal{B}$ since $\mathcal{B}$ is complete. This straightforwardly implies that $(\hat{b}_n)_{n\in\mathbb{N}}$ converges to $\hat{b}$, the equivalence class of b, and so $\hat{\mathcal{B}}$ is complete. $\square$

Chapter 4

Barcodes and action selectors in symplectic topology

As mentioned before, the terminology of barcodes was brought into symplectic topology by Polterovich and Shelukhin [75] quickly followed by Usher-Zhang's work [94]. Again, note that germs of this theory were already present in the work of Barannikov [9] and Usher [92, 93].

Action selectors were introduced by Viterbo [96] for Lagrangian submanifolds in a cotangent bundle using generating functions theory. After this construction, it was adapted to many contexts by Oh [67], Schwarz [79], Leclercq [58] and others... Even if actions selectors appeared before the notion of barcodes in symplectic topology, they can be defined naturally using the theory of persistence modules.

4.1 Morse case

Let M be a compact manifold together with a Morse function $f : M \to \mathbb{R}$. For all $t \in \mathbb{R}$, we define $V^t(f) = H_*(f, \{f < t\}, \mathbb{Z}/2)$ the Morse homology of sublevel sets of f with coefficients in $\mathbb{Z}/2$. Since we saw in Remark 2.1.6 that the differential decreases the value of the function, the generators form a subcomplex and hence this homology is well defined. Moreover, we set $i_t^s : V^s(f) \to V^t(f)$ to be the maps induced by the inclusion of subcomplexes. Then $(V(f), i)$ is a persistence module and its spectrum $S(V)$ coincides with the critical values of the Morse function f.

For a given degree $r \in \mathbb{Z}$, one can set $V_r^t(f) = H_r(f, \{f < t\}, \mathbb{Z}/2)$. In this case, the spectrum of V_r is contained in the set of values of f at the critical points of index r or $r+1$. This equips $V(f)$ with a structure of persistence module of $\mathbb{Z}$-graded vector spaces, and $V(f) = \oplus V_r(f)$.

The barcode $\mathcal{B}(V(f))$ is then a finite barcode, and the endpoints of the different bars are the critical values of f. A finite bar in degree r is a bar between a critical value

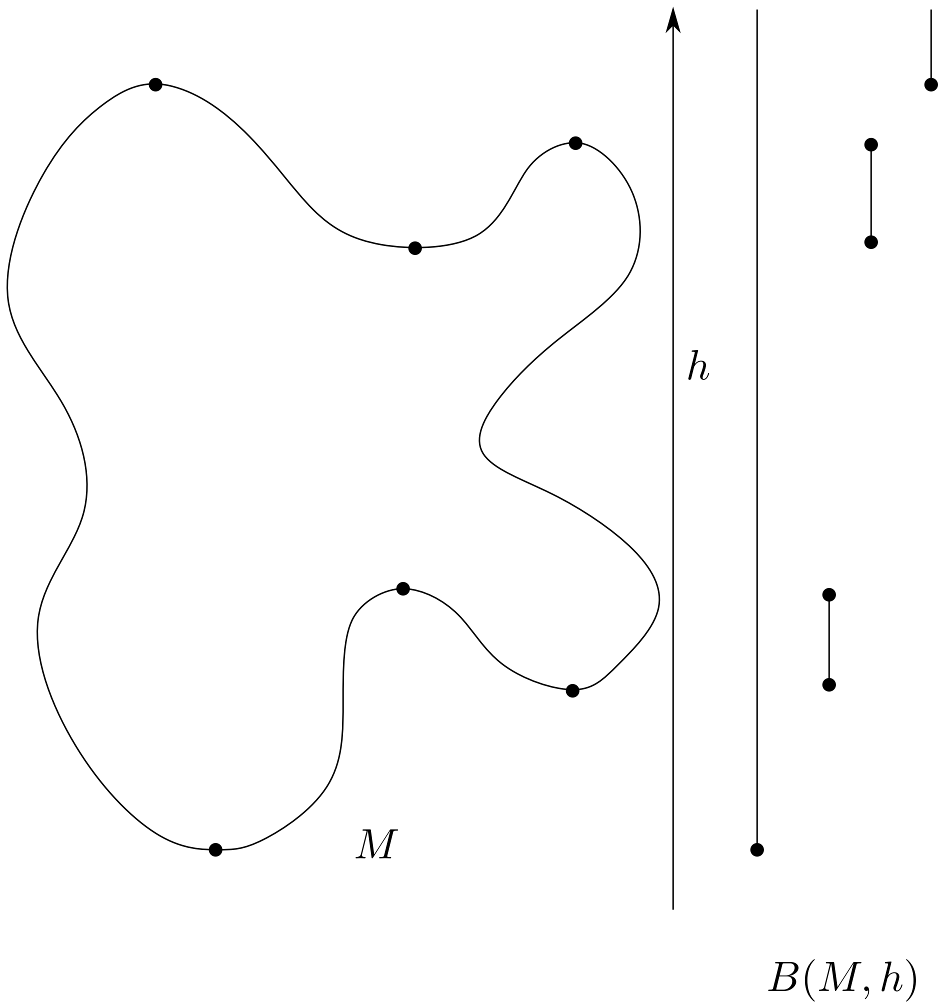

Figure 4.1 – Barcodes of M with the height function h

of f corresponding to a point of index r and a critical value of f corresponding to a critical point of index $r+1$. Indeed, a given non-zero class α in $H_r(f, \{f < t\}, \mathbb{Z}/2)$, $t \in \mathbb{R}$ disappears from the sublevel homology $H_r(f, \{f < s\}, \mathbb{Z}/2)$, for $s > t$ when there is $\beta \in H_{r+1}(f, \{f < s\}, \mathbb{Z}/2)$ such that $\alpha = \partial\beta$. If there is no such s, then α is non-zero in $H_*(f, \mathbb{Z}/2)$, and its critical value is the starting point of a semi-infinite bar.

Moreover, in terms of persistence modules, one can also naturally identify V^∞ from Definition 3.1.1 with the total Morse homology $H_*(f, \mathbb{Z}/2)$ with an isomorphism $\psi : V^\infty \to H_*(f, \mathbb{Z}/2)$. For all $\alpha \in H_*(f, \mathbb{Z}/2) \setminus 0$, we can redefine the *action selector or spectral invariant* for α using the vocabulary of persistence modules:

$$l(\alpha, f) = \inf\{t \in \mathbb{R}, \quad \psi^{-1}(\alpha) \in \operatorname{Im} i^t\},$$

where we recall that i^t is the structure map from V^t to V^∞. This corresponds to the selector for persistence modules introduced in Section 3.1. These spectral invariants are exactly the starting points of semi-infinite bars in the barcode $\mathcal{B}(V(f))$.

One of the main sources of motivation for studying barcodes in symplectic geometry is that, thanks to Remark 2.1.5 and in the view of Example 3.1.11, for all f and g Morse functions on M,

$$d_{inter}(V(f), V(g)) \leq \|f - g\|,$$

where $\|\cdot\|$ denotes the uniform norm.

This can be directly interpreted in terms of Morse homology. Indeed, to prove that the Morse homology of a compact manifold does not depend on the choice of the function, we have to construct a chain map. For two Morse functions f and g, this chain map is defined by counting anti-gradient trajectories of a homotopy between f and g. We know that this map induces an isomorphism on homology and that it cannot change the value of a critical point by more than $\|f-g\|$. So there is a chain map between $C_*(f, \{f < t\}, \mathbb{Z}/2)$ and $C_*(g, \{g < t + \|f - g\|\}, \mathbb{Z}/2)$ and another one between $C_*(g, \{g < t\}, \mathbb{Z}/2)$ and $C_*(f, \{f < t = \|f - g\|\}, \mathbb{Z}/2)$, both being inclusions.

Finally we once again obtain

$$d_{inter}(V(f), V(g)) \leq \|f - g\|.$$

We see here that the barcode associated to a Morse function is C^0-continuous.

4.2 Lagrangian Floer cohomology

Let $(M, \omega = d\lambda)$ be a Liouville domain, and L, L' two closed exact Lagrangian submanifolds intersecting transversely, together with two primitive functions $f_L : L \to \mathbb{R}$ and $f_{L'} : L' \to \mathbb{R}$ such that $df_L = \lambda_{|L}$ and $df_{L'} = \lambda_{|L'}$. We assume that the Floer cohomology is well defined. For all $\kappa \in \mathbb{R}$, we define

$$CF^{*,\kappa}(L, L') = \mathrm{span}_{\mathbb{Z}/2} \left\{ z \in \chi(L, L'), \quad \mathcal{A}_{L,L'}(z) < \kappa \right\} \subset CF^*(L, L').$$

Let us recall that, for all $x \in CF^{*,\kappa}(L, L')$, Remark 2.2.16 tells:

$$\mathcal{A}_{L,L'}(\partial x) < \mathcal{A}_{L,L'}(x) < \kappa.$$

This means that $CF^{*,\kappa}(L, L')$ is in fact a subcomplex of $CF^*(L, L')$, and consequently we can define:

$$HF^{*,\kappa}(L, L') = H^*(CF^{*,\kappa}(L, L')).$$

Moreover, the inclusions of cochain complexes, i.e. $\forall \kappa' < \kappa \in \mathbb{R}$,

$$CF^{*,\kappa'}(L, L') \subset CF^{*,\kappa}(L, L')$$

induce maps $i_{\kappa}^{\kappa'}$ in cohomology which commute for $\kappa_1 < \kappa_2 < \kappa_3$, thus satisfying the property required for structure maps. Finally, $((HF^{*,\kappa}(L,L'))_{\kappa\in\mathbb{R}}, i)$ has the structure of a finite $\mathbb{Z}$-graded persistence module. We denote its associated graded barcode

$$\mathcal{B}^*(L,L') = \mathcal{B}\left((HF^{*,\kappa}(L,L'))_{\kappa\in\mathbb{R}}, i\right).$$

Since $\chi(L,L')$ is finite, $\mathcal{B}(L,L')$ is a finite barcode.

It is easy to recover the cohomology from the barcode. Indeed, by definition

$$\lim_{\substack{\rightarrow \\ \kappa\rightarrow\infty}} CF^{*,\kappa}(L,L') = CF^*(L,L')$$

and then

$$\lim_{\substack{\rightarrow \\ \kappa\rightarrow\infty}} HF^{*,\kappa}(L,L') = HF^*(L,L').$$

This means that $HF^*(L,L')$ corresponds to the bars that survive when κ goes to infinity, i.e.

Proposition 4.2.1. *The graded rank of $HF(L,L')$ is equal to the graded number of semi-infinite bars in $\mathcal{B}(L,L')$.*

As for Morse homology, and barcodes in general, we now define selectors. This selector, denoted by $l(\cdot,L,L')$, is defined as the action selector of Definition 3.1.5 applied to the persistence module $HF^{\kappa}(L,L')$. Let us give an explicit definition.

Definition 4.2.2. *To any $\alpha \in HF^*(L,L')\setminus\{0\}$, we associate*

$$l(\alpha,L,L') = \inf\{\kappa\in\mathbb{R},\quad \alpha\in \operatorname{Im} i^{\kappa} : HF^{*,\kappa}(L,L') \rightarrow HF^*(L,L')\}.$$

As for Morse homology, these numbers are exactly all the different starting points of the semi-infinite bars, i.e. each semi-infinite bar corresponds to some non-zero $\alpha \in HF^*(L,L')$, and the starting point of this particular semi-infinite bar is given by $l(\alpha,L,L')$.

We now have to discuss what happens when we do not assume transversality. This actually does not make any fundamental change, we only have to acknowledge the resulting perturbations. Thus for a Hamiltonian perturbation H and an almost-complex structure J_t, the vector spaces of our new persistence module are

$$HF^{*,\kappa}(L,L';J_t,H) = H^*(CF^{*,\kappa}(L,L';J_t,H)).$$

We denote

$$\mathcal{B}^*(L,L';J_t,H) = \mathcal{B}\left((HF^{*,\kappa}(L,L';J_t,H))_{\kappa\in\mathbb{R}}, i\right),$$

its associated barcode. This is well-defined since the fact that $H^*(CF^{*,\kappa}(L,L';J_t,H))$ is a subcomplex still holds. Thus, given a non-zero cohomology class $\alpha \in HF^*(L,L';H,J)$, we can define the corresponding action selector as we did before:

$$l(\alpha, L, L'; J_t, H) = \inf\{\kappa \in \mathbb{R}, \quad \alpha \in i^\kappa : HF^{*,\kappa}(L, L'; J_t, H) \to HF^*(L, L'; J_t, H)\}. \quad (4.1)$$

The following proposition gives classical properties of these action selectors as found in [96, 79, 67, 58].

Proposition 4.2.3. *For every pair of closed exact Lagrangian submanifolds in a Liouville domain, and every non-zero class $\alpha \in HF(L, L'; J_t, H)$, the action selector $l(\alpha, L, L'; J_t, H)$ satisfies:*

- $l(\alpha, L, L'; J_t, H) \in Spec(L, L'; H)$,
- $l(\alpha, L, L'; J_t, H)$ *does not depend on J_t hence will be denoted $l(\alpha, L, L'; H)$,*
- $|l(\alpha, L, L'; H) - l(\alpha, L, L'; H')| \leq \|H - H'\|$, *where $\| \cdot \|$ denotes the Hofer norm.*

The first property is called the spectrality property, and the third one the Lipschitz continuity property. These are classical results when studying action selectors and thus we will not prove them here. However, we can say that the first two properties directly follow from the definition. The third one is a direct consequence of the construction of continuation maps used to prove that the cohomology does not depend on the choice of the Hamiltonian perturbation.

These action selectors satisfy the so-called Lagrangian splitting formula which is a direct consequence of the Künneth formula 2.6; see for example [31] or [47].

Proposition 4.2.4. *Let (M, ω) and (M', ω') be symplectic manifolds as before, and $(L_0, L_1) \subset M$, $(L'_0, L'_1) \subset M'$ two pairs of closed exact Lagrangian submanifolds. Let H and H' be two Hamiltonian perturbations to achieve transversality. Then, for $\alpha \in HF(L_0, L_1; J_t, H)$ and $\alpha' \in HF(L'_0, L'_1; J'_t, H')$ two non-zero cohomology classes,*

$$l(\alpha \otimes \alpha'; L_0 \times L'_0, L_1 \times L'_1; H \oplus H') = l(\alpha, L_0, L_1; H) + l(\alpha', L'_0, L'_1; H'),$$

where $\alpha \otimes \alpha'$ is defined by the Künneth formula 2.6.

As we did for barcodes in Morse homology, the continuation maps also give the continuity of the barcodes:

Proposition 4.2.5. *Let L, L' be two closed exact Lagrangian submanifolds in a Liouville domain, and let H, K be two Hamiltonians together with time dependent almost-complex structure J and J' such that the graded barcodes $\mathcal{B}^*(L, L'; J_t, H)$ and $\mathcal{B}^*(L, L'; J'_t, K)$ are well-defined. Then,*

$$d_{bottle}(\mathcal{B}(L, L'; J_t, H), \mathcal{B}(L, L'; J'_t, K)) \leq \|H - K\|,$$

where $\|.\|$ *denotes the Hofer distance.*

Note that this bound does not depend on choice of the almost complex structures J and J'.

The proof of this proposition is a straightforward translation to our context of a well-known result. This was proven by Polterovich-Shelukhin [75] and Usher-Zhang [94] in full generality.

Proof. This proposition comes from the construction of the continuation map from $CF(L, L'; J_t, H)$ to $CF(L, L'; J'_t, K)$. Let u be a continuation strip as defined in Subsection 2.2.7 between two generators $x \in CF(L, L'; J_t, H)$ and $y \in CF(L, L'; J'_t, K)$. Let us recall that x is actually a X_H-flow line $x(t)_{t\in[0,1]}$ from L to L' and y is a X_K-flow line $y(t)_{t\in[0,1]}$ from L to L'. We then have

$$\begin{aligned}\mathcal{A}^H_{L,L'}(x) - \mathcal{A}^K_{L,L'}(y) &= E(u) + \int K(y(t)) - \int H(x(t))dt \\ &\geq \int K(y(t)) - \int H(x(t))dt \\ &\geq \int_0^1 \min(K_t - H_t)dt = -\int_0^1 \max(H_t - K_t)dt.\end{aligned}$$

We thus obtain $\mathcal{A}^K_{L,L'}(y) \leq \mathcal{A}^H_{L,L'}(x) + \int_0^1 \max(H_t - K_t)dt$.

Taking the continuation in the other direction, we get by the same computation, $\mathcal{A}^H_{L,L'}(x) \leq \mathcal{A}^K_{L,L'}(y) + \int_0^1 \max(K_t - H_t)dt$.

Let us denote Ψ the continuation map from $CF(L, L'; J_t, H)$ to $CF(L, L'; J'_t, K)$ and Ψ' the continuation map from $CF(L, L'; J'_t, K)$ to $CF(L, L'; J_t, H)$. The previous computation shows that the map satisfies

$$\Psi : CF^{\kappa}(L, L'; J_t, H) \to CF^{\kappa+\mathcal{E}^+(H,K)}(L, L'; J'_t, K)$$

$$\Psi' : CF^{\kappa}(L, L'; J'_t, K) \to CF^{\kappa+\mathcal{E}^+(K,H)}(L, L'; J_t, H),$$

where $\mathcal{E}^+(H, K)$ is Usher's notation for $\int_0^1 \max(H_t - K_t)dt$. We recall that we saw in Subsection 2.2.7 that both compositions of Ψ and Ψ' are chain homotopic to the identity. For a good choice of homotopy between H and K, this chain homotopy respect the filtration [75, 94]. Then this corresponds to Definition 3.1.8 of the interleaving distance between the associated persistence modules $HF^{\kappa}(L, L'; J_t, H)$ and $HF^{\kappa+\mathcal{E}^+(H,K)}(L, L'; J'_t, K)$. The interleaving distance here is $\mathcal{E}^+(H, K) + \mathcal{E}^+(H, K) = \|H - K\|$, which concludes the proof of this proposition. $\square$

In the following chapters, we do not really care about the Hamiltonian perturbation. The fact that given any two closed exact Lagrangian submanifolds, the Hamiltonian perturbation can be made as small as one wishes by Remark 2.2.26, together with the Proposition 4.2.3 allows us to define, for a a non-zero class in $HF(L, L'; J_t, H)$

$$l(a; L, L') = \lim_{H \in \mathcal{H} \to 0} l(a; L, L'; H),$$

where $\mathcal{H}$ is the set of Hamiltonians satisfying the transversality requirements. If L and L' intersect transversely, it equals the action selector defined in Definition 4.2.2.

We can also use for barcodes the perturbation data notation as in Subsection 2.2.8, i.e. denoting $\mathcal{D}$ the pair (H, J) where H is the Hamiltonian perturbation and J the regular almost complex structure, the barcode can be written

$$\mathcal{B}(L, L'; \mathcal{D}).$$

Following Proposition 4.2.5, given two closed exact Lagrangian submanifolds L, L' in a Liouville domain $(M, \omega = d\lambda)$ with two primitive functions $f_L : L \to \mathbb{R}$ and $f_{L'} : L' \to \mathbb{R}$ such that $df_L = \lambda_{|L}$ and $df_{L'} = \lambda_{|L'}$, the map

$$H \mapsto \mathcal{B}(L, L'; H, J)$$

is continuous with respect to the Hofer distance. Since the space of barcodes is complete by Proposition 3.3.5, we can take the limit of $\mathcal{B}(L, L'; \mathcal{D})$ as the Hamiltonian part of the perturbation goes to zero and thus define

$$\mathcal{B}(L, L') = \lim_{H \to 0} \mathcal{B}(L, L'; H, J).$$

For two exact Lagrangian submanifolds L and L', we denote $\hat{B}(L, L')$ the image of $\hat{B}(L, L')$ in $\hat{\mathcal{B}}$.

4.3 Product structure in Lagrangian Floer cohomology

This section follows Auroux presentation in [7] together with the books of Oh [68] and Seidel [83].

The Floer cochain complex can be equipped with a product operation. Let L_0, L_1 and L_2 be three Lagrangian submanifolds of a symplectic manifold (M, ω). Under suitable assumptions, we aim to define a product operation from the Floer complexes $CF(L_1, L_2)$ and $CF(L_0, L_1)$ to $CF(L_0, L_2)$, i.e. a map

$$CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2).$$

The idea is to define this product in the same spirit as the construction of the differential. It will indeed count specific pseudo-holomorphic strips, using the relation between the index of a curve and the dimension of the associated moduli space.

We will firsttly define the moduli spaces, i.e. the pseudo-holomorphic curves we are interested in. To do so let us define what positive and negative punctures with asymptotic markers are.

Let S be the standard closed Riemannan disc in $\mathbb{C}$, with boundary S^1. Let us denote Γ a finite set of punctures in ∂S. These punctures will be of two types: the negative, or output, punctures denoted Γ_-, and the positive, or input, punctures denoted Γ_+.

A cylindrical end at a puncture $z \in \Gamma$ is a biholomorphism

$$u_z : (0, +\infty) \times [0, 1] \to U(z) \setminus \{z\},$$

if z is a positive puncture and

$$u_z : (-\infty, 0) \times [0, 1] \to U(z) \setminus \{z\},$$

if z is a negative puncture. In both cases, $U(z)\setminus\{z\}$ denotes a punctured open neighborhood of z in S. The idea of this setting is that the punctures of the punctured Riemann disc will be sent on generators of the different cochain complexes. The cylindrical ends will then help to achieve transversality by introducing perturbations of the almost-complex structure as well as a Hamiltonian perturbation.

Let us now introduce the moduli spaces we are interested in to define the product operation. They will correspond to moduli spaces of punctured Riemann disc: $\Sigma = S \setminus \Gamma$. Let L_0, L_1 and L_2 be three exact Lagrangian submanifolds of a Liouville domain (M, ω). We assume for now that they are pairwise transverse. Let $p \in \chi(L_0, L_1)$, $p' \in \chi(L_1, L_2)$ and $q \in \chi(L_0, L_2)$, and let J_t be a time dependent almost-complex structure on M. Let Σ be the closed Riemann disc with 3 boundary punctures z, z' and z_0, such that z_0 is a negative puncture and z, z' are positive punctures. We denote γ'_0 the boundary arc from z_0 to z, γ'_1 the one from z to z' and γ'_2 from z' to z_0.

Definition 4.3.1. *Let $[u]$ be a homotopy class. The space $\mathcal{M}(p, p', q; [u]; J_t)$ is the space of J-holomorphic maps $u : \Sigma \to M$ in the class of [u] which maps z, z' and z_0 to p, p' and q respectively and γ'_0, γ'_1 and γ'_2 to L_0, L_1 and L_2 respectively.*

Remark 4.3.2. As in the definition of the moduli spaces for the differential, in more general cases, i.e. when working with non-exact Lagrangian submanifolds, one has to add a condition of finite energy. In our context, as will be seen in the Section 4.4, the energy of such maps is constant when p, p' and q are fixed.

As the idea is exactly the same as for the moduli spaces used in the definition of the differential, we only discuss briefly the question of the dimension of the moduli spaces $\mathcal{M}(p, p', q; [u]; J_t)$. Assuming that the transversality properties hold, these are smooth

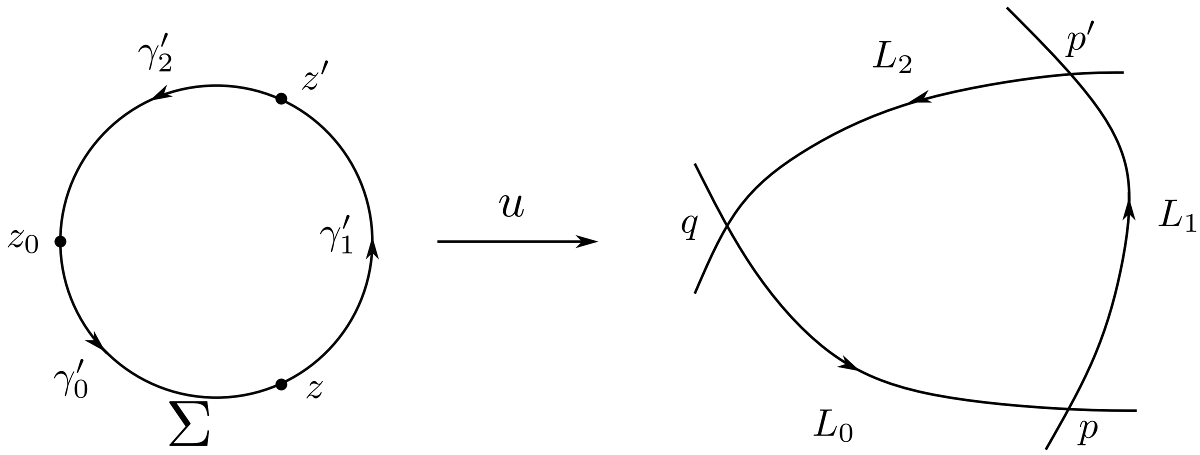

Figure 4.2 – Map $u : \Sigma \to M$, giving a J-holomorphic map from p, p' to q.

manifolds. Their dimension is given by the Fredholm index of the linearized Cauchy-Riemann operator D_u. As for Expression 2.3, this Fredholm index can be understood in terms of Maslov index for on appropriate Lagrangian loop.

Let us fix an orientation of γ_i for $i \in \{0, 1, 2\}$ such that the concatenation of these paths goes counterclockwise, as in Figure 4.2. After trivialization, the path $u(\gamma_i)$ induces a path, which will be denoted γ_i' in $\Lambda(n)$ given by the tangent space of L_i. We denote l_p the canonical short path from $T_p(L_0)$ to $T_p(L_1)$, $l_{p'}$ the canonical short path from $T_{p'}(L_1)$ to $T_{p'}(L_2)$ and l_q the canonical short path from $T_q(L_2)$ to $T_q(L0)$. The appropriate loop is then the concatenation

$$\gamma_1' \sharp l_{p'} \sharp \gamma_2 \sharp l_q \sharp \gamma_0 \sharp l_p.$$

As in the construction setting for the differential, the Maslov index of this loop in $\Lambda(n)$ is equal to the index $\operatorname{ind}(u)$. Working in well-behaved conditions, i.e. when $2c_1(M) = 0$, and μ_{L_0}, μ_{L_1} and μ_{L_2} vanish, we saw that we can associate a well-defined degree to the generators of the different chain complexes. These degrees then relate to the Maslov index of our Lagrangian loop and thus to the index of u in the following way [68] [83]:

$$\operatorname{ind}(u) = \deg(q) - \deg(p) - \deg(p').$$

Remark 4.3.3. Since we are working with $\mathbb{Z}/2$ as coefficient field, questions of orientation are not relevant in our context.

We can now define the product operation.

Definition 4.3.4. *The Floer product is the $\mathbb{Z}/2$-linear map*

$$CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2),$$

defined by

$$\mu^2(p', p) = \sum_{\substack{q \in \chi(L_0, L_2) \\ [u]:\mathrm{ind}([u])=0}} (\sharp \mathcal{M}(p, p', q; [u]; J_t))q.$$

Remark 4.3.5. A Gromov's compactness argument tells that the moduli spaces in this sum are compact. Together with the fact that they are 0-dimensional manifolds, this ensures that this sum is finite hence well-defined.

The following property ensures that this product on chain complexes induces a well-defined product on cohomology. The proof can be found in [68] or [83].

Proposition 4.3.6. *The Floer product satisfies the following Leibniz rule:*

$$\partial \mu^2(p', p) = \mu^2(\partial p', p) + \mu^2(p', \partial p),$$

where the signs do not matter, since we are working with $\mathbb{Z}/2$ *coefficients.*

This implies that the product defined on chain complexes induces a well-defined product

$$HF(L_1, L_2) \otimes HF(L_0, L_1) \to HF(L_0, L_2).$$

To prove this proposition, with the same idea as for $\partial^2 = 0$, we have to study the boundary of a compactified moduli space of dimension 1. The intersection points p, p' and q being generators of the Floer complexes as above, and $[u]$ being a homotopy class with $ind([u]) = 1$, the moduli spaces we are looking for are the $\mathcal{M}(p, p', q; [u]; J_t)$, which are indeed 1-dimensional. Gromov compactness tells that these moduli spaces admit a compactification $\overline{\mathcal{M}(p, p', q; [u]; J_t)}$.

Since we are working with conditions under which sphere or disc bubbling cannot appear, the only phenomenon we have to deal with is strip-breaking. The idea of this strip-breaking is the same as before: the energy can concentrate at one of the three punctures. As in Section 2.2, transversality implies that a non-constant strip has a positive index, and that no strip has a negative index. Moreover, the sum of the indices of the different discs in case of strip-breaking has to be equal to 1. This leaves us with only three possibilities: an index 0 disc with three punctures together with an index 1 strip, with boundary on two of the three Lagrangian submanifolds. In fact, these limit configurations are the subject of another gluing theorem: they are the boundary of our dimension 1 moduli spaces. More precisely, we have:

$$\partial \overline{\mathcal{M}(p, p', q; [u]; J_t)} = \bigcup_{\substack{r \in \chi(L_0, L_2) \\ [v]+[v']=[u] \\ ind([v])=1-ind([v'])=0}} \mathcal{M}(p', p, r; [v]; J_t) \times \widehat{\mathcal{M}}(r, q; [v']; J_t)$$

$$\cup \bigcup_{\substack{r\in\chi(L_1,L_2)\\ [v]+[v']=[u]\\ ind([v])=1-ind([v'])=0}} \mathcal{M}(r,p,q;[v];J_t)\times\widehat{\mathcal{M}}(p',r;[v'];J_t)$$
$$\cup \bigcup_{\substack{r\in\chi(L_0,L_1)\\ [v]+[v']=[u]\\ ind([v])=1-ind([v'])=0}} \mathcal{M}(p',r,q;[v];J_t)\times\widehat{\mathcal{M}}(p,r;[v'];J_t).$$

Writing down this gluing theorem this way allows to identify which configuration contributes to the coefficient of q in each term of the Leibniz formula. The first union contributes to the coefficient of q in $\partial(p'\cdot p)$, the second one to the coefficient of q in $(\partial p')\cdot p$, and the third one to the coefficient of q in $p'\cdot(\partial p)$. Since $\overline{\mathcal{M}(p,p',q;[u];J_t)}$ is a compact smooth 1-dimensional manifold with boundary, its boundary consists of an even number of set of points. Because we are working with $\mathbb{Z}/2$ coefficients, we immediately obtain:

$$\begin{aligned}0=\sharp\partial\overline{\mathcal{M}(p,p',q;[u];J_t)} &= \sum_{\substack{r\in\chi(L_0,L_2)\\ [v]+[v']=[u]\\ ind([v])=1-ind([v'])=0}} \sharp\mathcal{M}(p',p,r;[v];J_t)\sharp\widehat{\mathcal{M}}(r,q;[v'];J_t)\\ &+ \sum_{\substack{r\in\chi(L_1,L_2)\\ [v]+[v']=[u]\\ ind([v])=1-ind([v'])=0}} \sharp\mathcal{M}(r,p,q;[v];J_t)\sharp\widehat{\mathcal{M}}(p',r;[v'];J_t)\\ &+ \sum_{\substack{r\in\chi(L_0,L_1)\\ [v]+[v']=[u]\\ ind([v])=1-ind([v'])=0}} \sharp\mathcal{M}(p',r,q;[v];J_t)\sharp\widehat{\mathcal{M}}(p,r;[v'];J_t).\end{aligned}$$

Since we have the wished equality for each q, we finally get the Leibniz formula.

The Floer product is not associative on the level of cochain complexes, but it is associative in cohomology. To sketch the proof of this, we will need to introduce higher products and the relations between them. Before doing so, we sketch the way to deal with transversality issues, the one concerning intersections of Lagrangian submanifolds and the one concerning moduli spaces.

As seen above, the main tool to achieve transversality is to use time-dependent almost-complex structures and Hamiltonian perturbations. We will have to proceed more carefully than for the differential, taking care of what happens near the punctures. This is why we introduced the notion of cylindrical ends.

To achieve transversality, we have to choose a family of ω-compatible time-dependent complex structures J_t and a family of Hamiltonians H such that both families only depend on the $[0,1]$ coordinate in each cylindrical end. Then we perturb the Cauchy-Riemann equation so that it reduces on each cylindrical end to a perturbed equation with a specific Hamiltonian on each end. Let us denote $H_{i,j}$ the resulting Hamiltonian perturbation on the cylindrical end at the intersection between L_i and L_j, and $J_{i,j}$ the corresponding time-dependent complex structure. The cylindrical end satisfying the equation

$$\frac{\partial u}{\partial s} + J_t(u)\left(\frac{\partial u}{\partial t} - X_H(t,u)\right) = 0,$$

with boundaries mapping to L_i and L_j, the strip u does not converge any longer to the intersection point between L_i and L_j. It rather converges to $H_{i,j}$ flow lines from L_i to L_j, i.e. to the perturbed intersection points $L_i \cap (\phi^1_{H_{i,j}})^{-1}(L_j)$.

Applying the arguments above to these perturbed intersection points and these moduli space of perturbed pseudo holomorphic curves, we finally obtain a well-defined product map between the "perturbed" Floer complexes:

$$CF(L_1, L_2; H_{1,2}; J_{1,2}) \otimes CF(L_0, L_1; H_{0,1}; J_{0,1}) \to CF(L_0, L_2; H_{0,2}; J_{0,2}).$$

The Leibniz rule is still true for this product if one is careful enough to choose the appropriate differential.

We now discuss the associativity of this product in homology. To do so, we will briefly introduce higher products, i.e., given $k+1$ exact Lagrangian submanifolds $L_0, ..., L_k$ in a Liouville domain (M, ω), a map

$$\mu^k : CF(L_{k-1}, L_k) \otimes \cdots \otimes CF(L_0, L_1) \to CF(L_0, L_k).$$

This map is $(2-k)$-graded when it is possible to define a grading on the Floer complexes. The construction of these higher products is exactly the same as before: we count pseudo-holomorphic curves interpolating between intersection points. In fact, the case $k = 1$ corresponds to the differential and the case $k = 2$ to the product. There are also compactness and transversality issues for these products. Once again it is possible to take care of these using perturbations, but this issue will not be addressed here. We will assume that transversality and compactness hold. The results and construction to be stated now remain true under the perturbations to achieve transversality and compactness.

Given a homotopy class $[u]$, the moduli spaces to be considered consist of maps u in the class $[u]$ from a Riemann disc equipped with k positive punctures and 1 negative one to the symplectic manifold, with boundaries matching the approriate Lagrangian submanifolds. Let us denote $p_i \in \chi(L_i - 1, L_i)$, for $i \in \{1, ...k\}$ and $q \in \chi(L_0, L_k)$. For a time-dependent almost-complex structure J, the moduli space is denoted $\mathcal{M}(p_1, ..., p_k, q; [u]; J)$ and its dimension is

$$k - 2 + \text{ind}([u]) = k - 2 + \deg(q) - \sum_{i=1}^{n} \deg(p_i),$$

when the degree is defined. Note that the $k+1$ punctures on the Riemann disc are not fixed, they are allowed to vary. Therefore the $k+2 = (k+1) - 3$ term in the previous equality comes from the fact that there is $k+1$ punctures and that a biholomorphism on the disc acts 3-transitively on the boundary.

The operation $\mu^k : CF(L_{k-1}, L_k) \otimes \cdots \otimes CF(L_0, L_1) \to CF(L_0, L_k)$ is then defined as the $\mathbb{Z}/2$-linear map such that

$$\mu^k(p_k, ...p_1) = \sum_{\substack{q \in \chi(L_0, L_k) \\ [u]:\mathrm{ind}([u])=2-k}} (\sharp \mathcal{M}(p_1, ...p_k, q; [u]; J_t))q.$$

The study of the compactification and boundaries of the different moduli spaces concludes, among other things, that there are relations between the different products. These relations are the object of the following proposition which we will admit. A proof can be found in [83]. Let us just recall that we are working with closed exact Lagrangian submanifolds intersecting transversely.

Proposition 4.3.7. *The operations μ^k satisfy the following relations:*

$$\sum_{j=1}^{k} \sum_{i=0}^{k-j} \mu^{k+1-j}(p_k, ..., p_{i+j+1}, \mu^j(p_{i+j}, ...p_{i+1}), p_i, ..., p_1) = 0.$$

Since we are working with $\mathbb{Z}/2$ coefficients, the signs do not matter.

Remark 4.3.8. When $k = 1$, the relation writes as $\partial^2 = 0$, and when $k = 2$, it corresponds to the Leibniz rule.

The relation of interests here is the one for $k = 3$. Indeed, with $\mathbb{Z}/2$ coefficients, it writes

$$\begin{aligned} \mu^2(\mu^2(p_3, p_2), p_1) + \mu^2(p_3, \mu^2(p_2, p_1)) &= \partial\mu^3(p_3, p_2, p_1) + \mu^3(\partial p_3, p_2, p_1) \\ &\quad + \mu^3(p_3, \partial p_2, p_1) + \mu^3(p_3, p_2, \partial p_1). \end{aligned} \tag{4.2}$$

This means that the Floer product is not associative on the Floer complexes, but it is up to the explicit homotopy just written above. More interestingly this also means that the Floer product is in associative in cohomology. This homotopy will also be essential when we will define filtrations.

Moreover, we have the following property [83].

Proposition 4.3.9. *Let L and L' be two closed exact Lagrangian submanifolds in M. The product*

$$CF(L', L) \otimes CF(L', L') \to CF(L', L)$$

is cohomologically unital. This unit is given by the image of the fundamental class $[L']$ of L' in $HF(L', L')$.

Adequate choices of Hamiltonian perturbations make it possible to have an isomorphism on the level of cochain complexes. Let us pick $\varepsilon > 0$ and choose a ε-small perturbation f for the pair (L', L') as explained in Remark 2.2.34. By abuse of notation, f both denotes the Morse function on the Lagrangian L and its extension to a Hamiltonian on M. We also choose a ε-small perturbation H for the pair (L, L'). The choice of the perturbation f implies that there is a unique representative of the fundamental class $[L']$ in $CF^0(L', L'; f, J)$ (see Remark 2.2.34).

In [72] Piunikhin, Salamon and Schwarz defined the so-called PSS morphism. Let g be a Morse function on L and K a Hamiltonian perturbation for the pair (L', L'). The PSS morphism

$$\mathrm{PSS} : CM^*(L', g) \to CF^*(L', L'; K)$$

is a homotopy equivalence. Indeed one can construct a PSS inverse morphism, PSS^{-1} which is the inverse of PSS up to homotopy. This is proven by Biran-Cornea in [11].

In our case we have $K = g$ denoted f. So we have the following commutative diagram:

$$\begin{tikzcd} CF(L', L; H) \otimes CF(L', L'; f) \arrow[r] & CF(L', L; H_f) \, , \\ CF(L', L; H) \otimes CM(L'; f) \arrow[u, "\mathrm{Id} \otimes \mathrm{PSS}"] \arrow[ur] & \end{tikzcd}$$

where H_f denotes the perturbation $f \sharp H$ of H by the Hamiltonian f.

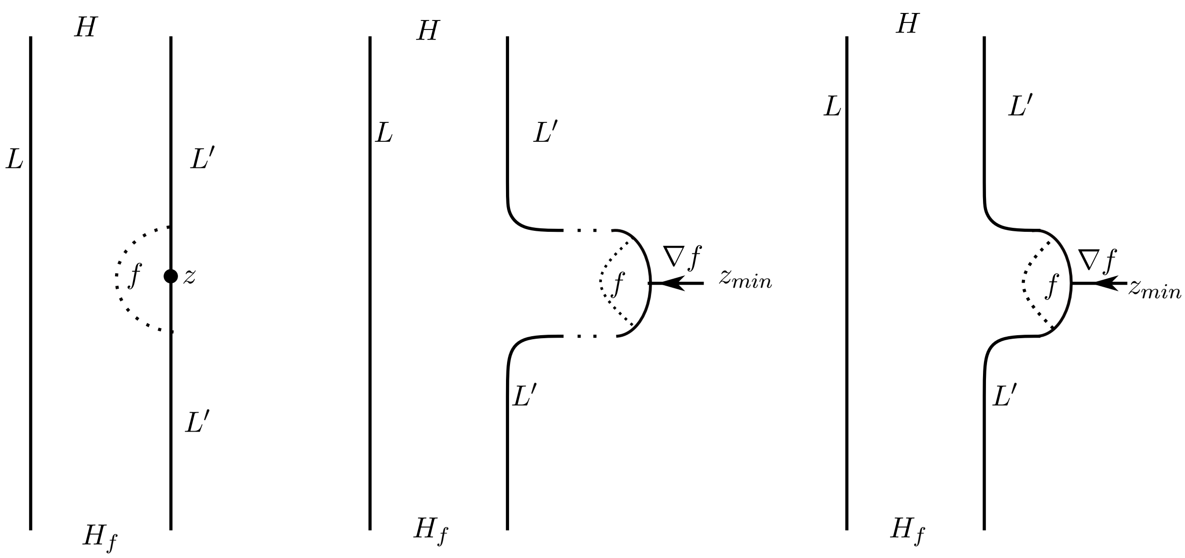

Figure 4.3 – Product by z z_min and the map ψ

Let us recall that since we work with a Morse function f presenting a unique minimum, there is only one element in degree 0 in both $CM(L', f)$ and $CF(L', L'; f)$. We denote them $z_M \in CM^0(L', f)$ and $z \in CF^0(L', L'; f)$. Since PSS is a homotopy equivalence we have $\mathrm{PSS}(z_M) = z$. Consequently we have the following commutative diagram

$$\begin{tikzcd} CF(L', L; H) \otimes z \arrow[r] & CF(L', L; H_f)\ . \\ CF(L', L; H) \otimes z_M \arrow[u] \arrow[ur, "\psi"'] & \end{tikzcd}$$

By a classical gluing theorem in Floer theory [77, 37, 6], the map ψ consists of counting strip with the perturbation H near the positive puncture, perturbation H_f near the negative puncture, with a marked point z' on the L boundary with a perturbation f near z'. On this marked point z' there is also a f-gradient trajectory from z_M to z'. See Figure 4.3. Since z_M is a unique minimum, the gradient trajectory is unique and thus does not impose any condition. For ε-small enough, the continuation data on the strip satisfies the condition explained in Remark 2.2.31 and thus we obtain an isomorphism on the cochain complexes. This can be summarized by the following proposition.

Proposition 4.3.10. *Let L and L' be two closed exact Lagrangian submanifolds in M and $\varepsilon > 0$. Let f a Hamiltonian perturbation for (L', L') defined as in Remark 2.2.34 and H a Hamiltonian perturbation for (L', L). Assume that f and H are ε-small. Then for ε small enough, the following map is an isomorphism:*

$$\mu^2(\cdot, z) : CF(L', L; H) \to CF(L', L; H_f),$$

where z is the unique representative of the image (Proposition 2.2.32) of the fundamental class $[L']$ in $CF(L', L'; f)$ and $H_f = f \sharp H$.

4.4 Product in filtered Lagrangian Floer cohomology

In this section, we focus on the action during a product on Floer complexes. Regarding the degree, results are the same as those in non-filtered Floer cohomology. However we need to understand precisely how we can bound the shift of action in order to define this structure on filtered Floer cohomology.

We will actually be working in the case we need, which turns out to be a nice and easy one.

Let (M, ω) be a $2n$-dimensional exact symplectic manifold.

Let L_0, L_1, L_2 be three pairwise transverse closed exact Lagrangian submanifolds in M. We assume that the product is well defined.

Since these Lagrangian submanifolds are exact, they come with three primitive functions (defined up to a constant; see Remark 1.3.5) $f_i : L_i \mapsto \mathbb{R}$, such that $df_i = \lambda_{|L_i}$ for $i \in \{0, 1, 2\}$. Let $p_1 \in \chi(L_0, L_1)$, $p_2 \in \chi(L_1, L_2)$ and $z \in CF^*(L_0, L_2)$ such that $\mu^2(p_2, p_1) = z$. Note that z is a formal sum of $(q_j)_j \in \chi(L_0, L_2)$.

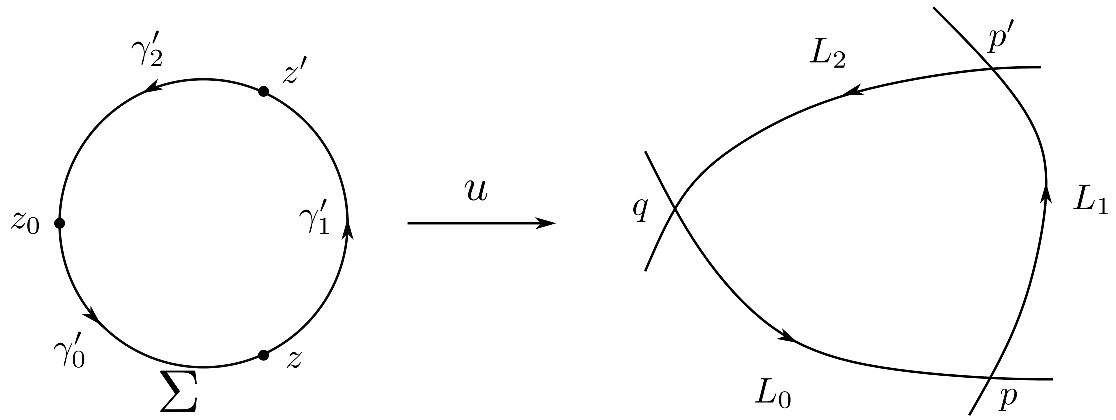

Figure 4.4 – Product in Floer cohomology

Let us recall that

$$\mathcal{A}_{L_0,L_1}(p_1) = f_0(p_1) - f_1(p_1),$$

$$\mathcal{A}_{L_1,L_2}(p_2) = f_1(p_2) - f_2(p_2),$$

$$\mathcal{A}_{L_0,L_2}(q_i) = f_0(q_i) - f_2(q_i).$$

Let $u : \Sigma \to (M; L_0, L_1, L_2)$ be a pseudo-holomorphic curve with punctures asymptotic to (p_1, p_2, q_j) as defined earlier in Section 4.3 for the product in Lagrangian Floer cohomology. Let us denote for $i \in \{0; 1; 2\}$ the paths $\gamma_i : [0; 1] \to L_i$ such that $\gamma_i([0; 1]) = u(\mathbb{D}^2, \partial\mathbb{D}^2) \cap L_i$. We set the orientations of the γ_i for $i \in \{0; 1; 2\}$ such that their concatenation $\gamma_0 \sharp \gamma_1 \sharp \gamma_2$ turns counterclockwise as in Figure 4.4.

Since ω is exact, equal to $d\lambda$, Stokes' theorem gives

$$\begin{aligned} \mathrm{Area}(u) &= \int_{\mathbb{D}^2} u^*\omega \\ &= \int_{\gamma_1} \lambda_{L_1} + \int_{\gamma_2} \lambda_{L_2} + \int_{\gamma_0} \lambda_{L_0}. \end{aligned}$$

Moreover, all the L_i being exact Lagrangian submanifolds, with associated functions f_i, we get:

$$\forall i \in \{0, 1, 2\}, \int_{\gamma_i} \lambda_i = f_i(\gamma_i(1)) - f_i(\gamma_i(0)).$$

Then,

$$\begin{aligned} \mathrm{Area}(u) &= f_0(p_1) - f_0(q_j) + f_1(p_2) - f_1(p_1) + f_2(q_j) - f_2(p_2) \\ &= \mathcal{A}_{L_0,L_1}(p_1) + \mathcal{A}_{L_1,L_2}(p_2) - \mathcal{A}_{L_0,L_2}(q_j). \end{aligned}$$

Since the area of u is positive, we have

$$\mathcal{A}_{L_2,L_0}(q_j) < \mathcal{A}_{L_0,L_1}(p_1) + \mathcal{A}_{L_1,L_2}(p_2).$$

Let us recall that

$$\mathcal{A}_{L_2,L_0}(z) = \max_j \{\mathcal{A}_{L_2,L_0}(q_j)\}.$$

We immediately get

$$\mathcal{A}_{L_2,L_0}(z) < \mathcal{A}_{L_0,L_1}(p_1) + \mathcal{A}_{L_1,L_2}(p_2).$$

As done in previous sections, we now have to discuss the case where we do not assume the transversality properties, and hence where we need a perturbation data $\mathcal{D}$. The argument is exactly the same as for Inequality 2.10, as the perturbation data has to be taken into account in the same way when computing $E(u)$. Since the perturbation data $\mathcal{D}$ can be chosen as small as desired, as before, for all $\varepsilon > 0$, we can find $\mathcal{D}$ such that all our cohomologies are well-defined and

$$E(u) \leq \mathcal{A}_{L_0,L_1}(p_1) + \mathcal{A}_{L_1,L_2}(p_2) - \mathcal{A}_{L_0,L_2}(z) + \varepsilon.$$

We then straightforwardly obtain

$$\mathcal{A}_{L_2,L_0}(z) \leq \mathcal{A}_{L_2,L_1}(p_1) + \mathcal{A}_{L_1,L_0}(p_2) + \varepsilon,$$

for $p_1 \in \chi(L_0, L_1)$, $p_2 \in \chi(L_1, L_2)$ and $z \in CF^*(L_0, L_2; \mathcal{D})$ with $\mu^2(p_2, p_1) = z$.

This means that the product preserves the filtration and immediately implies the following lemma which will be essential for the upcoming discussions.

Lemma 4.4.1. *Let L_0, L_1, L_2 be three closed exact Lagrangian submanifolds in (M, ω) exact, together with a ε-small perturbation data collection $\mathcal{D}$, and let $p_2 \in CF^k(L_1, L_2; \mathcal{D})$, with action b. Let us assume that the product $\mu^2(p_2, \cdot) : CF^*(L_0, L_1; \mathcal{D}) \mapsto CF^{*+k}(L_0, L_2; \mathcal{D})$ is well defined.*

Then, we have a morphism of persistence modules:

$$\mu^2(p_2, \cdot) : CF^{*,t}(L_0, L_1; \mathcal{D}) \to CF^{*+k,t+b+\varepsilon}(L_0, L_2; \mathcal{D}), \quad \forall t \in \mathbb{R}$$

$$\mu^2(p_2, \cdot) : CF^*(L_0, L_1; \mathcal{D}) \to CF^{*+k}(L_0, L_2; \mathcal{D})[b + \varepsilon].$$

Lemma 4.4.2. *Let L_0, L_1, L_2 be three closed exact Lagrangian submanifolds in (M, ω) exact together with a perturbation data ε-small $\mathcal{D}$. Let $p_1 \in CF^k(L_0, L_1; \mathcal{D})$, with action a, $p_2 \in CF^k(L_1, L_0; \mathcal{D})$, with action b. The following maps obtained by composition*

$$\mu^2(p_2, \mu^2(p_1, \cdot)) : CF(L_2, L_0; \mathcal{D}) \to CF(L_2, L_0; \mathcal{D})[a + b + 3\varepsilon],$$

$$\mu^2(p_1, \mu^2(p_2, \cdot)) : CF(L_2, L_1; \mathcal{D}) \to CF(L_2, L_1; \mathcal{D})[a + b + 3\varepsilon],$$

are well-defined and filtered chain homotopic to the maps

$$\mu^2(\mu^2(p_2, p_1), \cdot) : CF(L_2, L_0; \mathcal{D}) \to CF(L_2, L_0; \mathcal{D})[a + b + 3\varepsilon],$$

$$\mu^2(\mu^2(p_1, p_2), \cdot) : CF(L_2, L_1; \mathcal{D}) \to CF(L_2, L_1; \mathcal{D})[a + b + 3\varepsilon].$$

Proof. The composition maps are well-defined and filtered by the preceding lemma. Since we saw with Equality 4.2 that the product in Lagrangian Floer cohomology is associative, we only have to check that the associator behaves correctly with respect to the filtration. Let us recall that for our chain complexes we have

$$\begin{aligned} \mu^2(\mu^2(p_2, p_1), q) + \mu^2(p_2, \mu^2(p_1, q)) &= \partial\mu^3(p_2, p_1, q) + \mu^3(\partial p_2, p_1, q) \\ &\quad + \mu^3(p_2, \partial p_1, q) + \mu^3(p_2, p_1, \partial q), \end{aligned}$$

where q is an element of $CF(L_2, L_0)$. Then, the exact same computation as for μ^2 gives us

$$\mathcal{A}_{L_2,L_0}(\mu^3(p_2, p_1, q)) \leq \mathcal{A}_{L_0,L_1}(p_1) + \mathcal{A}_{L_1,L_0}(p_2) + \mathcal{A}_{L_2,L_0}(q) + 3\varepsilon.$$

Moreover, as seen in Lemma 2.2.7, the differential decreases the action, so that

$$\begin{aligned} &\max\{\mathcal{A}_{L_2,L_0}(\partial\mu^3(p_2, p_1, q)), \mathcal{A}_{L_2,L_0}(\mu^3(\partial p_2, p_1, q)), \\ &\quad \mathcal{A}_{L_2,L_0}(\mu^3(p_2, \partial p_1, q)), \mathcal{A}_{L_2,L_0}(\mu^3(p_2, p_1, \partial q))\} \\ &\quad \leq \mathcal{A}_{L_0,L_1}(p_1) + \mathcal{A}_{L_1,L_0}(p_2) + \mathcal{A}_{L_2,L_0}(q) + 3\varepsilon. \end{aligned}$$

This means that the homotopy defined from μ^3 between the two different compositions preserves the filtration, which concludes the proof of this lemma. □

The following lemma will be a key argument in the proof of Section 5.3.

Lemma 4.4.3. *Let L, L' be two closed exact Lagrangian submanifolds in (M, ω) together with ε-small perturbation data f and let H behave as in Proposition 4.3.10. Denote $H_f = f \sharp H$. Let $z \in CF^0(L', L'; f, J)$ be as in the same proposition. The multiplication map*

$$m_2(\cdot, z) : CF^*(L', L; H) \to CF^*(L', L; H_f)[2\varepsilon]$$

are filtered chain-homotopic to the standard inclusion and hence induce 2ε-shift maps on the persistence modules.

Proof. Let us recall that Proposition 4.3.10 tells us the multiplication by z is an isomorphism of cochain complexes and hence induces the standard inclusion of persistence modules. We now just need the energy estimate.

Since the Hamiltonian part of the perturbations are ε-small, the action of z is smaller than ε and H_f is ε close to H. Consequently, using the same argument as the one implying Lemma 4.4.1, this map induces a $\varepsilon + \varepsilon = 2\varepsilon$-shift of action. This concludes the proof of this lemma. □

4.5 Spectral norm and exact Lagrangians in a cotangent bundle

Given a closed exact Lagrangian submanifold L together with a non-degenerate Hamiltonian H, the spectral norm $\gamma_L(H)$ is defined as

$$\gamma_L(H) = l([L], L, L; H) + l([L], L, L; \overline{H}),$$

where $[L]$ denotes the image of the fundamental class $[L]$ through the isomorphism of Proposition 2.2.32. It is equal to the diameter of the spectrum $\mathrm{Spec}(L, L; H)$. This is called the Lagrangian spectral norm or Viterbo norm as its first version was introduced by Viterbo in [96]. A similar version also exists in Hamiltonian Floer homology.

Let L and L' be two closed exact Lagrangian submanifolds in a symplectic manifold M as before, together with a Hamiltonian perturbation H. Then, in the same spirit, we set

$$\gamma(L, L'; H) = \mathrm{Diam}(\mathrm{Spec}^*(L, L'; H)),$$

where $\mathrm{Spec}^*(L, L'; H)$ is the set of action selectors for $HF(L, L'; H, J)$. We denote by $\mathrm{Diam}(\cdot)$ the diameter (i.e. $\max - \min$). Note that this definition is only interesting when the cohomology $HF(L, L'; H, J)$ has dimension at least 2. By Proposition 4.2.4, and with the same notations we immediately get

$$\gamma(L_0 \times L_0', L_1 \times L_1'; H \oplus H') = \gamma(L_0, L_1; H) + \gamma(L_0', L_1'; H'). \tag{4.3}$$

Consequently, if $M = M'$, $L_0 = L_0'$, $L_1 = L_1'$ and $H = H'$,

$$\gamma(L_0 \times L_0, L_1 \times L_1; H \oplus H) = 2\gamma(L_0, L_1; H). \tag{4.4}$$

Remark 4.5.1. Let L and L' be two closed exact Lagrangian submanifolds in a Liouville domain (M, ω) together with a Hamiltonian H and a function $f : M \to \mathbb{R}$. The third point of Proposition 4.2.3 together with the definition of γ tells us that

$$|\gamma(L, L'; H + f) - \gamma(L, L'; H)| \leq 2(\max f - \min f).$$

Note that we do not need any transversality assumptions for the intersections between L and L'. Indeed, by the continuity of the spectral invariants, $\gamma(L, L'; H)$ is defined for all

H. So we can set

$$\gamma(L, L') = \lim_{H \in \mathcal{H} \to 0} \mathrm{Diam}(\mathrm{Spec}^*(L, L'; H)),$$

where $\mathcal{H}$ is the set of Hamiltonians satisfying the transversality requirements.

Remark 4.5.2. Given two Lagrangian submanifolds L and L', we can actually define γ directly since the spectrum is defined without any transversality assumptions.

The question of the continuity of γ with respect to the C^0-distance is a fundamental one. This has been proved for specific symplectic manifolds in [96, 86, 15, 48, 88]. We will also prove its continuity in our context in Section 5.4, and thus we will not discuss it more here.

An important question is the relation between two C^0-close Lagrangian submanifolds. It is the object of Arnold's famous Nearby Lagrangian Conjecture. The results on this conjecture will be useful for both another, more precise, definition of γ and for the arguments of Section 5.3. Let us start by stating this conjecture.

Conjecture 4.5.3. *Let M be a closed manifold. Any closed exact Lagrangian submanifold in T^*M is Hamiltonian isotopic to the zero section.*

Together with Weinstein's theorem, this means that in a symplectic manifold M together with a closed Lagrangian submanifold $L \subset M$, any exact closed Lagrangian submanifold $L' \subset M$ is Hamiltonian isotopic to L if L' is C^0-close enough to L.

For most cases, this conjecture is still open and subject to a lot of research. Indeed, proving this conjecture would allow us to understand much better the symplectic topology of general symplectic manifolds by using well-known symplectic tools in cotangent bundles.

This conjecture has been fully proved in special cases. Hind [44] proved the following theorem:

Theorem 4.5.4. *The Nearby Lagrangian Conjecture is true in T^*S^2.*

For T^*S^1, there is not much to discuss and it is also true. Rizell, Goodman and Ivrii [28], Abouzaid-Kragh [2] or Abouzaid [1] proved it for $T^*\mathbb{T}^2$.

In more general context, important progress has been made by Fukaya, Seidel and Smith [40], proving that when the Maslov class vanishes, the projection

$$\pi : L' \to L \subset T^*L$$

induces an isomorphism on homology. This result was improved later by Abouzaid-Kragh in the following theorem [2].

Theorem 4.5.5. *Let L be a closed manifold together with L' an exact closed Lagrangian submanifold in T^*L. Then, there exists an integer $i = i_{L'} \in \mathbb{Z}$ such that for every exact closed Lagrangian submanifold K in T^*L, there are chain-level quasi-isomorphisms* *in both directions between $CF^*(L', K)$ and $CF^{*+i}(L, K)$ and between $CF^*(K, L')$ and $CF^{*-i}(K, L)$. These quasi-isomorphisms are compatible with the product structure in Floer cohomology.*

The quasi-isomorphisms in fact result from the product by a carefully chosen cohomology class.

Let L_0 and L_1 be two closed exact Lagrangian submanifolds in $(T^*L, \omega = d\lambda)$ exact, together with two primitives functions f_{L_0} and f_{L_1} such that $df_{L_i} = \lambda_{|L_i}$, for $i \in \{0, 1\}$. As mentioned before, this theorem allows us to propose an other definition of $\gamma(L_0, L_1)$. Indeed, previous Theorem 4.5.5 and Proposition 2.2.32 respectively tell that we have the two following isomorphisms

$$HF^*(L, L) \underset{\sigma}{\xrightarrow{\sim}} HF^*(L_0, L_1),$$

$$H_*(L) \underset{\theta}{\xrightarrow{\sim}} HF^{n-*}(L, L).$$

By abuse of notation, we denote

$$[L] = \sigma \circ \theta([L]) \in HF^0(L_0, L_1),$$

$$[pt] = \sigma \circ \theta([pt]) \in HF^n(L_0, L_1).$$

It is known that the following lemma holds by the same argument as in [64, 59].

Lemma 4.5.6.

$$\gamma(L_0, L_1) = l\left([L], L_0, L_1\right) - l\left([pt], L_0, L_1\right).$$

Let us give the basic properties of γ. Following Definition 2.2.2, we have $\mathcal{A}_{L_0,L_1} = -\mathcal{A}_{L_1,L_0}$. Together with the fact that the two complexes $CF(L_0, L_1)$ and $CF(L_1, L_0)$ are dual to each other, we have

$$l\left([pt], L_0, L_1\right) = -l\left(\sigma' \circ \theta([L]), L_1, L_0\right),$$

where σ' is the isomorphism from $HF(L, L)$ to $HF(L_1, L_0)$ given by Theorem 4.5.5. We thus obtain

$$\gamma(L_0, L_1) = l\left([L], L_0, L_1\right) + l\left(\sigma' \circ \theta([L]), L_1, L_0\right).$$

Consequently, for all L_0 and L_1 exact in T^*L,

$$\gamma(L_0, L_1) = \gamma(L_1, L_0). \tag{4.5}$$

Moreover, for all L_0, L_1, L_2 closed exact Lagrangian submanifolds in T^*L with primitive functions $f_{L_0}, f_{L_1}, f_{L_2}$, it also satisfies the triangle inequality

$$\gamma(L_0, L_1) \leq \gamma(L_0, L_2) + \gamma(L_2, L_1). \tag{4.6}$$

Indeed, if $x \in CF(L_2, L_1)$ and $y \in CF(L_0, L_2)$ both represent the fundamental class in their respective homology, so does $\mu^2(x, y)$ in $CF(L_0, L_1)$ (see Section 5.3). Together with Lemma 4.4.1, we immediately obtain this triangle inequality.

Chapter 5

Continuity of the barcode

5.1 Results and idea of the proof

5.1.1 Main theorem and consequences

The object of this chapter is to prove the following theorem which will be the key to prove our results concerning the Dehn twist. It shows a certain local Lipschitz continuity on barcodes associated to Lagrangian submanifolds. We will always assume that the considered Lagrangian submanifolds are connected.

Theorem 5.1.1. *Let M be a Liouville domain. Let L and L' be two closed exact Lagrangian submanifolds, and assume that $H^1(L', \mathbb{R}) = 0$. Then there exist $K \geq 0$ and $l > 0$ such that for all φ and ψ in $\mathrm{Symp}(M, \omega)$, if $d_{C^0}(\varphi, \psi) \leq l$, we have*

$$d_{bottle}(\hat{B}(\varphi(L'), L), \hat{B}(\psi(L'), L)) \leq K d_{C^0}(\varphi, \psi).$$

The fact that we have a uniform Lipschitz continuity with respect to the C^0 distance immediately implies the following corollary.

Corollary 5.1.2. *The map $\varphi \mapsto \hat{B}(\varphi(L'), L)$ continuously extends to a map $\overline{\mathrm{Symp}}(M, \omega) \to \hat{\mathcal{B}}$.*

Since L and L' are closed, the number of semi-infinite bars of $\mathcal{B}(\varphi(L'), L)$ stays finite for all $\varphi \in \mathrm{Symp}(M, \omega)$. This extension to the closure requires to work with $\mathcal{B}$ as defined in Definition 3.3.3 which is the completion of the space of barcodes $\mathcal{B}_f$ by Proposition 3.3.5. As we will see in the proof, we will then have to work with $\hat{\mathcal{B}}$ to deal with possible overall shifts.

From Theorem 5.1.1, we obtain the two following theorems. They are direct consequences of the continuity of barcodes together with its Corollary 5.1.2.

Theorem 5.1.3. *Let M be a Liouville domain. Let L and L' be two exact compact Lagrangian submanifolds, and assume that $H^1(L', \mathbb{R}) = 0$. Consider two symplectomorphisms φ and ψ. If these two symplectomorphisms are in the same connected component of $\overline{\mathrm{Symp}}(M, \omega)$, then the two barcodes $\hat{B}(\varphi(L'), L)$ and $\hat{B}(\psi(L'), L)$ are in the same connected component of $\hat{\mathcal{B}}$.*

The following theorem is an immediate consequence of Theorem 5.1.3.

Theorem 5.1.4. *Let M be a Liouville domain. Let L and L' be two closed exact Lagrangian submanifolds, and assume that $H^1(L', \mathbb{R}) = 0$. Consider two symplectomorphisms φ and ψ. If these two symplectomorphisms are isotopic in $\overline{\mathrm{Symp}}(M, \omega)$, then there is a continuous path of barcodes from $\hat{B}(\varphi(L'), L)$ to $\hat{B}(\psi(L'), L)$.*

This continuous path can be constructed in the following way. Let us denote $(\phi^t)_{t\in[0,1]}$ the path in $\overline{\mathrm{Symp}}(M, \omega)$ from φ to ψ. For each $t \in [0, 1]$, Corollary 5.1.2 allows to associate a barcode $\hat{B}^t$ to ϕ^t. The path of barcodes is then the path $(\hat{B}^t)_{t\in[0,1]}$.

Remark 5.1.5. In smooth symplectic topology, the two previous theorems would be equivalent. However, in C^0 symplectic topology we do not know whether $\overline{\mathrm{Symp}}(M, \omega)$ is locally path-connected, thus it is not known whether the connected components of $\overline{\mathrm{Symp}}(M, \omega)$ are path-connected. Consequently Theorem 5.1.3 implies Theorem 5.1.4 but the reciprocal implication is far from clear.

5.1.2 Sketch of the proof

In order to prove Theorem 5.1.1, we prove the two following propositions. The first one bounds the bottleneck distance by the spectral norm γ.

Proposition 5.1.6. *Let L and L' be two closed exact Lagrangian submanifolds in a Liouville domain (M, ω), with $H^1(L', \mathbb{R}) = 0$. There exists $\delta > 0$, independant of L, such that for all φ and $\psi \in \mathrm{Symp}(M, \omega)$ satisfying $d_{C^0}(\varphi, \psi) \leq \delta$, then*

$$d_{bottle}(\hat{\mathcal{B}}(\varphi(L'), L), \hat{\mathcal{B}}(\psi(L'), L)) \leq \tfrac{1}{2}\gamma(L', \psi^{-1} \circ \varphi(L')).$$

The second proposition asserts that $\gamma(L', \varphi(L'))$ goes to zero, as φ goes to identity.

Proposition 5.1.7. *There exist constants $l \geq 0$ and $\kappa \geq 0$ such that for all $\varphi \in \mathrm{Symp}(M, \omega)$ satisfying $d_{C^0}(\varphi, \mathrm{Id}_M) \leq l$, we have*

$$\gamma(L', \varphi(L')) \leq \kappa d_{C^0}(\varphi, \mathrm{Id}_M).$$

Proof of Theorem 5.1.1. With these two propositions, Theorem 5.1.1 is straightforward. Let φ and ψ be in $\mathrm{Symp}(M, \omega)$ such that $d_{C^0}(\varphi, \psi) \leq l$. We can assume without loss of generality that $l \leq \delta$. (See the choice of l in Section 5.4.)

Indeed we have

$$\begin{aligned}
d_{bottle}(\hat{\mathcal{B}}(\varphi(L'),L),\hat{\mathcal{B}}(\psi(L'),L)) &\leq \tfrac{1}{2}\gamma(L',\psi^{-1}\circ\varphi(L'))\\
&\leq \tfrac{1}{2}\kappa d_{C^0}(\psi^{-1}\circ\varphi,\mathrm{Id}_M)\\
&= \tfrac{1}{2}\kappa \sup_{x\in M} d(\psi^{-1}(x),\varphi^{-1}(x))\\
&\leq \tfrac{1}{2}\kappa d_{C^0}(\psi,\varphi).
\end{aligned}$$

Setting $K=\frac{1}{2}\kappa$, this proves Theorem 5.1.1.

□

Let us now briefly sketch the proof of Proposition 5.1.6 and Proposition 5.1.7 and set up some conventions. Proposition 5.1.6 will be implied by the case where $\psi=\mathrm{Id}_M$.

Let us fix $\varepsilon_0>0$, $\varepsilon'\ll\varepsilon_0$ and assume that all the Hamiltonian parts of the perturbation data at stake in this proof are of C^2-norm smaller than ε'.

Let us fix such a perturbation data collection $\mathcal{D}$ such that $HF^t(\varphi(L'),L;\mathcal{D})$, $HF^t(L',L;\mathcal{D})$, $HF^t(\varphi(L'),L';\mathcal{D})$, $HF^t(L',L';\mathcal{D})$ and $HF^t(\varphi(L'),\varphi(L');\mathcal{D})$ are well defined.

Remark 5.1.8. In the case of $HF(L',L';\mathcal{D})$, we require that the Hamiltonian perturbation is defined in the following way (see also Remark 2.2.34). Let f be a $\varepsilon'/2$-small Morse function defined on L' with a unique maximum and a unique minimum. We extend it to a Hamiltonian H which is supported on a ε_0-small tubular neighbourhood of L'. This construction implies that there is only one element in $CF^n(L',L';\mathcal{D})$ and only one in $CF^0(L',L';\mathcal{D})$. We perform the same construction in the case of $HF(\varphi(L'),\varphi(L');\mathcal{D})$.

We aim to find two morphisms of persistence modules $A=\{A^t\}_{t\in\mathbb{R}}$ and $B=\{B^t\}_{t\in\mathbb{R}}$ together with $\delta,\delta'\in\mathbb{R}$:

$$A^t: CF^t(\varphi(L'),L;\mathcal{D})\longmapsto CF^{t+\delta}(L',L;\mathcal{D}),$$

$$B^t: CF^t(L',L;\mathcal{D})\longmapsto CF^{t+\delta'}(\varphi(L'),L;\mathcal{D}),$$

such that these maps are filtered and their compositions are chain homotopic to shifts of persistence modules:

$$sh_{\varphi(L')}:\mathcal{B}(\varphi(L'),L;\mathcal{D})\longmapsto\mathcal{B}(\varphi(L'),L;\mathcal{D})[\delta+\delta'+\varepsilon']$$

$$sh_{L'}:\mathcal{B}(L',L)\longmapsto\mathcal{B}(L',L)[\delta+\delta'+\varepsilon'].$$

If they indeed satisfy the above conditions, these maps A and B provide a $\delta+\delta'+\varepsilon'$-matching. Then, to achieve the proof, we will only have to bound the shift $\delta+\delta'+\varepsilon'$ by the C^0 distance between φ and Id_M. We will prove that this shift is in fact equal to

$\frac{1}{2}\gamma(L';\varphi(L');\mathcal{D})+\varepsilon'$, and use this to get the bound. This is the purpose of Section 5.3. Proving that this bound goes to zero when φ C^0-converges to the identity is the purpose of the last Section 5.4.

Following Kislev and Shelukhin's idea [51], these maps A and B will come from the multiplication in Floer cohomology:

- A corresponds to the multiplication by a specific class $[x]$ in $HF(L',\varphi(L');\mathcal{D})$.
- B corresponds to the multiplication by a specific class $[y]$ in $HF(\varphi(L'),L';\mathcal{D})$.

These choices will be achieved using Abouzaid-Kragh's Theorem 4.5.5 [2]. This result requires the Lagrangian submanifolds to be in a cotangent space. To obtain this requirement, we will consider a symplectomorphism φ C^0-close enough to the identity so that $\varphi(L')$ is included in a Weinstein neighbourhood of L'. We thus obtain two cohomologies which could be different: the one computed in M and the one computed in T^*L'. Consequently, for the sake of our argument, we will first prove that we have the isomorphisms

$$HF(L',\varphi(L');\mathcal{D},M)\cong HF(L',\varphi(L');\mathcal{D},T^*L'),$$

$$HF(\varphi(L'),L';\mathcal{D},M)\cong HF(\varphi(L'),L';\mathcal{D},T^*L').$$

Of course we will also prove that these isomorphisms respect the filtration. We will in fact only give the details for one of these isomorphisms since the proofs are identical for both. By abuse of notation, we denote by $\mathcal{D}$ both the perturbation data in M and its image in T^*L. This is the purpose of the following Section 5.2.

Remark 5.1.9. Now that the proof is sketched, we can explain the conditions required for the two Lagrangian submanifolds L' and L in Theorem 5.1.1. These are both exactness conditions. In the previous chapters, to define Lagrangian Floer cohomology, the product and the action filtration, we require the considered Lagrangian submanifolds to be exact. This exactness condition is also required for Theorem 4.5.5 that will be used to construct the maps A and B.

The condition $H^1(L',\mathbb{R})=0$ guarantees that, for any symplectomorphism $\varphi\in\mathrm{Symp}(M,\omega)$, $\varphi(L')$ is an exact Lagrangian submanifold as well. With these conditions, we are sure that all the above mentioned objects used in the following proof will be well defined.

This implies that, when working with $\varphi\in\mathrm{Ham}(M,\omega)$, we can drop the condition $H^1(L',\mathbb{R})=0$ for the weaker condition that L' is exact. Indeed, the image of an exact Lagrangian submanifold by a Hamiltonian diffeomorphism is always exact.

The following sections are dedicated to the proof of Proposition 5.1.6 and Proposition 5.1.7.

5.2 Equality of the barcodes in M and in T^*L'

If the symplectomorphism φ is C^0-close enough to the identity, then $\varphi(L')$ is included in a Weinstein neighbourhood of L'. This will contribute to the definition of the constants δ and $l \in \mathbb{R}$ at stake in Proposition 5.1.6 and Proposition 5.1.7. By abuse of notation, we also denote $L', \varphi(L')$ their respective images in T^*L' by a Weinstein embedding. Denoting $\mathcal{D}$ a perturbation data in T^*L', we also denote $\mathcal{D}$ its pull-back by the chosen Weinstein embedding. Let us recall that $HF^s(\varphi(L'), L'; \mathcal{D}, M)$ is the filtered cohomology computed in M and $HF^s(\varphi(L'), L'; \mathcal{D}, T^*L')$ the filtered cohomology computed in T^*L'. We aim to prove that these two cohomologies are isomorphic and that this isomorphism respects the filtration.

This section is thus dedicated to the proof of the following Lemma 5.2.1. The idea for this is to localize the Floer trajectories near L'. Indeed, this will imply that the Floer trajectories in M and T^*L' are in $1:1$ correspondence, and thus the two cochain complexes are isomorphic.

Lemma 5.2.1. *If φ is C^0-close to the identity, then for an arbitrary choice of data (see Remark 1.3.5) there exists $C \in \mathbb{R}$ such that for all $s \in \mathbb{R}$*

$$HF^s(\varphi(L'), L'; \mathcal{D}, M) \cong HF^s(\varphi(L'), L'; \mathcal{D}, T^*L')[C].$$

We can actually choose the primitive functions of the 1-forms λ_M and $\lambda_{T^*L'}$ restricted to the Lagrangian submanifolds such that the shift C is equal to 0.

Proof. The idea here is to retract the Lagrangian submanifold $\varphi(L')$ by the negative Liouville flow. This will decrease the diameter of the spectrum and thus allow to have small enough energy estimates on the moduli spaces.

Let us choose two Weinstein's tubular neighbourhoods U and W of L' such that $U \Subset W$. We denote $\psi : W \to T^*L'$, the symplectic embedding provided by Weinstein's theorem. Let us suppose that φ is close to Id_M, so that we have $\varphi(L')$ included in U. We have two Liouville forms on W. The first one is the Liouville form λ_M restricted to W. The second one is the Liouville form obtained from the Liouville form $\lambda_{T^*L'}$ on T^*L': $\psi^*\lambda_{T^*L'}$. Let us recall that $\psi^*\lambda_{T^*L'} - \lambda_M$ is closed on W. Since $H^1(L', \mathbb{R}) = 0$ we have $H^1(W, \mathbb{R}) = 0$. Then $\psi^*\lambda_{T^*L'} - \lambda_M$ is exact on W and consequently there exists a function $F : W \to \mathbb{R}$ such that $\psi^*\lambda_{T^*L'} = (\lambda_M)_{|W} + dF$.

Let us pick a cut-off function $\beta : W \to \mathbb{R}$ such that β is constant, equal to 1 on U and equal to 0 near the boundary of W. By abuse of notation, we denote F the function defined on M equal to βF on W and continuously extended by 0 outside of W. The 1-form $(\lambda_M + dF)$ is a Liouville form on M equal to $\psi^*\lambda_{T^*L'}$ on U. We thus obtain a globally defined negative Liouville flow (i.e. the flow of the negative Liouville vector field) on M which preserves U and matches with the negative Liouville flow on T^*L'.

In T^*L', let us denote $\varphi^t_{-\mathcal{L}}$ the negative Liouville flow. When we apply this flow to $\psi(\varphi(L'))$ for $t \in \mathbb{R}^+$, we obtain a smooth path $(L'_t)_{t\in\mathbb{R}^+}$ of Lagrangian submanifolds in T^*L'. We can now consider the smooth path of Lagrangian submanifolds in M given by $(L_t)_{t\in\mathbb{R}^+} = (\psi^{-1}(L'_t))_{t\in\mathbb{R}^+}$.

Lemma 5.2.2. *For all $t \in \mathbb{R}^+$ we have $\mathrm{Spec}(L'_t, L'; T^*L') = \mathrm{Spec}(L_t, L'; M) + C_t$, where $C_t \in \mathbb{R}$. Moreover, we can choose the primitive functions of the 1-forms λ_M and $\lambda_{T^*L'}$ restricted to the Lagrangian submanifolds such that C_t is equal to 0 for all t.*

Proof. Fix $t \in \mathbb{R}^+$. Let x be in $\chi(L_t, L') \subset M$, with action $\mathcal{A}_{L'_t,L'}(x)$. Denote $x' = \psi(x)$ which is consequently in $\chi(L_t, L') \subset T^*L'$ with action $\mathcal{A}_{L'_t,L'}(x')$. Set $C_t = \mathcal{A}_{L'_t,L'}(x') - \mathcal{A}_{L_t,L'}(x)$.

For any other $y \in \chi(L_t, L')$ together with $y' = \psi(y) \in \chi(L'_t, L')$, let us denote γ_1 a path from x to y in L' and γ_2 a path from y to x in L. We denote γ'_1 and γ'_2 their respective images by ψ. From Definition 2.2.2 and Lemma 2.2.7, we have

$$\mathcal{A}_{L_t,L'}(y) - \mathcal{A}_{L_t,L'}(x) = \int_{\gamma_1} \lambda_M + \int_{\gamma_2} \lambda_M,$$

$$\mathcal{A}_{L'_t,L'}(y') - \mathcal{A}_{L'_t,L'}(x') = \int_{\gamma'_1} \lambda_{T^*L'} + \int_{\gamma'_2} \lambda_{T^*L'}.$$

Denoting $\gamma_1\sharp\gamma_2$ the concatenation of γ_1 and γ_2 we get

$$\begin{aligned}
\mathcal{A}_{L'_t,L'}(y') &= \mathcal{A}_{L'_t,L'}(x') + \int_{\gamma'_1\sharp\gamma'_2} \lambda_{T^*L'} \\
&= \mathcal{A}_{L'_t,L'}(x') + \int_{\gamma_1\sharp\gamma_2} \psi^*\lambda_{T^*L'} \\
&= \mathcal{A}_{L'_t,L'}(x') + \int_{\gamma_1\sharp\gamma_2} \lambda_M + dF \\
&= \mathcal{A}_{L'_t,L'}(x') + \int_{\gamma_1\sharp\gamma_2} \lambda_M \quad \text{since } \gamma_1\sharp\gamma_2 \text{ is a loop} \\
&= \mathcal{A}_{L'_t,L'}(x') + \mathcal{A}_{L_t,L'}(y) - \mathcal{A}_{L_t,L'}(x) \\
&= \mathcal{A}_{L_t,L'}(y) + C_t.
\end{aligned}$$

Since this is true for any $t \in \mathbb{R}^+$ and any pair (y, y') such as before, we can conclude that

$$\forall t \in \mathbb{R}^+, \quad \mathrm{Spec}(L'_t, L'; T^*L') = \mathrm{Spec}(L_t, L'; M) + C_t.$$

Now, for all t, choosing two primitive functions on L'_t and L_t such that $\mathcal{A}_{L'_t,L'}(x') = \mathcal{A}_{L_t,L'}(x)$ gives $C_t = 0$, which finishes this proof.

□

From now on, assume that the primitives on the Lagrangian submanifolds chosen so that for all $t \in \mathbb{R}$, $C_t = 0$. Since in T^*L' we have $(\varphi^t_{-\mathcal{L}})^*\omega = e^{-t}\omega$, and $(\varphi^t_{-\mathcal{L}})$ is equal to the identity on L', we get

$$\mathrm{Spec}(L'_t, L'; T^*L') = e^{-t}\mathrm{Spec}(L'_0, L'; T^*L'). \tag{5.1}$$

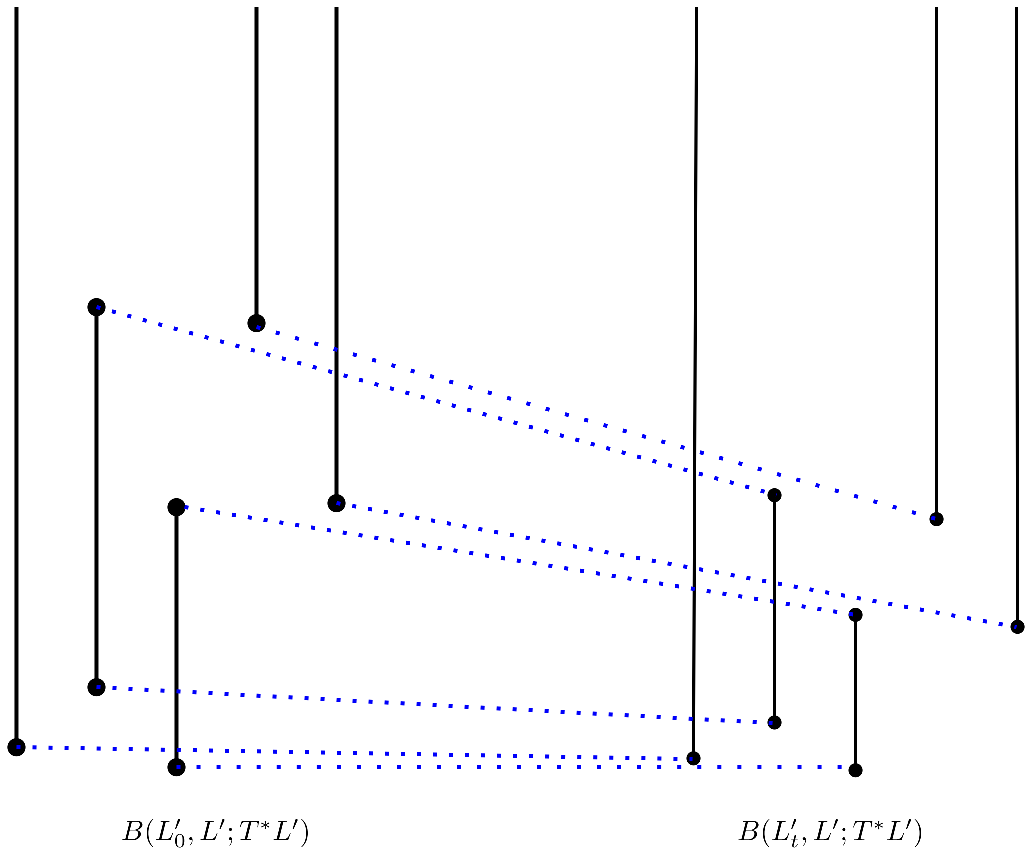

Figure 5.1 – Evolution of the barcode during the Liouville retraction

Lemma 5.2.3. *For T large enough, there is a canonical identification between the cochain complexes $CF(L'_T, L'; \mathcal{D}, T^*L')$ and $CF(L'_T, L'; \mathcal{D}, M)$ given by the Weinstein's neighbourhood embedding.*

Corollary 5.2.4. *For T large enough there is a canonical isomorphism*

$$HF^s(L'_T, L'; \mathcal{D}, T^*L') \cong HF^s(L'_T, L'; \mathcal{D}, M)$$

holding for all $s \in \mathbb{R}$.

Proof. These two cochain complexes are generated by the perturbed intersection points, which are identified by Weinstein's neighbourhood embedding. To prove this lemma, we thus have to show that for T large enough, the differential is the same, i.e. that the J-

holomorphic curves between two intersection points agree. To do so we will show that if T is large enough, no such J-holomorphic curves can go outside of W.

Since we are working with a Liouville domain, which is always tame, Sikorav's proposition 4.3.1 and its corollary in [89] are verified. Consequently, there exists a constant $\kappa \in \mathbb{R}$, such that for any compact subset K, any compact connected J-holomorphic curve u such that $u \cap K \neq \emptyset$, and $\partial u \subset K$ satisfies

$$u \subset U(K, \kappa\mathcal{A}(u)), \tag{5.2}$$

where $U(K, \kappa\mathcal{A}(u))$ is the $\kappa\mathcal{A}(u)$-neighbourhood of K. Let us fix $\delta > 0$ small enough such that we can find a compact neighbourhood K of L' such that $U(K, \delta) \subset W$.

Let us denote Γ_t, the diameter of the spectrum $\mathrm{Spec}(L'_t, L'; T^*L')$, which is equal by Lemma 5.2.2 to the diameter of the spectrum $\mathrm{Spec}(L'_t, L'; M)$. From the previous equality 5.1, we have $\Gamma_t = e^{-t}\Gamma_0$. Set $t_\delta = \ln(\frac{\Gamma_0\kappa}{\delta})$. We then have

$$\forall t \geq t_\delta, \Gamma_t \leq \frac{\delta}{\kappa}.$$

Let us recall that following Lemma 2.2.7, the area of a J-holomorphic strip between two intersection points is equal to the difference of action between the two intersection points. This area is thus bounded by the diameter of the spectrum Γ_t. Let us fix $T > t_\delta$. A J-holomorphic strip u between two generators of $CF(L'_T, L'; \mathcal{D}, M)$ satisfies $\mathcal{A}(u) \leq \frac{\delta}{\kappa}$. Inclusion 5.2 then becomes

$$u \subset U(K, \kappa\mathcal{A}(u)) = U(K, \delta) \subset W.$$

This means that the J-holomorphic strips defining the differential of the chain complex $CF(L'_T, L'; \mathcal{D}, M)$ stay in W. They are identified by the embedding ψ to the J-holomorphic strips defining the differential of the chain complex $CF(L'_T, L'; \mathcal{D}, T^*L')$. Consequently the differential of the two chain complexes behave well with respect to the embedding ψ. This concludes the proof of this lemma. □

Remark 5.2.5. In this lemma, we only dealt with J-holomorphic curves computing the differential. However, we can conduct the exact same proof with other moduli spaces. This implies that the μ^k-operations are preserved by the isomorphism given by Lemma 5.2.3.

Remark 5.2.6. Since the paths $(L_t)_{t\in\mathbb{R}}$ and $(L'_t)_{t\in\mathbb{R}}$ are smooth, the associated barcode paths are continuous according to the expression 1.2 and Proposition 4.2.5. Let us denote $B_t = \mathcal{B}(L_t, L'; \mathcal{D}, M)$ and $B'_t = \mathcal{B}(L'_t, L'; \mathcal{D}, T^*L')$. The previous lemma tells that for T large enough, $B_T = B'_T$.

Moreover let $(B_t)_{t\in[0;1]}$ be a continuous path of barcodes such that there is a positive continuous function $f : \mathbb{R} \to \mathbb{R}$ which satisfies

$$\forall t \in [0;1], \mathrm{Spec}(B_t) = f(t)\mathrm{Spec}(B_0).$$

Since there is no bifurcation in the spectrum, B_t is a dilation by $f(t)$ of B_0.

Lemma 5.2.7. *Let L be a closed exact Lagrangian submanifold in a Weinstein neighbourhood U of L' with associated embedding ψ. For all $t \in [-T, 0]$ let us denote $L'^t = \varphi^t_{-\mathcal{L}} \circ \psi(L)$. Assume that for all $t \in [-T, 0]$ $L'^t \subset U$. Let us denote $L^t = \psi^{-1}(L'^t)$. Then for all $s \in \mathbb{R}$*

$$HF^s(L'^t, L'; (\varphi^t_{-\mathcal{L}})_*\mathcal{D}, T^*L') \cong HF^{se^{-t}}(\psi(L), L'; \mathcal{D}, T^*L'),$$

$$HF^s(L^t, L'; (\varphi^t_{-\mathcal{L}})_*\mathcal{D}, M) \cong HF^{se^{-t}}(L, L'; \mathcal{D}, M).$$

Proof. The symplectic invariance given by Equality 2.7 tells that there is a function $f : \mathbb{R} \to \mathbb{R}$ such that

$$HF^s(L'^t, \varphi^t_{-\mathcal{L}}(L'); (\varphi^{-t}_{-\mathcal{L}})_*\mathcal{D}, T^*L') \cong HF^{f(s)}(\psi(L), L'; \mathcal{D}, T^*L'),$$

$$HF^s(L^t, \psi^{-1} \circ \varphi^t_{-\mathcal{L}}(L'); (\varphi^t_{-\mathcal{L}})_*\mathcal{D}, M) \cong HF^{f(s)}(L, L'; \mathcal{D}, M).$$

We can indeed write the second isomorphism since the negative Liouville flow has been globally defined on M. Since $\varphi^t_{-\mathcal{L}}(L') = L'$ for all t, we have

$$HF^s(L'^t, L'; (\varphi^t_{-\mathcal{L}})_*\mathcal{D}, T^*L') \cong HF^{f(s)}(\psi(L), L'; \mathcal{D}, T^*L'),$$

$$HF^s(L^t, L'; (\varphi^t_{-\mathcal{L}})_*\mathcal{D}, M) \cong HF^{f(s)}(L, L'; \mathcal{D}, M).$$

Moreover Equality 5.1 tells that $f(s) = se^{-t}$. We then have

$$HF^s(L'^t, L'; (\varphi^t_{-\mathcal{L}})_*\mathcal{D}, T^*L') \cong HF^{se^{-t}}(\psi(L), L'; \mathcal{D}, T^*L').$$

The same computation as in Lemma 5.2.2 gives

$$HF^s(L^t, L'; (\varphi^t_{-\mathcal{L}})_*\mathcal{D}, M) \cong HF^{se^{-t}}(L, L'; \mathcal{D}, M).$$

□

Applying this lemma to L_T together with Lemma 5.2.3 and the fact that $\varphi^{-T}_{-\mathcal{L}}(L'_T) = \varphi(L')$, we finally obtain

$$HF^s(\varphi(L'), L'; \mathcal{D}, M) \cong HF^s(\varphi(L'), L'; \mathcal{D}, T^*L')$$

for all $s \in \mathbb{R}$. This concludes the proof of Lemma 5.2.1.

Let us assume, once and for all that φ is sufficiently close to identity, so that $\varphi(L')$ is inside the Weinstein neighbourhood of L' and reciprocally.

5.3 Bounding the bottleneck distance by the spectral norm

In this section, we will bound the bottleneck distance by the spectral norm γ by proving the following proposition.

Proposition 5.3.1. *Let L and L' be two closed exact Lagrangian submanifolds in a Liouville domain (M,ω). There exists $\delta > 0$, independant of L, such that for all $\varphi \in \mathrm{Symp}(M,\omega)$ satisfying $d_{C^0}(\varphi, \mathrm{Id}_M) \leq \delta$, then there exists $C \in \mathbb{R}$ such that*

$$d_{bottle}(\mathcal{B}(L', L), \mathcal{B}(\varphi(L'), L)[C]) \leq \tfrac{1}{2}\gamma(L', \varphi(L')).$$

In [51], Kislev and Shelukhin proved a similar statement in a different setting. In their case, $L = L'$ is a weakly monotone Lagrangian submanifold in a closed symplectic manifold and φ is a Hamiltonian diffeomorphism. The following proof of Proposition 5.3.1 is an adaptation of their proof to our setting.

We choose $\delta > 0$ so that, for all $\varphi \in \mathrm{Symp}(M,\omega)$, if $d_{C^0}(\varphi, \mathrm{Id}_M) \leq \delta$ then $\varphi(L')$ is included in a Weinstein neighbourhood of L'. We will denote this Weinstein neighbourhood $W(L')$.

We can now prove Proposition 5.1.6 required to prove Theorem 5.1.1.

Proof of Proposition 5.1.6. To prove this proposition, we will apply Proposition 5.3.1 to the symplectomorphism $\psi^{-1} \circ \varphi$. As in Proposition 5.3.1, we choose $\delta > 0$ so that, for all $\phi \in \mathrm{Symp}(M,\omega)$, if $d_{C^0}(\phi, \mathrm{Id}_M) \leq \delta$ then $\phi(L')$ is included in $W(L')$, a Weinstein neighbourhood of L'. Let us assume that $d_{C^0}(\varphi, \psi) \leq \delta$.

$$\begin{aligned} d_{C^0}(\varphi,\psi) &= \max\left\{\sup_{x\in M} d(\varphi(x),\psi(x)), \quad \sup_{x\in M} d(\varphi^{-1}(x),\psi^{-1}(x))\right\} \\ &\geq \sup_{x\in M} d(\varphi^{-1}(x),\psi^{-1}(x)) \\ &= \sup_{x\in M} d(\psi^{-1}\circ\varphi(x), x) \\ &= d_{C^0}(\psi^{-1}\circ\varphi, \mathrm{Id}_M). \end{aligned}$$

So we get

$$d_{C^0}(\psi^{-1}\varphi, \mathrm{Id}_M) \leq \delta.$$

We introduced the set of barcodes quotiented by an overall shift $\hat{\mathcal{B}}$ to get rid of the shift in the inequality of Proposition 5.3.1. Indeed, when working with the barcodes in $\hat{\mathcal{B}}$, this inequality becomes

$$d_{bottle}(\hat{B}(L',L),\hat{B}(\varphi(L'),L)) \leq \tfrac{1}{2}\gamma(L',\varphi(L')).$$

By invariance of the barcode under the action of a symplectomorphism, we have

$$d_{bottle}(\hat{\mathcal{B}}(\varphi(L'),L),\hat{\mathcal{B}}(\psi(L'),L)) = d_{bottle}(\hat{\mathcal{B}}(L',\psi^{-1}(L)),\hat{\mathcal{B}}(\psi^{-1}\circ\varphi(L'),\psi^{-1}(L))).$$

By the previous inequality and Proposition 5.3.1, we then have

$$d_{bottle}(\hat{\mathcal{B}}(L',\psi^{-1}(L)),\hat{\mathcal{B}}(\psi^{-1}\circ\varphi(L'),\psi^{-1}(L))) \leq \tfrac{1}{2}\gamma(L',\psi^{-1}\circ\varphi(L')),$$

which concludes the proof of this proposition.

□

Let us now prove Proposition 5.3.1 and the desired bound. We start by introducing the interleaving maps.

Definition of the interleaving maps

As explained in Remark 5.1.9, the condition $H^1(L',\mathbb{R})=0$ guarantees that for all $\varphi \in \mathrm{Symp}(M,\omega)$, $\varphi(L')$ is an exact Lagrangian submanifold. Hence, we can apply Abouzaid-Kragh's Theorem 4.5.5 [2], thus obtaining two isomorphisms

$$HF(L',L';\mathcal{D},T^*L') \xrightarrow[\sigma]{\sim} HF(L',\varphi(L');\mathcal{D},T^*L'),$$

$$HF(\varphi(L'),\varphi(L');\mathcal{D},T^*L') \xrightarrow[\sigma']{\sim} HF(\varphi(L'),L';\mathcal{D},T^*L').$$

Moreover, by Proposition 2.2.32 applied to L' and $\varphi(L')$, and Poincaré duality there is an isomorphism

$$\theta : H_*(L') \to HF^{n-*}(L',L';\mathcal{D},T^*L').$$

By symplectic invariance of Floer cohomology (see equality 2.7), we have

$$HF^*(\varphi(L'),\varphi(L');\phi^*\mathcal{D},T^*L') \cong HF^*(L',L';\mathcal{D},T^*L').$$

Consequently we also have an isomorphism

$$\theta' : H_*(L') \to HF^{n-*}(\varphi(L'),\varphi(L');\mathcal{D},T^*L').$$

Let us choose $c \in HF(L',L';\mathcal{D},T^*L')$ to be the class $\theta([L'])$, and $c' \in HF(\varphi(L'),\varphi(L');\mathcal{D},T^*L')$ the class $\theta'([L'])$. Moreover assume that the gradings are chosen so that c and c' are both of degree 0.

Lemma 5.2.1 provides two isomorphisms ζ and ζ' between $HF(L',\varphi(L');T^*L')$ and

$HF(L', \varphi(L'); M)$ and between $HF(\varphi(L'), L'; T^*L')$ and $HF(\varphi(L'), L'; M)$. We can now choose two cycles $x \in CF(L', \varphi(L'); M)$ and $y \in CF(\varphi(L'), L'; M)$ such that

$$[x] = \zeta(\sigma(c))$$

$$[y] = \zeta'(\sigma'(c')).$$

Let us choose two primitive functions $f' : L' \to \mathbb{R}$ and $g : \varphi(L') \to \mathbb{R}$ such that $df' = \lambda_{|L'}$, $dg = \lambda_{|\varphi(L')}$ and such that we can find

- z such that $[z] = c \in HF(L', L'; \mathcal{D})$ with $\mathcal{A}(z) \leq \varepsilon'/2 \ll \varepsilon_0$
- z' such that $[z'] = c' \in HF(\varphi(L'), \varphi(L'); \mathcal{D})$ with $\mathcal{A}(z') \leq \varepsilon'/2 \ll \varepsilon_0$.

According to the previous choices of degree, we actually have $[z] \in HF^0(L', L'; \mathcal{D})$ and $[z'] \in HF^0(\varphi(L'), \varphi(L'); \mathcal{D})$.

Lemma 5.3.2. *The multiplication maps*

$$m_2(\cdot, z) : CF^*(L', L; \mathcal{D}) \to CF^*(L', L; \mathcal{D})[\varepsilon']$$

$$m_2(\cdot, z') : CF^*(\varphi(L'), L; \mathcal{D}) \to CF^*(\varphi(L'), L; \mathcal{D})[\varepsilon']$$

are filtered chain-homotopic to the standard inclusions and hence induce the ε'-shift maps on the persistence modules.

Proof. This lemma is an immediate consequence of Lemma 4.4.3. □

By abuse of notation, to make the following expressions clearer, we denote $[L'] = \zeta \circ \sigma \circ \theta([L']) \in HF^0(L', \varphi(L'); \mathcal{D}, M)$ and $[\varphi(L')] = \zeta' \circ \sigma' \circ \theta'([L']) \in HF^0(\varphi(L'), L'; \mathcal{D}, M)$. Now, let us choose $x \in CF^0(L', \varphi(L'); \mathcal{D})$ and $y \in CF^0(\varphi(L'), L'; \mathcal{D})$ as above such that:

$$l([L']; L', \varphi(L'); \mathcal{D}) \leq \mathcal{A}(x) = a \leq l([L']; L', \varphi(L'); \mathcal{D}) + \varepsilon',$$

$$l([\varphi(L')]; \varphi(L'), L'; \mathcal{D}) \leq \mathcal{A}(y) = b \leq l([\varphi(L')]; \varphi(L'), L'; \mathcal{D}) + \varepsilon',$$

which is possible by definition of $l([L']; L', \varphi(L'); \mathcal{D})$ and $l([\varphi(L')]; \varphi(L'), L'; \mathcal{D})$.

Moreover, by definition of x, y, we have $[\mu^2(y, x)] = [z] \in HF^0(L', L'; \mathcal{D})$. Indeed, up to the appropriate isomorphisms, the cycles x, y, z all represent the same class $[L]$ in their respective cochain complexes. With our particular choice of perturbation data for the pair (L', L') as explained in Remark 5.1.8, the cycle z is the only representative of his class. The same argument holds for z'. Consequently, we have the following lemma.

Lemma 5.3.3.

$$\mu^2(y, x) = z \in CF^0(L', L'; \mathcal{D}),$$

$$\mu^2(x, y) = z' \in CF^0(\varphi(L'), \varphi(L'); \mathcal{D}).$$

Remark 5.3.4. If we choose to work with φ being a Hamiltonian diffeomorphism and not only a symplectomorphism, the definition of x and y is much easier. In this case, it is achieved without Abouzaid-Kragh's result [2] of Theorem 4.5.5.

Indeed, continuation morphisms give the isomorphisms:

$$HF^*(\varphi(L'), L') \cong HF^*(L', L') \cong HF^*(\varphi(L'), \varphi(L')) \cong HF^*(L', \varphi(L')).$$

Since these continuations morphisms are compatible with the product structure on Lagrangian Floer cohomology, we can directly define x and y, and it is easy to see that the product by these elements will not be constant equal to 0. Indeed we easily have

$$[\mu^2(y, x)] = [z]$$

$$[\mu^2(x, y)] = [z'].$$

Moreover, the two multiplication operators $m_2(\cdot, z)$ and $m_2(\cdot, z')$ are still filtered chain-homotopic to the standard inclusion.

Bounding the bottleneck distance

Now that our objects are defined, we can adapt the Kislev-Shelukhin method [51] to our context. The point here is to carefully study the shifts of action induced by the the multiplication by the elements introduced above. Let us start with the two following lemmas.

Lemma 5.3.5. *The maps*

$$\mu^2(\cdot, x) : CF^*(\varphi(L'), L; \mathcal{D}) \to CF^*(L', L; \mathcal{D})[a + \varepsilon']$$
$$\mu^2(\cdot, y) : CF^*(L', L; \mathcal{D}) \to CF^*(\varphi(L'), L; \mathcal{D})[b + \varepsilon']$$

are well-defined and induce filtered maps of chain complexes.

Lemma 5.3.6. *The maps*

$$\mu^2(\mu^2(\cdot, y), x) : CF^*(L', L; \mathcal{D}) \to CF^*(L', L; \mathcal{D}))[a + b + 3\varepsilon']$$
$$\mu^2(\mu^2(\cdot, x), y) : CF^*(\varphi(L'), L; \mathcal{D}) \to CF^*(\varphi(L'), L; \mathcal{D})[a + b + 3\varepsilon']$$

are well-defined and filtered chain homotopic to the multiplication operators:

$$\mu^2(\cdot, \mu^2(y, x)) : CF^*(L', L; \mathcal{D}) \to CF^*(L', L; \mathcal{D})[a + b + 3\varepsilon']$$
$$\mu^2(\cdot, \mu^2(x, y)) : CF^*(\varphi(L'), L; \mathcal{D}) \to CF^*(\varphi(L'), L; \mathcal{D})[a + b + 3\varepsilon']$$

Proof. These two lemmas directly follow from the discussion on the product structure. Lemma 4.4.1 gives the first one and Lemma 4.4.2 the second one. □

Remark 5.3.7. In Kislev and Shelukhin's paper [51], there is another term in the previous equality which is a boundary. This additional term induces a shift in action by a constant β which vanishes in our case.

We now have the relation between the previous maps and the multiplication by z or z': the maps

$$\mu^2(\cdot,\mu^2(y,x)) : CF^*(L',L;\mathcal{D}) \to CF^*(L',L';\mathcal{D})[a+b+3\varepsilon']$$
$$\mu^2(\cdot,\mu^2(x,y)) : CF^*(\varphi(L'),L;\mathcal{D}) \to CF^*(\varphi(L'),L;\mathcal{D})[a+b+3\varepsilon']$$

are equal to the multiplication operators:

$$\mu^2(\cdot,z) : CF^*(L',L;\mathcal{D}) \to CF^*(L',L;\mathcal{D})a+b+3\varepsilon']$$
$$\mu^2(\cdot,z') : CF^*(\varphi(L'),L;\mathcal{D}) \to CF^*(\varphi(L'),L;\mathcal{D})[a+b+3\varepsilon']$$

Following Lemma 5.3.2, we obtain on the level of filtered homology the shifts of persistence modules morphisms

$$sh_{L'} : CF^*(L',L;\mathcal{D}) \to CF^*(L',L;\mathcal{D})[a+b+4\varepsilon']$$
$$sh_{\varphi(L')} : CF^*(\varphi(L'),L;\mathcal{D}) \to CF^*(\varphi(L'),L;\mathcal{D})[a+b+4\varepsilon'].$$

Let us recall that the barcodes are C^2-continuous by Proposition 4.2.5. We can take the limit as the Hamiltonian part of the perturbation data goes to zero as explained after Proposition 4.2.5 and assume that

$$a < l([L'];L',\varphi(L')) + 2\varepsilon',$$

$$b < l([L'];\varphi(L'),L') + 2\varepsilon'.$$

Consequently we have shift maps of barcodes without the perturbation data:

$$sh_{L'} = \mu^2(\cdot,x)\circ\mu^2(\cdot,y) : \mathcal{B}(L',L;) \to \mathcal{B}(L',L)[\gamma(L',\varphi(L'))+6\varepsilon']$$
$$sh_{\varphi(L')} = \mu^2(\cdot,y)\circ\mu^2(\cdot,x) : \mathcal{B}(\varphi(L'),L) \to \mathcal{B}(\varphi(L'),L)[\gamma(L',\varphi(L'))+6\varepsilon'].$$

Indeed, as discussed in Section 4.5,

$$l([L'];L',\varphi(L')) + l([L'];\varphi(L'),L') = \gamma(L',\varphi(L')) \geq 0.$$

For readability reasons, we denote

$$\alpha = l([L']; L', \varphi(L')) \text{ and } \bar{\alpha} = l([L']; \varphi(L'), L').$$

With this expression, the multiplication operators appear as maps between persistence modules:

$$\mu^2(\cdot, x) : \mathcal{B}(\varphi(L'), L) \to \mathcal{B}(L', L)[\alpha + 3\varepsilon']$$
$$\mu^2(\cdot, y) : \mathcal{B}(L', L) \to \mathcal{B}(\varphi(L'), L)[\bar{\alpha} + 3\varepsilon'].$$

Let us recall that, by Lemma 4.5.6, we have $\gamma(L', \varphi(L')) = \alpha + \bar{\alpha}$. Consequently, the previous multiplication operators can be written as

$$\mu^2(\cdot, x) : \mathcal{B}(\varphi(L'), L) \to \mathcal{B}(L', L)[\tfrac{1}{2}(\alpha - \bar{\alpha})][\tfrac{1}{2}\gamma(L', \varphi(L')) + 3\varepsilon']$$
$$\mu^2(\cdot, y) : \mathcal{B}(L', L)[\tfrac{1}{2}(\alpha - \bar{\alpha})] \to \mathcal{B}(\varphi(L'), L)[\tfrac{1}{2}\gamma(L', \varphi(L')) + 3\varepsilon']$$

Together with the previous identity of persistence modules, this is the exact definition of the fact that $\mathcal{B}(L', L)$ and $\mathcal{B}(\varphi(L'), L)[\frac{1}{2}(\alpha - \bar{\alpha})]$ are $\frac{1}{2}\gamma(L', \varphi(L')) + 3\varepsilon'$-interleaved. Taking the limit as ε' goes to zero, we get

$$d_{bottle}(\mathcal{B}(L', L), \mathcal{B}(\varphi(L'), L)[\tfrac{1}{2}(\alpha - \bar{\alpha})]) \leq \tfrac{1}{2}\gamma(L', \varphi(L')). \tag{5.3}$$

Setting $C = \frac{1}{2}(\alpha - \bar{\alpha})$, this concludes the proof of Proposition 5.3.1.

5.4 Bounding the spectral norm by the C^0-distance

We will now prove Proposition 5.1.7. This proof is an adaptation to our context of a lemma and a proof of Buhovsky-Humilière-Seyfaddini [15]. In their paper, they proved the same result for a Lagrangian submanifold Hamiltonian isotopic to the zero section in a cotangent bundle. Here we are working with L' being a closed exact Lagrangian submanifold in M. Let us denote $W(L')$ a Weinstein neighbourhood of L'. By definition, if $\varphi \in \mathrm{Symp}(M, \omega)$ is close enough to Id_M, then $\varphi(L') \subset W(L')$. By abuse of notation, we also respectively denote B and $\varphi(L')$ the images by a Weinstein embedding of respectively B and $\varphi(L')$ in T^*L', where B is a ball in $W(L')$.

Lemma 5.4.1. *Let B be a ball in L'. Let $\mathrm{Symp}_B(M, \omega) := \{\varphi \in \mathrm{Symp}(M, \omega) | \quad \varphi(L') \cap T^*B = 0_B)\}$. There exist $\delta > 0$ and $C > 0$ such that for any $\varphi \in \mathrm{Symp}_B(M, \omega)$, if $d_{C^0}(\mathrm{Id}_M, \varphi) \leq \delta$, then $\gamma(L', \varphi(L')) \leq C d_{C^0}(\mathrm{Id}_M, \varphi)$.*

Proof. Let $\varepsilon > 0$ and let us choose an ε-small smooth function $f : L' \to \mathbb{R}$ whose critical points are all contained in B. Let $\pi : T^*L' \to L'$ be the natural projection and $\rho : T^*L' \to$

$[0;1]$ be a compactly supported function in T^*L' equal to 1 on $T^*_R L'$, for R large.

Denote $H = \rho\pi^* f$ and with $r \ll R$, we have

$$\phi^t_H(q,p) = (q, p + tdf(q)), \forall t \in [0;1], \forall (q,p) \in T^*_r L'.$$

Since f has no critical point outside of B, and L' is compact, let us denote $\eta > 0$, the minimum of $\|df(q)\|$, for $q \in L' \setminus B$.

Let us fix $\delta > 0$ such that, for all $\varphi \in \mathrm{Symp}_B(M,\omega)$, if $d_{C^0}(\mathrm{Id}_M, \varphi) \leq \delta$ then $\varphi(L') \subset W(L')$. From now on in this proof, we assume $d_{C^0}(\mathrm{Id}_M, \varphi) \leq \delta$.

Let φ be in $\mathrm{Symp}_B(M,\omega)$ and $d = d_{C^0}(\mathrm{Id}_M, \varphi)$.

Since $B \cap \varphi(L')$ is connected, the following lemma tells that two points $(q_1, 0)$ and $(q_2, 0)$ in $B \cap \varphi(L')$ have the same action α.

Lemma 5.4.2. *Let (M,ω) be an exact symplectic manifold, with λ such that $d\lambda = \omega$. Let L and L' be two closed exact Lagrangian submanifolds together with primitive functions f_L and $f_{L'}$ such that $df_L = \lambda_{|L}$ and $df_{L'} = \lambda_{|L'}$. Assume there is a ball B such that $L \cap L' \cap B$ is connected. Then, all points in $(L \cap L' \cap B)$ have the same action.*

Proof. In $(L \cap L' \cap B)$, we have that $d(f_L - f_{L'}) = 0$. This means that $f_L - f_{L'}$ is locally constant. Moreover $(L \cap L' \cap B)$ is connected so the function $f_L - f_{L'}$ is constant on $(L \cap L' \cap B)$. Since for a point $p \in (L \cap L')$, the action $\mathcal{A}_{L,L'}(p)$ is equal to $f_L(p) - f_{L'}(p)$, this concludes the proof of this lemma. □

Let us recall that the action is defined up to a constant. We can thus choose the action $\alpha = 0$.

Fix $\varepsilon' > 0$ and let $f_{d,\varepsilon'} = (\frac{d}{\eta} + \varepsilon')f$. We then have, for $q \in L' \setminus B$,

$$\begin{aligned} \|df_{d,\varepsilon'}(q)\| &= \left(\frac{d}{\eta} + \varepsilon'\right)\|df(q)\| \\ &\geq d + \varepsilon'\eta, \end{aligned}$$

and for all $q \in L'$

$$\|f_{d,\varepsilon'}(q)\| \leq \left(\frac{d}{\eta} + \varepsilon'\right)\varepsilon.$$

Let $H_{d,\varepsilon'} = \rho\pi^* f_{d,\varepsilon'}$.

Consequently we have

$$\phi^1_{H_{d,\varepsilon'}}\varphi(L') = \left\{(q, p + df_{d,\varepsilon'}(q)), (q,p) \in \varphi(L')\right\} =: \varphi(L') + L_{f_{d,\varepsilon'}}.$$

Therefore,

$$(\varphi(L') + L_{f_{d,\varepsilon'}}) \cap L' = \left\{(q,0), df_{d,\varepsilon'}(q) = 0\right\}.$$

To achieve the proof of Lemma 5.4.1, we now need to bound the difference of action between the intersection points of $(\varphi(L') + L_{f_{d,\varepsilon'}})$ and L'.

In T^*L', the action of a critical point $x = (q, 0)$ is given by $\mathcal{A}_{T^*L'}(x) = f(q)$, if we choose 0 to be the primitive of $\lambda_{std|B\times\{0\}} = 0_{|B\times\{0\}}$. We have to make a choice of primitive here because the action is defined up to a constant. With that choice, the action in T^*L equals the action in M for the points in $B \cap \varphi(L')$.

Let us denote $\psi : W(L') \to T^*L'$ the symplectomorphism provided by Weinstein's neighbourhood theorem.

In T^*B, $\psi^*\lambda - \lambda_{std}$ is closed, and since it is simply connected domain, it is also exact. Consequently we can write:

$$\psi^*\lambda - \lambda_{std} = dF,$$

where $F : T^*B \to \mathbb{R}$.

The action of an intersection point x in $W(L')$ is given by:

$$\mathcal{A}_{\varphi(L'),L'\subset M}(x) = f_{\varphi(L')}(x) - f_{L'}(x).$$

The previous equality leads to:

$$\begin{aligned}
\mathcal{A}_{\varphi(L'),L'\subset M}(x) &= F_{|\psi(\varphi(L'))}(\psi(x)) + f_{d,\varepsilon'}(\psi(x)) - F_{|B}(\psi(x)) \\
&= f_{d,\varepsilon'}(\psi(x)).
\end{aligned}$$

Our choice for the action $\alpha = 0$ and our choice of $0_{L'}$ to be the primitive of $\lambda_{std|L'}$ comes to sense now. Without these choices, we would have $\mathcal{A}_{\varphi(L'),L'\subset M}(x) = f_{d,\varepsilon'}(\psi(x)) + C$, where $C \in \mathbb{R}$ does not depend on x. This does not actually matter since we are only interested in differences of action, it is just more convenient this way.

Since $\|f_{d,\varepsilon'}(q)\| \leq (\frac{d}{\eta} + \varepsilon')\varepsilon$, and $\gamma(L', \varphi(L') + L_f)$ is a difference between the actions of two intersection points, we get

$$\gamma(L', \varphi(L') + L_{f_{d,\varepsilon'}}) \leq 2\left(\frac{d}{\eta} + \varepsilon'\right)\varepsilon.$$

Moreover, the third point of Proposition 4.2.3 or Remark 4.5.1 yields

$$|\gamma(L', \varphi(L') + L_{f_{d,\varepsilon'}}) - \gamma(L', \varphi(L'))| \leq 2\left(\max(f_{d,\varepsilon'}) - \min(f_{d,\varepsilon'})\right).$$

Consequently we get

$$\gamma(L', \varphi(L')) \leq 4\left(\frac{d}{\eta} + \varepsilon'\right)\varepsilon.$$

As ε' goes to zero, we finally have

$$\gamma(L', \varphi(L')) \quad \leq \quad 4\frac{d}{\eta}\varepsilon$$

$$\leq \quad 4\frac{\varepsilon}{\eta} d_{C^0}(\mathrm{Id}_M, \varphi).$$

Setting $C = 4\frac{\varepsilon}{\eta}$ concludes the proof of this lemma.

□

In order to finish the proof of Proposition 5.1.7, we need to reduce to Lemma 5.4.1. Indeed in the hypothesis, we do not have such a ball B. To do so, we will use and adapt to our context a trick from [15]. This trick consists in doubling the coordinates and introducing the following auxiliary map:

$$\Phi : \varphi \times \varphi^{-1} = M \times M \to M \times M,$$

where $M \times M$ is equipped with the natural symplectic form $\omega \oplus \omega$.

We will now prove the following lemma:

Lemma 5.4.3. *For any ball B in M, there is a smaller ball $B' \subset B$ with the following property. There exists $\Delta > 0$ such that for any $\varphi \in \mathrm{Symp}(M, \omega)$ with $d_{C^0}(\varphi, \mathrm{Id}_M) < \Delta$, we can find a symplectomorphism $\Psi \in \mathrm{Symp}(M \times M, \omega \oplus \omega)$ satisfying:*

1. *supp$(\Psi) \subset B \times B$ and supp$(\Phi \circ \Psi) \subset M \times M \backslash B' \times B'$,*

2. *$d_{C^0}(\Psi, \mathrm{Id}_{M\times M}) < C_B d_{C^0}(\varphi, \mathrm{Id}_M)$ and $d_{C^0}(\Phi \circ \Psi, \mathrm{Id}_{M\times M}) < C'_B d_{C^0}(\varphi, \mathrm{Id}_M)$, where C_B and C'_B do not depend on φ.*

Proof. Let B be a non empty open ball in M.

The following lemma gives us the existence of a Hamiltonian diffeomorphism f on $M \times M$ which locally switches the coordinates. This lemma and its proof come from [15], so we will not prove it here.

Lemma 5.4.4. *For any non empty open ball $B'' \subset B$ there exists a Hamiltonian diffeomorphism f on $M \times M$, such that*

- *f is the time-1 map of a Hamiltonian supported in $B \times B$,*
- *for all $(x, y) \in B'' \times B''$, we have $f(x, y) = f(y, x)$.*

For the rest of the proof of Lemma 5.4.3, we pick a ball B'' and a Hamiltonian diffeomorphism f as provided by the previous Lemma 5.4.4. Let B' be a ball such that its closure is included in B'' and let us denote $\Gamma = \varphi \times \mathrm{Id}_M$. Let

$$\Psi = \Gamma^{-1} \circ f^{-1} \circ \Gamma \circ f.$$

Let us pick $\Delta > 0$ small enough such that if $d_{C^0}(\varphi, \mathrm{Id}_M) < \Delta$, we have

- $\Gamma^{-1}(supp(f)) \subset B \times B$

- $\varphi(B') \subset B''$.

Since $\Gamma^{-1}(supp(f)) = supp(\Gamma^{-1} \circ f^{-1} \circ \Gamma)$, and $supp(f) \subset B \times B$, we conclude that $supp(\Psi) \subset B \times B$.

Moreover, for all $(x, y) \in B' \times B'$, we have

$$\begin{aligned}
\Phi \circ \Psi(x,y) &= \Phi \circ \Gamma^{-1} \circ f^{-1} \circ \Gamma \circ f(x,y) \\
&= \Phi \circ \Gamma^{-1} \circ f^{-1} \circ \Gamma(y,x) \\
&= \Phi \circ \Gamma^{-1} \circ f^{-1}(\varphi(y),x) \\
&= \Phi \circ \Gamma^{-1}(x,\varphi(y)) \\
&= \Phi(\varphi^{-1}(x),\varphi(y)) \\
&= (x,y).
\end{aligned}$$

This concludes the proof of the first point of the lemma.

Let us recall that $\Psi = \Gamma^{-1} \circ f^{-1} \circ \Gamma \circ f$ and $\Gamma = \varphi \times \mathrm{Id}_M$. This means that, by a triangle inequality,

$$\begin{aligned}
d_{C^0}(\Psi, \mathrm{Id}_{M\times M}) &\leq d_{C^0}(\Gamma^{-1}, \mathrm{Id}_{M\times M}) + d_{C^0}(f^{-1} \circ \Gamma \circ f, \mathrm{Id}_{M\times M}) \\
&\leq C_B d_{C^0}(\varphi, \mathrm{Id}_M),
\end{aligned}$$

where C_B depends only on the Lipschitz constants of both f and f^{-1}.

Moreover, by another triangle inequality, we get

$$d_{C^0}(\Phi \circ \Psi, \mathrm{Id}_{M\times M}) \leq C'_B d_{C^0}(\varphi, \mathrm{Id}_M),$$

where once again C'_B depends only on the Lipschitz constants of both f and f^{-1}.

Finally, the symplectomorphism Ψ is exact, as a composition of exact symplectomorphisms.

□

This Lemma 5.4.3 together with Lemma 5.4.1 will allow to conclude the proof of Proposition 5.1.7 and thus Theorem 5.1.1. Indeed, we have proven that $\mathcal{B}(L', L)$ and $\mathcal{B}(\varphi(L'), L)[\frac{1}{2}(\alpha - \bar{\alpha})]$ are $\frac{1}{2}\gamma(L', \varphi(L'))$-interleaved. We now just have to find two constants $\kappa > 0$ and $l > 0$ such that if $d_{C^0}(\varphi, \mathrm{Id}_M) \leq l$, then $\gamma(L', \varphi(L')) \leq \kappa d_{C^0}(\varphi, \mathrm{Id}_M)$.

Let us pick a point $x \in L'$ and a ball B_x centered on x. Lemma 5.4.3 provides a smaller ball B' also centered on x. Pick a smaller ball B_0, centered on x and whose closure is included in B'. The same lemma also provides $l_0 > 0$ and a symplectomorphism Ψ such that $l_0 < \Delta$ and if $d_{C^0}(\varphi, \mathrm{Id}_M) < l_0$, then $\Phi \circ \Psi(L' \cap B_0 \times L' \cap B_0) \cap T^*(B' \times B') = L' \cap B_0 \times L' \cap B_0$.

Then, let us pick another ball B_1 centered on $y \in L' \times L'$ whose closure is included in $M \backslash B \times B$. Since $d_{C^0}(\Psi, \mathrm{Id}_{M\times M}) \leq C_B d_{C^0}(\varphi, \mathrm{Id}_M)$ and $supp(\Psi) \subset B \times B$, we can find

$l_1 > 0$ such that if $d_{C^0}(\varphi, \mathrm{Id}_M) < l_1$, then Ψ and B_1 satisfy the conditions of Lemma 5.4.3.

Let us choose $l > 0 = \min\{\delta, l_0, l_1\}$. Then we have, using successively Proposition 4.2.4 and its consequence of Equality 4.3 and the triangle inequality 4.6 and the symmetry of γ 4.5, for all φ such that $d_{C^0}(\varphi, \mathrm{Id}_M) < l$:

$$\begin{aligned}
\gamma(L', \varphi(L')) &= \frac{1}{2}\gamma(L' \times L', \Phi(L' \times L')) \\
&= \frac{1}{2}\gamma(\Psi^{-1}\Phi^{-1}(L' \times L'), \Psi^{-1}(L' \times L')) \\
&\leq \frac{1}{2}\gamma(L' \times L', \Psi^{-1}(L' \times L')) + \frac{1}{2}\gamma(\Psi^{-1}\Phi^{-1}(L' \times L'), L' \times L') \\
&= \frac{1}{2}\gamma(L' \times L', \Psi^{-1}(L' \times L')) + \frac{1}{2}\gamma(L' \times L', \Phi\Psi(L', L')).
\end{aligned}$$

For the second equality, the same argument as in Lemma 5.2.2 indeed tells that $\gamma(L' \times L', \Phi(L' \times L')) = \gamma(\Psi^{-1}\Phi^{-1}(L' \times L'), \Psi^{-1}(L' \times L'))$, when composing by $\Psi^{-1}\Phi^{-1}$. A similar argument holds for the first equality and for the last one.

Choosing $B_0 \times B_0$ for the ball in Lemma 5.4.1, we can apply it to $\Phi \circ \Psi$ for all φ such that $d_{C^0}(\varphi, \mathrm{Id}_M) < l$. We then get that there is a constant $C_1 > 0$ such that for all these φ, we have:

$$\begin{aligned}
\gamma(L' \times L', \Phi \circ \Psi(L' \times L')) &\leq C_0 d_{C^0}(\Phi \circ \Psi, \mathrm{Id}_{M \times M}) \\
&\leq C_0 C'_B d_{C^0}(\varphi, \mathrm{Id}_M).
\end{aligned}$$

Moreover, for all such φ, Lemma 5.4.3 gives for Ψ:

$$\begin{aligned}
\gamma(L' \times L', \Psi^{-1}(L' \times L')) &\leq C_1 d_{C^0}(\Psi^{-1}, \mathrm{Id}_{M \times M}) \\
&\leq C_1 C_B d_{C^0}(\varphi, \mathrm{Id}_M).
\end{aligned}$$

Putting all this together, we get:

$$\gamma(L', \varphi(L')) \leq \frac{1}{2}(C_0 C'_B + C_1 C_B) d_{C^0}(\varphi, \mathrm{Id}_M).$$

By setting $\kappa = \frac{1}{2}(C_0 C'_B + C_1 C_B)$, we get that for all φ such that $d_{C^0}(\varphi, \mathrm{Id}_M) \leq l$, then $\gamma(L', \varphi(L')) \leq \kappa d_{C^0}(\varphi, \mathrm{Id}_M)$. □

Taking into account the discussions in Section 5.1.2, the proof of Theorem 5.1.1 is now complete, and this Chapter 5 finished.

Chapter 6

The Dehn-Seidel twist in C^0-symplectic grometry

6.1 The Dehn-Seidel twist

6.1.1 The classical Dehn twist on surfaces

In this section, we will briefly recall the definition of the Dehn twist for surfaces, and then state a few results, still for surfaces, to explain our interest for this particular diffeomorphism. As it will appear, this is an important map in the study of the mapping class group. In the case of surfaces, it comes from the work of Dehn and for example is extensively discussed in [34].

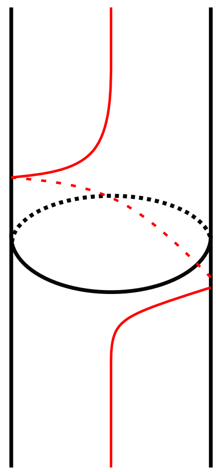

Figure 6.1 – Dehn twist in T^*S^1

We consider the annulus $T_1^*S^1 = S^1 \times [-1, 1]$. We denote $\tau : T_1^*S^1 \to T_1^*S^1$ the map given by

$$\tau(\theta, t) = (\theta + 2\pi f(t), t),$$

where $f : [-1, 1] \to \mathbb{R}^+$ is a smooth function equal to 0 near -1 and equal to 1 near 1. The isotopy class of this map does not depend on the choice of f. This map is called a *positive twist map.* Note that this map τ is an area and orientation preserving homeomorphism which fixes $\partial T_1^* S^1$ pointwise. Moreover, one could choose to have -2π in the formula instead of $+2\pi$, this is called a *negative twist.*

Now that we have our model, we can describe the Dehn twist for surfaces. Let Σ be an oriented surface, let l be a simple smooth closed curve in Σ, and let N be a tubular neighbourhood of l, together with an orientation preserving homeomorphism $\psi : T_1^* S^1 \to N$. We define a *Dehn twist along* l to be the map $\tau_l : \Sigma \to \Sigma$ such that

$$\tau_l(x) = \begin{cases} \psi \circ \tau \circ \psi^{-1}(x) & \text{if } x \in N \\ x & \text{if } x \in \Sigma \setminus N. \end{cases}$$

The map τ_l depends on both the choices of the neighbourhood N and the homeomorphism ψ. However, the isotopy class of τ_l does not depend on either of these choices. In fact τ_l depends only on the isotopy class of the simple closed curve l. By abuse of notation, we denote τ_l the element of the mapping class group $\mathrm{MCG}(\Sigma)$ called *the* Dehn twist along l which is well-defined.

The first interesting property of the Dehn twist is the following, assessing that we are not studying an irrelevant object.

Proposition 6.1.1. *Let l be a simple closed curve in a surface Σ. If l is not homotopic to a point or a puncture of Σ, then the Dehn twist τ_l is a non-trivial element of* $\mathrm{MCG}(\Sigma)$.

In fact, by studying intersection numbers, one can show that Dehn twist has infinite order. Dehn twists are actually a central piece in the study of the mapping class group of surfaces, as stated by the following theorem of Dehn and Lickorish. We denote Σ_g the surface of genus g.

Theorem 6.1.2. *For $g \geq 0$, the mapping class group* $\mathrm{MCG}(\Sigma_g)$ *is generated by finitely many Dehn twists along non-separating simple closed curves.*

Of course, there are relations between the different Dehn twists, depending on the intersection numbers of the curves along which they are defined, but they will not be discussed here.

Before moving on to higher dimensions, let us recall that in dimension 2 the area-preserving diffeomorphisms are exactly the symplectomorphisms. Moreover, a one parameter version of Moser's trick tells us that

$$\mathrm{MCG}^\omega(\Sigma) \cong \mathrm{MCG}(\Sigma).$$

Since the mapping class group is generated by Dehn twists, so does the symplectic mapping class group. This is part of the motivation to study higher dimensional Dehn twists.

We actually have the following isomorphisms:

$$\mathrm{MCG}^{\omega}(\Sigma) \cong \mathrm{MCG}(\Sigma) \cong \mathrm{MCG}(\Sigma, C^0) \cong \mathrm{MCG}^{\omega}(\Sigma, C^0).$$

The surjectivity of the second isomorphism is due to the fact that any homeomorphism is a limit of diffeomorphisms, which is for example proven in [57], together with the fact that the group of homeomorphisms is locally contractible [22]. Its injectivity comes from the local contractibility of the group of diffeomorphisms [56]. Finally, the third isomorphism was proven by Fathi [35]. As we will see, the situation is drastically different in higher dimension.

6.1.2 Presentation of the Dehn-Seidel twist

The Dehn-Seidel twist is a generalization of the Dehn twist to higher dimensions. It was defined by Arnold [4] and later deeply studied by Seidel [84, 81] and many others. This section presents the definition of this morphism. We will also state its main properties to be used for the following discussions and comment its importance for the mapping class group.

To describe the Dehn-Seidel twist, we start by describing a model, as we did for surfaces. Given $n \in \mathbb{N}^*$, we consider

$$T_1^*S^n = \{\xi \in T^*S^n, \quad |\xi| \leq 1\},$$

equipped with the standard round metric and the standard symplectic structure which we will denote $\omega_{T_1^*S^n}$. In coordinates, $T_1^*S^n$ can be written as

$$T_1^*S^n = \{(u,v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, \quad |u| \leq 1, |v| = 1, \langle u, v\rangle = 0\},$$

and $\omega_{T_1^*S^n} = \sum_i du_i \wedge dv_i$. Let us consider the function $H(u,v) = |u|$ on $T^*S^n \setminus S^n$. It induces an Hamiltonian circle action $\sigma = (\sigma_t)_{t \in S^1}$ which is the normalized geodesic flow. This means that the flow transports the cotangent vectors at unit speed along the corresponding geodesic, independently of the norm of the vector. Since all the geodesics of S^n are closed and 2π-periodic, this defines a circle action. Since the geodesic flow is well-known in coordinates, we get an explicit expression for σ:

$$\sigma_t(u,v) = \left(\cos(2\pi t)u - \sin(2\pi t)v|u|, \cos(2\pi t)v + \sin(2\pi t)\frac{u}{|u|}\right),$$

for $t \in [0,1]$, and $(u,v) \in T^*S^n \setminus S^n$. When $t = 1/2$, σ corresponds to the antipodal map: $\sigma_{1/2}(u,v) = (-u,-v)$. Note that this antipodal map extends continuously to the zero-section, but it is not the case for other $t \in (0,1/2)$. Let us now take a smooth function

$\rho : \mathbb{R} \to \mathbb{R}$ such that

$$\rho(t) = \begin{cases} 1/2 & \text{if } t \leq 1/3 \\ 0 & \text{if } t \geq 2/3, \end{cases}$$

and define a diffeomorphism τ on $T_1^* S^n$:

$$\tau(\xi) = \sigma_{\rho(|\xi|)}(\xi).$$

This diffeomorphism is well defined thanks to the smooth extension of the antipodal map to the zero-section. Note also that it is equal to identity near $\partial T_1^* S^n$. This map τ is called a *model Dehn twist*. An explicit computation of the expression σ shows that this map is a symplectomorphism.

Remark 6.1.3. For different choices of ρ, satisfying ρ equal to 0 near 1 and equal to 1/2 near 0, one gets different maps which are isotopic in $\mathrm{Aut}(T_1^* S^n, \partial T_1^* S^n)$. This results from the fact that the map $\sigma_{\rho(|\xi|)}$ can be seen as the time 1 map of a Hamiltonian flow $r_\rho(H)$, where r_ρ is function defined on $\mathbb{R}$. Thus, taking two different functions ρ_0 and ρ_1, one obtains an isotopy between the two resulting Dehn twists which can be expressed using ρ_0 and ρ_1.

We now want to embed our local model into a symplectic manifold, by matching the zero-section with a Lagrangian sphere. Let (M, ω) be a symplectic manifold with boundary, together with a Lagrangian sphere $l : S^n \to M$.

By Weinstein's neighbourhood theorem, there exists an embedding $i : T_1^* S^n \to M$ and $c > 0$ such that $i_{|S^n} = l$ and $i^* \omega = c\omega_{T_1^* S^n}$. We can define a Dehn-Seidel twist along l, denoted τ_l, as

$$\tau_l(p) = \begin{cases} i\tau i^{-1}(p) & \text{if } p \in i(T^* S^n) \\ p & \text{elsewhere.} \end{cases}$$

The map τ being a symplectomorphism equal to the identity near the boundary, τ_l is a symplectomorphism as well. Moreover, the class $[\tau_l]$ of this map in $\pi_0(\mathrm{Aut}(M, \partial M \omega))$ is independent of the choice of ρ. Concerning the dependence in i, it only depends on the Lagrangian isotopy class of $i_{|S^n}$. Thus τ_l is called *the Dehn-Seidel twist along l*.

Remark 6.1.4. One can check that when $n = 1$, this definition corresponds to the positive Dehn twist we described for surfaces.

Before discussing the properties of τ_l, we shall briefly present another way to define of the map τ which is more geometric and directly constructs the Dehn-Seidel twist from the common Dehn twist on surfaces. This can be found in [78].

Let us recall that the geodesics of S^n are closed. For each geodesic γ of S^n, γ is a great circle, and we consider $A_\gamma = T^*\gamma \cap T_1^*S^n \cong S^1 \times [-1,1]$. For this expression to make sense, we identify the tangent and cotangent bundles through the round metric on S^n. We can define the Dehn twist τ_γ on this annulus as in Subsection 6.1.1, so that τ_γ equals the antipodal map on the zero section and is compactly supported.

Moreover, since we have $T_1^*S^n = \bigcup_\gamma A_\gamma$, we can define the map τ as

$$\tau(q) = \tau_\gamma(q) \text{ for } q \in A_\gamma.$$

Since all τ_γ are equal to the antipodal map on the zero section of their respective A_γ, this map is well-defined on the zero-section of $T_1^*S^n$. Away from the zero-section, for each point q in $T_1^*S^n \setminus S^n$, there is a unique geodesic γ such that q is in the annulus A_γ. Finally, the map τ is well defined and we get the model Dehn-Seidel twist.

One can wonder whether the symplectic isotopy class of the Dehn-Seidel twist only depends on the image L of l. Since for $n \leq 3$, $\pi_0(\mathrm{Diff}^+ S^n) = 1$, it is clear that this is true for these dimensions. Indeed, for two Lagrangian spheres l and l' from S^n to M, if l and l' are isotopic, then $[\tau_l] = [\tau_{l'}]$. However this is an open question in higher dimensions.

Let l be a Lagrangian sphere in M, with image L. Seidel proved [84, 80] that this Dehn-Seidel twist corresponds to a symplectic version of Picard-Lefschetz transformations, and hence, their action on homology is given by the following formula:

$$(\tau_l)_*(x) = \begin{cases} x - (-1)^{\frac{n(n-1)}{2}}(x \cdot [L])[L] & \text{if } x \in H_n(M,\mathbb{Z}) \\ x & \text{if } x \in H_k(M,\mathbb{Z}), k \neq n. \end{cases}$$

Let us recall that

$$[L]\cdot[L] = (-1)^{\frac{n(n-1)}{2}}\chi(L),$$

$\chi(L)$ being the Euler characteristic. For n even, $\chi(L) = 2$. A direct computation then shows that τ_l^2 acts trivially on $H_n(M,\mathbb{Z})$ and hence on the whole homology. If n is odd, then, as long as $[L]$ is not a torsion class, $(\tau_l)_*$ has infinite order. This is the main reason why we will study the square of the Dehn-Seidel twist in M^{2n} symplectic manifolds, with n even. Indeed, since in the other cases the Dehn-Seidel twist does not act trivially on the homology, we directly know that it is not in the connected component of the identity.

In his thesis [84], Seidel proved that, when n equals 2, the square of the Dehn-Seidel twist is isotopic to the identity in $\mathrm{Diff}(M)$. Although we are sure this does not hold for odd n, we do not know whether this is true for larger even n. Moreover, as for the Dehn twist on surfaces, there are some relations between Dehn twists coming from different Lagrangian spheres. However, the question of the connectedness of τ_l^2 to the identity is a non trivial question. It will be the subject of the following sections.

For now, we will give a few facts on these diffeomorphisms that will be used in the following discussions.

Remark 6.1.5. Let L be a Lagrangian sphere in M. Then for any Lagrangian sphere L' in M, $\tau_l(L')$ is a Lagrangian sphere as well.

It can be checked that the Dehn-Seidel twist is in fact an exact symplectomorphism.

Remark 6.1.6. Let (l_1, l_2) be an A_2-configuration of Lagrangian spheres. Seidel [81] proved also that $\tau_{l_1}(L_2)$ and $\tau_{l_2}^{-1}(L_1)$ are isotopic as Lagrangian submanifolds to the Polterovich surgery [73] of L_1 and L_2, $L_1 \sharp L_2$.

6.2 Seidel's theorem

Since the square of the Dehn-Seidel twist has been proved to be isotopic to the identity in $\mathrm{Diff}(M^{2n})$ when $n = 2$ [84], it is a natural question to ask whether this is true in higher dimensions. Since this map is symplectic, it is also natural to wonder whether this also holds in $\mathrm{Symp}(M, \omega)$, or whether this is a purely smooth (non-symplectic) result. Even if the answer to the first question is still unknown, regarding the second one, Seidel proved in [81] a stronger result, by considering images of Lagrangian submanifolds instead of directly considering the Dehn twist. Let us state Seidel's theorem.

Theorem 6.2.1 (Seidel [81])**.** *Let (M^{2n}, ω) be a compact symplectic manifold with contact type boundary, with n even, which satisfies $[\omega] = 0$ and $2c_1(M, \omega) = 0$. Assume that M contains an A_3-configuration (l_∞, l', l) of Lagrangian spheres. Then M contains infinitely many symplectically knotted Lagrangian spheres. More precisely, if one defines $L'^{(k)} = \tau_l^{2k}(L')$ for $k \in \mathbb{Z}$, then all the $L'^{(k)}$ are isotopic as smooth submanifolds of M, but no two of them are isotopic as Lagrangian submanifolds.*

Since no two of these Lagrangian submanifolds are isotopic as Lagrangian submanifolds, the following corollary is immediate.

Corollary 6.2.2. *τ_l^2 is not in the identity component of $\mathrm{Symp}_c(T^*S^n)$, the compactly supported symplectomorphisms of T^*S^n.*

Indeed, if this symplectomorphismmorphism was in the identity component of $\mathrm{Symp}_c(T^*S^n)$, its conjugation by the embedding j would also be in the identity component of $\mathrm{Symp}_c(M, \omega)$, and thus, all the Lagrangian spheres in Theorem 6.2.1 would be isotopic as Lagrangian submanifolds.

The proof of this theorem deeply relies on the isotopy invariance of Floer homology together with the action of the Dehn-Seidel twist on the Maslov index. The proof we will give for the analogous result in C^0 symplectic topology also relies on barcodes and consequently on Floer cohomology. However there are some technical difficulties to adapt Seidel's proof to our context.

Remark 6.2.3. A similar result holds when working with odd n. However, one should not consider the square of the Dehn-Seidel twist, but the cube of the composition of two Dehn-Seidel twists defined along different but intersecting Lagrangian submanifolds [81].

We introduced the notion of Milnor fibres after Definition 2 as these are examples of manifolds satisfying the conditions required for Theorem 6.2.1.

We state here the following technical lemma, which was a key argument in Seidel's proof [81] and which will be useful in the following computations.

Lemma 6.2.4. *There is a unique $\mathcal{L}^\infty$ grading $\tilde{\tau}_l$ of τ_l which acts trivially on the part of $\mathcal{L}^\infty$ which lies over $M \setminus \mathrm{Im}(i)$. It satisfies $\tilde{\tau}_l \tilde{L} = \tilde{L}[1-n]$ for any grading $\tilde{L}$ of L.*

See Appendix A for explanations on this grading.

6.3 Long exact sequence in Floer cohomology

As mentioned in the previous section, Floer cohomology will be essential to our proof. It is actually possible to compute the action of the Dehn-Seidel twist on the Floer cohomology of certain exact Lagrangian submanifolds. It is the object of the following theorem, also proved by Seidel [82].

Theorem 6.3.1. *Let $l : S^n \to M$ be a Lagrangian sphere in (M^{2n}, ω) with image L. For any two exact Lagrangian submanifolds $L_0, L_1 \in M$, there is a long exact sequence of Floer cohomology groups:*

$$\begin{tikzcd} HF(\tau_l(L_0), L_1) \arrow[rr, "0"] & & HF(L_0, L_1) \arrow[dl, "n"] \\ & HF(L, L_1) \otimes HF(L_0, L) \arrow[ul, "1-n"] & \end{tikzcd}$$

Now that this theorem is stated, we make some computations of this long exact sequence, in order to use it in our context.

Proposition 6.3.2. *Let (M^{2n}, ω) be a connected Liouville domain, $n > 2$, $2c_1(M, \omega) = 0$. Assume that M contains an A_2-configuration of Lagrangian spheres (l, l'). Let $x = L \cap L'$. Choose an ∞-Maslov covering on M and $\mathcal{L}^\infty$-gradings $\tilde{L}, \tilde{L}'$ of L and L' such that $\tilde{I}(x, \tilde{L}', \tilde{L}) = 0$.*

Then, for all $k \in \mathbb{Z}$,

$$HF^*(\tilde{\tau}_l{}^2(\tilde{L}'), \tilde{L}') \ncong HF^{*+k}(\tilde{L}', \tilde{L}').$$

Proof. Since our Lagrangian submanifolds are spheres of dimension > 1, they are exact. We can thus apply Seidel's Theorem 6.3.1 to $L_1 = L_0 = L'$ to get an exact triangle in

Floer cohomology:

$$\begin{tikzcd}
HF(\tau_l(L'), L') \arrow[rr, "0"] & & HF(L', L') \arrow[dl, "n"] \\
& HF(L, L') \otimes HF(L', L) \arrow[ul, "1-n"] &
\end{tikzcd}$$

Seidel's exact triangle in our particular case leads to a long exact sequence in Floer cohomology:

$$\begin{tikzcd}
HF^0(\tilde{\tau}_l(\tilde{L}'), \tilde{L}') \arrow[r] & HF^0(\tilde{L}', \tilde{L}') \arrow[r, "\star"] & \bigoplus\limits_{k+l=n} HF^k(\tilde{L}, \tilde{L}') \otimes HF^l(\tilde{L}', \tilde{L}) \arrow[dll] \\
HF^1(\tilde{\tau}_l(\tilde{L}'), \tilde{L}') \arrow[r] & HF^1(\tilde{L}', \tilde{L}') \arrow[r] & \bigoplus\limits_{k+l=n+1} HF^k(\tilde{L}, \tilde{L}') \otimes HF^l(\tilde{L}', \tilde{L}) \arrow[dll, dashed] \\
HF^{n-1}(\tilde{\tau}_l(\tilde{L}'), \tilde{L}') \arrow[r] & HF^{n-1}(\tilde{L}', \tilde{L}') \arrow[r] & \bigoplus\limits_{k+l=2n-1} HF^k(\tilde{L}, \tilde{L}') \otimes HF^l(\tilde{L}', \tilde{L}) \arrow[dll] \\
HF^n(\tilde{\tau}_l(\tilde{L}'), \tilde{L}') \arrow[r] & HF^n(\tilde{L}', \tilde{L}') \arrow[r] & \bigoplus\limits_{k+l=2n} HF^k(\tilde{L}, \tilde{L}') \otimes HF^l(\tilde{L}', \tilde{L}) \arrow[dll] \\
\cdots & &
\end{tikzcd}$$

Following Equality A.4 regarding the properties of the cohomology of graded Lagrangian submanifolds, we obtain

$$HF(\tilde{L}', \tilde{L}') = \mathbb{Z}/2^{[0]} \oplus \mathbb{Z}/2^{[n]}.$$

Moreover, we can choose particular gradings for L and L' such that, together with the Poincaré duality (A.3) we have:

$$HF(\tilde{L}', \tilde{L}) = \mathbb{Z}/2^{[0]} \text{ and } HF(\tilde{L}, \tilde{L}') = \mathbb{Z}/2^{[n]}.$$

Consequently we have

$$\bigoplus_{k+l=j} HF^k(\tilde{L}, \tilde{L}') \otimes HF^l(\tilde{L}', \tilde{L}) = \begin{cases} \mathbb{Z}/2 & \text{if} \quad j = n \\ 0 & \text{else} \end{cases}$$

We now have to discuss the arrow $\star$. Since $HF^0(\tilde{L}', \tilde{L}')$ and $\bigoplus\limits_{k+l=n} HF^k(\tilde{L}, \tilde{L}') \otimes HF^l(\tilde{L}', \tilde{L})$

are both of dimension 1, this arrow is either a bijection or zero. We will check both cases.

Let us start with the case when the arrow $\star$ is a bijection. Our long exact sequence then becomes

$$\begin{tikzcd}
0 \arrow[r] & HF^0(\tilde{\tau}_l(\tilde{L}'), \tilde{L}') \arrow[r, hook] & \mathbb{Z}/2 \arrow[r, leftrightarrow] & \mathbb{Z}/2 \arrow[dll, two heads] \\
 & HF^1(\tilde{\tau}_l(\tilde{L}'), \tilde{L}') \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, dashed] \\
 & HF^{n-1}(\tilde{\tau}_l(\tilde{L}'), \tilde{L}') \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
 & HF^{n}(\tilde{\tau}_l(\tilde{L}'), \tilde{L}') \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[dll] \\
 & \dots & &
\end{tikzcd}$$

We conclude that

$$HF(\tilde{\tau}_l(\tilde{L}'), \tilde{L}') = \mathbb{Z}/2^{[n]}.$$

We now consider the three Lagrangian submanifolds: L, L' and $\tau_l(L')$. The submanifolds L and L' are still two Lagrangian spheres in (M, ω). Moreover $\tau_l(L')$ is an exact Lagrangian submanifold as well according to Remark 6.1.5.

We can now apply Seidel's Theorem 6.3.1 to $L = L$, $L_0 = \tau_l(L')$ and $L_1 = L'$. We thus get an exact triangle in Floer cohomology:

$$\begin{tikzcd}
HF(\tau_l^2(L'), L') \arrow[rr, "0"] & & HF(\tau_l(L'), L') \arrow[dl, "n"] \\
 & HF(L, L') \otimes HF(\tau_l(L'), L) \arrow[ul, "1-n"] &
\end{tikzcd}$$

This exact triangle leads to a long exact sequence in Floer cohomology:

$$
\begin{tikzcd}
HF^{1-n}(\tilde{\tau}_l{}^2(\tilde{L}'),\tilde{L}') \arrow[r] & HF^{1-n}(\tilde{\tau}_l(\tilde{L}'),\tilde{L}') \arrow[r] & \bigoplus\limits_{k+l=1} HF^k(\tilde{L},\tilde{L}')\otimes HF^l(\tilde{\tau}_l(\tilde{L}'),\tilde{L}) \arrow[dll] \\
HF^{2-n}(\tilde{\tau}_l{}^2(\tilde{L}'),\tilde{L}') \arrow[r] & HF^{2-n}(\tilde{\tau}_l(\tilde{L}'),\tilde{L}') \arrow[r] & \bigoplus\limits_{k+l=2} HF^k(\tilde{L},\tilde{L}')\otimes HF^l(\tilde{\tau}_l(\tilde{L}'),\tilde{L}) \arrow[dll, dashed] \\
HF^{n-1}(\tilde{\tau}_l{}^2(\tilde{L}'),\tilde{L}') \arrow[r] & HF^{n-1}(\tilde{\tau}_l(\tilde{L}'),\tilde{L}') \arrow[r] & \bigoplus\limits_{k+l=2n-1} HF^k(\tilde{L},\tilde{L}')\otimes HF^l(\tilde{\tau}_l(\tilde{L}'),\tilde{L}) \arrow[dll] \\
HF^{n}(\tilde{\tau}_l{}^2(\tilde{L}'),\tilde{L}') \arrow[r] & HF^{n}(\tilde{\tau}_l(\tilde{L}'),\tilde{L}') \arrow[r] & \bigoplus\limits_{k+l=2n} HF^k(\tilde{L},\tilde{L}')\otimes HF^l(\tilde{\tau}_l(\tilde{L}'),\tilde{L}) \arrow[dll] \\
\dots & &
\end{tikzcd}
$$

Using Equality A.2 and Lemma 6.2.4 together with the fact that $HF(\tilde{L}',\tilde{L}) = \mathbb{Z}/2^{[0]}$, we can compute $HF(\tilde{\tau}_l(\tilde{L}'),\tilde{L})$:

$$
\begin{aligned}
HF^*(\tilde{\tau}_l(\tilde{L}'),\tilde{L}) &= HF^*(\tilde{L}',\tilde{\tau}_l{}^{-1}(\tilde{L})) \\
&= HF^*(\tilde{L}',\tilde{L}[n-1]) \\
&= HF^{*+n-1}(\tilde{L}',\tilde{L})
\end{aligned}
$$

We thus obtain

$$
HF^*(\tilde{\tau}_l(\tilde{L}'),\tilde{L}) = \begin{cases} \mathbb{Z}/2 & \text{if} \quad * = 1-n \\ 0 & \text{else.} \end{cases}
$$

We also have

$$
\bigoplus_{k+l=j} HF^k(\tilde{L},\tilde{L}')\otimes HF^l(\tilde{\tau}_l(\tilde{L}'),\tilde{L}) = \begin{cases} \mathbb{Z}/2 & \text{if} \quad j = 1 \\ 0 & \text{else.} \end{cases}
$$

Our long exact sequence then becomes:

$$\begin{tikzcd}
0 \arrow[r] & HF^{1-n}(\tilde{\tau_l}^2(\tilde{L'}),\tilde{L'}) \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[dll] \\
& HF^{2-n}(\tilde{\tau_l}^2(\tilde{L'}),\tilde{L'}) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, dashed] \\
& HF^{n-1}(\tilde{\tau_l}^2(\tilde{L'}),\tilde{L'}) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
& HF^{n}(\tilde{\tau_l}^2(\tilde{L'}),\tilde{L'}) \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[dll] \\
& \dots & &
\end{tikzcd}$$

So finally, we get:

$$HF(\tilde{\tau_l}^2(\tilde{L'}),\tilde{L'}) = \mathbb{Z}/2^{[n]} \oplus \mathbb{Z}/2^{[2-n]}.$$

We now have to check the case where the arrow $\star$ is zero. In this case our long exact sequence then becomes:

$$\begin{tikzcd}
0 \arrow[r] & HF^{0}(\tilde{\tau_l}(\tilde{L'}),\tilde{L'}) \arrow[r, hook] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2 \arrow[dll, two heads] \\
& HF^{1}(\tilde{\tau_l}(\tilde{L'}),\tilde{L'}) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, dashed] \\
& HF^{n-1}(\tilde{\tau_l}(\tilde{L'}),\tilde{L'}) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] \\
& HF^{n}(\tilde{\tau_l}(\tilde{L'}),\tilde{L'}) \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[dll] \\
& \dots & &
\end{tikzcd}$$

We conclude that

$$HF(\tilde{\tau_l}(\tilde{L'}),\tilde{L'}) = \mathbb{Z}/2^{[0]} \oplus \mathbb{Z}/2^{[1]} \oplus \mathbb{Z}/2^{[n]}.$$

As above, we apply Seidel's Theorem 6.3.1 to $L = L$, $L_0 = \tau_l(L')$ and $L_1 = L'$. The exact triangle and the computation of the previous case lead to the following exact sequence:

$$
\begin{array}{l}
0 \longrightarrow HF^{1-n}(\tilde{\tau}_l{}^2(\tilde{L}'), \tilde{L}') \longrightarrow 0 \longrightarrow \mathbb{Z}/2 \\
\qquad HF^{2-n}(\tilde{\tau}_l{}^2(\tilde{L}'), \tilde{L}') \longrightarrow 0 \longrightarrow 0 \\
\qquad HF^{0}(\tilde{\tau}_l{}^2(\tilde{L}'), \tilde{L}') \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \\
\qquad HF^{1}(\tilde{\tau}_l{}^2(\tilde{L}'), \tilde{L}') \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \\
\qquad HF^{n-1}(\tilde{\tau}_l{}^2(\tilde{L}'), \tilde{L}') \longrightarrow 0 \longrightarrow 0 \\
\qquad HF^{n}(\tilde{\tau}_l{}^2(\tilde{L}'), \tilde{L}') \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \\
\qquad \qquad \dots
\end{array}
$$

So finally, we obtain

$$HF(\tilde{\tau}_l{}^2(\tilde{L}'), \tilde{L}') = \mathbb{Z}/2^{[n]} \oplus \mathbb{Z}/2^{[1]} \oplus \mathbb{Z}/2^{[0]} \oplus \mathbb{Z}/2^{[2-n]}.$$

This concludes the proof of Proposition 6.3.2. □

6.4 Connectedness to the identity in high dimensions

The computation of Floer cohomology of Proposition 6.3.2 indicates that we may have to split the cases for $n = 2$ and $n \geq 4$. We will discuss in this section the case when $n \geq 4$. The case $n = 2$ will be discussed in the following Section 6.5. We will now prove Theorem A stated in the introduction. For the reader's convenience, we repeat it here.

Theorem 6.4.1. *Let (M^{2n}, ω) be a $2n$-dimensionnal Liouville domain, n even, $n \geq 4$, $2c_1(M, \omega) = 0$. Assume that M contains an A_2-configuration of Lagrangian spheres (l, l').*

Then, τ_l^2 is not in the connected component of the identity in $\overline{\mathrm{Symp}}(M, \omega)$.

Proof. Let us assume that τ_l^2 is in the connected component of the identity in $\overline{\mathrm{Symp}}(M, \omega)$. Theorem 5.1.3 implies then that $\hat{B}(L', L')$ and $\hat{B}(\tau_l^2(L'), L')$ are in the same connected component in $\hat{\mathcal{B}}$, i.e. up to an overall shift, they have their semi-infinite bars in the same degree.

Moreover, since $HF(L', L') = \mathbb{Z}/2^{[0]} \oplus \mathbb{Z}/2^{[n]}$, Proposition 4.2.1 implies that $B^0 = \hat{B}(L', L')$ has only two semi-infinite bars, one in degree 0 and one in degree n. Consequently, the barcode $\hat{B}(\tau_l^2(L'), L')$ has two semi-infinite bars, one in degree 0 and one in degree n.

However, let us recall that Proposition 6.3.2 gives, for all $k \in \mathbb{Z}$,

$$HF(\tilde{\tau}_l(\tilde{L}'), \tilde{L}'[k]) \ncong \mathbb{Z}/2^{[0]} \oplus \mathbb{Z}/2^{[n]}.$$

This means that $\hat{B}(\tau_l^2(L'), L')$ cannot have the semi-infinite bars in the same degree as for $\hat{B}(L', L')$. This contradicts the fact that they should be in the same connected component and thus concludes the proof of this theorem. □

The following statements correspond to Corollary B and Corollary C stated in the introduction. For the reader's convenience, we repeat them here. The first one is a straightforward consequence of Theorem 6.4.1.

Corollary 6.4.2. *Let (M^{2n}, ω) be a $2n$-dimensional Liouville domain, n even, $n \geq 4$, $2c_1(M, \omega) = 0$. Assume that M contains an A_2-configuration of Lagrangian spheres (l, l').*

Then, τ_l^2 is not isotopic to the identity in $\overline{\mathrm{Symp}}(M, \omega)$. In particular, $\mathrm{MCG}^{\omega}(M, C^0)$ is non-trivial.

Let us recall that, according to the discussion held in the introduction, Corollary 6.4.2 does not imply Theorem 6.4.1. Indeed, whether $\overline{\mathrm{Symp}}(M, \omega)$ is locally path-connected remains an open question.

The following theorem is also a consequence of Theorem 6.4.1. Indeed the subspace $\overline{\mathrm{Ham}}(M, \omega) \subset \overline{\mathrm{Symp}}(M, \omega)$ is connected as it is the closure of the connected space $\mathrm{Ham}(M, \omega)$.

Theorem 6.4.3. *Let (M^{2n}, ω) be a $2n$-dimensional Liouville domain, n even, $n \geq 4$, $2c_1(M, \omega) = 0$. Assume that M contains an A_2-configuration of Lagrangian spheres (l, l').*

Then, τ_l^2 does not belong to $\overline{\mathrm{Ham}}(M, \omega)$.

6.5 In dimension 4

When working in dimension 4, we cannot use the computation of Proposition 6.3.2. However, we can apply Hind's Theorem 4.5.4 on the nearby Lagrangian conjecture [44]. Nevertheless, we also have to use Seidel's Theorem 6.2.1. Consequently, in dimension 4, we require an A_3-configuration instead of an A_2-configuration as in higher dimensions.

Theorem 6.5.1. *Let (M^4, ω) be a compact connected 4-dimensional submanifold with contact type boundary, $[\omega] = 0$, $2c_1(M, \omega) = 0$. Assume that M contains an A_3-configuration of Lagrangian spheres (l, l', l_∞).*

Then, τ_l^2 does not belong to $\overline{\mathrm{Ham}}(M, \omega)$.

Using Hind's proof of the Nearby Lagrangian Conjecture of Theorem 4.5.4 in the case of T^*S^2 [44], the proof is quite straightforward and does not require the use of barcodes.

Proof. Let us assume that there is a sequence of Hamiltonian diffeomorphisms $(\varphi_n)_{n\in\mathbb{N}}$ of M which C^0-converges to τ_l^2. Then, for N large enough, we have that $\varphi_N(L')$ is included in a Weinstein neighbourhood of $\tau_l^2(L')$.

Moreover, $\tau_l^2(L')$ is a Lagrangian sphere and so its Weinstein neighbourhood is, by definition, symplectomorphic to a neighbourhood of the zero section in T^*S^2. Consequently, we are under the condition of application of the Nearby Lagrangian Conjecture as in Theorem 4.5.4, in the case of T^*S^2.

We get that $\varphi_N(L')$ is Lagrangian isotopic to $\tau_l^2(L')$, which contradicts Seidel's result in Theorem 6.2.1. □

For the reader's convenience, we repeat here the statement of Theorem E. It is the counterpart in dimension 4 of Corollary 6.4.2.

Theorem 6.5.2. *Let (M^4,ω) be a compact connected 4-dimensionnal submanifold with contact type boundary, $[\omega]=0$, $2c_1(M,\omega)=0$. Assume that M contains an A_3-configuration of Lagrangian spheres (l,l',l_∞).*

Then, τ_l^2 is not isotopic to the identity in $\overline{\mathrm{Symp}}(M,\omega)$.

Note that none of Theorem 6.5.1 and Theorem 6.5.2 imply the other.

Proof. As above for the case $n=2$, this proof heavily relies on the proof of the Nearby Lagrangian conjecture for T^*S^2 as in Theorem 4.5.4 [44].

Let us assume that τ_l^2 is connected to identity in $\overline{\mathrm{Symp}}(M,\omega)$. This means that we can find a continuous path $(\varphi^t)_{t\in[0;1]}\subset\overline{\mathrm{Symp}}(M,\omega)$ such that $\varphi^0=\mathrm{Id}_M$ and $\varphi^1=\tau_l^2$.

Since for all $t\in[0;1]$, φ^t is in $\overline{\mathrm{Symp}}(M,\omega)$, we can find sequences $\varphi_n^t\in\mathrm{Symp}(M,\omega)$ such that

$$\forall t\in(0;1),\ \lim_{n\to\infty}\varphi_n^t=\varphi^t.$$

Let us choose a Weinstein neighbourhood $W(L')$ of L' together with $\varepsilon>0$ such that, for all $\varphi\in\mathrm{Symp}(M,\omega)$, if $d_{C^0}(\varphi,\mathrm{Id}_M)<\varepsilon$, then $\varphi(L')\subset W(L')$.

The path φ^t being continuous, we can find a finite sequence $(\varphi^{t_i})_{i\in[\![0;N]\!]}\subset\overline{\mathrm{Symp}}(M,\omega)$ such that $\varphi^{t_0}=\mathrm{Id}_M$, $\varphi^{t_N}=\tau_l^2$ and

$$\forall i\in[\![1;N]\!],\quad d_{C^0}(\varphi^{t_{i-1}},\varphi^{t_i})<\tfrac{\varepsilon}{3}.$$

Moreover, for each $(t_i)_{i\in[\![0;N-1]\!]}$, we can find n_i such that $d_{C^0}(\varphi^{t_i},\varphi_{n_i}^{t_i})<\frac{\varepsilon}{3}$. We choose $\varphi_0^{t_0}=\mathrm{Id}_M$ and $\varphi_N^{t_N}=\tau_l^2$.

Consequently, we get a sequence $(\varphi_{n_i}^{t_i})_{i\in[\![0;N]\!]}\subset\mathrm{Symp}(M,\omega)$ such that $\varphi_{n_0}^{t_0}=\mathrm{Id}_M$,

$\varphi_{n_N}^{t_N} = \tau_l^2$ which satisfies $\forall i \in [\![1;N]\!]$,

$$\begin{array}{rcl} d_{C^0}((\varphi_{n_{i-1}}^{t_{i-1}})^{-1} \circ \varphi_{n_i}^{t_i}, \mathrm{Id}_M) & \leq & d_{C^0}(\varphi_{n_{i-1}}^{t_{i-1}}, \varphi_{n_i}^{t_i}) \\ & \leq & d_{C^0}(\varphi_{n_{i-1}}^{t_{i-1}}, \varphi^{t_{i-1}}) + d_{C^0}(\varphi^{t_{i-1}}, \varphi^{t_i}) + d_{C^0}(\varphi^{t_i}, \varphi_{n_i}^{t_i}) \\ & < & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{array}$$

Then

$$\begin{array}{rcl} \varphi_{n_i}^{t_i}(L') & = & \varphi_{n_{i-1}}^{t_{i-1}} \circ (\varphi_{n_{i-1}}^{t_{i-1}})^{-1} \circ \varphi_{n_i}^{t_i}(L') \\ & \subset & \varphi_{n_{i-1}}^{t_{i-1}}(W(L')) \cong W(\varphi_{n_i}^{t_i}(L')), \end{array}$$

where $W(\varphi_{n_i}^{t_i}(L'))$ denotes a Weinstein neighbourhood of $\varphi_{n_i}^{t_i}(L')$.

Applying now Hind's Theorem 4.5.4, we obtain that for all $i \in [\![1;N]\!]$, $\varphi_{n_i,t_i}(L')$ is Lagrangian isotopic to $\varphi_{n_{i-1},t_{i-1}}(L')$. Gluing these paths together, we finally get that L' is Lagrangian isotopic to $\tau_l^2(L')$. This contradicts Seidel's result of Theorem 6.2.1 and concludes this proof. $\square$

Remark 6.5.3. One can directly prove Corollary 6.4.2 without the use of barcodes. Indeed, an argument similar to the one for the case $n = 2$ holds for the proof of this corollary. The idea is to find a sequence $(\varphi_n)_{n\in\mathbb{N}}$ such that the cohomology $HF(\varphi(L'), L')$ remains constant. We can then conclude in the same way.

We could have also proved this result using Theorem 5.1.4 to construct a continuous path of barcodes between $\hat{B}(L', L')$ and $\hat{B}(\tau_l^2(L'), L')$. This path together with Corollary 3.3.13 telling that the degree of the semi-infinite bars cannot change along a continuous path leads to a contradiction.

6.6 Further remarks

We now discuss what we would have obtained if we avoided the use of "high-technology" results, i.e. results of [40, 1, 2] concerning the Nearby Lagrangian conjecture. Of course, we obtain weaker, but still interesting results and new questions.

The main issue is the following. Let (M, ω) be a Liouville domain together with L and L' two exact Lagrangian submanifolds. Let $\varphi \in \overline{\mathrm{Ham}}(M, \omega)$ be defined as the C^0-limit of a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset \mathrm{Ham}(M, \omega)$.

Remark 5.3.4 tells that when we are working with Hamiltonian diffeomorphisms instead of symplectomorphisms, we can bound the bottleneck distance without using the Nearby Lagrangian conjecture's result of Theorem 4.5.5. The rest of the proof of Theorem 5.1.1 goes through. Consequently we can define a barcode $B \in \hat{\mathcal{B}}$ as the limit of the barcodes $\hat{B}(\varphi_n(L'), L)$. Assuming that $\varphi(L')$ is a smooth exact Lagrangian submanifold, we can define the barcode $\hat{B}(\varphi(L'), L))$. However, there is no reason for B to be equal to $\hat{B}(\varphi(L'), L))$.

Nevertheless, we can still associate a continuous loop of barcodes to a continuous loop in $\overline{\mathrm{Ham}}$. This loop is given by the following proposition.

Proposition 6.6.1. *Let M be a Liouville domain. Let L and L' be two exact compact Lagrangian submanifolds. Let φ be a Hamiltonian diffeomorphism together with a loop $(\varphi^t)_{t\in[0,1]} \subset \overline{Ham}(M, \omega)$ such that $\varphi^0 = \varphi^1 = \varphi$. Then we can associate a loop in $\hat{\mathcal{B}}$ to this loop in $\overline{Ham}(M, \omega)$.*

Proof. Since, for all $t \in [0; 1]$, φ_t is in $\overline{\mathrm{Ham}}(M, \omega)$, we can find sequences $\varphi_{n,t} \in \mathrm{Ham}(M, \omega)$, one for each t such that:

$$\forall t \in (0; 1), \lim_{n\to\infty} \varphi_{n,t} = \varphi_t.$$

For each $t \in (0, 1)$, the sequence $\varphi_{n,t}$ Cauchy converges. Then the associated barcode sequence $B_{n,t} = \hat{B}(\varphi_{n,t}(L'), L')$ is also Cauchy for n large enough, by Theorem 5.1.1. This consequently allows, using Corollary 5.1.2, to define

$$B_t = \lim_{n\to\infty} \hat{B}(\varphi_{n,t}(L'), L').$$

By setting $B_0 = B_1 = \hat{B}(\varphi(L'), L')$, we get a loop of barcodes $(B_t)_{t\in[0,1]}$.

Lemma 6.6.2. *$(B_t)_{t\in[0;1]}$ is a continuous loop of barcodes with base point $\hat{B}(\varphi(L'), L')$.*

This lemma is a straightforward consequence of Theorem 5.1.1 in this Hamiltonian case. This completes the proof of this proposition.

□

At this point, there is no available tool to study $\pi_1(\overline{\mathrm{Ham}}(M, \omega))$. However, we strongly hope that this proposition will provide such a useful tool. Moreover, we can apply the same approach for $\overline{\mathrm{Symp}}(M, \omega)$ using Theorem 5.1.1, and thus we can hope to obtain results concerning $\pi_1(\overline{\mathrm{Symp}}(M, \omega))$.

It would maybe help us to define a kind of "C^0 Seidel's morphism". However this is still an ongoing project as we need to better understand the topology of the relevant space of barcodes.

Remark 6.6.3. We could apply the same argument for the higher homotopy groups $\pi_k(\overline{\mathrm{Ham}}(M, \omega))$ and $\pi_k\overline{\mathrm{Symp}}(M, \omega))$. We would then have to study the higher homotopy groups of the relevant space of barcodes.

Appendix A

Absolute grading in Floer cohomology

As mentioned before, the degree of a generator of the Floer complex is defined up to an overall shift. While this does not matter in most of the contexts we will consider, it can be useful to have an absolute grading. This is possible and it was in fact introduced by Seidel [81]. Even if we will not really use this notion in our proof, this absolute grading is a key point in the proof of his theorem related to the Dehn-Seidel twist. As this notion will come handy later in a computation, we will briefly present it here. We will not give any proof as they all can be found in the same paper from Seidel [81]. This section is following his work and Audin's work in [5].

Let us first describe the linear setting. Let us recall that we have a mapping

$$det^2 : U(n) \to \Lambda(n) \to S^1,$$

which induces an isomorphism on the π_1. For $N \in \mathbb{N}$, consider the N-fold covering of S^1 and pull it back by the previous mapping. We obtain a covering of $\Lambda(n)$:

$$\Lambda^N(n) = \left\{(L, z) \in \Lambda(n) \times S^1 | \quad det^2(L) = z^N\right\}.$$

These coverings are always connected and correspond to the elements of $H^1(\Lambda(n), \mathbb{Z}/N)$.

Let (M, ω) be a connected symplectic manifold as above. We denote $\mathcal{L}(M)$ the Lagrangian Grassmanian bundle over M associated to the tangent bundle TM.

Seidel proved [81] that there exists a covering $\mathcal{L}^N(M)$ of $\mathcal{L}(M)$ which restricts on each fiber to the covering $\Lambda^N(n) \to \Lambda(n)$ if and only if N divides $2N_M$, where N_M is the minimal Chern number of M. This covering is unique if and only if $H^1(M, \mathbb{Z}/N) = 0$.

Let us take a Lagrangian immersion $l : L \to M$, together with its Gauss mapping $\Gamma : L \to \mathcal{L}(M)$. In the same paper, Seidel proved that such a Lagrangian immersion can be lifted to $\mathcal{L}^N(M)$ if and only if N divides the minimal Chern number of L. Such a lift is called a N-grading of the immersion l. A Lagrangian submanifold satisfying $H_1(L;\mathbb{Z}/N)=0$ always admits a N-grading.

When $2c_1(M,\omega)=0$, it is possible to define the ∞-covering $\mathcal{L}^\infty(M)$ which restricts on each fiber to a covering $\Lambda^\infty(n)$ coming from an ∞-covering of S^1:

$$\Lambda^\infty(n)=\left\{(L,t)\in\Lambda(n)\times\mathbb{R}|\quad det^2(L)=e^{2i\pi t}\right\}.$$

In that case, a Lagrangian immersion l in M such that $H^1(L)=0$ always admits an ∞-grading. This is our context of interest.

We now have to link this notion with the Maslov index. Let us assume, following Seidel's notations that $\tilde{L}_0$ and $\tilde{L}_1$ are two $\mathbb{Z}/N$-graded Lagrangian submanifolds in a symplectic manifold (M,ω) which intersect transversaly at a point x. This gives two elements of $\Lambda^N(n)$ $\tilde{L}_0(x)$ and $\tilde{L}_1(x)$. Let us choose two paths $\tilde{\lambda}_0$ and $\tilde{\lambda}_1$ from $[0,1]$ to $\Lambda^N(n)$ such that $\tilde{\lambda}_0(0)=\tilde{\lambda}_1(0)$ and $\tilde{\lambda}_0(1)=\tilde{\Lambda}_0(x)$ and $\tilde{\lambda}_1(1)=\tilde{L}_1(x)$. We denote λ_0 and λ_1 the projection of these two paths on $\Lambda(n)$. We set

$$\tilde{I}(\tilde{L}_0,\tilde{L}_1;x)=\frac{1}{2}n-\mu(\lambda_0,\lambda_1),$$

where $\mu(\lambda_0,\lambda_1)\in\frac{1}{2}\mathbb{Z}$ is the Maslov index for a pair of paths [95, 76]. This $\tilde{I}(\tilde{L}_0,\tilde{L}_1;x)$ is an integer whose class in $\mathbb{Z}/N$ is independent of all choices.

Moreover, given two points $p,q\in\chi(L_0,L_1)$ together with a strip u connecting p to q, we have

$$\mu(u)=\tilde{I}(\tilde{L}_0,\tilde{L}_1;q)-\tilde{I}(\tilde{L}_0,\tilde{L}_1;q)\bmod\mathbb{Z}/N.$$

Denoting $\tilde{L}[k]$ the graded Lagrangian $\tilde{L}$ whose grading has been shifted by $k\in\mathbb{Z}/N$, one can show that

$$\tilde{I}(\tilde{L}_0[k],\tilde{L}_1[l];x)=\tilde{I}(\tilde{L}_0,\tilde{L}_1;x)-k+l.$$

Consequently, we have the following useful property [81].

$$HF^*(\tilde{L}_0[k],\tilde{L}_1[l])\cong HF^{*-k+l}(\tilde{L}_0[k],\tilde{L}_1[l]). \tag{A.1}$$

In addition we have the invariance under the action of a graded symplectomorphisms $\tilde{\varphi}$:

$$HF^*(\tilde{\varphi}(\tilde{L}_0),\tilde{\varphi}(\tilde{L}_1))\cong HF^*(\tilde{L}_0,\tilde{L}_1), \tag{A.2}$$

the Poincaré duality

$$HF^*(\tilde{L}_1,\tilde{L}_0)\cong HF^{n-*}(\tilde{L}_0,\tilde{L}_1). \tag{A.3}$$

Finally, when Proposition 2.2.32 holds, we have a graded counterpart:

$$HF^*(\tilde{L}, \tilde{L}) \cong \bigoplus_{i \in \mathbb{Z}} H^{*+iN}(L, \mathbb{Z}/2). \tag{A.4}$$

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