Families of genus 3 hyperelliptic curves whose jacobians are $2-2-2$ isogenous.

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The situation

Genus 1 Classical to determine 2-isogenies between elliptic curves.

Genus 2 Construction known since Richelot (19th century). Other explicit formulae for the curves given by Bost and Mestre.

Genus 3 Smith has given examples of hyperelliptic curves in genus 3 that are $2-2-2$ isogenous thanks to the existence of non trivial factorization of $f(x)-g(z)$ leading to a correspondence between hyperelliptic curves $y^2=f(x)$ and $y^2=g(x)$.

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Goal (of part one)

Find all hyperelliptic curves of genus 3 for which there is an hyperelliptic curve and a $2-2-2$ isogeny between their jacobians.

Factorization of the multiplication by 2

The factorization of the multiplication by two is given by:

$$\begin{array}{ccc}
\mathbb{C}^3/(\mathbb{Z}^3+\Omega\mathbb{Z}^3) & \xrightarrow{\varphi} & \mathbb{C}^3/(\mathbb{Z}^3+2\Omega\mathbb{Z}^3) \\
 & \searrow^{[2]} & \downarrow z \mapsto z \\
 & & \mathbb{C}^3/(\mathbb{Z}^3+\Omega\mathbb{Z}^3)
\end{array}$$

where the $2-2-2$ isogeny φ is

$$\varphi : \begin{array}{ccc} \mathbb{C}^3/(\mathbb{Z}^3+\Omega\mathbb{Z}^3) & \longrightarrow & \mathbb{C}^3/(\mathbb{Z}^3+2\Omega\mathbb{Z}^3) \\ z & \longmapsto & 2z \end{array}$$

Factorization of the multiplication by 2

The general situation is given by the action of $\mathrm{Sp}_6(\mathbb{Z})$:

$$\varphi = \beta \circ \alpha_\Gamma : \mathbb{C}^3/(\mathbb{Z}^3+\Omega\mathbb{Z}^3) \xrightarrow[\sim]{\alpha_\Gamma} \mathbb{C}^3/(\mathbb{Z}^3+\Omega'\mathbb{Z}^3) \xrightarrow{\beta} \mathbb{C}^3/(\mathbb{Z}^3+2\Omega'\mathbb{Z}^3)$$

$$[2] : \mathbb{C}^3/(\mathbb{Z}^3+\Omega'\mathbb{Z}^3) \longrightarrow \mathbb{C}^3/(\mathbb{Z}^3+\Omega'\mathbb{Z}^3), \qquad \mathbb{C}^3/(\mathbb{Z}^3+2\Omega'\mathbb{Z}^3) \xrightarrow{z \mapsto z} \mathbb{C}^3/(\mathbb{Z}^3+\Omega'\mathbb{Z}^3)$$

where

$$\Omega' = \Gamma \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}$$

$$\begin{array}{rccc} \alpha_\Gamma : & \mathbb{C}^3/(\mathbb{Z}^3+\Omega\mathbb{Z}^3) & \longrightarrow & \mathbb{C}^3/(\mathbb{Z}^3+\Omega'\mathbb{Z}^3) \\ & z & \longmapsto & {}^t(C\Omega + D)^{-1} z \end{array}$$

is the action of $\mathrm{Sp}_6(\mathbb{Z})$.

The kernels: two geometric situations

In order to determine all curves, it's easier to determine the possible kernels. They must be:

- isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$,
- totally isotropic for the Weil pairing modulo 2.

Proposition

There are 135 such groups.

The kernels split into two categories:

- 105 "tractable" groups *i.e.* generated by differences of 2 points,
- 30 groups in "Fano disposition".

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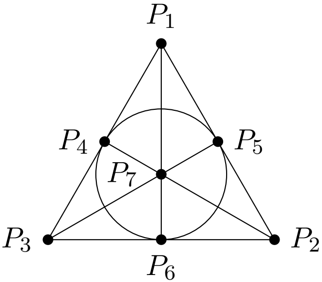

- P_8 is not represented: elements are sets of points modulo complementary.
- Elements of the group: the 7 lines, of order 2, and the identity element.
- The group law: the addition of two distinct lines is the third which has a common intersection point.

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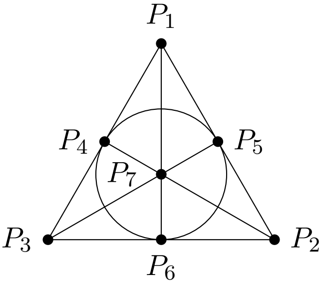

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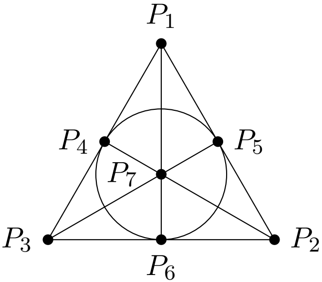

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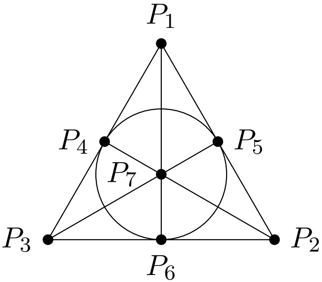

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Thêta constants

Definition (half characteristics)

We call thêta constant with half characteristics $\eta', \eta'' \in (\frac{1}{2}\mathbb{Z}/\mathbb{Z})^3$ the function

$$\vartheta \begin{bmatrix} \eta' \\ \eta'' \end{bmatrix} (0, \Omega) = \exp(i\pi {}^t\eta' \Omega \eta' + 2i\pi {}^t\eta' \eta'') \vartheta(\Omega\eta' + \eta'', \Omega)$$

It is said even if $4 {}^t\eta'\eta''$ is even or odd otherwise.

Odd thêta constants are zero. On the other side:

Theorem (genus 3)

An abelian variety is the jacobian of an hyperelliptic curve iff exactly one even thêta constant, among 36, is zero.

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Thomae formula, duplication formula

Link between thêta constants and Weierstrass points:

Theorem (Thomae)

$$\vartheta[\eta_S](0,\Omega)^4 = \begin{cases} c(-1)^{|S\cap U|} \prod_{\substack{i\in S\Delta U \\ j\notin S\Delta U}} \frac{1}{x_i - x_j} & \text{if } |S\Delta U| = g+1, \\ 0 & \text{if } |S\Delta U| \neq g+1. \end{cases}$$

To compute thêta constants of periods 2Ω with those of Ω, we have:

Proposition (Duplication formula, Riemann)

$$2^g \vartheta\begin{bmatrix}\eta' \\ \eta''\end{bmatrix}(0,2\Omega)^2 = \sum_{\delta\in(\frac{1}{2}\mathbb{Z}/\mathbb{Z})^g} (-1)^{4\,{}^t\eta'\delta} \vartheta\begin{bmatrix}0 \\ \eta''+\delta\end{bmatrix}(0,\Omega)\vartheta\begin{bmatrix}0 \\ \delta\end{bmatrix}(0,\Omega).$$

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Functional equation of ϑ

The functional equation is verified on the subset $\Gamma_{1,2} \subset \mathrm{Sp}_{2g}(\mathbb{Z})$.

Theorem (Functional equation, Riemann)

$$\vartheta\left({}^t(C\Omega+D)^{-1}z,(A\Omega+B)(C\Omega+D)^{-1}\right)$$
$$=\zeta_\Gamma\left(\det(C\Omega+D)\right)^{\frac{1}{2}}\exp\left(i\pi\,{}^tz(C\Omega+D)^{-1}Cz\right)\vartheta(z,\Omega).$$

Proposition

The set of matrix $\left\{\left(\begin{smallmatrix} I & \Delta \\ 0 & I\end{smallmatrix}\right), \Delta \text{ diagonal with coefficients in } \{0,1\}\right\}$, *and* $\Gamma_{1,2}$ *span* $\mathrm{Sp}_{2g}(\mathbb{Z})$.

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Computational aspects and results

- We split the thêta constants in 8 subsets:
  - The set S_1 of the 8 thêta constants $\begin{bmatrix}\eta' \\ 0\end{bmatrix}$.
  - The 7 sets $S_2, \dots S_8$ of even thêta constants with $\eta'' \neq 0$ fixed.
- Fixing 3 out of 8 Weierstrass points x_i and we can compute the products for these 8 subsets.
- We multiply 2 by 2 with simplifications: we can assume that thêta constants are positively dependant, so all the square roots can be taken positive.
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"Tractable" kernels

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$$\text{“Trace”}_{\mathfrak{S}_3 \hookrightarrow \mathfrak{S}_6}(x_1x_2x_3 + x_1x_2x_4) = \sum_{\sigma \in \mathfrak{S}_3 \hookrightarrow \mathfrak{S}_6} \epsilon(\sigma)(x_1x_2x_3 + x_1x_2x_4)^{\sigma}$$

  where $\mathfrak{S}_3 \hookrightarrow \mathfrak{S}_6 : \sigma \mapsto \sigma(\{1,2\},\{3,4\},\{5,6\})$.

- The set S_1 gives another family *(f-2)* with a polynomial of total degree 16, degree 4 in each variable and 19591 monomials. The stabilizing group is $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathfrak{S}_4$, letting invariant the transpositions (12), (34), (56) et (78) and acting by $\mathfrak{S}_4$ on these pairs.

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$$\text{"Trace"}_{\mathfrak{S}_4 \hookrightarrow \mathfrak{S}_8}(x_1x_2x_5x_7) = \sum_{\sigma \in \mathfrak{S}_4 \hookrightarrow \mathfrak{S}_8} \epsilon(\sigma)(x_1x_2x_5x_7)^\sigma$$

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- S_1 gives another family *(f-4)* : total degree 24, degree 6 in each variable and 215601 monomials!
Smith gives an example in this family: in $\mathbb{Q}(\sqrt{7})$, $\alpha_7 = \frac{1+\sqrt{7}}{2}$, the family, in t, of curves

$$y^2 = \frac{x^7}{7} - \alpha_7 t x^5 - \alpha_7 t x^4 - (2\alpha_7+5)t^2x^3 - (4\alpha_7+6)t^2x^2 + ((3\alpha_7-2)t^3 - (\alpha_7+3)t^2)x + \alpha_7 t^3.$$

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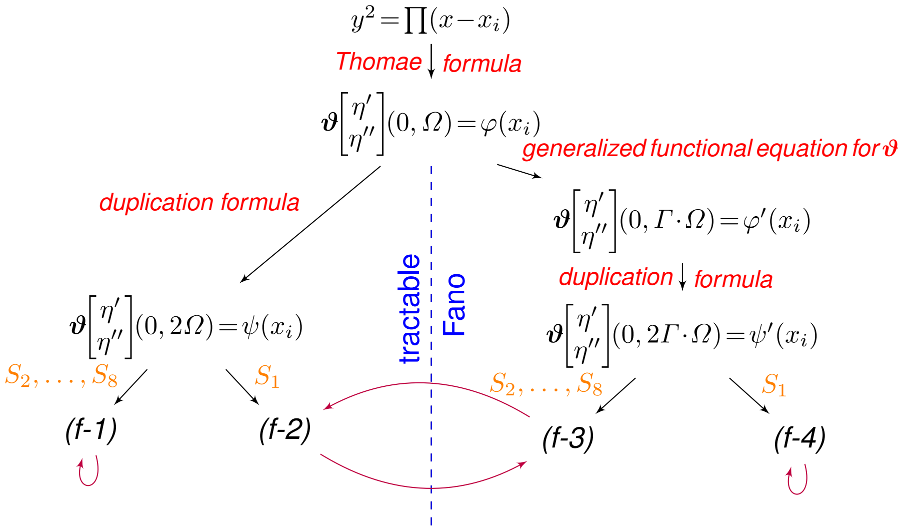
Part one summary



Tractable and Fano curves

Goal (of part 2)

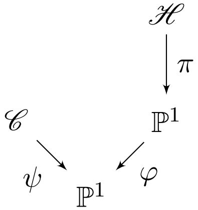
Find equations of "tractable" curves in family (f-2), of "Fano" curves in family (f-3) and correspondences between them.

Recillas' trigonal construction

$$\begin{array}{ccc} & & \mathscr{H} \\ & & \downarrow \pi \\ \mathscr{C} & & \mathbb{P}^1 \\ & \psi \searrow \quad \swarrow \varphi & \\ & \mathbb{P}^1 & \end{array}$$

The diagram of the trigonal construction

Recillas' trigonal construction



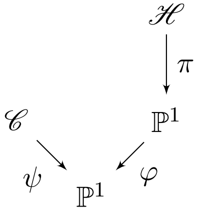
The diagram of the trigonal construction

- π is a degree 2 covering, φ has degree 3 and ψ has degree 4.
- The trigonal φ identifies pairs of Weierstrass points.

Proposition

There is a $3-2$ correspondence between $\mathscr{C}$ and $\mathscr{H}$ that induce a $2-2-2$ isogeny between their jacobians.

Recillas' trigonal construction



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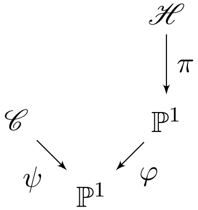

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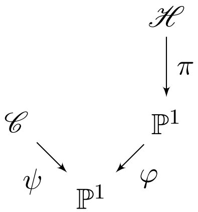

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- φ identifies pairs of Weierstrass points at $0, \infty, 1$ and e.
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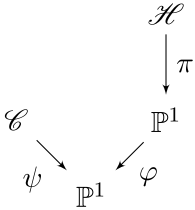

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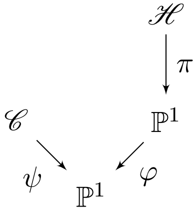

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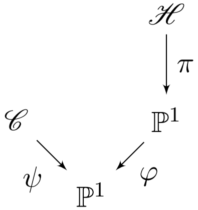

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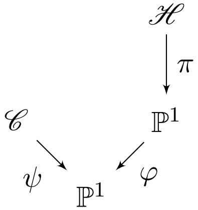

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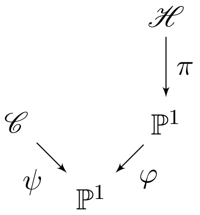

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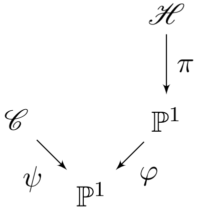

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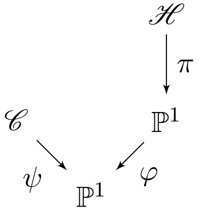

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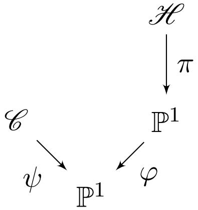

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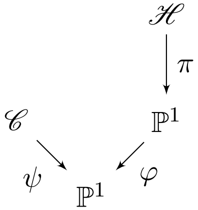

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Family *(f-3)*

Theorem

The following are equivalent.

1. *The curve $\mathscr{C}$ is hyperelliptic .*
2. *There is only one trigonal (ie. $d = e$).*
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With the $3-2$ correspondence, we can compute the kernel of *the dual isogeny* between $\operatorname{Jac}\mathscr{C}$ and $\operatorname{Jac}\mathscr{H}$.

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Weierstrass model and definition field

Let $[f,g](x)$ be the polynomial $f'(x)g(x)-f(x)g'(x)$ and f_1,f_2,f_3 and f_4 the four polynomials which correspond to the pairs of identified points in the equation of $\mathscr{H}$.

Proposition (Fields of definition)

1. *We have a Weierstrass equation of $\mathscr{C}$ above a conic, defined over the base field.*
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The equations

- This field extension can be parametrized by

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Characterization of the Weierstrass model of $\mathscr{C}$

Theorem (Geometrical characterization of family *(f-3)*)

1. *The j-invariants of $\{\mathscr{W}_5, \ldots, \mathscr{W}_8\}$ and $\{\mathscr{W}_1, \ldots, \mathscr{W}_4\}$ are* ***equal****.*
2. *An hyperelliptic curve is in (f-3) iff there is a partition 4-4 of its Weierstrass points with a* ***common cross-ratio****.*

Corollary

An hyperelliptic curve $\mathscr{C}$ is in family (f-3) iff it appears in a trigonal construction.

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Characterization of the Weierstrass model of $\mathscr{C}$

Corollary

1. *The curve* $\mathscr{C}$ *can be deduced of* $\mathscr{H}$ *by the 4 Weierstrass points* $(0, 1, d, \infty)$*, fixed points of* φ*, and by the homography*
$$\mathscr{J} : w \mapsto \frac{(\overline{x}_{1,3} + x_{3,4})w - d(\overline{x}_{1,4} + x_{3,4})}{(x_{1,4} + x_{3,4})w - (x_{1,3} + x_{3,4})}$$
where $x_{i,j}$ *and* $\overline{x}_{i,j}$ *are the roots of* $[f_i, f_j](x)$.
2. *Conversely, if* $\infty, 0, 1, \lambda_4, \dots \lambda_8$ *are the Weierstrass points of* $\mathscr{C}$*,*
$$A = -\frac{(\lambda_5\lambda_6 - \lambda_6\lambda_7 - \lambda_5 + \lambda_6)(\lambda_4 - 1)\lambda_4}{\lambda_4\lambda_5\lambda_6 - \lambda_4\lambda_6\lambda_7 - \lambda_4\lambda_6 + \lambda_4\lambda_7 - \lambda_5\lambda_7 + \lambda_6\lambda_7}$$
$$B = \frac{(\lambda_4 - 1)(\lambda_4\lambda_5 - \lambda_4\lambda_7 - \lambda_5\lambda_7 + \lambda_6\lambda_7}{\lambda_4\lambda_5\lambda_6 - \lambda_4\lambda_6\lambda_7 - \lambda_4\lambda_6 + \lambda_4\lambda_7 - \lambda_5\lambda_7 + \lambda_6\lambda_7}$$
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Correspondence and hyperelliptic involutions

The $3-2$ correspondence given by the trigonal construction does not preserve the hyperelliptic involutions. However:

Theorem

It exists, over the field extension of degree 8, a correspondence $4-3$ between Weierstrass models of curves $\mathscr{H}$ et $\mathscr{C}$, which respects hyperelliptic involutions, i.e. we have polynomials $P(x,X)$ and $Q(x,X)$ s.t.

$$\begin{cases} y^2 = f(x) & (\mathscr{H}) \\ 0 = P(x,X) & \\ yY = Q(x,X) & \\ Y^2 = F(X) & (\mathscr{C}), \end{cases}$$

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This correspondence associates to Weierstrass points $0, 1, d$, and ∞ of $\mathscr{C}$, one pair of identified Weierstrass points of $\mathscr{H}$.
To the points $\mathscr{W}_5, \dots, \mathscr{W}_8$, it associates 0 or 2 pairs of Weierstrass points of $\mathscr{H}$.

More precisely, we have constants s.t.

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