RECOVERING THE HAMILTONIAN FROM SPECTRAL DATA

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ABSTRACT. We show that the contributions to the Gutzwiller formula with observables associated to the iterates of a given elliptic non-degenerate periodic trajectory γ and to certain families of observables localized near γ determine the quantum Hamiltonian in a formal neighborhood of the trajectory γ, that is, the full Taylor expansion of its total symbol near γ. We also treat the "bottom of a well" case both for general and Schrödinger operators, and give some analog classical results.

CONTENTS

1. INTRODUCTION AND MAIN RESULTS

It is well known that spectral properties of semiclassical Hamiltonians and dynamical properties of their principal symbols are linked. Even when there is no precise information "eigenvalue by eigenvalue" of the spectrum, the so-called Gutzwiller trace formula provides information on averages of the spectrum at scale of the Planck constant $\hbar$. More precisely, let $H(x, \hbar D_x)$ be a self-adjoint semiclassical elliptic pseudodifferential operator on a compact manifold X of dimension $n+1$, whose symbol $H(x, \xi)$ is proper (as a map from T^*X into $\mathbb{R}$). Let E be a regular value of H and γ a non-degenerate periodic trajectory of primitive period T_γ lying on the energy surface $H = E$.

Consider the Gutzwiller trace (see [11])

$$(1.1) \qquad \mathrm{Tr}\left(\psi_r\left(\frac{H(x, \hbar D_x) - E}{\hbar}\right)\right) = \sum_i \psi_r\left(\frac{E_i(\hbar) - E}{\hbar}\right),$$

where for $r \in \mathbb{Z}^*$, ψ_r is a C^∞ function whose Fourier transform is compactly supported with support in a small enough neighborhood of rT_γ and is identically

one in a still smaller neighborhood containing rT_γ. As shown in [14], [15] (1.1) has an asymptotic expansion

$$(1.2) \qquad\qquad e^{i\frac{S_\gamma}{\hbar} + \frac{\pi}{2}\sigma_\gamma} \sum_{k=0}^{\infty} a_k^r \hbar^k.$$

In [8] it was shown how to compute the terms of this expansion to <u>all</u> orders in terms of a microlocal Birkhoff canonical form for H in a formal neighborhood of γ. Following earlier results by Zelditch [19, 20] and Guillemin [7] in the context of the high part of the spectrum of the Laplacian, it was also proved in [8] that the family of constants $(a_k^r)_{(k,r)\in\mathbb{N}\times\mathbb{Z}^*}$ determines the microlocal (and hence, a fortiori, the classical) Birkhoff canonical form for H in a formal neighborhood of γ; see also [12] for another proof based on trace formulas for monodromy operators.

In the case of the bottom of a well, the determination of the Birkhoff form by the low part of the spectrum has been shown in [9]. When it is known in addition that $H(x, \hbar D_x)$ is a Schrödinger operator $-\hbar^2 \Delta + V$, it has been shown that the low part of the spectrum determines the Taylor expansion of the potential V when this one is even in all variables [10] or in dimension one under generic assumptions [4], and the potential itself in one dimension under generic assumptions [3]. But in the general case the Gutzwiller formula will determine only the normal form of the Hamiltonian, that is to say $H(x, \hbar D_x)$ only modulo unitary operators, and its principal symbol only modulo symplectomorphisms. Of course it cannot determine more, as the spectrum, and a fortiori the trace, is insensitive to unitary conjugation.

The aim of this paper is to address the question of determining the *true* Hamiltonian from more precise spectral data, namely from the Gutzwiller trace formula with observables.

Let us point out that, aside from the intrinsic interest (in particular for physical purposes) of determining a true Hamiltonian and not only its symplectomorphically (with unknown symplectomorphism in general) conjugated normal form, we will get as a bi-product of our results the missing spectral information needed to complete the Schrödinger case, discussed in the preceding paragraph and treated in [3, 4, 10]. Let us also remark that the original problem developed by M. Kac in "Can one hear the shape of a drum?" [13] also involves the true Hamiltonian through the boundary of the domain on which the Laplacian acts.

It is well known that, for any pseudodifferential operator $O(x, \hbar D_x)$ of symbol $\mathcal{O}(x, \xi)$, there is a result equivalent to (1.2) for the quantity

(1.3)
$$\mathrm{Tr}\left(O(x, \hbar D_x)\psi_r\left(\frac{H(x, \hbar D_x) - E}{\hbar}\right)\right) = \sum_i \langle \varphi_j, O(x, \hbar D_x)\varphi_j\rangle \psi_r\left(\frac{E - E_i}{\hbar}\right)$$

(here φ_j is meant as the eigenvector of eigenvalue E_j and ψ_r is as before) under the form of an asymptotic expansion of the form

$$(1.4) \qquad\qquad e^{i\frac{S_\gamma}{\hbar} + \frac{\pi}{2}\sigma_\gamma} \sum_{k=0}^{\infty} a_k^r(\mathcal{O})\hbar^k,$$

where a_k^r are distributions supported on γ.

We will show in the present paper that the knowledge of the coefficients $a_k^r(O)$ for O belonging to some family of observables localized near γ is enough to determine the full Taylor expansion of the total symbol of $H(x, \hbar D_x)$ near γ, in other words

$H(x, \hbar D_x)$ microlocally in a formal neighborhood of γ, when γ is non-degenerate elliptic (including the case where γ is reduced to a point (bottom of a well)).

Let us first remark that, besides the fact that not all observables will be needed to conclude, the formula (1.3) gives access only to diagonal matrix elements of observables between eigenvectors and not to the full knowledge of them. Nevertheless the asymptotics of (1.3) surprisingly determine the full Taylor expansion of the symbol of the Hamiltonian.

Let us also remark that the trace formula with any observable microlocalized in a small enough neighborhood of γ determines obviously its primitive period T_γ. We will assume that any multiple of T_γ is isolated in the set of the periods of all the periodic trajectories on the same energy shell (let us remark that in case this condition is not fulfilled, our results remain valid by taking observables microlocalized in a neighborhood of the non-degenerate elliptic γ). Moreover it is known ([6, 7]) that the coefficients of the trace formula determine the Poincaré angles modulo $2\pi\mathbb{Z}$ and we prove in Appendix B that, in the case where γ is not reduced to one point, any realization of the Poincaré angles as real numbers leads to a different Birkhoff normal form but gives an explicit symplectomorphism that conjugates one to another: hence, our reconstruction of the "true" Hamiltonian is independent of the choice of the realization. We also show that in the "bottom of a well" case, the θ_i's are determined by the spectrum.

Therefore the only knowledge we will require will be the fact that there exists a geometric periodic trajectory γ, possibly of dimension zero, which is elliptic non-degenerate (see the definition below) and whose set of periods is isolated in the set of periods of the same energy shell.

We will be concerned with three cases:

(1) γ is a curve;
(2) the general "bottom of a well" case (γ reduced to a point);
(3) the "bottom of a well" case when the Hamiltonian is a Schrödinger operator.

Our results will also be of three different kinds:

a. The knowledge of the coefficients of the trace formula for (1), or of some of the diagonal matrix elements (expectation values) between eigenvectors of the Hamiltonian for (2),(3), for a family of observables satisfying some algebraic properties on γ determine some Fermi coordinates (see the definition below). It is the content of Theorems 1.3, 1.9, 1.13.

b. The knowledge of the coefficients of the trace formula for (1), or of some of the diagonal matrix elements (expectation values) between eigenvectors of the Hamiltonian for (2), (3), for another family of observables, expressed on any (not necessarily the one determined by a.) Fermi system of coordinates, determines the full Taylor expansion of the total symbol of the Hamiltonian expressed on these Fermi coordinates (Theorems 1.4, 1.10, 1.14).

c. The combination of the two preceding cases, where the family of observables defined in a. drives the knowledge of the full Hamiltonian. More precisely, the knowledge of the quantities expressed in a. determines a family of observables, which is precisely the one defined in b. expressed in the Fermi system determined in a., the trace coefficients or some of the diagonal matrix elements of which determine the full Taylor expansion of the Hamiltonian on a determined system of coordinates (Corollaries 1.5, 1.11, 1.15).

Finally we obtain analog classical results as a byproduct of the quantum results in Section 4.

Definition 1.1. A periodic trajectory of the Hamiltonian flow generated by $H(x, \xi)$ is said to be non-degenerate elliptic if its linearized Poincaré map has eigenvalues $(e^{\pm i\theta_i})_{1 \leq i \leq n}$, $\theta_j \in \mathbb{R}$, and the rotation angles θ_i $(1 \leq i \leq n)$ and π are independent over $\mathbb{Q}$.

Definition 1.2 (Fermi coordinates). We will denote by "Fermi coordinates" any system of local coordinates of $T^*\mathcal{M}$ near γ, $(x, t, \xi, \tau) \in T^*(\mathbb{R}^n \times \mathbb{S}^1)$, such that $\gamma = \{x = \xi = \tau = 0\}$ and on which the principal symbol H_p of $H(x, \hbar D_x)$ can be written for any chosen realization of the Poincaré angles $\theta_i \in \mathbb{R}$ as:

$$(1.5) \qquad H_p(x, t, \xi, \tau) = H_0(x, t, \xi, \tau) + H_2,$$

where

$$(1.6) \qquad H_0(x, t, \xi, \tau) = E + \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2} + \tau$$

and

$$(1.7) \qquad H_2 = O\left((x^2 + \xi^2 + |\tau|)^{\frac{3}{2}}\right).$$

The existence of such local coordinates, guaranteed by the Weinstein tubular neighborhood theorem ([18]), was proved in [7, 8, 19] under the hypothesis of non-degeneracy mentioned earlier. However the construction of Fermi coordinates involves the knowledge of the quadratic part of H_p in a neighborhood of γ. Our first result shows that a system of Fermi coordinates can be determined by γ only at the classical level and some quantum spectral quantities (constructed out of a system of local coordinates near γ and some quantum spectral quantities).

Theorem 1.3. *Let $P_p^k, k = 0, 1, \ldots, 2n^2 + n, p \in \mathbb{Z}$, be any pseudodifferential operators whose respective principal symbols $\mathcal{P}_p^k$ satisfy*

$$(1.8) \qquad \begin{aligned} &\mathcal{P}_p^0(x, t, \xi, \tau) = e^{-2i\pi pt}\tau \quad and \\ &\mathcal{P}_p^k(x, t, \xi, \tau) = e^{-2i\pi pt}\mathcal{R}^k(x, \xi), \ k = 1, \ldots, 2n^2 + n, \end{aligned}$$

*in a local symplectic system of coordinates $(x, t, \xi, \tau) \in T^*(\mathbb{R}^n \times \mathbb{S}^1)$ such that $\gamma = \{x = \xi = \tau = 0\}$, with the property that $\mathcal{R}^k(0) = \nabla\mathcal{R}^k(0) = 0$ **and the Hessians $d^2\mathcal{R}^k(0)$ are linearly independent**.*

An example of such symbols is given by the family

$$(1.9) \qquad \begin{cases} \mathcal{Q}^1_{ijp}(x, t, \xi, \tau) &= e^{-2i\pi pt}x_i\xi_j, \\ \mathcal{Q}^2_{ijp}(x, t, \xi, \tau) &= e^{-2i\pi pt}x_ix_j, \\ \mathcal{Q}^3_{ijp}(x, t, \xi, \tau) &= e^{-2i\pi pt}\xi_i\xi_j, \\ \mathcal{Q}_p(x, t, \xi, \tau) &= e^{-2i\pi pt}\tau. \end{cases}$$

Then the knowledge of the coefficients $(a_1^l(P_p^k))_{0 \leq k \leq 2n^2+n, p \in \mathbb{Z}}$ in (1.3)-(1.4) determines (in a constructive way) an explicit system of Fermi coordinates near γ.

Theorem 1.4. *Let γ be a non-degenerate elliptic periodic trajectory of the Hamiltonian flow generated by the principal symbol H_p of $H(x, \hbar D_x)$ on the energy shell $H_p^{-1}(E)$, and let $(x, t, \xi, \tau) \in \mathbb{R}^n \times \mathbb{S}^1 \times \mathbb{R}^{n+1}$ be a system of Fermi coordinates near γ.*

For $(m, n, p) \in \mathbb{N}^{2n} \times \mathbb{Z}$, let O_{mnp}, O_p be any pseudodifferential operators whose total Weyl symbols (in this system of coordinates) $\mathcal{O}_{mnp}$, $\mathcal{O}_p$ satisfy

$$(1.10) \quad \begin{cases} \mathcal{O}_{mnp}(x, t, \xi, \tau) = e^{-i2\pi pt} \prod_{j=1}^{n} \left(\frac{x_j + i\xi_j}{\sqrt{2}} \right)^{m_j} \left(\frac{x_j - i\xi_j}{\sqrt{2}} \right)^{n_j} \\ \qquad\qquad\qquad + \sum_{2l+N=|m|+|n|+1} O(\hbar^l (x^2 + \xi^2 + |\tau|)^{\frac{N}{2}}), \\ \mathcal{O}_p(x, t, \xi, \tau) = e^{-i2\pi pt} \tau + \sum_{2l+N=3} O(\hbar^l \left(x^2 + \xi^2 + |\tau| \right)^{\frac{N}{2}}) \end{cases}$$

in a neighborhood of γ. Then the knowledge of the coefficients $a_k^l(O_{mnp})$ and $a_k^l(O_q)$ in (1.3)-(1.4) for $k \leq N$ and m, n, p, q satisfying

(1) $|m| + |n| \leq N$,
(2) $\forall j = 1, \ldots, n$, $m_j = 0$ **or** $n_j = 0$,
(3) $p \in \mathbb{Z}$, $q \in \mathbb{Z}^*$,

determines the Taylor expansion near γ up to order M_1 in (x, ξ) and M_2 in τ, of the total Weyl symbol, in this system of Fermi coordinates, of $H(x, \hbar D_x)$ up to order l in $\hbar$ at the condition that $2l + M_1 + 2M_2 \leq N$.

Concatenating the two preceding results we get the coordinate free statement:

Corollary 1.5. *Let P_p^k be as in Theorem 1.3. Then the knowledge of the coefficients $a_1^l(Q_{ijp}^k)$, $a_1^l(Q_p)$ for $p \in \mathbb{Z}$, $l \in \mathbb{Z}, 1 \leq i, j \leq n, k \in \{1, 2, 3\}$ determine observables O_{mnp}, O_q out of which the coefficients $a_k^l(O_{mnp})$ and $a_k^l(O_q)$, for $k \leq N$ and m, n, p, q satisfying conditions $(1), (2), (3)$ in Theorem 1.4, determine modulo a function vanishing to infinite order on γ the full symbol of $H(x, \hbar D_x)$ in a determined system of local coordinates near γ.*

Remark 1.6. It is easy to see that condition (2) implies that the number of observables in the transverse to γ directions (for each Fourier coefficient in t) needed for determining $H(x, \hbar D_x)$ up to order N is a polynomial function of N of degree $n - 1$, while the number of all polynomial functions in (x, ξ, τ) of order N is a polynomial in N of higher degree $2n$. The fact that not all observables are needed can be understood by the fact that we know that the Hamiltonian we are looking for is conjugated to the normal form by a unitary operator and not by any operator (see the discussion after Theorem 2.1). At the classical level this is a trace of the fact that we are looking for a symplectomorphism, and not any diffeomorphism (see Section 4).

Remark 1.7. The asymptotic expansion of the trace (1.3) involves only the microlocalization of $H(x, \hbar D_x)$ in a formal neighborhood of γ. Therefore there is no hope to recover from spectral data more precise information than the Taylor expansion of its symbol near γ. The rest of the symbol concerns spectral data of order $\hbar^\infty$.

Let us now consider the case where γ is reduced to one point, namely the "bottom of a well" case. Let us assume that the principal symbol H_p of $H(x, \hbar D_x)$ has a global non-degenerate minimum at $z_0 \in T^*\mathcal{M}$, and let $d^2 H_p(z_0)$ be the Hessian of H at z_0. Let us define the matrix Ω defined by $d^2 H_p(z_0)(\cdot, \cdot) =: \omega_{z_0}(\cdot, \Omega^{-1}\cdot)$ where $\omega_{z_0}(\cdot, \cdot)$ is the canonical symplectic form of $T^*\mathcal{M}$ at z_0. The eigenvalues of Ω are purely imaginary; let us denote them by $\pm i\theta_j$ with $\theta_j > 0$, $j = 1, \ldots, n$. Let us assume moreover that $\theta_j, j = 1, \ldots, n$, are rationally independent.

Definition 1.8. By extension of Definition 1.2, we will also denote by Fermi coordinates any system of Darboux coordinates $(x, \xi) \in T^*\mathbb{R}^n$ centered at z_0 such that:

$$(1.11) \qquad H_p(x, \xi) = H_p(z_0) + \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2} + O((x, \xi)^3).$$

The existence of such local coordinates will be proved in Section 3, once again by using the knowledge of the quadratic part of H_p near z_0. Our next result shows that one can explicitly construct Fermi coordinates out of the knowledge of some quantum spectral quantities.

Theorem 1.9. *Let* $P^k, k = 1, \ldots, 2n^2 + n$, *be **any** pseudodifferential operators whose principal symbols* $\mathcal{P}^k$ *are such that* $\mathcal{P}^k(z_0) = \nabla\mathcal{P}^k(z_0) = 0$ ***and the Hessians*** $d^2\mathcal{P}^k(z_0)$ ***are linearly independent.*** *An example of such symbols is given by the family*

$$(1.12) \qquad \begin{cases} \mathcal{Q}_{ij}^1(x, \xi) & = & x_i\xi_j, \\ \mathcal{Q}_{ij}^2(x, \xi) & = & x_ix_j, \\ \mathcal{Q}_{ij}^3(x, \xi) & = & \xi_i\xi_j, \end{cases}$$

$1 \le i, j \le n$, $k \in \{1, 2, 3\}$, *in any system* $(x, \xi) \in T^*\mathbb{R}^n$ *of Darboux coordinates centered at* z_0.

Then, for any $\epsilon = \epsilon(\hbar) > 0$, $\hbar = o(\epsilon(\hbar))$ *(e.g.* $\epsilon = \hbar^{1-\eta}, \eta > 0$*), the knowledge of the spectrum of* $H(x, \hbar D_x)$ *in* $[H_p(z_0), H_p(z_0) + \epsilon]$ *and the diagonal matrix elements of* P^k *between the corresponding eigenvectors of* $H(x, \hbar D_x)$ *determines (in a constructive way) an explicit system of Fermi coordinates.*

Theorem 1.10. *For* $(m, n) \in \mathbb{N}^{2n}$, *let* O_{mn} *be any pseudodifferential operator whose total Weyl symbol* $\mathcal{O}_{mn}$ *satisfies*

$$(1.13)$$
$$\mathcal{O}_{mn}(x, \xi) = \prod_{j=1}^{n} \left(\frac{x_j + i\xi_j}{\sqrt{2}} \right)^{m_j} \left(\frac{x_j - i\xi_j}{\sqrt{2}} \right)^{n_j} + \sum_{\substack{2l+N= \\ |m|+|n|+1}} O\left(\hbar^l \left(x^2 + \xi^2 \right)^{\frac{N}{2}} \right)$$

in a neighborhood of z_0 *and in a system* $(x, \xi) \in T^*\mathbb{R}^n$ *of Fermi coordinates centered at* z_0.

Then the knowledge of the spectrum of $H(x, \hbar D_x)$ in $[H_p(z_0), H_p(z_0) + \epsilon]$ with $\hbar^{1-\alpha} = O(\epsilon)$ for some $\alpha > 0$, and the diagonal matrix elements of O_{mn} between the corresponding eigenvectors of $H(x, \hbar D_x)$, for:

  (1) $|m| + |n| \le N$,
  (2) $\forall j = 1, \ldots, n$, $m_j = 0$ ***or*** $n_j = 0$,

determines the Taylor expansion up to order N of the full symbol of $H(x, \hbar D_x)$ at z_0 in the coordinates (x, ξ).

Corollary 1.11. *The diagonal matrix elements of the operators* P^k *as in Theorem 1.9 determine observables* O_{mn} *whose diagonal matrix elements as in Theorem 1.10 determine, modulo a function vanishing to infinite order at* z_0, *the full symbol of* $H(x, \hbar D_x)$, *in a determined system of local coordinates near* z_0.

Remark 1.12. Although we will not prove it here, let us remark that Theorem 1.10 (and also Theorem 1.4) is also valid in the framework of quantization of Kählerian manifolds.

In the case where $H(x, \hbar D_x)$ is a Schrödinger operator $-\hbar^2 \Delta + V$, it is known [10] that the (actually classical) normal form determines the Taylor expansion of the potential in the case where the latter is invariant, for each $i = 1, \ldots, n$, by the symmetry $x_i \to -x_i$. The same result holds without the symmetry assumption in the case $n = 1$, with assumption $V'''(0) \neq 0$, as has been shown in [4].

Now let $H = -\hbar^2 \Delta + V$ be a Schrödinger operator and let q_0 be a global non-degenerate minimum of V. Let us assume that the square-roots $(\theta_i)_{1 \leq i \leq n}$ of the eigenvalues of $d^2 V(q_0)$ are linearly independent over the rationals. In that precise case, we will denote by Fermi coordinates any system of Darboux coordinates $(x, \xi) \in T^* \mathbb{R}^n$, in which the (principal or total, both notions are equivalent here) symbol H of our Schrödinger operator can be written as

$$(1.14) \qquad H(x, \xi) = V(q_0) + \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2} + R(x),$$

where $R(x) = O(x^3)$. The existence of such local coordinates will also be proved in Section 3, and Theorem 1.13 below proves that one can explicitly construct Fermi coordinates out of any system of local coordinates centered at q_0.

Theorem 1.14 shows that the matrix elements of only a **finite number** of observables are necessary to recover the full Taylor expansion of the potential in the general case.

Theorem 1.13. *Let $P^k, k = 1, \ldots, \frac{n(n+1)}{2}$, be **any** pseudodifferential operators whose principal symbols are potentials $\mathcal{P}^k$ such that $\mathcal{P}^k(q_0) = \nabla \mathcal{P}^k(q_0) = 0$ **and the Hessians** $d^2 \mathcal{P}^k(q_0)$ **are linearly independent** (an example of such potentials is the family $\mathcal{Q}_{ij}^2(x) = x_i x_j$ in a local system of coordinates centered at q_0).*

Then, for any $\epsilon = \epsilon(\hbar) > 0, \hbar = o(\epsilon)$, the knowledge of the spectrum of $H(x, \hbar D_x)$ in $[V(q_0), V(q_0) + \epsilon]$ and the diagonal matrix elements of $P^k, k = 1, \ldots, \frac{n^2 + n}{2}$, between the corresponding eigenvectors of $H(x, \hbar D_x)$ determines (in a constructive way) an explicit system of Fermi coordinates.

Theorem 1.14. *Let $(x, \xi) \in T^* \mathbb{R}^n$ be a system of Fermi coordinates centered at $(q_0, 0)$.*

Then the knowledge of the spectrum of $H(x, \hbar D_x)$ in $[V(q_0), V(q_0) + \epsilon]$ with $\hbar^{1-\alpha} = O(\epsilon)$ for some $\alpha > 0$, and the diagonal matrix elements of the $2^n - 1$ observables $O_{m0}, m = (m_1, \ldots, m_n) \in \{0, 1\}^n \setminus \{0\}$, defined in Theorem 1.10, between the corresponding eigenvectors of $H(x, \hbar D_x)$ determines the full Taylor expansion of V at q_0 in the coordinates x.

Corollary 1.15. *The diagonal matrix elements of the operators P^k as in Theorem 1.13 determine $2^n - 1$ observables O_{m0} whose diagonal matrix elements as in Theorem 1.14 determine the potential V up to a function vanishing to infinite order at q_0.*

Remark 1.16. Note that since we are dealing with observables localized near the bottoms of the wells, the hypothesis that z_0 in Theorems 1.9 and 1.10 and q_0 in Theorems 1.13 and 1.14 are *global* minima can be released and the corresponding results can be formulated in a straightforward way.

The proof of Theorem 1.4 relies on two results having their own interest per se: **Proposition 2.13** which shows that the coefficients of the trace formula determine the matrix elements $\langle \varphi_j, O(x, \hbar D_x)\varphi_j \rangle$, where φ_j are the eigenvectors of the normal form of the Hamiltonian, and **Proposition 2.14** which states that the knowledge of the matrix elements of the conjugation of a given known self-adjoint operator by a unitary one determines, in a certain sense, the latter.

As a byproduct of Proposition 2.14 we obtain also a purely classical result, somehow an analog of it: the averages on Birkhoff angles associated to Birkhoff coordinates of the same classical observables than the ones in Theorem 1.4 determine the Taylor expansion of the (true) Hamiltonian. This is the content of **Theorem 4.2** below.

The paper is organized as follows. Section 2 is devoted to the proof of Theorems 1.4, 1.10, and 1.14. In Section 3, we give an explicit construction of some Fermi coordinates out of any system of local coordinates in both the periodic and "bottom of the well" case: this is the content of Theorems 1.3, 1.9, and 1.13. In Section 4 we show the classical equivalent of our quantum formulation.

Through the whole paper, $[[l, m]]$, $l < m$, will stand for the set of integers $\{l, \ldots, m\}$ and we we will assume, without loss of generality, that the period of γ is equal to 1.

2. Recovering the Hamiltonian in some given Fermi coordinates

Let us start this section by observing that, by microlocalization near γ, it is enough, in order to prove Theorem 1.4, to prove Theorem 2.1 below, which is nothing but the same statement expressed in a local Fermi system of coordinates.

The proof of Theorem 2.1 will need a construction of the quantum Birkoff normal form, given in Subsection 2.1. The rest of the proof is then a consequence of Proposition 2.13 (Subsection 2.2) and Proposition 2.14 (Subsection 2.3). Subsection 2.4 contains the proof of the analogs of Theorem 1.4 when γ is reduced to a single point, both in the general and "Schrödinger" cases (Theorems 1.10 and 1.14).

Theorem 2.1. *Let $H(x, \hbar D_x)$ be a self-adjoint semiclassical elliptic pseudodifferential operator on $L^2(\mathbb{R}^n \times \mathbb{S}^1)$. Let $(x, t, \xi, \tau) \in T^*(\mathbb{R}^n \times \mathbb{S}^1)$ be the canonical symplectic coordinates and let us assume that $\gamma = \mathbb{S}^1 = \{x = \xi = \tau = 0\}$ is a nondegenerate elliptic periodic orbit of the Hamiltonian flow generated by the principal symbol H_p of $H(x, \hbar D_x)$ on the energy shell $H_p^{-1}(E)$.*

Let us assume moreover that H_p can be written in these coordinates as

$$(2.1) \qquad\qquad H_p(x, t, \xi, \tau) = H_0(x, t, \xi, \tau) + H_2,$$

where

$$(2.2) \qquad\qquad H_2 = O\left((x^2 + \xi^2 + |\tau|)^{\frac{3}{2}} \right)$$

and H_0 is equal to

$$(2.3) \qquad\qquad H_0(x, t, \xi, \tau) = E + \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2} + \tau.$$

For $(m, n, p) \in \mathbb{N}^{2n} \times \mathbb{Z}$, let O_{mnp}, O_p be any pseudodifferential operators whose total Weyl symbols $\mathcal{O}_{mnp}$, $\mathcal{O}_p$ satisfy

$$(2.4) \quad \begin{cases} \mathcal{O}_{mnp}(x, t, \xi, \tau) & = & e^{-i2\pi pt} \prod_{j=1}^{n} \left(\frac{x_j + i\xi_j}{\sqrt{2}} \right)^{m_j} \left(\frac{x_j - i\xi_j}{\sqrt{2}} \right)^{n_j} \\ & & + \displaystyle\sum_{2l+N=|m|+|n|+1} O(\hbar^l \left(x^2 + \xi^2 + |\tau| \right)^{\frac{N}{2}}), \\ \mathcal{O}_p(x, t, \xi, \tau) & = & e^{-i2\pi pt} \tau + \displaystyle\sum_{2l+N=3} O(\hbar^l \left(x^2 + \xi^2 + |\tau| \right)^{\frac{N}{2}}) \end{cases}$$

in a neighborhood of γ. Then the knowledge of the coefficients $a_k^l(O_{mnp})$ and $a_k^l(O_q)$ in (1.3)-(1.4) for $k \leq N$ and m, n, p, q satisfying

(1) $|m| + |n| \leq N$,
(2) $\forall j = 1, \ldots, n$, $m_j = 0$ **or** $n_j = 0$,
(3) $p \in \mathbb{Z}$, $q \in \mathbb{Z}^*$,

determines the Taylor expansion near γ of the full symbol (in the system of coordinates (x, t, ξ, τ)) of $H(x, \hbar D_x)$ up to order N.

The proof of Theorem 2.1 will be divided into three steps: first, we will prove in Proposition 2.2 the existence of the quantum Birkhoff normal form in a form convenient for our computations, especially concerning the discussion of orders. In Proposition 2.13, we will show that the trace formula with any observable O determines the matrix elements of O in the eigenbasis of the Hamiltonian. Finally, in Proposition 2.14, we will show that these matrix elements (associated to the family of operators defined in Theorem 2.1) determine $H(x, \hbar D_x)$ in a formal neighborhood of $x = \xi = \tau = 0$, which will lead to Theorem 2.1.

Let us first fix some notation and standard results. For $i \in [[1, n]] := \{1, \ldots, n\}$, we define the following operators on $L^2(\mathbb{R}^n \times \mathbb{S}^1)$:

- $a_i = \frac{1}{\sqrt{2}}(x_i + \hbar \partial_{x_i})$,
- $a_i^* = \frac{1}{\sqrt{2}}(x_i - \hbar \partial_{x_i})$,
- $D_t = -i\hbar \partial_t$,
- $P_i := \frac{1}{2}\left(-\hbar \partial_{x_i}^2 + x_i^2 \right) = a_i^* a_i + \frac{\hbar}{2}$.

For $\mu \in \mathbb{N}^n$, $\nu \in \mathbb{Z}$ we will denote by $|\mu, \nu\rangle$ the common eigenvectors of $P_1, \ldots, P_n$ and D_t:

$$(2.5) \qquad P_i|\mu, \nu\rangle = (\mu_i + \frac{1}{2})\hbar|\mu, \nu\rangle \text{ and } D_t|\mu, \nu\rangle = 2\pi\hbar\nu|\mu, \nu\rangle.$$

These vectors are explicitly constructed as follows:

$$(2.6) \quad |0, 0\rangle(x, t) := \frac{1}{(\pi\hbar)^{\frac{n}{4}}} e^{\frac{-x^2}{2\hbar}}, \quad |\mu, \nu\rangle(x, t) := e^{i2\pi\nu t} \prod_{i=1}^{n} \frac{1}{\sqrt{\mu_i! \hbar^{|\mu|}}} a_i^{*\mu_i} |0, 0\rangle(x, t).$$

We will also need the notation

$$(2.7) \qquad\qquad\qquad |\mu\rangle(x) := |\mu, 0\rangle(x, 0).$$

We will not need the explicit expressions of $|\mu, \nu\rangle(x, t)$ and $|\mu\rangle(x)$ in terms of rescaled Hermite functions, but rather use the following identities:

$$(2.8) \quad \begin{cases} a_i|\mu, \nu\rangle = \sqrt{\mu_i \hbar}|\mu_1, \ldots, \mu_{i-1}, \mu_i - 1, \mu_{i+1}, \ldots, \mu_n, \nu\rangle, \\ a_i^*|\mu, \nu\rangle = \sqrt{(\mu_i + 1)\hbar}|\mu_1, \ldots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \ldots, \mu_n, \nu\rangle, \\ [a_i, a_j^*] = \delta_{ij}\hbar, \ [a_i, a_j] = 0. \end{cases}$$

We shall write $|\mu| := \sum_{i=1}^{n} \mu_i$, $z_i = \frac{x_i + i\xi_i}{\sqrt{2}}$, $p_i = \frac{x_i^2 + \xi_i^2}{2}$ and denote by $\mathrm{Op}^W(f)$ the pseudodifferential operator whose total Weyl symbol is f. We have

$$(2.9) \qquad \mathrm{Op}^W(z_i) = a_i, \ \mathrm{Op}^W(\bar{z}_i) = a_i^*, \ \mathrm{Op}^W(z_i \bar{z}_i) = P_i, \ \text{and} \ \mathrm{Op}^W(\tau) = D_t.$$

Finally, we will denote by a, a^* or P the n-tuple of operators a_i, a_i^*, P_i, $i \in [[1, n]]$, and denote for j the n-tuple of nonnegative integers $X^j = \prod_{i=1}^{n} X_i^{j_i}$.

2.1. Construction of the quantum Birkhoff normal form. Our construction of the normal form, inspired by [8], is the content of the following proposition.

Proposition 2.2. *Let $H(x, \hbar D_x)$ be a self-adjoint semiclassical elliptic pseudodifferential operator on $L^2(\mathbb{R}^n \times \mathbb{S}^1)$, whose principal symbol is*

$$(2.10) \qquad\qquad H_p(x, t, \xi, \tau) = H_0(p, \tau) + H_2,$$

where $H_0(p, \tau) = \sum_{i=1}^{n} \theta_i p_i + \tau$ and H_2 vanishes to the third order on $x = \xi = \tau = 0$.

Then for any $N \geq 3$, there exists a self-adjoint semiclassical elliptic pseudodifferential operator $\widetilde{W}_{\leq N}$ and a smooth function $h(p_1, \ldots, p_n, \tau, \hbar)$ satisfying microlocally in a neighborhood of $x = \xi = \tau = 0$ the following statement:

$$\forall M > 0, \ \exists C_N = C_N(M) > 0, \forall (\mu, \nu, \hbar) \in \mathbb{N}^n \times \mathbb{Z} \times [0, 1[, |\mu\hbar| + |\nu\hbar| < M,$$

$$(2.11) \quad \left\| \left(e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} - h(P_1, \ldots, P_n, D_t, \hbar) \right) |\mu, \nu\rangle \right\| \leq C_N (|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}}.$$

The operators $\widetilde{W}_{\leq N}$ can be computed recursively in the form

$$(2.12) \qquad\qquad \widetilde{W}_{\leq N} = W_{\leq N} + (D_t^2 + \sum_{i=1}^{n} P_i)^{N+1},$$

where

$$(2.13) \qquad \begin{cases} W_{\leq N} = \sum_{3 \leq q \leq N} W_q, \\ W_q := \sum_{2p + |j| + |k| + 2m = q} \alpha_{pjkm}(t) \hbar^p \mathrm{Op}^W(z^j \bar{z}^k) D_t^m \end{cases}$$

with α_{pjkm} smooth and W_q symmetric.

Remark 2.3 (Important convention). We are only interested in recovering the Hamiltonian in a formal neighborhood of γ: every asymptotic expansion is meant microlocally and we will be rewriting equations such as (2.11) simply as:
(2.14)
$$\left\| \left(e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} - h(P_1, \ldots, P_n, D_t, \hbar) \right) |\mu, \nu\rangle \right\| = O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}} \right).$$

By abuse of notation, we will identify the same way any operator with its version microlocalized near γ.

Remark 2.4. We introduce $\widetilde{W}_{\leq N}$ in order to gain ellipticity and self-adjointness as was done in Lemma 4.5 of [8].

The proof of Proposition 2.2 will need several preliminaries:

Definition 2.5. We will say that a pseudodifferential operator A on $L^2(\mathbb{R}^n \times \mathbb{S}^1)$ is "polynomial of order $r \in \mathbb{N}$" (PO(r)) if there exists $\alpha_{pjkm} \in C^\infty(\mathbb{S}^1, \mathbb{C})$ such that

$$(2.15) \qquad A = \sum_{2p+|j|+|k|+2m=r} \alpha_{pjkm}(t) \hbar^p \mathrm{Op}^W(z^j \bar{z}^k) D_t^m.$$

These operators have the following properties.

Proposition 2.6. *Let A be a pseudodifferential operator on $L^2(\mathbb{R}^n \times \mathbb{S}^1)$. Then there exists a family of operators A_r, $r \in \mathbb{N}$, such that for any $r \in \mathbb{N}$, A_r is PO(r) and*

$$(2.16) \qquad \forall N \in \mathbb{N}, \left\| \left(A - \sum_{r=0}^{N} A_r \right) |\mu, \nu\rangle \right\| = O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}} \right).$$

Let us define a notion of suitable asymptotic equivalence.

Definition 2.7. Let us introduce for any operator A the notation $\lfloor A \rfloor_r$ and $\lfloor A \rfloor_{\leq N}$ which represent respectively the terms of order r and of order smaller than or equal to N in the expansion (2.16).

If A and B are two operators, we will write $A \sim B$ if $\lfloor A \rfloor_r = \lfloor B \rfloor_r$ for any $r \in \mathbb{N}$. Also, if $(A_n)_{n \in \mathbb{N}}$ is a family of operators, we will write

$$(2.17) \qquad A \sim \sum_{n=0}^{+\infty} A_n$$

if, for any $N \in \mathbb{N}$, $\lfloor A_n \rfloor_{\leq N}$ is zero for n sufficiently large and the finite sum

$$(2.18) \qquad \sum_{n=0}^{+\infty} \lfloor A_n \rfloor_{\leq N} = \lfloor A \rfloor_{\leq N}.$$

Proof of Proposition 2.6. Let a be the total Weyl symbol of A. Let us define the family $(\alpha_{pjkm})_{(p,m,j,k) \in \mathbb{N}^2 \times (\mathbb{N}^n)^2}$ of functions on $\mathbb{S}^1$ by the Taylor expansion of a near $z = \bar{z} = \tau = \hbar = 0$, for any $N \in \mathbb{N}$:
$$(2.19)$$

$$a(z, t, \bar{z}, \tau, \hbar) = \sum_{r=0}^{N} \sum_{\substack{2p+|j|+|k| \\ +2m=r}} \alpha_{pjkm}(t) \hbar^p z^j \bar{z}^k \tau^m + \sum_{p=0}^{\frac{N+1}{2}} O\left(\hbar^p (|z|^2 + |\tau|)^{\frac{N+1}{2} - p} \right).$$

For any $r \in \mathbb{N}$, let us note that $(z, t, \bar{z}, \tau, \hbar) \mapsto \sum_{2p+|j|+|k|+2m=r} \alpha_{pjkm}(t) \hbar^p z^j \bar{z}^k \tau^m$ is the total symbol of a pseudodifferential operator A_r, which is PO(r). And by (2.5), (2.9), and (2.19) (see [8]):

$$(2.20) \qquad \forall N \in \mathbb{N}, \left\| \left(A - \sum_{r=0}^{N} A_r \right) |\mu, \nu\rangle \right\| = \sum_{p=0}^{\frac{N+1}{2}} \hbar^p O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2} - p} \right)$$

$$= O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}} \right).$$

This concludes the proof. $\qquad \square$

The following lemma will be crucial for our computations.

Lemma 2.8. *Let F and G be $PO(r)$ and $PO(r')$ respectively. Then $\frac{[F,G]}{i\hbar}$ is $PO(r + r' - 2)$.*

Proof. The proof of Lemma 2.8 will be a direct consequence of the two following lemmas, whose proofs will be given at the end of this proof.

Lemma 2.9. *Any monomial operator of order r, that is, of the form $\alpha(t)\hbar^p b_1 \ldots b_l D_t^m$, where:*

- *for $j \in [[1,l]]$, $b_j \in \{a_1, a_1^*, \ldots, a_n, a_n^*\}$,*
- *$2p + l + 2m = r$,*

is $PO(r)$.

Lemma 2.10. *If F and G are monomials of order r and r' respectively, then $\frac{[F,G]}{i\hbar}$ is $PO(r + r' - 2)$.*

Indeed, any $PO(r)$ is a finite sum of monomials of the same order, hence if F and G are $PO(r)$ and $PO(r')$ respectively, then $\frac{[F,G]}{i\hbar}$ is a finite sum of quantities of type $\frac{[\widetilde{F},\widetilde{G}]}{i\hbar}$, where $\widetilde{F}$ and $\widetilde{G}$ are monomials of order r and r' respectively. Any of those quantities are $PO(r + r' - 2)$ by Lemmas 2.9 and 2.10, and a finite sum of $PO(r + r' - 2)$ is $PO(r + r' - 2)$. Lemma 2.8 is proved. □

Let us now prove Lemmas 2.9 and 2.10.

Proof of Lemma 2.9. Since for any $i, j \in [[1,n]]$, $i \neq j$, a_i and a_i^* commute with both a_j and a_j^*, it is sufficient to prove that any ordered product $b_1 \ldots b_l$, where $l \geq 1$ and $b_j \in \{a_1, a_1^*\}$ for any $j \in [[1,l]]$, is $PO(r)$. For any such ordered product, let us introduce the integer $k(b_1 \ldots b_l) = \sharp\{m \in [[1,l]], b_m = a_1^*\}$.

We will proceed by induction on l. Let us define for any positive integer l the following assertion:

(A_l) "Any ordered product $b_1 \ldots b_l$, where $b_j \in \{a_1, a_1^*\}$ for any $j \in [[1,l]]$, is the sum of the operator $\mathrm{Op}^W(z_1^{l-k}\bar{z}_1^k)$ (where $k = k(b_1 \ldots b_l)$) and of a linear combination of the operators $\hbar^p \mathrm{Op}^W(z_1^j \bar{z}_1^m)$ with $p \geq 1$, $2p + j + m = l$, and $j - m = l - 2k$."

If $l = 1$, there is nothing to prove since $a_1 = \mathrm{Op}^W(z_1)$ and $a_1^* = \mathrm{Op}^W(\bar{z}_1)$. If $l = 2$,

$$\begin{cases} a_1^2 = \mathrm{Op}^W(z_1^2), \\ a_1^{*2} = \mathrm{Op}^W(\bar{z}_1^2), \\ a_1 a_1^* = P_1 + \frac{\hbar}{2} = \mathrm{Op}^W(z_1 \bar{z}_1) + \frac{\hbar}{2}, \\ a_1^* a_1 = \mathrm{Op}^W(z_1 \bar{z}_1) - \frac{\hbar}{2}, \end{cases}$$

and therefore the assertion is proved for $l = 2$.

Now, let l be a positive integer, and let us assume (A_k) up to order $k = l$. Let $B = b_1 \ldots b_{l+1}$ be an ordered product, where $b_j \in \{a_1, a_1^*\}$ for any $j \in [[1, l+1]]$. If $b_j = b_{j+1}$ for any $j \in [[1,l]]$, then $B = \mathrm{Op}^W(z_1^{l+1})$ or $B = \mathrm{Op}^W(\bar{z}_1^{l+1})$. Otherwise, one can assume that $b_1 = a_1$, and that $j_0 = \max\{j \in [[1, l+1]], b_j = a_1\}$ satisfies $1 \leq j_0 \leq l$. Then, we have $[a_1^{j_0}, a_1^*] = j_0 \hbar a_1^{j_0 - 1}$, so that:
(2.21)
$$b_1 \ldots b_{l+1} = a_1^{j_0} a_1^* b_{j_0+2} \ldots b_{l+1} = a_1^* a_1^{j_0} b_{j_0+2} \ldots b_{l+1} + \hbar j_0 a_1^{j_0 - 1} b_{j_0+2} \ldots b_{l+1}.$$

Therefore, if one sets $k := k(b_1 \ldots b_{l+1})$, since $\binom{l+1}{k} = \binom{l}{k} + \binom{l}{k-1}$:

$$
(2.22) \quad \binom{l+1}{k} b_1 \ldots b_{l+1} = \binom{l}{k} a_1^{j_0} a_1^* b_{j_0+2} \ldots b_{l+1} + \binom{l}{k-1} a_1^* a_1^{j_0} b_{j_0+2} \ldots b_{l+1}
$$
$$
+ \hbar \binom{l}{k-1} j_0 a_1^{j_0-1} b_{j_0+2} \ldots b_{l+1}.
$$

(A_{l-1}) gives us that $a_1^{j_0-1} b_{j_0+2} \ldots b_{l+1}$ is a linear combination of the operators $\hbar^p \mathrm{Op}^W(z_1^j \bar{z}_1^m)$ with $2p + j + m = l - 1$ and $j - m = l + 1 - 2k$.

Let us now observe that $\binom{l+1}{k} \mathrm{Op}^W(z^{l+1-k} \bar{z}^k)$ is a sum of $\binom{l+1}{k}$ ordered monomials, which we divide into two parts: the $\binom{l}{k}$ ordered monomials whose first term is a_1, whose sum forms precisely $\binom{l}{k} a_1 \mathrm{Op}^W(z^{l-k} \bar{z}^k)$, and the $\binom{l}{k-1}$ others, whose sum forms $\binom{l}{k-1} a_1^* \mathrm{Op}^W(z^{l+1-k} \bar{z}^{k-1})$. More precisely:

$$
(2.23)
$$
$$
\binom{l+1}{k} \mathrm{Op}^W(z^{l+1-k} \bar{z}^k) = \binom{l}{k} a_1 \mathrm{Op}^W(z^{l-k} \bar{z}^k) + \binom{l}{k-1} a_1^* \mathrm{Op}^W(z^{l+1-k} \bar{z}^{k-1})
$$

so that (A_l), for ordered products $a_1^{j_0-1} a_1^* b_{j_0+2} \ldots b_{l+1}$ and $a_1^{j_0} b_{j_0+2} \ldots b_{l+1}$, gives us, by equation that $\binom{l+1}{k} b_1 \ldots b_{l+1}$ is the sum of

$$
\binom{l}{k} a_1 \mathrm{Op}^W(z^{l-k} \bar{z}^k) + \binom{l}{k-1} a_1^* \mathrm{Op}^W(z^{l+1-k} \bar{z}^{k-1}) = \binom{l+1}{k} \mathrm{Op}^W(z^{l+1-k} \bar{z}^k)
$$

and a linear combination of the operators $\hbar^p \mathrm{Op}^W(z_1^j \bar{z}_1^m)$ with $p \geq 1$, $2p + j + m = l + 1$, and $j - m = l + 1 - 2k$ $\qquad \square$

Proof of Lemma 2.10. It is sufficient to remark that if F and G are of the form:

$$
F = \alpha(t) b_1 \ldots b_l D_t^m \text{ and } G = \beta(t) b_1' \ldots b_{l'}' D_t^{m'},
$$

where:

- α and β are smooth,
- $l + 2m = r$, $l' + 2m' = r'$,
- for $j \in [[1, l]]$, for $j' \in [[1, l']]$, $b_j, b_{j'}' \in \{a_1, a_1^*\}$,

then $\frac{[F,G]}{i\hbar}$ is a finite sum of monomials of order $r + r' - 2$ since, by Lemma 2.9, each of them is $\mathrm{PO}(r + r' - 2)$. With those assumptions on F and G, we get:

$$
(2.24)
$$
$$
\frac{[F,G]}{i\hbar} = \frac{[\alpha(t) b_1 \ldots b_l D_t^m, \beta(t) b_1' \ldots b_{l'}' D_t^{m'}]}{i\hbar}
$$
$$
= \alpha(t)\beta(t) \frac{[b_1 \ldots b_l, b_1' \ldots b_{l'}']}{i\hbar} D_t^{m+m'} + \alpha(t) b_1 \ldots b_l \frac{[D_t^m, \beta(t)]}{i\hbar} b_1' \ldots b_{l'}' D_t^{m'}
$$
$$
- \beta(t) b_1' \ldots b_{l'}' \frac{[D_t^{m'}, \alpha(t)]}{i\hbar} b_1 \ldots b_l D_t^m.
$$

Therefore it is sufficient to prove that $\frac{[b_1 \ldots b_l, b_1' \ldots b_{l'}']}{i\hbar}$, $\frac{[D_t^m, \beta(t)]}{i\hbar}$, and $\frac{[D_t^{m'}, \alpha(t)]}{i\hbar}$ are respectively: $\mathrm{PO}(l + l' - 2)$, $\mathrm{PO}(2m - 2)$, and $\mathrm{PO}(2m' - 2)$ (with the convention that a $\mathrm{PO}(j)$ with $j < 0$ is 0). For the two last, it is quite obvious, since:

$$
(2.25) \qquad \frac{[D_t^m, \beta(t)]}{i\hbar} = \sum_{k=0}^{m-1} \binom{m}{k} (i\hbar)^{m-k-1} \beta^{(m-k)}(t) D_t^k.
$$

Now, for $j \in [[1, l']]$, let us set $\epsilon_j = 1$ if $b'_j = a_1^*$ and $\epsilon_j = -1$ otherwise. Since $[a_1, a_1^*] = \hbar$, we get

$$
b_1 \ldots b_l b'_1 \ldots b'_{l'} = b'_1 b_1 \ldots b_l b'_2 \ldots b'_{l'} + \frac{\epsilon_1 + 1}{2} \hbar \sum_{\substack{k=1 \\ b_k = a_1}}^{l} b_1 \ldots b_{k-1} b_{k+1} \ldots b_l b'_2 \ldots b'_{l'}
$$

$$
+ \frac{\epsilon_1 - 1}{2} \hbar \sum_{\substack{j=1 \\ b_k = a_1^*}}^{l} b_1 \ldots b_{k-1} b_{k+1} \ldots b_l b'_2 \ldots b'_{l'}.
$$

Hence by induction on $j \in [[1, l']]$:

(2.26)

$$
\frac{[b_1 \ldots b_l, b'_1 \ldots b'_{l'}]}{i\hbar} = -i \sum_{j=1}^{l'} \frac{\epsilon_j + 1}{2} \sum_{\substack{k=1 \\ b_k = a_1}}^{l} b'_1 \ldots b'_{j-1} b_1 \ldots b_{k-1} b_{k+1} \ldots b_l b'_{j+1} \ldots b'_{l'}
$$

$$
- i \sum_{j=1}^{l'} \frac{\epsilon_j - 1}{2} \sum_{\substack{k=1 \\ b_k = a_1^*}}^{l} b'_1 \ldots b'_{j-1} b_1 \ldots b_{k-1} b_{k+1} \ldots b_l b'_{j+1} \ldots b'_{l'}.
$$

The right-hand side of (2.26) is a finite sum of monomials of order $l + l' - 2$, hence it is $PO(l + l' - 2)$ by Lemma 2.9, and Lemma 2.10 is proved. □

Proposition 2.11. *Let G be $PO(r)$. Then there exist F, $PO(r)$, and $G_1 = G_1(P_1, \ldots, P_n, D_t, \hbar)$ such that:*

(2.27)
$$
\frac{[H_0(P, D_t), F]}{i\hbar} = G + G_1.
$$

Moreover, F is symmetric if G is symmetric, $G_1 = 0$ if r is odd, and G_1 is a homogeneous polynomial function of total order $\frac{r}{2}$ if r is even.

Remark 2.12. If $F = \sum_{2p+|j|+|k|+2m=r} \alpha_{pjkm}(t) \hbar^p \mathrm{Op}^W(z^j \bar{z}^k) D_t^m$, one can choose

(2.28)
$$
\int_{\mathbb{S}^1} \alpha_{pjjm}(t) dt = 0.
$$

Indeed, any $\mathrm{Op}^W(z^j \bar{z}^j) D_t^m$ commutes with $H_0(P, D_t, \hbar)$. It is the choice we will make through this article.

Proof of Proposition 2.11. Let us first assume that G is a monomial of order r: $G = \beta(t) b_1 \ldots b_l D_t^m$, where:

- α is smooth,
- $l + 2m = r$,
- for $j \in [[1, l]]$, $b_j \in \{a_1, a_1^*, \ldots, a_n, a_n^*\}$,

and let us look for F of the form: $F = \alpha(t)b_1 \ldots b_l D_t^m$. We have

$$(2.29) \quad \begin{aligned} \frac{[H_0, F]}{i\hbar} &= \frac{[H_0, \alpha(t)b_1 \ldots b_l D_t^m]}{i\hbar} \\ &= \alpha(t) \sum_{s=1}^{n} \theta_s \frac{[P_s, b_1 \ldots b_l]}{i\hbar} D_t^m + \frac{[D_t, \alpha(t)]}{i\hbar} b_1 \ldots b_l D_t^m \\ &= \alpha(t) \sum_{s=1}^{n} \theta_s \frac{[P_s, b_1 \ldots b_l]}{i\hbar} D_t^m + \alpha'(t)b_1 \ldots b_l D_t^m. \end{aligned}$$

If we set $k_s = \sharp\{m \in [[1, l]], b_m = a_s^*\}$ and $j_s = \sharp\{m \in [[1, l]], b_m = a_s\}$ for $s \in [[1, n]]$, we deduce from (2.26) that

$$(2.30) \quad \frac{[P_s, b_1 \ldots b_l]}{i\hbar} = i(j_s - k_s)b_1 \ldots b_l.$$

Hence,

$$(2.31) \quad \frac{[H_0, F]}{i\hbar} = i \sum_{s=1}^{n} \theta_s(j_s - k_s)\alpha(t)b_1 \ldots b_l D_t^m + \alpha'(t)b_1 \ldots b_l D_t^m,$$

where $\frac{[H_0, F]}{i\hbar} = G$ admits a solution if there exists α such that

$$(2.32) \quad i \sum_{s=1}^{n} \theta_s(j_s - k_s)\alpha(t) + \alpha'(t) = \beta(t).$$

If $(c_p(\alpha))_{p\in\mathbb{Z}}$ and $(c_p(\beta))_{p\in\mathbb{Z}}$ are the Fourier coefficients of α and β, it is sufficient that, for $p \in \mathbb{Z}$, $c_p(\alpha)$ is a solution of:

$$(2.33) \quad i\left(\sum_{s=1}^{n} \theta_s(j_s - k_s) + 2\pi p\right) c_p(\alpha) = c_p(\beta)$$

and

$$(2.34) \quad c_p(\alpha) \underset{p \to +\infty}{=} O\left(\frac{1}{|p|^{\infty}}\right).$$

If the n-tuples j and k are different, the non-degeneracy condition on the θ_i's together with the fact that $c_p(\beta) \underset{p \to +\infty}{=} O\left(\frac{1}{|p|^{\infty}}\right)$ (because β is smooth) gives the existence of $c_p(\alpha)$, satisfying (2.33) and (2.34).

If r is odd, j and k cannot be equal, hence Proposition 2.11 is proved in this case (r odd and G monomial).

If r is even and $j = k$, there exists a family $(c_p(\alpha))_{p\in\mathbb{Z}^*}$ satisfying (2.33) and (2.34). Hence, if α is the smooth function with Fourier coefficients $c_p(\alpha)$ for $p \neq 0$ and $c_0(\alpha) = 0$, we get

$$(2.35) \quad \frac{[H_0, F]}{i\hbar} = G + c_0(\beta)b_1 \ldots b_l D_t^m.$$

From the proof of Lemma 2.9, we know that $c_0(\beta)b_1 \ldots b_l D_t^m$ can be reordered as the sum: $G_1(P, D_t, \hbar) := c_0(\beta) \sum_{2p+2|k|=l} a_{p,k}\hbar^p P^k D_t^m$. Therefore, Proposition 2.11 is proved in the case where r is even and G is monomial.

The general case is easily deduced from the case where G is monomial, since G is a finite sum of monomials of the same order. Also, the form of F allows us to conclude immediately that F is symmetric if G is so. $\qquad\square$

Now we have everything we need for the proof by induction of Proposition 2.2.

Proof of Proposition 2.2. Microlocally near $x = \xi = \tau = 0$, $H(x, \hbar D_x)$ satisfies, in the sense of Definition 2.7,

$$(2.36) \quad H := H(x, \hbar D_x) \sim H_0(P_1, \ldots, P_n, D_t) + \sum_{q \geq 3} H_q, \quad H_q := \lfloor H(x, \hbar D_x) \rfloor_q.$$

Let us set $W_{\leq 2} = 0$ and construct $(W_q)_{q \geq 3}$ and $(H^q)_{q \geq 3}$ by induction such that:
- for $q \geq 3$, W_q is PO(q) and H^q is zero if q is odd, a homogeneous polynomial function of total order $\frac{q}{2}$ if q is even, and
- for any $q \geq 3$:

$$\frac{i}{\hbar}[W_q, H_0] + H_q + \left\lfloor \frac{i}{\hbar}[W_{\leq q-1}, H - H_0] + \sum_{l \geq 2} \frac{i^l}{\hbar^l l!}[\overbrace{W_{\leq q-1}, \ldots, W_{\leq q-1}}^{l \text{ times}}, H] \right\rfloor_q$$

$$= H^q(P, D_t, \hbar).$$

The existence of such a family is guaranteed by Proposition 2.11.

Let us set, for any $N \geq 3$, $\widetilde{W}_{\leq N} := \sum_{q=3}^{N} W_q + (|D_t|^2 + \sum_{i=1}^{n} P_i)^{\frac{N+1}{2}}$. As H^{2q} is a homogeneous polynomial function of total order q for any $q \geq 2$, we can choose, by Borel's lemma, a smooth function h such that, for any $N \geq 2$ and in a neighborhood of $p = \tau = 0$:

$$(2.37) \qquad \left| h(p, \tau, \hbar) - H_0(p, \tau) - \sum_{q=2}^{N} H^{2q}(p, \tau, \hbar) \right| = O\left((|p| + |\tau| + |\hbar|)^{N+1} \right).$$

We have, for any $N \geq 3$:

$$e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \sim H + \frac{i}{\hbar}[\widetilde{W}_{\leq N}, H] + \sum_{l \geq 2} \frac{i^l}{\hbar^l l!}[\overbrace{\widetilde{W}_{\leq N}, \ldots, \widetilde{W}_{\leq N}}^{l \text{ times}}, H]$$

$$\sim H + \frac{i}{\hbar}[W_{\leq N}, H_0] + \frac{i}{\hbar}[W_{\leq N}, H - H_0]$$

$$+ \sum_{l \geq 2} \frac{i^l}{\hbar^l l!}[\overbrace{\widetilde{W}_{\leq N}, \ldots, \widetilde{W}_{\leq N}}^{l \text{ times}}, H] + \frac{i}{\hbar}[\widetilde{W}_{\leq N} - W_{\leq N}, H].$$

Since W_q is PO(q) and H_0 is PO(2) for any $q \leq N$, Lemma 2.8 gives us that

$$(2.38) \qquad\qquad\qquad \left\lfloor \frac{i}{\hbar}[W_{\leq N}, H_0] \right\rfloor_q = \frac{i}{\hbar}[W_q, H_0].$$

Since the expansion of $H - H_0$ in PO(r) contains no term of order less than or equal to 2, Lemma 2.8 also gives for $q \leq N$:

$$(2.39) \qquad\qquad\qquad \left\lfloor \frac{i}{\hbar}[W_{\leq N}, H - H_0] \right\rfloor_q = \lfloor \frac{i}{\hbar}[W_{\leq q-1}, H - H_0] \rfloor_q.$$

Lemma 2.8 finally gives us that since the expansion of $H(x, \hbar D_x)$ in PO(r) contains no term of order less than or equal to 1 and the expansion of $\widetilde{W}_{\leq N}$ contains no

term of order less than or equal to 2 for $q \leq N$:

$$(2.40) \qquad \left\lfloor \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq N}, \ldots, \widetilde{W}_{\leq N}}^{l \text{ times}}, H] \right\rfloor_q = \left\lfloor \sum_{l \geq 2} \frac{i^l}{\hbar^l l!} [\overbrace{W_{\leq q-1}, \ldots, W_{\leq q-1}}^{l \text{ times}}, H] \right\rfloor_q,$$

and since the expansion of $\widetilde{W}_{\leq N} - W_{\leq N}$ contains no term of order less than or equal to $N + 1$:

$$(2.41) \qquad \lfloor \frac{i}{\hbar} [\widetilde{W}_{\leq N} - W_{\leq N}, H] \rfloor_q = 0.$$

Therefore for any $q \leq N$:

$$(2.42) \qquad \left\lfloor e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \right\rfloor_q = H^q(P, D_t, \hbar) = \lfloor h(P, D_t, \hbar) \rfloor_q.$$

Finally Proposition 2.6 gives us:

$$(2.43) \qquad \left\| \left(e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} - h(P, D_t, \hbar) \right) |\mu, \nu\rangle \right\| = O\left(|\mu\hbar| + |\nu\hbar| \right)^{\frac{N+1}{2}},$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

2.2. **Recovering the matrix elements from the trace formula.** The next result is the first inverse result needed for the proof of Theorem 2.1.

Proposition 2.13. *Let O be a pseudodifferential operator whose principal symbol vanishes on γ.*

(1) *There exists a smooth function f vanishing at $(0,0,0)$ such that for any $N \geq 3$:*

$$(2.44) \quad \langle \mu, \nu | e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} | \mu, \nu \rangle = f\left((\mu + \frac{1}{2})\hbar, 2\pi\nu\hbar, \hbar \right) + O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N}{2}} \right).$$

Moreover let ϕ_l be a Schwartz function for any integer l whose Fourier transform is compactly supported in $(l-1, l+1)$, and let $(a_j^l(O))_{l \geq 0}$ be provided by the trace formula (1.4). Then

(2) *The Taylor expansion of f up to order N is entirely determined by the family $(a_j^l(O))$, $0 \leq j \leq N$, $l \in \mathbb{N}$.*

Proof. Let us first prove point (1). Let us consider a monomial $G = \alpha(t)b_1 \ldots b_l D_t^m$, where:

- α is smooth,
- $l + 2m = r$,
- for $j \in [[1, l]]$, $b_j \in \{a_1, a_1^*, \ldots, a_n, a_n^*\}$.

Let us set $k_i = \sharp\{m \in [[1, l]], b_m = a_i^*\}$ and $j_i = \sharp\{m \in [[1, l]], b_m = a_i\}$ for $i \in [[1, n]]$.

If $j \neq k$ or $\alpha \notin \mathbb{C}$, then $\langle \mu, \nu | G | \mu, \nu \rangle = 0$ for any $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{Z}$. If now $j = k$ and $\alpha \in \mathbb{C}$, then there exist complex numbers α_l $(0 \leq l_i \leq j_i$ for $i \in [[1, n]])$, such that:

$$(2.45) \qquad G = \sum_{0 \leq l_i \leq j_i} \alpha_l \hbar^{|l|} P_1^{j_1 - l_1} \ldots P_n^{j_n - l_n} D_t^m, \quad \alpha_0 = \alpha.$$

Therefore for any $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{Z}$:

$$(2.46) \qquad \langle \mu, \nu | G | \mu, \nu \rangle = \sum_{0 \leq l_i \leq j_i} \alpha_l \hbar^{|l|} \left(\left(\mu + \frac{1}{2} \right) \hbar \right)^{j-l} (2\pi\nu\hbar)^m.$$

Hence, if G is PO(r), then for any $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{Z}$:

- $\langle \mu, \nu | G | \mu, \nu \rangle = 0$ if r is odd.
- If r is even, there exists a homogeneous polynomial function g of order $\frac{r}{2}$ such that

$$(2.47) \qquad \langle \mu, \nu | G | \mu, \nu \rangle = g \left((\mu + \frac{1}{2})\hbar, 2\pi\nu\hbar, \hbar \right).$$

By Proposition 2.6 and Borel's lemma, we get that for any operator A there exists a function g such that for any $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{Z}$:

$$(2.48) \qquad \langle \mu, \nu | A | \mu, \nu \rangle = g \left((\mu + \frac{1}{2})\hbar, 2\pi\nu\hbar, \hbar \right) + O \left((|\mu\hbar| + |\nu\hbar|)^\infty \right).$$

Hence, the only point which remains to be proved is that the function f in point (1) does not depend on N. It is therefore sufficient to prove that for any $q \leq N-1$,

$$(2.49) \qquad \left\lfloor e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \right\rfloor_q = \left\lfloor e^{\frac{i\widetilde{W}_{\leq q+1}}{\hbar}} O e^{\frac{-i\widetilde{W}_{\leq q+1}}{\hbar}} \right\rfloor_q.$$

But (2.49) is a direct consequence of Lemma 2.8. Indeed,

$$(2.50) \qquad e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \sim O + \sum_{l \geq 1} \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq N}, \ldots, \widetilde{W}_{\leq N}}^{l \text{ times}}, O]$$

and since the principal symbol of O vanishes on γ, Lemma 2.8 gives us for any $l \geq 1$ and any $q \leq N-1$:

$$(2.51) \qquad \left\lfloor \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq N}, \ldots, \widetilde{W}_{\leq N}}^{l \text{ times}}, O] \right\rfloor_q = \left\lfloor \frac{i^l}{\hbar^l l!} [\overbrace{\widetilde{W}_{\leq q+1}, \ldots, \widetilde{W}_{\leq q+1}}^{l \text{ times}}, O] \right\rfloor_q.$$

Let us now move on to the proof of point (2). Since $\hat{\phi}_l$ is supported near a single period of the flow, one can microlocalize the trace formula with observables near γ:

$$(2.52)$$
$$2\pi \text{Tr} \left(O\phi_l \left(\frac{H-E}{\hbar} \right) \right) = \text{Tr} \left(O \int_{\mathbb{R}} \hat{\phi}_l(t) \rho(P_1 + \cdots + P_n + |\zeta|) e^{it\frac{H-E}{\hbar}} dt \right) + O(\hbar^\infty),$$

where $\rho \in C_0^\infty(\mathbb{R})$ is compactly supported and $\rho = 1$ in a neighborhood of $p = \tau = 0$. Therefore we can conjugate (2.52) by the microlocally unitary operator $e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}}$:

$$2\pi \text{Tr} \left(O\phi_l \left(\frac{H-E}{\hbar} \right) \right)$$
$$= \text{Tr} \left(e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} O e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}} \int_{\mathbb{R}} \hat{\phi}_l(t) \rho(P_1 + \cdots + P_n + |\zeta|) e^{it\frac{e^{\frac{i\widetilde{W}_{\leq N}}{\hbar}} H e^{\frac{-i\widetilde{W}_{\leq N}}{\hbar}}}{\hbar} - E} dt \right)$$
$$+ O(\hbar^\infty).$$

Thanks to Proposition 2.2, we can lighten the r.h.s. for any $(\mu, \nu) \in \mathbb{N}^n \times \mathbb{Z}$:

(2.53)

$$\int_{\mathbb{R}} \hat{\phi}_l(t) \rho(P_1 + \cdots + P_n + |\zeta|) e^{it \frac{e^{\frac{i \widetilde{W}_{\leq N}}{\hbar}} H e^{\frac{-i \widetilde{W}_{\leq N}}{\hbar}}}{\hbar} - E} dt |\mu, \nu\rangle$$
$$= \left(\int_{\mathbb{R}} \hat{\phi}_l(t) \rho \left((|\mu| + \frac{n}{2} + |2\pi\nu|) \hbar \right) e^{it \frac{h((\mu + \frac{1}{2})\hbar, \nu\hbar, \hbar) - E + O(|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}}}{\hbar}} dt \right) |\mu, \nu\rangle.$$

As $\hat{\phi}_l$ is smooth and compactly supported, together with the non-degeneracy condition on the θ_i's, we can assure that if we choose a sufficiently small support for ρ, we have for any $\eta > 0$:

$$\left(\int_{\mathbb{R}} \hat{\phi}_l(t) \rho \left((|\mu| + \frac{n}{2} + |2\pi\nu|) \hbar \right) e^{it \frac{h((\mu + \frac{1}{2})\hbar, \nu\hbar, \hbar) - E + O(|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}}}{\hbar}} dt \right) |\mu, \nu\rangle$$
$$= \left(\int_{\mathbb{R}} \hat{\phi}_l(t) \rho \left((|\mu| + \frac{n}{2} + |2\pi\nu|) \hbar^\eta \right) e^{it \frac{h((\mu + \frac{1}{2})\hbar, \nu\hbar, \hbar) - E + O(|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}}}{\hbar}} dt \right) |\mu, \nu\rangle$$
$$+ O(\hbar^\infty).$$

Hence, choosing $\eta < \frac{1}{2}$:

$$2\pi \text{Tr} \left(O \phi_l \left(\frac{H - E}{\hbar} \right) \right) + O(\hbar^\infty)$$
$$= \sum_{\mu, \nu} \langle \mu, \nu | e^{\frac{i \widetilde{W}_{\leq N}}{\hbar}} O e^{\frac{-i \widetilde{W}_{\leq N}}{\hbar}} |\mu, \nu\rangle \times \int_{\mathbb{R}} \hat{\phi}_l(t) \rho \left((|\mu| + \frac{n}{2} + |\nu|) \hbar^\eta \right) e^{it(2\pi\nu + \theta.(\mu + \frac{1}{2}))}$$
$$\dots \exp \left(\frac{it}{\hbar} \sum_{1 \leq q \leq N-2} H^q \left((\mu + \frac{1}{2})\hbar, \nu\hbar, \hbar \right) + O \left((|\mu| + |\nu|)^{\frac{N+1}{2}} \hbar^{\frac{N-1}{2}} \right) \right) dt$$
$$= \sum_{\mu, \nu} \int_{\mathbb{R}} \hat{\phi}_l(t) \rho \left((|\mu| + \frac{n}{2} + |2\pi\nu|) \hbar^\eta \right) e^{it(2\pi\nu + \theta.(\mu + \frac{1}{2}) + \frac{H^2(0,0,\hbar)}{\hbar})}$$
$$\times \left(1 + \sum_{i \geq 1}^{\frac{N-1}{2}} \hbar^i Q_i(\mu + \frac{1}{2}, \nu, t) \right)$$
$$\times \sum_{p \geq 1}^{\frac{N+1}{2}} \sum_{|k|+m \leq p} b_{k,m,p-|k|-m}(\mu + \frac{1}{2})^k (2\pi\nu)^m \hbar^p dt + O(\hbar^{\frac{N+1}{2}}),$$

where for any $i \leq \frac{N-1}{2}$, Q_i is a determined polynomial function of degree in $(\mu + \frac{1}{2}, \nu)$ less than or equal to $i + 1$, which depends on the H^q's and the Taylor expansion of exp, and the $b_{k,m,s}$ $((k, m, s) \in \mathbb{N}^{n+2} \setminus \{0\})$ come from the Taylor expansion at $(0, 0, 0)$ of the function f defined in the first point of Proposition 2.13, *i.e.* for any $N \geq 1$:

(2.54) $$f(x, y, z) = \sum_{1 \leq |k|+m+s \leq N} b_{k,m,s} x^k y^m z^s + O \left((|x| + |y| + |z|)^{N+1} \right).$$

Now, let us set:

$$(2.55) \qquad \forall t \in \mathbb{R}^*, \forall \alpha \in (\mathbb{R}\backslash \tfrac{2\pi}{t}\mathbb{Z})^n, g(t,\alpha) := \frac{e^{i\frac{t}{2}(\alpha_1+\cdots+\alpha_n)}}{\prod\limits_{i=1}^{n}\left(1 - e^{it\alpha_i}\right)}.$$

By the non-degeneracy condition on the θ_i's, g is well defined on the compact support of $\hat{\phi}_l$ around a single period, which is precisely l.

Therefore we get from the Poisson formula and the Riemann-Lebesgue lemma that the quantity $X_p(l)$ below can be computed recursively on $p \leq \frac{N+1}{2}$ from the $a_j^l(O), j = 0, \dots, p$:

$$(2.56) \qquad \begin{aligned} X_p(l) &= \sum_{|k|+m\leq p} b_{k,m,p-|k|-m} \left[\left(-i\frac{\partial}{\partial t}\right)^m \left(\hat{\phi}_l(t)\left(\frac{-i}{t}\right)^k \frac{\partial^k g}{\partial \alpha^k}(t,\alpha)\right)\right](l,\theta) \\ &= \sum_{|k|+m\leq p} b_{k,m,p-|k|-m} \left[\left(-i\frac{\partial}{\partial t}\right)^m \left(-i\frac{\partial}{t\partial \alpha}\right)^k g\right](l,\theta) \end{aligned}$$

since $\hat{\phi}_l$ is identically 1 around l.

Now, let us set $x_i(t,\alpha) = e^{i\frac{t\alpha_i}{2}}$ for any $i \in [[1,n]]$, any $t \in \mathbb{R}$, and any $\alpha \in (\mathbb{R}\backslash \tfrac{2\pi}{t}\mathbb{Z})^n$, and also define the holomorphic function h on $\mathbb{C}\backslash\{-1,1\}$ by $h(z) = \frac{z}{1-z^2}$ for $z \in \mathbb{C}\backslash\{-1,1\}$. We have for any $k \in \mathbb{N}^n$:

$$(2.57) \qquad \left(-i\frac{\partial}{t\partial \alpha}\right)^k g = \prod_{i=1}^{n}\left(-i\frac{\partial}{t\partial \alpha_i}\right)^{k_i}(h \circ x_i).$$

For any $i \in [[1,n]]$, an easy induction on $k_i \in \mathbb{N}$ leads to the following, since $h(z) = \frac{1}{2}\left(\frac{1}{1-z} - \frac{1}{1+z}\right)$ for any $z \in \mathbb{C}\backslash\{-1,1\}$ and $-i\frac{\partial x_i}{t\partial \alpha_i} = \frac{1}{2}x_i$:

$$(2.58) \qquad \left(-i\frac{\partial}{t\partial \alpha_i}\right)^{k_i}(h \circ x_i) = \frac{k_i!}{2^{k_i+1}}\left(\frac{x_i}{(1-x_i)^{k_i+1}} + \frac{x_i}{(1+x_i)^{k_i+1}}\right).$$

Now, since $-i\frac{\partial x_i}{\partial t} = \frac{\alpha_i}{2}x_i$, an induction on $s_i \in \mathbb{N}$ shows that

$$(2.59) \qquad \begin{aligned} &\left(-i\frac{\partial}{\partial t}\right)^{s_i}\left(-i\frac{\partial}{t\partial \alpha_i}\right)^{k_i}(h \circ x_i) \\ &= \frac{(k_i+s_i)!\alpha_i^{s_i}}{2^{k_i+s_i+1}}\left(\frac{x_i}{(1-x_i)^{k_i+s_i+1}} + \frac{x_i}{(1+x_i)^{k_i+s_i+1}}\right). \end{aligned}$$

Let us now introduce for any n-tuple s such that $|s| = m$, the multinomial coefficient

$$\binom{m}{s} = \frac{m!}{s_1!\dots s_n!}.$$

We have

$$(2.60) \qquad \left(-i\frac{\partial}{\partial t}\right)^m\left(-i\frac{\partial}{t\partial \alpha}\right)^k g = \sum_{|s|=m}\binom{m}{s}\prod_{i=1}^{n}\left(-i\frac{\partial}{\partial t}\right)^{s_i}\left(-i\frac{\partial}{t\partial \alpha_i}\right)^{k_i}(h \circ x_i).$$

Let us use Kronecker's theorem, whose hypothesis is precisely the non-degeneracy condition on the θ_i's: for any n-tuple $(x_1, \dots, x_n) \in \mathbb{S}_1^n$, one can find a sequence of integers $(l_p)_{p\in\mathbb{Z}}$, such that:

$$\forall j \in [[1,n]], \ x_j(l_p,\theta) \underset{p\to+\infty}{\longrightarrow} x_j.$$

Therefore, setting

$$(2.61) \quad u^{(k,m)} = \sum_{|s|=m} \binom{m}{s} \prod_{i=1}^{n} \frac{(k_i+s_i)!\theta_i^{s_i}}{2^{k_i+s_i+1}} \left(\frac{x_i}{(1-x_i)^{k_i+s_i+1}} + \frac{x_i}{(1+x_i)^{k_i+s_i+1}} \right)$$

for any $(x_1,\ldots,x_n) \in (\mathbb{S}_1\backslash\{-1,1\})^n$ and $(k,m) \in \mathbb{N}^{n+1}$, we have that (2.56), (2.59), and (2.60) together with Kronecker's theorem allows us to conclude that $X_p := \sum_{|k|+m \le p} b_{k,m,p-|k|-m} u^{(k,m)}$ is determined by the $a_j^l(O)$, $j = 0,\ldots,p$.

Hence, the only thing which remains to be proved is that, if one chooses x_i tending to 1 in a way convenient to us, the $|u^{(k,m)}|$'s will be neglectable in comparison to each other. More precisely, let x_i tend to 1 in such a way that

$$(2.62) \qquad \forall i \in [[1,n-1]], |1-x_i| = o\left(|1-x_{i+1}|^p\right).$$

We have that $s_1 = m$ gives the leading order in (2.61) and therefore

$$(2.63) \qquad (1-x_1)^m u^{(k,m)} \sim C\prod_{i=1}^{n} \frac{1}{(1-x_i)^{k_i+1}}$$

for some $C > 0$. Hence, if one sets $\widetilde{m} = (m,0,\ldots,0)$, then

$$(2.64) \qquad u^{(k,m)} = o\left(u^{(k',m')}\right) \text{ if } k + \widetilde{m} < k' + \widetilde{m'},$$

where $<$ is the lexicographical order on $\mathbb{N}^n$. Therefore, for any $p \in \mathbb{N}$ and $(k,m) \in \mathbb{N}^{n+1}$ such that $|k_0| + m_0 \le p$, the following quantity can be recursively determined from X_p:

$$(2.65) \qquad X_{k_0,m_0} = \sum_{k'+\widetilde{m'}=k+\widetilde{m}} b_{k,m,p-|k|-m} u^{(k,m)}.$$

Reversing for example the roles of $i = 1$ and $i = 2$ in (2.62), and observing that $k_2 + m \ne k_2' + m'$ if $k + \widetilde{m} = k' + \widetilde{m'}$ and $(k,m) \ne (k',m')$, one determines $b_{k,m,p-|k|-m}$ from (2.65) recursively on m. Finally, each $b_{k,m,s}$ with $|k|+m+s \le N$ is determined by the $a_j^l(O)$, with $j = 0,\ldots,N$ and $l \in \mathbb{N}$ and the point (2) is proved, which ends the proof of Proposition 2.13. $\qquad\square$

2.3. Recovering the Hamiltonian from matrix elements.

In order to finish the proof of Theorem 2.1 we will show how the knowledge of the diagonal matrix elements of a given known self-adjoint operator conjugated by a unitary one determines the latter (in the framework of asymptotic expansion).

Let $\widetilde{W}_{\le N}$ be as in Proposition 2.2 and let O_{mnp}, O_p be as in Theorem 2.1. By Proposition 2.13, there exist smooth functions f_{mnp} and f_p vanishing at $(0,0,0)$ if $(m,n) \ne (0,0)$ such that for any $N \ge 3$:

$$(2.66)$$
$$\langle \mu,\nu|e^{\frac{i\widetilde{W}_{\le N}}{\hbar}} O_{mnp} e^{\frac{-i\widetilde{W}_{\le N}}{\hbar}}|\mu,\nu\rangle = f_{mnp}\left((\mu+\frac{1}{2})\hbar, 2\pi\nu\hbar, \hbar\right) + O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{N}{2}}\right)$$

and

$$(2.67)$$
$$\langle \mu,\nu|e^{\frac{i\widetilde{W}_{\le N}}{\hbar}} O_p e^{\frac{-i\widetilde{W}_{\le N}}{\hbar}}|\mu,\nu\rangle = f_p\left((\mu+\frac{1}{2})\hbar, 2\pi\nu\hbar, \hbar\right) + O\left((|\mu\hbar|+|\nu\hbar|)^{\frac{N}{2}}\right).$$

Proposition 2.14. *The Taylor expansions, at the origin, of the functions f_{mnp}, f_q up to order $N - 1$, $N \geq 3$, for $(m, n, p, q) \in \mathbb{N}^{2n} \times \mathbb{Z}^2$ satisfying conditions*

  (1) $0 < |m| + |n| \leq N$,
  (2) $\forall j = 1, \ldots, n, \ m_j = 0 \ \textbf{or} \ n_j = 0$,
  (3) $p \in \mathbb{Z}, \ q \in \mathbb{Z}^*$,

determine completely $W_{\leq N}$.

Proof of Proposition 2.14. Let us write

$$
(2.68) \quad
\begin{aligned}
W_N &= \sum_{2l+|j|+|k|+2s=N} \alpha_{ljks}(t) \hbar^l \mathrm{Op}^W(z^j \bar{z}^k) D_t^s \\
&:= \sum_{2l+|j|+|k|+2s=N} \sum_{d \in \mathbb{Z}} \alpha_{ljksd} \hbar^l e^{i2\pi dt} \mathrm{Op}^W(z^j \bar{z}^k) D_t^s,
\end{aligned}
$$

where every α_{ljjs0} is chosen to be zero by the convention of Remark 2.12.

Since $W_2 = 0$ we can proceed by induction on $N \geq 3$: let's assume $W_{\leq N-1}$ is already determined. Let $(m, n, p, q) \in \mathbb{N}^{2n} \times \mathbb{Z} \times \mathbb{Z}^*$ be such that

$$
(2.69) \quad 0 < |m| + |n| \leq N, \quad \forall i \in [[1, n]], \ m_i n_i = 0.
$$

Let us also state the following lemma, whose proof will be given after the end of the present proof.

Lemma 2.15. *Let $(j, k, s, d) \in \mathbb{N}^{2n+1} \times \mathbb{Z}$, such that $|j| + |k| + 2s = N$. If $j + m = k + n$, then*

$$
(2.70) \quad
\begin{aligned}
\langle \mu, \nu | [e^{i2\pi pt} \mathrm{Op}^W(z^j \bar{z}^k) D_t^s, O_{mnp}] | \mu, \nu \rangle &= -\hbar g_{jks}\left(\left(\mu + \tfrac{1}{2}\right) \hbar, \nu\hbar \right) \\
&+ O\left(\hbar^2 (|\mu\hbar| + |\nu\hbar|)^{\frac{N+|m|+|n|}{2}-2} + \hbar(|\mu\hbar| + |\nu\hbar|)^{\frac{N+|m|+|n|-1}{2}} \right),
\end{aligned}
$$

where

$$
g_{jks}\left(\left(\mu + \frac{1}{2}\right) \hbar, \nu\hbar \right) = (2\pi\nu\hbar)^s (\mu\hbar)^{\max(j,k)} \left(\sum_{i=1}^n \frac{k_i m_i - j_i n_i}{\mu_i \hbar} + \frac{ps}{\nu\hbar} \right)
$$

and $\max(j, k) = (\max(j_i, k_i))_{1 \leq i \leq n}$. If $j + m \neq k + n$ or $d \neq p$, then

$$
(2.71) \quad
\begin{aligned}
\langle \mu, \nu | [e^{i2\pi dt} \mathrm{Op}^W(z^j \bar{z}^k) D_t^s, O_{mnp}] | \mu, \nu \rangle &= O\left(\hbar^2 (|\mu\hbar| + |\nu\hbar|)^{\frac{N+|m|+|n|}{2}-2} \right) \\
&+ O\left(\hbar(|\mu\hbar| + |\nu\hbar|)^{\frac{N+|m|+|n|-1}{2}} \right).
\end{aligned}
$$

We also have, if $j = k$:

$$
(2.72) \quad
\begin{aligned}
\langle \mu, \nu | [e^{i2\pi qt} \mathrm{Op}^W(z^j \bar{z}^k) D_t^s, O_q] | \mu, \nu \rangle &= -2\pi\hbar q(1+s) \left((\mu + 1/2) \hbar \right)^j (\nu\hbar)^s \\
&+ O\left(\hbar^2 (|\mu\hbar| + |\nu\hbar|)^{\frac{N-2}{2}} + \hbar(|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}} \right),
\end{aligned}
$$

and if $j \neq k$ or $d \neq q$:

$$
(2.73) \quad
\begin{aligned}
\langle \mu, \nu | [e^{i2\pi dt} \mathrm{Op}^W(z^j \bar{z}^k) D_t^s, O_q] | \mu, \nu \rangle &= O\left(\hbar^2 (|\mu\hbar| + |\nu\hbar|)^{\frac{N-2}{2}} \right) \\
&+ O\left(\hbar(|\mu\hbar| + |\nu\hbar|)^{\frac{N+1}{2}} \right).
\end{aligned}
$$

By equation (2.66), the Taylor expansion of function f_{mnp} up to order $N - 1$ determines modulo $o\left((|\mu\hbar| + |\nu\hbar|)^N \right)$:

$$
(2.74) \quad \langle \mu, \nu | e^{\frac{i \widetilde{W}_{\leq 2N}}{\hbar}} O_{mnp} e^{\frac{-i \widetilde{W}_{\leq 2N}}{\hbar}} | \mu, \nu \rangle - \langle \mu, \nu | O_{mnp} | \mu, \nu \rangle.
$$

Since $\widetilde{W}_{\leq 2N}$ is a sum of polynomial operators of order greater than 3, we get from Lemma 2.8 that

(2.75)

$$\sum_{l \geq 2} \frac{i^l}{\hbar^l l!} \langle \mu, \nu | [\overbrace{\widetilde{W}_{\leq 2N}, \ldots, \widetilde{W}_{\leq 2N}}^{l \text{ times}}, O_{mnp}] | \mu, \nu \rangle = O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+|m|+|n|-1}{2}}\right).$$

Hence, using the notation of Lemma 2.15, (2.74) is equal, modulo known terms and $O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+|m|+|n|-1}{2}}\right) + O\left(\hbar(|\mu\hbar| + |\nu\hbar|)^{\frac{N+|m|+|n|}{2}-2}\right)$, to

(2.76)

$$\sum_{\substack{|j|+|k|+2s=N+1 \\ j+m=k+n}} i\alpha_{0jksp} g_{jks}\left(\left(\mu + \frac{1}{2}\right)\hbar, \nu\hbar\right).$$

Let us define the set $\Gamma = \{(j, k, s) \in \mathbb{N}^{2n+1} \mid |j| + |k| + 2s = N, \ j + m = k + n\}$ and let us choose $\mu_1(\hbar), \ldots, \mu_n(\hbar), \nu(\hbar)$ such that, as $\hbar$ tends to 0,

(2.77)

$$\nu^{\frac{N-2}{N-1}} \ll \mu_1 \ll \cdots \ll \mu_n \ll \nu \ll \hbar^{-\frac{1}{3}}.$$

Let us also define $i_0 := \min\{i \in [[1, n]], m_i \neq n_i\}$ (it exists since $(m, n) \neq (0, 0)$ and $m_i n_i = 0$ for any $i \in [[1, n]]$). Let us also remark that $j_{i_0} n_{i_0} - k_{i_0} m_{i_0}$ never vanishes on Γ. We have by (2.77) that, for $(j, k, s) \in \Gamma$,

(2.78)

$$g_{jks}\left(\left(\mu + \frac{1}{2}\right)\hbar, \nu\hbar\right) \underset{\hbar \to 0}{\sim} \frac{j_{i_0} n_{i_0} - k_{i_0} m_{i_0}}{\mu_{i_0}\hbar}(2\pi\nu\hbar)^s \prod_{i=1}^{n}(\mu_i\hbar)^{\max(j_i, k_i)}.$$

Let us now define a strict total order $\prec$ on Γ by

(2.79)

$$(j, k, s) \prec (j', k', s')$$
$$\Updownarrow$$
$$(\max(j_1, k_1), \ldots, \max(j_n, k_n), s) < (\max(j'_1, k'_1), \ldots, \max(j'_n, k'_n), s'),$$

where $<$ is the lexicographical order on $\mathbb{N}^{n+1}$. $\prec$ is asymmetric since the sign of $m_i - n_i$ determines whether $\max(j_i, k_i)$ is equal to j_i or k_i for $i \in [[1, n]]$. (2.77) and (2.78) give that

(2.80) $(j, k, s) \prec (j', k', s') \Rightarrow g_{jks}\left(\left(\mu + \frac{1}{2}\right)\hbar, \nu\hbar\right) \underset{\hbar \to 0}{\ll} g_{j'k's'}\left(\left(\mu + \frac{1}{2}\right)\hbar, \nu\hbar\right)$

and for any $(j, k, s) \in \Gamma$:

$$O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+|m|+|n|-1}{2}}\right) + O\left(\hbar(|\mu\hbar| + |\nu\hbar|)^{\frac{N+|m|+|n|}{2}-2}\right)$$
$$\ll g_{jks}\left(\left(\mu + \frac{1}{2}\right)\hbar, \nu\hbar\right).$$

Therefore, the Taylor expansion up to order $N - 1$ of the functions f_{mnp} determines the coefficients $(\alpha_{0jksp})_{|j|+|k|+2s=N, j+m=k+n}$ by induction on $(\Gamma, <)$.

Let (m, n, p) run over all the possible values in $\mathbb{N}^{2n} \times \mathbb{Z}$ while satisfying condition (2.69). We claim that one can determine every function α_{0jks} with $|j| + |k| + 2s = N$ and $j \neq k$. Indeed, for any $(j, k, s) \in \mathbb{N}^{2n+1}$ such that $|j| + |k| + 2s = N$ and $j \neq k$, let us choose for any $i \in [[1, n]]$:

(2.81)

$$n_i = \max(j_i - k_i, 0) \text{ and } m_i = \max(k_i - j_i, 0);$$

then $j + m = k + n$ and $(m, n) \neq (0, 0)$ while for any $i \in [[1, n]]$, $m_i = 0$ or $n_i = 0$. Finally,

$$|m| + |n| = \sum_{i=1}^{n} |j_i - k_i| \leq |j| + |k| \leq N.$$

Let us remark that condition $j \neq k$ is always satisfied if N is odd and $|j| + |k| + 2s = N$. If N is even, the Taylor expansion up to order $\frac{N}{2}$ of the function f_q determines modulo known terms and $O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+2}{2}}\right) + O\left(\hbar(|\mu\hbar| + |\nu\hbar|)^{\frac{N-2}{2}}\right)$:

$$(2.82) \qquad \sum_{2|j|+2s=N} i\alpha_{0jjsq}2\pi q(1+s)\left((\mu+1/2)\hbar\right)^j (\nu\hbar)^s.$$

Let us choose $\mu_1(\hbar), \ldots, \mu_n(\hbar), \nu(\hbar)$ such that, as $\hbar$ tends to 0,

$$(2.83) \qquad \nu^{\frac{N-2}{N}} \ll \mu_1 \ll \cdots \ll \mu_n \ll \nu \ll \hbar^{-\frac{1}{2}}.$$

We have, for any (j, s, q) such that $2|j| + 2s = N$:

$$O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+2}{2}}\right) + O\left(\hbar(|\mu\hbar| + |\nu\hbar|)^{\frac{N-2}{2}}\right) \ll \left((\mu+1/2)\hbar\right)^j (\nu\hbar)^s.$$

Thus, every α_{0jjsq} is determined by induction on the set $\{2|j| + 2s = N\}$ ordered by the lexicographical order. Hence, letting q run over $\mathbb{Z}^*$, we finally determine every α_{0jksd} with $|j| + |k| + 2s = N$ and $d \neq 0$ if $j = k$, hence the principal symbol of W_N.

Let us now choose $1 \leq l_0 < \frac{N}{2}$ and assume that we already determined the functions α_{ljks} with $2l + |j| + |k| + 2s = N$ and $l < l_0$. Let $(m, n, p, q) \in \mathbb{N}^{2n} \times \mathbb{Z} \times \mathbb{Z}^*$ be such that

$$(2.84) \qquad 0 < |m| + |n| \leq N - 2l_0, \quad \forall i \in [[1, n]], \ m_i n_i = 0.$$

The Taylor expansion of f_{mnp} up to order $N - 1 - l_0$ determines modulo known terms $O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+|m|+|n|-1}{2}}\right) + O\left(\hbar^{l_0+1}(|\mu\hbar| + |\nu\hbar|)^{\frac{N-2l_0+|m|+|n|}{2}-2}\right)$:

$$\sum_{\substack{2l_0+|j|+|k|+2s=N \\ j+m=k+n}} i\alpha_{l_0jksp}\hbar^{l_0}g_{jks}\left(\left(\mu+\frac{1}{2}\right)\hbar, \nu\hbar\right).$$

Let us choose $\mu_1(\hbar), \ldots, \mu_n(\hbar), \nu(\hbar)$ such that, as $\hbar$ tends to 0:

$$(2.85) \qquad \nu^{\frac{N-l_0-2}{N-l_0-1}} \ll \mu_1 \ll \cdots \ll \mu_n \ll \nu \ll \hbar^{-\frac{1}{2l_0+3}}.$$

Then for any (j, k, s) such that $2l_0 + |j| + |k| + 2s = N$ and $j + m = k + n$, we have: $O\left((|\mu\hbar| + |\nu\hbar|)^{\frac{N+|m|+|n|-1}{2}}\right) + O\left(\hbar^{l_0+1}(|\mu\hbar| + |\nu\hbar|)^{\frac{N-2l_0+|m|+|n|}{2}-2}\right) \ll \hbar^{l_0}g_{jks}\left(\left(\mu+\frac{1}{2}\right)\hbar, \nu\hbar\right)$. Therefore, every α_{l_0jksp} with $2l_0 + |j| + |k| + 2s = N$ and $j + m = k + n$ is determined just like before. Letting (m, n, p) run over all the possible values in $\mathbb{N}^{2n} \times \mathbb{Z}$ while satisfying (2.84), we determined every α_{l_0jksp} with $2l_0 + |j| + |k| + 2s = N - 1$ and $j \neq k$. The Taylor expansion of f_q up to order $\frac{N}{2}$ determines the remaining α_{l_0jjsq}, and, finally, every function α_{l_0jks} where (j, k, s) satisfies $2l_0 + |j| + |k| + 2s = N - 1$, which concludes our proof by induction. $\square$

Proof of Lemma 2.15. The principal symbol of $\frac{1}{i\hbar}[e^{i2\pi dt}\mathrm{Op}^W z^j\bar{z}^k D_t^s, O_{mnp}]$ is

$$\begin{aligned}\sigma_{jkds}(z,t,\bar{z},\tau) &= \left\{e^{i2\pi dt}z^j\bar{z}^k\tau^s, \mathcal{O}_{mnp}\right\}\\
(2.86)\qquad &= \left\{e^{i2\pi dt}z^j\bar{z}^k\tau^s, e^{-i2\pi pt}z^m\bar{z}^n\right\}\\
&\quad + O\left((|z|^2+\tau)^{\frac{|m|+|n|+N-1}{2}}\right).\end{aligned}$$

Hence,

$$\begin{aligned}
\sigma_{jkds}(z,t,\bar{z},\tau) &= -i\sum_{i=1}^n \frac{\partial}{\partial z_i}(e^{i2\pi dt}z^j\bar{z}^k\tau^s)\frac{\partial}{\partial\bar{z}_i}(e^{-i2\pi pt}z^m\bar{z}^n)\\
&\quad + i\sum_{i=1}^n \frac{\partial}{\partial\bar{z}_i}(e^{i2\pi dt}z^j\bar{z}^k\tau^s)\frac{\partial}{\partial z_i}(e^{-i2\pi pt}z^m\bar{z}^n)\\
(2.87)\qquad &\quad - \frac{\partial}{\partial\tau}(e^{i2\pi dt}z^j\bar{z}^k\tau^s)\frac{\partial}{\partial t}(e^{-i2\pi pt}z^m\bar{z}^n)\\
&\quad + O\left((|z|^2+\tau)^{\frac{|m|+|n|+N-1}{2}}\right)\\
&= -ie^{i2\pi(d-p)t}z^{j+m}\bar{z}^{k+n}\tau^s\left(\sum_{i=1}^n \frac{j_in_i - k_im_i}{z_i\bar{z}_i} - \frac{2\pi ps}{\tau}\right)\\
&\quad + O\left((|z|^2+\tau)^{\frac{|m|+|n|+N-1}{2}}\right).
\end{aligned}$$

Let us remark that if $j + m = k + n$, then $j + m = k + n = \max(j,k)$. Let us also remark that if $j_in_i - k_im_i \neq 0$, then $j_i + m_i \neq 0$ and $k_i + n_i \neq 0$. Therefore, the last line in (2.87) can be reduced to a polynomial expression modulo $O\left((|z|^2+\tau)^{\frac{|m|+|n|+N-1}{2}}\right)$. $\frac{1}{\hbar}[e^{i2\pi dt}\mathrm{Op}^W z^j\bar{z}^k D_t^s, O_{mnp}]$ has the same principal symbol as the polynomial operator obtained when replacing each z_i by a_i, $\bar{z}_i$ by a_i^*, and τ by D_t in this polynomial expression. Since the expansion in $\mathrm{PO}\frac{1}{\hbar}[e^{i2\pi dt}\mathrm{Op}^W z^j\bar{z}^k D_t^s, O_{mnp}]$ starts at order $N + |m| + |n| - 2$, we can hence conclude that the asymptotic expansions (2.70) and (2.71) are verified.

Now, the principal symbol of $\frac{1}{i\hbar}[e^{i2\pi dt}\mathrm{Op}^W z^j\bar{z}^k D_t^s, O_q]$ is

$$\begin{aligned}
\tilde{\sigma}_{jkds}(z,t,\bar{z},\tau) &= \left\{e^{i2\pi dt}z^j\bar{z}^k\tau^s, \mathcal{O}_q\right\}\\
&= \left\{e^{i2\pi dt}z^j\bar{z}^k\tau^s, e^{-i2\pi qt}\tau\right\}\\
(2.88)\qquad &= \frac{\partial}{\partial t}(e^{i2\pi dt}z^j\bar{z}^k\tau^s)\frac{\partial}{\partial\tau}(e^{-i2\pi qt}\tau)\\
&\quad - \frac{\partial}{\partial\tau}(e^{i2\pi dt}z^j\bar{z}^k\tau^s)\frac{\partial}{\partial t}(e^{-i2\pi qt}\tau)\\
&= i2\pi(d+sq)e^{i2\pi(d-q)t}z^j\bar{z}^k\tau^s
\end{aligned}$$

modulo $O\left((|z|^2+\tau)^{\frac{N+1}{2}}\right)$. Hence, (2.72) and (2.73) are verified just as before. $\square$

Theorem 2.1 is, as has already been said, a direct consequence of Propositions 2.13 and 2.14.

2.4. "Bottom of a well".

In this subsection, we treat the "bottom of a well" analogs of Theorem 1.4, namely Theorems 1.10 and 1.14. The proof of Theorem 1.10 is a line-by-line analog of the proof of Proposition 2.14: we omit it here.

However, Theorem 1.14, which needs fewer assumptions in the particular case of a Schrödinger operator, deserves a proper proof.

Proof of Theorem 1.14. In a system of Fermi coordinates, the (principal and total) symbol of our Schrödinger operator can be written as

$$(2.89) \qquad H(x,\xi) = V(q_0) + \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2} + R(x), \quad R(x) = O(x^3).$$

Let $H_0(x,\xi) = \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2}$. Let us state the following lemma, which is a classical analog of Proposition 2.11 (we therefore omit its proof) and uses the hypothesis of rational independence of the θ_i's.

Lemma 2.16. *Let $G \in \mathcal{C}^\infty(T^*(\mathbb{R}^n), \mathbb{R})$ be a homogeneous polynomial of degree $k \geq 3$. There exists a unique couple of functions $G_1 \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ and $F \in \mathcal{C}^\infty(T^*(\mathbb{R}^n), \mathbb{R})$ such that*

$$(2.90) \qquad \forall (x,\xi) \in T^*(\mathbb{R}^n), \ \{H_0, F\}(x,\xi) = G(x,\xi) - G_1(p)$$

and F is polynomial with no diagonal term when written as a function of $(z, \bar{z})$ (i.e. of the form $z^l \bar{z}^l$).
   Moreover:

   (1) *F is a homogeneous polynomial of degree k and is entirely determined by the extradiagonal terms of G, i.e. of the form $z^l \bar{z}^m$ ($l \neq m$) with $z = (x + i\xi)/\sqrt{2}$.*
   (2) *G_1 is a homogeneous polynomial of degree $\frac{k}{2}$ if k is even, zero otherwise. Moreover, $G_1(z\bar{z})$ is equal to the sum of the diagonal terms of G.*

Just as in the proof of Proposition 2.2, one shows recursively, using Lemma 2.16, the existence of a family of real numbers $(\alpha_{lm})_{l,m\in\mathbb{N}}$ such that if the functions $(F_N)_{N\geq 3}$ are defined for $N \geq 3$ by

$$(2.91) \qquad F_N(z, \bar{z}) = \sum_{|l|+|m|=N} \alpha_{lm} z^l \bar{z}^m,$$

then there exist homogeneous polynomials $H^i \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ of degree i satisfying, for $N \geq 3$:

$$(2.92) \qquad H \circ \exp \chi_{F_{\leq N}}(x,\xi) = \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor} H^i(p) + O((x,\xi)^{N+1}).$$

Here $p = p(x,\xi) = (\frac{x_i^2 + \xi_i^2}{2})_{1 \leq i \leq n}$, $F_{\leq N} = \sum_{k=1}^{N} F_k$, and $\chi_{F_{\leq N}}$ is the vector field

$$(2.93) \qquad \chi_{F_{\leq N}} = \sum_{i=1}^{n} \frac{\partial F_{\leq N}}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial F_{\leq N}}{\partial x_i} \frac{\partial}{\partial \xi_i}.$$

$\sum_{i=1}^{+\infty} H^i$ (well defined modulo a flat function) is the classical Birkhoff normal form of H.

Let us define $a_k = \frac{1}{k!} \frac{\partial^{|k|} R}{\partial x^k}(0)$ for $k \in \mathbb{N}^n$ and $|k| \geq 3$. We observe that, for $k \in \mathbb{N}^n$:

$$(2.94) \qquad x^k = \left(\frac{z + \bar{z}}{\sqrt{2}} \right)^k = \frac{1}{\sqrt{2}^{|k|}} \sum_{\substack{(l,m) \in \mathbb{N}^n \\ l+m=k}} \prod_{j=1}^n \binom{k_j}{m_j} z^l \bar{z}^m.$$

Let us also define $\mathcal{K} = \{ k \in \mathbb{N}^n, |k| \geq 3 \} \setminus 2\mathbb{N}^n$. By Lemma 2.16, there exists a unique homogeneous polynomial of degree $|k| \geq 3$ with no diagonal terms, such that:

$$(2.95) \qquad \{H_0, I_k\}(x, \xi) = \begin{cases} x^k & \text{if } k \in \mathcal{K}, \\ x^k - \frac{1}{\sqrt{2}^{|k|}} \prod_{j=1}^n \binom{k_j}{k_j/2} |z|^k & \text{if } k \in 2\mathbb{N}^n. \end{cases}$$

Functions $(F_N)_{N \geq 3}$ and $(H^i)_{i \geq 1}$ are constructed recursively as follows: let $N \geq 2$ and assume that we have already constructed $F_3, \ldots, F_N$ ($F_2 = 0$) and $H_1, \ldots, H^{\lfloor \frac{N}{2} \rfloor}$ ($H_1(p) = \sum_{i=1}^n \theta_i p_i$). Let us set

$$(2.96) \qquad G_{N+1}(x, \xi) = H \circ \exp \chi_{F_{\leq N}}(x, \xi) - \sum_{i=1}^{\lfloor \frac{N}{2} \rfloor} H^i(p) + O(\|(x,\xi)\|^{N+1})$$

and define F_{N+1} and, if N is odd, $H^{\frac{N+1}{2}}$ by Lemma 2.16:

$$(2.97) \qquad \{H_0, F_{N+1}\}(x, \xi) = \begin{cases} G_{N+1}(x, \xi) & \text{if } N \text{ is even}, \\ G_{N+1}(x, \xi) - H^{\frac{N+1}{2}}(p) & \text{if } N \text{ is odd}. \end{cases}$$

We remark that, in our case, $(x, \xi) \mapsto G_{N+1}(x, \xi) - \sum_{|k|=N+1} a_k x^k$ is a sum of terms that depend only on $F_{\leq N}$, $(H^i)_{1 \leq i \leq \lfloor \frac{N}{2} \rfloor}$, and $(a_k)_{|k| \leq N}$. Therefore, we get by induction that the function(s)

$$(2.98)$$
$$F_{N+1} - \sum_{|k|=N+1} a_k I_k \quad \text{and, when } N \text{ is odd,} \quad H^{\frac{N+1}{2}}(p) - \sum_{|l|=\frac{N+1}{2}} \frac{a_{2l}}{2^{|l|}} \prod_{j=1}^n \binom{2l_j}{l_j} p^l$$

depend only on $(a_k)_{|k| \leq N}$.

Now, let us define, for $k \in \mathbb{N}^n$, $(l_k, m_k) \in \mathbb{N}^{2n}$ by their components: for $i \in [[1, n]]$, $(l_k)_i = \lfloor \frac{k_i}{2} \rfloor$, $(m_k)_i = k_i - \lfloor \frac{k_i}{2} \rfloor$. $k \mapsto (l_k, m_k)$ is a bijective correspondence between $\mathcal{K}$ and the set Λ defined by:

$$(2.99) \qquad \Lambda = \{ (l, m) \in \mathbb{N}^{2n} \mid m - l \in \{0, 1\}^n \setminus \{0\}, |l| + |m| \geq 3 \}.$$

Moreover, for $k \in \mathcal{K}$, I_k is entirely determined by (2.95) and is equal in $z, \bar{z}$ coordinates to:

$$(2.100) \qquad I_k(z, \bar{z}) = \frac{1}{\sqrt{2}^{|k|}} \sum_{\substack{(l,m) \in \mathbb{N}^n \\ l+m=k}} \frac{\prod_{j=1}^n \binom{k_j}{m_j}}{\theta.(l-m)} z^l \bar{z}^m.$$

Therefore, if $k \in \mathcal{K}$, $|k| = N + 1$, we get by (2.98) that

$$
(2.101) \qquad \alpha_{l_k m_k} - \frac{a_k}{\sqrt{2}^{|k|}} \frac{\prod\limits_{j=1}^{n} \binom{k_j}{\lfloor k_j/2 \rfloor}}{\theta.(l_k - m_k)}
$$

depends only on $(a_k)_{|k| \leq N}$.

Now if $k \in 2\mathbb{N}^n$ and $|k| = N + 1$, and if we write $H^{\frac{N+1}{2}}(p) = \sum\limits_{|l| = \frac{N+1}{2}} b_l p^l$, then we get by (2.98) that

$$
(2.102) \qquad b_{k/2} - \frac{a_k}{\sqrt{2}^{|k|}} \prod_{j=1}^{n} \binom{k_j}{k_j/2}
$$

depends only on $(a_k)_{|k| \leq N}$.

Therefore we get by (2.101) and (2.102) that the family $(a_k)_{|k|=N+1}$ can be determined from the terms of order $N + 1$ in the Taylor expansion of the classical Birkhoff normal form, the family $(\alpha_{l_k m_k})_{|k|=N+1}$, and the family $(a_k)_{|k| \leq N}$.

So we just proved, by induction, that for any $N \geq 3$, $(a_k)_{|k| \leq N}$ is determined by the Taylor expansion of the classical Birkhoff normal form up to order N and the family $(\alpha_{lm})_{(l,m) \in \Lambda, |l|+|m| \leq N}$.

As shown in [9, 10], the Taylor expansion of the classical Birkhoff normal form is determined by the spectrum of $H(x, \hbar D_x)$ in $[V(q_0), V(q_0) + \epsilon]$, $\epsilon > 0$. In fact it is obvious that one can take ϵ in the $\hbar$-dependent form given in Theorem 1.14 (and Theorem 1.10) since the proof goes along the trace formula argument, and eigenvalues above this value of ϵ give an $\hbar^\infty$ contribution to the trace formula.

Moreover we have for $N \geq 2$ and $m \in \{0,1\}^n \setminus \{0\}$

$$
\mathcal{O}_{m0} \circ \exp \chi_{F_{\leq N+1}}(x, \xi) = \mathcal{O}_{m0}(x, \xi) + \{F_{\leq N+1}, \mathcal{O}_{m0}\}(x, \xi) + O((x, \xi)^{N+|m|})
$$

$$
= z^m - \sum_{\substack{(l,k) \in \Lambda \\ |l|+|k|=N+1 \\ k-l=m}} \alpha_{lk} |z|^{2l} \sum_{i=1}^{n} k_i m_i + \cdots + O((x, \xi)^{N+|m|}),
$$

where ... stands for extradiagonal terms and terms which depend only on $(\alpha_{lk})_{(l,k) \in \Lambda, |l|+|m| \leq N}$. Therefore, the diagonal matrix elements of an observable O_{m0} are equal, modulo terms depending only on $(\alpha_{lk})_{(l,k) \in \Lambda, |l|+|k| \leq N}$, to

$$
(2.103) \qquad \sum_{\substack{(l,k) \in \Lambda \\ |l|+|k| \leq N+1 \\ k-l=m}} \alpha_{lk} |\mu \hbar|^l \sum_{i=1}^{n} k_i m_i + O(\hbar) + O(|\mu \hbar|^{\frac{N+|m|}{2}}).
$$

This shows, as in the proof of Theorem 2.1, that the α_{lm}, $(l, m) \in \Lambda$, are all determined, so the full Taylor expansion of R, hence of V, near q_0, is completely determined. $\qquad \square$

3. Explicit construction of Fermi coordinates

In this section we prove Theorems 1.3, 1.9, and 1.13, using Lemmas A.1, A.2, and A.4 on linear and bilinear algebra. We start by the "bottom of a well", toy model for the periodic trajectory case.

3.1. **General "bottom of a well" case.**

Proof of Theorem 1.9. Let $(x, \xi) \in T^*(\mathbb{R}^n)$ be a system of Darboux coordinates centered at z_0. $d^2 H_p(z_0)$ is a positive bilinear form on $T_{z_0}(T^*\mathcal{M})$, therefore, by Lemma A.1, there exists a local change of variable ϕ, symplectic and linear in the Darboux coordinates, such that

$$(3.1) \qquad H_p \circ \phi(x, \xi) = H_p(z_0) + \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2} + O(\|(x, \xi)\|^3).$$

We will prove that the diagonal matrix elements of the family of pseudodifferential operators P^k in the system of eigenvectors corresponding to eigenvalues of $H(x, \hbar D_x)$ in $[H_p(z_0), H_p(z_0) + \epsilon(\hbar)]$ provide an explicit construction of such a symplectomorphism ϕ (which is not unique).

We first start with the case where the family $(\mathcal{P}^k)_{1 \le k \le 2n^2 + n}$ is realized by the example (1.12).

Let S be the matrix of $d\phi_{z_0}$ in the basis $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial \xi_n})$. We have for $(i, j) \in [[1, n]]^2$ and $s \in \{1, 2, 3\}$:

$$
\begin{aligned}
(3.2) \qquad \mathcal{Q}_{i,j}^s \circ \phi(x, \xi) &= \left(\sum_{k=1}^{n} S_{i^s, 2k-1} x_k + S_{i^s, 2k} \xi_k \right) \left(\sum_{k=1}^{n} S_{j^s, 2k-1} x_k + S_{j^s, 2k} \xi_k \right) \\
&= \sum_{k,k'=1}^{n} S_{i^s, 2k-1} S_{j^s, 2k'-1} x_k x_{k'} + \sum_{k,k'=1}^{n} S_{i^s, 2k} S_{j^s, 2k'-1} \xi_k x_{k'} \\
&\quad + \sum_{k,k'=1}^{n} S_{i^s, 2k-1} S_{j^s, 2k'} x_k \xi_{k'} + \sum_{k,k'=1}^{n} S_{i^s, 2k} S_{j^s, 2k'} \xi_k \xi_{k'} \\
&= \sum_{k=1}^{n} [S_{i^s, 2k-1} S_{j^s, 2k-1} + S_{i^s, 2k} S_{j^s, 2k}] z_k \bar{z}_k + R,
\end{aligned}
$$

where, for $(i, j) \in [[1, n]]^2$,

$$
i^s = \begin{cases} 2i - 1 & \text{if } s \in \{1, 2\}, \\ 2i & \text{if } s = 3, \end{cases} \quad \text{and} \quad j^s = \begin{cases} 2j & \text{if } s \in \{1, 3\}, \\ 2j - 1 & \text{if } s = 2, \end{cases}
$$

and R is a linear combination of terms of the form $z_k z_{k'}$ $((k, k') \in [[1, n]])$ and $z_k \bar{z}_{k'}$ $((k, k') \in [[1, n]], k \ne k')$.

Let A_ϕ be any Fourier integral operator implementing locally $d\phi_{z_0}$ and $|\mu\rangle$ defined by (2.7). The condition that $A_\phi^{-1}|\mu\rangle$ belongs to the spectral interval defined in Theorem 1.9 reads as $|\mu\hbar| \le \epsilon$. We get from (3.2) that

$$(3.3) \quad \langle \mu | A_\phi \mathcal{Q}_{i,j}^s A_\phi^{-1} | \mu \rangle = \sum_{k=1}^{n} [S_{i^s, 2k-1} S_{j^s, 2k-1} + S_{i^s, 2k} S_{j^s, 2k}] \left(\mu_k + \frac{1}{2} \right) \hbar + O(\hbar),$$

with the term $O(\hbar)$ coming from the subsymbols contribution (let us recall that we are microlocalized in a bounded neighborhood of z_0). Therefore (3.3) for $|\mu\hbar| \le \epsilon$ with the condition $\hbar = 0(\epsilon)$ determines the values of $S_{i, 2k-1} S_{j, 2k-1} + S_{i, 2k} S_{j, 2k}$ for $(i, j) \in [[1, 2n]]^2$ and $k \in [[1, n]]$.

As claimed by Lemma A.2, the preceding quantities are independent of the choice of a symplectic matrix S satisfying (3.1). Since, as we already said, such a matrix S is not unique, it is not possible to determine S out of the preceding matrix elements. However, by Lemma A.2, the family $(S_{i, 2k-1} S_{j, 2k-1} + S_{i, 2k} S_{j, 2k})_{(i,j) \in [[1, 2n]]^2, k \in [[1, n]]}$

(determined by the preceding matrix elements) allows us to construct explicitly a suitable matrix S, hence a suitable symplectomorphism ϕ.

This ends the proof in the case where the family $(\mathcal{P}^k)_{1 \leq k \leq 2n^2+n}$ is realized by the example (1.12). Let us now consider the general case. The family of Hessian matrices $(d^2\mathcal{P}^k(z_0))_{1 \leq k \leq 2n^2+n}$ forms a basis of the space of $2n \times 2n$ symmetric matrices. Hence, each $d^2Q_{i,j}^s(z_0)$ for $(i,j) \in [[1,n]]^2$ and $s \in \{1,2,3\}$ is a linear combination of the matrices $d^2\mathcal{P}^k(z_0)$, $1 \leq k \leq 2n^2+n$. Since $\mathcal{P}^k(z_0) = \nabla\mathcal{P}^k(z_0) = 0$, there exists a family $(\lambda_{ijs}^k)_{(i,j,s,k) \in [[1,n]]^2 \times \{1,2,3\} \times [[1,2n^2+n]]}$ of complex numbers such that for any $(i,j,s) \in [[1,n]]^2 \times \{1,2,3\}$:

$$(3.4) \qquad \mathcal{Q}_{i,j}^s(x,\xi) = \sum_{m=1}^{2n^2+n} \lambda_{ijs}^k \mathcal{P}^k(x,\xi) + O(\|(x,\xi)\|^3)$$

and therefore

$$(3.5) \qquad \langle \mu | A_\phi Q_{i,j}^s A_\phi^{-1} | \mu \rangle = \sum_{k=1}^{2n^2+n} \lambda_{ijs}^k \langle \mu | A_\phi P^k A_\phi^{-1} | \mu \rangle + O(\hbar) + O(|\mu\hbar|^2).$$

Hence, the family $(S_{i,2k-1}S_{j,2k-1} + S_{i,2k}S_{j,2k})_{(i,j) \in [[1,2n]]^2, k \in [[1,n]]}$ is determined just as before, and this ends the proof in the general case. $\qquad \square$

3.2. The "Schrödinger case".

Proof of Theorem 1.13. Let $x \in \mathbb{R}^n$ be **any** system of local coordinates centered at $q_0 \in \mathcal{M}$, and let $(x,\xi) \in T^*(\mathbb{R}^n)$ be the corresponding Darboux coordinates centered at $(q_0, 0) \in T^*\mathcal{M}$. Since $d^2V(q_0)$ is a positive bilinear form on $T_{q_0}\mathcal{M}$, there exists, by Lemma A.4, a local change of variable u, linear and orthogonal in the Darboux coordinates, such that

$$(3.6) \qquad V \circ u(x) = \frac{1}{2}\sum_{i=1}^n \theta_i^2 x_i^2 + O(x^3),$$

where the θ_i^2's are the eigenvalues of $d^2V(q_0)$.

Let us denote by U the matrix of du_{q_0} written in the basis $(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$, and define a symplectomorphism ϕ locally by its expression in the Darboux coordinates: $\phi(x,\xi) = (Ux, U\xi)$.

If ϕ_0 is the symplectomorphism sending (x,ξ) to $(\frac{x_1}{\sqrt{\theta_1}}, \ldots, \frac{x_n}{\sqrt{\theta_n}}, \sqrt{\theta_1}\xi_1, \ldots, \sqrt{\theta_i}\xi_n)$ and H is the (principal and total) symbol of the considered Schrödinger operator, then

$$(3.7) \qquad H \circ \phi \circ \phi_0(x,\xi) = V(q_0) + \sum_{i=1}^n \theta_i \frac{x_i^2 + \xi_i^2}{2} + O(x^3).$$

Just as in the proof of Theorem 1.9, the diagonal matrix elements of the family of the pseudodifferential operators $(Q_{ij}^2)_{1 \leq i,j \leq n}$ in the system of eigenvectors corresponding to eigenvalues of $H(x, \hbar D_x)$ in $[V(q_0), V(q_0) + \epsilon(\hbar)]$ determine the family $(U_{ik}U_{jk})_{1 \leq i,j,k \leq n}$. An orthogonal matrix U such that (3.7) is verified is not unique, therefore it is not possible to determine the matrix U from the preceding diagonal matrix elements. However, by Lemma A.4, the family $(U_{ik}U_{jk})_{1 \leq i,j,k \leq n}$ does not depend on the suitable matrix U (*i.e.* orthogonal and satisfying (3.7)), and as we just saw it is determined by the preceding matrix elements. Therefore, one can determine the absolute values of the coefficients of any suitable matrix U, and also,

for any $k \in [[1, n]]$, an index $i_k \in [[1, n]]$, such that $U_{i_k k} \neq 0$. The choice of the sign of $U_{i_k k}$ then determines the sign of every other coefficient of the kth column. Therefore, one can determine the 2^n suitable matrices, corresponding to n choices of signs, as claimed by Lemma A.4. Choosing one of them determines (explicitly) a suitable symplectomorphism ϕ. $\qquad\square$

3.3. The periodic trajectory case.

Proof of Theorem 1.3. Let $X, H(x, \hbar D_x), E, \gamma$ be as in Theorem 1.3. We first recall [7, 8, 18, 19] that there exists a (non-unique) symplectomorphism ϕ from a neighborhood of $\mathbb{S}^1$ in $T^*(\mathbb{R}^n \times \mathbb{S}^1)$ in a neighborhood of γ in $T^*(X)$ such that

$$(3.8) \qquad H_p \circ \phi(x, t, \xi, \tau) = H_0 + H_2 \quad \text{and} \quad \gamma(t) = \phi(0, t, 0, 0),$$

with H_0 and H_2 defined as in (1.6) and (1.7).

It is easy to check that, in a local system of coordinates (x', ξ', t', τ') near γ such that $\gamma = \{x' = \xi' = \tau' = 0\}$, ϕ can be chosen in the following form:

$$(3.9) \qquad \phi(x, t, \xi, \tau) = \phi_S(x, t, \xi, \tau) = (S(t)(x, \xi), t, \tau + q_S(t, x, \xi)).$$

Here, for any $t \in \mathbb{S}^1$, $S(t)$ is a linear symplectic change of variable (identified with its matrix in our system of coordinates) and, in order for ϕ_S to be symplectic, $q_S(t, \cdot, \cdot)$ is the quadratic form associated to the matrix

$$(3.10) \qquad M(t) = {}^t S(t) J_n \dot{S}(t),$$

where $\dot{S}(t) = \frac{d}{dt} S(t)$ and J_n is the antisymmetric matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

For $(i, j) \in [[1, n]]^2$, $p \in \mathbb{Z}$, and $s \in \{1, 2, 3\}$, let A_S be any Fourier integral operator implementing ϕ_S. We have
$$(3.11)$$
$$\langle \mu, \nu | A_S P_p^k A_S^{-1} | \mu, \nu \rangle = \sum_{k=1}^n c_p \left(S_{i^s, 2k-1}^\sigma S_{j^s, 2k-1}^\sigma + S_{i^s, 2k}^\sigma S_{j^s, 2k}^\sigma \right) \left(\mu_k + \frac{1}{2} \right) \hbar + O(\hbar),$$

where $c_p(\cdot)$ maps a function to its pth Fourier coefficient, σ is the permutation defined by (A.5), S^σ is defined by conjugation by the permutation matrix associated to σ just as in (A.6), and where, for $(i, j) \in [[1, n]]^2$,

$$i^s = \begin{cases} 2i - 1 & \text{if } s \in \{1, 2\}, \\ 2i & \text{if } s = 3, \end{cases} \quad \text{and} \quad j^s = \begin{cases} 2j & \text{if } s \in \{1, 3\}, \\ 2j - 1 & \text{if } s = 2. \end{cases}$$

Now, just as in the proof of Proposition 2.13, the coefficients $\left(a_1^l(P_p^k)\right)_{l \in \mathbb{Z}}$ determine $c_p \left(S_{i^s, 2k-1}^\sigma S_{j^s, 2k-1}^\sigma + S_{i^s, 2k}^\sigma S_{j^s, 2k}^\sigma \right)$ for any $k \in [[1, n]]$. Therefore, if the coefficients $a_1^l(P_p^k)$ are given for any $l \in \mathbb{Z}$, $(i, j) \in [[1, n]]^2$, $p \in \mathbb{Z}$, and $s \in \{1, 2, 3\}$, then the functions

$$(3.12) \qquad A_{i,j,k} := S_{i, 2k-1}^\sigma S_{j, 2k-1}^\sigma + S_{i, 2k}^\sigma S_{j, 2k}^\sigma$$

are determined for any $(i, j) \in [[1, 2n]]^2$ and $k \in [[1, n]]$.

An easy adaptation of the proof of Lemma A.2 shows that, once the set of functions $(A_{i,j,k})_{(i,j) \in [[1,2n]]^2, k \in [[1,n]]}$ is given, one can construct explicitly a particular smooth function $\mathbb{S}^1 \ni t \mapsto S_0(t)$ with values in the set of symplectic matrices, such that equality (3.12) holds. We also get that any matrix S^σ such that equality (3.12) holds is related to S_0 by the equality $S^\sigma = S_0^\sigma U$, where $t \mapsto U(t)$ is a smooth

function that takes its values in the set of block diagonal matrices whose diagonal blocks are 2-by-2 rotations.

Now let us consider this particular S_0 and let U be any smooth function that takes its values in the set of block diagonal matrices whose diagonal blocks are 2-by-2 rotations. Let us finally define S by the relation $S^\sigma = S_0^\sigma U$. Since for any $t \in \mathbb{S}^1$, $q_S(t, \cdot, \cdot)$ is quadratic, we have

$$\mathcal{P}_p^0 \circ \phi_S(x, t, \xi, \tau) = e^{-2i\pi pt}\tau + e^{-2i\pi pt} q_S(t, x, \xi)$$

(3.13)
$$= e^{-2i\pi pt}\tau + e^{-2i\pi pt} \sum_{k=1}^n \left(\frac{\partial^2 q_S}{\partial x_k^2} + \frac{\partial^2 q_S}{\partial \xi_k^2} \right)(t) z_k \bar{z}_k + R,$$

where R is a linear combination of terms of the form $e^{-2i\pi pt} z_k z_{k'}$ $((k, k') \in [[1, n]])$ and $e^{-2i\pi pt} z_k \bar{z}_{k'}$ $((k, k') \in [[1, n]], k \neq k')$.

Just as before, the coefficients $\left(a_1^l(P_p^0) \right)_{l, p \in \mathbb{Z}}$ determine the family of functions (of t only) $\left(\frac{\partial^2 q_S}{\partial x_k^2} + \frac{\partial^2 q_S}{\partial \xi_k^2} \right)_{k \in [[1, n]]}$.

Now, we get from equation (3.10) that, for $k \in [[1, n]]$ and $t \in \mathbb{S}^1$:

$$\frac{\partial^2 q_S}{\partial x_k^2}(t) + \frac{\partial^2 q_S}{\partial \xi_k^2}(t) = \sum_{i=1}^n \dot{S}_{i+n,k}(t) S_{i,k}(t) + \dot{S}_{i+n,k+n}(t) S_{i,k+n}(t)$$

(3.14)
$$- \sum_{i=1}^n \dot{S}_{i,k}(t) S_{i+n,k}(t) + \dot{S}_{i,k+n}(t) S_{i+n,k+n}(t).$$

For $k \in [[1, n]]$ and $t \in \mathbb{S}^1$, let us denote by $U_k(t) = \begin{pmatrix} \cos\theta_k(t) & -\sin\theta_k(t) \\ \sin\theta_k(t) & \cos\theta_k(t) \end{pmatrix}$ the kth diagonal block of $U(t)$. Then, for $j \in [[1, 2n]]$, $k \in [[1, n]]$, and $t \in \mathbb{S}^1$:

(3.15)
$$\begin{pmatrix} S_{j,k}(t) \\ S_{j,k+n}(t) \end{pmatrix} = {}^t U_k(t) \begin{pmatrix} S_{0,j,k}(t) \\ S_{0,j,k+n}(t) \end{pmatrix}.$$

Therefore,

(3.16)
$$\begin{pmatrix} \dot{S}_{j,k}(t) \\ \dot{S}_{j,k+n}(t) \end{pmatrix} = {}^t U_k(t) \begin{pmatrix} \dot{S}_{0,j,k}(t) \\ \dot{S}_{0,j,k+n}(t) \end{pmatrix} + {}^t \dot{U}_k(t) \begin{pmatrix} S_{0,j,k}(t) \\ S_{0,j,k+n}(t) \end{pmatrix}.$$

Let us now observe that for $k \in [[1, n]]$, and any $t \in \mathbb{S}^1$:

(3.17)
$$\dot{U}_k(t) \, {}^t U_k(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, since $U_k(t)$ is an orthogonal matrix and $S_0(t)$ is a symplectic matrix for any $k \in [[1, n]]$ and $t \in \mathbb{S}^1$, we get from equations (3.16) and (3.17) that

(3.18)
$$\frac{\partial^2 q_S}{\partial x_k^2}(t) + \frac{\partial^2 q_S}{\partial \xi_k^2}(t) = \frac{\partial^2 q_{S_0}}{\partial x_k^2}(t) + \frac{\partial^2 q_{S_0}}{\partial \xi_k^2}(t) + 2\dot{\theta}_k(t).$$

Since the function $t \mapsto \frac{\partial^2 q_S}{\partial x_k^2}(t) + \frac{\partial^2 q_S}{\partial \xi_k^2}(t)$ has been determined above, and the function $t \mapsto \frac{\partial^2 q_{S_0}}{\partial x_k^2}(t) + \frac{\partial^2 q_{S_0}}{\partial \xi_k^2}(t)$ is entirely determined by the explicitly constructed function $t \mapsto S_0$, equation (3.18) then determines the function $\dot{\theta}_k$. Therefore, the function $t \mapsto U(t)$, hence the function $t \mapsto S^\sigma(t)$, is determined up to right multiplication by a constant block diagonal matrix U_0 whose diagonal block matrices

are 2-by-2 rotations. It is now sufficient to observe that if two functions $t \mapsto S_1(t)$ and $t \mapsto S_2(t)$ are related by the equation

$$(3.19) \qquad\qquad S_2^\sigma = S_1^\sigma U_0,$$

where U_0 is a constant matrix, then

$$(3.20) \qquad\qquad \phi_{S_2} = \phi_{S_1} \circ \phi_{U_0^{\sigma-1}}$$

and, if U_0 is a constant block diagonal matrix whose diagonal block matrices are 2-by-2 rotations,

$$(3.21) \qquad\qquad H^0 \circ \phi_{U_0^{\sigma-1}} = H_0.$$

Finally, the choice of U_0 in the determination of $t \mapsto S(t)$ does not change the validity of equation (3.8) for $\phi = \phi_S$, and Theorem 1.3 is proved. $\qquad\square$

4. Classical analogs

In this section we prove a classical result, analogous to our preceding quantum results, and motivated by the following straightforward lemma.

Lemma 4.1. *Let O be a polynomial operator on $L^2(\mathbb{R}^n \times \mathbb{S}^1)$ whose Weyl symbol, expressed in polar and cylindrical coordinates, is the function $\mathcal{O}(A, \tau, \varphi, \tau)$. Then*

$$(4.1) \qquad \langle \mu, \nu | O | \mu, \nu \rangle = \int_{\mathbb{T}^n \times \mathbb{S}^1} \mathcal{O}(\mu\hbar, \nu\hbar, \varphi, t) d\varphi dt + O(\hbar),$$

where $x_j + i\xi_j = \sqrt{A_j} e^{i\varphi_j}$ for any $j = 1, \ldots, n$.

We concatenate analogs of Theorems 1.3 and 1.4 in the following.

Theorem 4.2. *Let γ be a non-degenerate elliptic periodic trajectory of the Hamiltonian flow generated by a proper smooth Hamiltonian function H. Let $\mathcal{P}_p^k$ be functions satisfying*

$$(4.2) \qquad \begin{aligned} &\mathcal{P}_p^0(x, t, \xi, \tau) = e^{-2i\pi pt}\tau \quad and \\ &\mathcal{P}_p^k(x, t, \xi, \tau) = e^{-2i\pi pt}\mathcal{R}^k(x, \xi), \ k = 1, \ldots, 2n^2 + n, \end{aligned}$$

*in a local symplectic system of coordinates $(x, t, \xi, \tau) \in T^*(\mathbb{R}^n \times \mathbb{S}^1)$ such that $\gamma = \{x = \xi = \tau = 0\}$, with the property that $\mathcal{R}^k(0) = \nabla \mathcal{R}^k(0) = 0$ **and the Hessians $d^2\mathcal{R}^k(0)$ are linearly independent**.*

Let Φ be the formal (unknown a priori) symplectomorphism which leads to the Birkhoff normal form near γ and (A, τ, φ, t) the corresponding (formal and also unknown a priori) action-angle coordinates such that $\gamma = \{A = \tau = 0\}$. Let us define near $A = \tau = 0$ the "average" quantities

$$(4.3) \qquad \overline{\mathcal{P}_p^k}(A, \tau) := \int_{\mathbb{T}^n \times \mathbb{S}^1} \mathcal{P}_p^k \circ \Phi(A, \tau, \varphi, t) d\varphi dt.$$

Then the knowledge of $\nabla \overline{\mathcal{P}_p^k}(0, 0)$ for $k = 1, \ldots, 2n^2 + n$ determines (in a constructive way) an explicit system of Fermi coordinates near γ.

Moreover, now let $(x, t, \xi, \tau) \in T^*(\mathbb{R}^n \times \mathbb{S}^1)$ *be* **any** *system of Fermi coordinates near* γ *and let* $\mathcal{O}_{mnp}, \mathcal{O}_p$ *for* $(m, n, p) \in \mathbb{N}^{2n} \times \mathbb{Z}$ *be functions satisfying in a neighborhood of* γ:

(4.4)
$$\begin{cases} \mathcal{O}_{mnp}(x, t, \xi, \tau) = e^{-i2\pi pt} \prod_{j=1}^{n} \left(\frac{x_j + i\xi_j}{\sqrt{2}} \right)^{m_j} \left(\frac{x_j - i\xi_j}{\sqrt{2}} \right)^{n_j} + O\left((|A| + |\tau|)^{\frac{|m| + |n| + 1}{2}} \right), \\ \mathcal{O}_p(x, t, \xi, \tau) = e^{-i2\pi pt}\tau + O\left((|A| + |\tau|)^{\frac{3}{2}} \right). \end{cases}$$

Then the knowledge of the Birkhoff normal form near γ *and of the Taylor expansion at* $A = \tau = 0$ *up to order* N *of* $\overline{\mathcal{O}}_{mnp}, \overline{\mathcal{O}}_q$, *defined as in (4.3) for*

(1) $0 < |m| + |n| \le N$,
(2) $\forall j = 1, \ldots, n$, $m_j = 0$ **or** $n_j = 0$,
(3) $p \in \mathbb{Z}$, $q \in \mathbb{Z}^*$,

determines the Taylor expansion of the "true" Hamiltonian H *up to the same order in the picked-up system of Fermi coordinates.*

Proof. Let us first prove the second part of Theorem 4.2. Let $(x, t, \xi, \tau) \in T^*(\mathbb{R}^n \times \mathbb{S}^1)$ be a system of Fermi coordinates near γ. Let, for $N \ge 3$, F_N be the principal symbol of W_N. With the notation of Proposition 2.2 we write

(4.5)
$$F_N(z, t, \bar{z}, \tau) = \sum_{|j| + |k| + 2s = N} \alpha_{0jks}(t)e^{2i\pi pt} z^j \bar{z}^k \tau^s.$$

Let F satisfy

(4.6)
$$F \sim \sum_{N=3}^{+\infty} F_N.$$

With the notation of the proof of Proposition 2.2, we have

(4.7)
$$H \circ \exp(\chi_F)(x, t, \xi, \tau) \sim h(p, \tau, 0).$$

Let $N \ge 3$, $(m, n, p, q) \in \mathbb{N}^{2n} \times \mathbb{Z} \times \mathbb{Z}^*$, and $(j, k, s) \in \mathbb{N}^{2n+1}$ satisfy:

(4.8) $0 < |m| + |n| \le N$, $m_i n_i = 0 \; \forall i \in [[1, n]]$, $|j| + |k| + 2s = N$.

Then, as seen in the proof of Lemma 2.15, if σ_{jks}^1 and σ_{jks}^2 are the symbols of $\{\alpha_{0jks}(t)e^{2i\pi pt} z^j \bar{z}^k \tau^s, \mathcal{O}_{mnp}\}$ and $\{\alpha_{0jks}(t)e^{2i\pi pt} z^j \bar{z}^k \tau^s, \mathcal{O}_q\}$ respectively, we have, if $j + m = k + n$,

(4.9)
$$\int_{\mathbb{T}^n \times \mathbb{S}^1} \sigma_{jks}^1(A, \tau, \varphi, t) d\varphi dt = ic_p(\alpha_{0jks})A^{\max(j,k)}\tau^s \left(\sum_{i=1}^{n} \frac{k_i m_i - j_i n_i}{A_i} + \frac{2\pi ps}{\tau} \right)$$
$$+ O\left((|A| + |\tau|)^{\frac{N + |m| + |n| - 1}{2}} \right)$$

and if $j + m \ne k + n$,

(4.10)
$$\int_{\mathbb{T}^n \times \mathbb{S}^1} \sigma_{jks}^1(A, \tau, \varphi, t) d\varphi dt = O\left((|A| + |\tau|)^{\frac{N + |m| + |n| - 1}{2}} \right).$$

We also have, if $j = k$,

(4.11) $\displaystyle \int_{\mathbb{T}^n \times \mathbb{S}^1} \sigma_{jks}^2(A, \tau, \varphi, t) d\varphi dt = ic_q(\alpha_{0jks})2\pi q(1 + s)A^j \tau^s + O\left((|A| + |\tau|)^{\frac{N+1}{2}} \right)$

and if $j \neq k$,

$$(4.12) \qquad \int_{\mathbb{T}^n \times \mathbb{S}^1} \sigma_{jks}^2(A, \tau, \varphi, t) d\varphi dt = O\left((|A| + |\tau|)^{\frac{N+1}{2}}\right).$$

Now, let us set $F_2 = 0$ and assume that the function $F_{\leq N-1}$ has been determined for some $N \geq 3$. Then for $l \geq 2$, $\overbrace{\{F_{\leq N}, \{\ldots, \{F_{\leq N}, \mathcal{O}_{mnp}\}\}\}}^{l \text{ times}}$ and $\overbrace{\{F_{\leq N}, \{\ldots, \{F_{\leq N}, \mathcal{O}_q\}\}\}}^{l \text{ times}}$ are determined modulo $O\left((|A| + |\tau|)^{\frac{N+|m|+|n|-1}{2}}\right)$ and $O\left((|A| + |\tau|)^{\frac{N+1}{2}}\right)$ respectively.

Therefore, by equalities (4.9) and (4.10), $\overline{\mathcal{O}}_{mnp}(A, \tau)$ is equal, modulo $O\left((|A| + |\tau|)^{\frac{N+|m|+|n|-1}{2}}\right)$ and known terms, to

$$(4.13) \qquad \sum_{\substack{|j|+|k|+2s=N \\ j=m=k+n}} ic_p(\alpha_{0jks})A^{\max(j,k)}\tau^s \left(\sum_{i=1}^n \frac{k_i m_i - j_i n_i}{A_i} + \frac{2\pi ps}{\tau}\right)$$

and by equalities (4.11) and (4.12), $\overline{\mathcal{O}}_q(A, \tau)$ is equal, modulo known terms and $O\left((|A| + |\tau|)^{\frac{N+1}{2}}\right)$, to

$$(4.14) \qquad \sum_{2|j|+2s=N} ic_q(\alpha_{0jjs})2\pi q(1+s)A^j\tau^s.$$

Thus, just as in the proof of Proposition 2.14, let $(m, n, p, q) \in \mathbb{N}^{2n} \times \mathbb{Z} \times \mathbb{Z}^*$ run over all possible values under condition (4.8). We determine every function α_{0jks}, $|j| + |k| + 2s = N$, hence F_N, which concludes the proof of the second part of Theorem 4.2.

The proof of the first part of the theorem follows the same strategy with respect to Theorem 1.3 as the proof of the second part with respect to Proposition 2.14. $\square$

The last result of this paper will be the classical analog of Theorems 1.10 and 1.14.

Let us recall [10] that in the case where $H(x, \xi) = \xi^2 + V(x)$ the classical normal form determines the Taylor expansion of the potential when the latter is invariant by the symmetry $x_i \to -x_i$ for each $i \in [[1, n]]$. In the general case the Taylor expansion of the averages, in the sense of (4.3), of a finite number of classical observables are necessary to recover the full potential.

Let us assume $H \in \mathcal{C}^\infty(T^*\mathcal{M}, \mathbb{R})$ has a global minimum at $z_0 \in T^*\mathcal{M}$, and let $d^2 H_p(z_0)$ be the Hessian of H at z_0. Let us define the matrix Ω defined by $d^2 H_p(z_0)(\cdot, \cdot) =: \omega_{z_0}(\cdot, \Omega^{-1}\cdot)$, where $\omega_{z_0}(\cdot, \cdot)$ is the canonical symplectic form of $T^*\mathcal{M}$ at z_0. The eigenvalues of Ω being purely imaginary, we denote them by $\pm i\theta_j$ with $\theta_j > 0$, $j \in [[1, n]]$. Let us assume that $\theta_j, j \in [[1, n]]$, are rationally independent.

Theorem 4.3. *The statement of Theorem* 4.2 *remains valid verbatim by replacing* γ *by* z_0 *and ignoring the variables* t, τ.

Let us now enunciate the classical analog of Theorem 1.14 in the case of a Schrödinger operator with potential $V \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$:

Theorem 4.4. *Let q_0 be a global non-degenerate minimum of V on $\mathcal{M}$. Let us assume that the square roots of the eigenvalues of $d^2V(q_0)$ are linearly independent over the rationals.*

*Let $\mathcal{P}^k$, $k = 1, \ldots, \frac{n(n+1)}{2}$, be smooth functions on $\mathcal{M}$ such that $\mathcal{P}^k(q_0) = \nabla\mathcal{P}^k(q_0) = 0$ **and the Hessians** $d^2\mathcal{P}^k(q_0)$ **are linearly independent** (an example of such potentials is the family $\mathcal{Q}_{ij}(x) = x_i x_j$ in a local system of coordinates centered at q_0).*

Then the knowledge of the Birkhoff normal form near q_0 and of the Taylor expansion at $A = 0$ up to the (finite number of) "average" $\overline{\mathcal{P}^k}(A)$ determines (in a constructive way) an explicit system of Fermi coordinates.

Moreover, let $(x, \xi) \in T^*\mathbb{R}^n$ be any system of Fermi coordinates centered at $(q_0, 0)$ and let $\mathcal{O}_{m0}$ be defined in Theorem 1.14.

Then the knowledge of the Taylor expansion at $A = 0$ up to order $N \geq 3$ of the (finite number of) "average" quantities $\overline{\mathcal{O}_{m0}}$ as in (4.3) together with the Birkhoff normal form itself, determines the Taylor expansion up to order N of V at q_0 in the picked-up system of coordinates.

In line with the proof of Theorem 4.2 the proofs of Theorems 4.4 and 4.3 are easy adaptations of the proofs of Theorems 1.14 and 1.10. We omit them here.

APPENDIX A. LEMMAS ON LINEAR AND BILINEAR ALGEBRA

Lemma A.1. *Let q be a positive quadratic form on $\mathbb{R}^{2n}$. Then there exists a canonical endomorphism ϕ on $\mathbb{R}^{2n}$, and an n-tuple of positive real numbers $(\lambda_1, \ldots, \lambda_n)$, defined as the imaginary part of the eigenvalues of the positive imaginary part of the endomorphism defined by*

$$(A.1) \qquad \langle \cdot; a(\cdot)\rangle_q = \omega(\cdot, \cdot),$$

where $\langle \cdot; \cdot \rangle_q$ is the scalar product associated to q and ω is the canonical symplectic form on $\mathbb{R}^{2n}$, and such that:

$$(A.2) \qquad \forall (x, \xi) \in \mathbb{R}^{2n}, \ q(\phi(x, \xi)) = \sum_{i=1}^n \lambda_i(x_i^2 + \xi_i^2).$$

Moreover, if the real numbers $\lambda_1, \ldots, \lambda_n$ are pairwise different, and ϕ' is an endomorphism of $\mathbb{R}^{2n}$, then ϕ' is canonical and satisfies (A.2) if and only if there exists an orthogonal isomorphism u on $\mathbb{R}^{2n}$ whose restriction to the plane spanned by $(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_i})$ (for any $i \in [[1, n]]$) is a rotation, such that $\phi' = \phi \circ u$.

Proof of Lemma A.1. a is antisymmetric with respect to q, and therefore there exists a q-orthonormal basis of $\mathbb{R}^{2n}$ $(u_1, \ldots, u_n, v_1, \ldots, v_n)$ and an n-tuple of positive real numbers $(\lambda_1, \ldots, \lambda_n)$ such that, for $j \in [[1, n]]$,

$$(A.3) \qquad \lambda_j a(u_j) = -v_j \text{ and } \lambda_j a(v_j) = u_j.$$

Now let us set, for $j \in [[1, n]]$,

$$(A.4) \qquad \tilde{u}_j = \sqrt{\lambda_j} u_j \text{ and } \tilde{v}_j = \sqrt{\lambda_j} v_j.$$

Then $(\tilde{u}_1, \ldots, \tilde{u}_n, \tilde{v}_1, \ldots, \tilde{v}_n)$ is a q-orthogonal basis of $\mathbb{R}^{2n}$ satisfying, for $j \in [[1, n]]$, $q(\tilde{u}_j) = \lambda_j$ and $q(\tilde{v}_j) = \lambda_j$, and the preceding properties together with

(A.1) imply that it is also a symplectic basis, which concludes the proof of the first part of Lemma A.1.

To prove the second part of Lemma A.1, let us consider another symplectic and orthogonal basis $(u'_1, \ldots, u'_n, v'_1, \ldots, v'_n)$, where, for $j \in [[1, n]]$, the q-norm of u'_j and v'_j is λ_j. Then, by (A.2), for any $j \in [[1, n]]$, $a(u'_j)$ is orthogonal to any vector of the basis but v'_j and $\langle v'_j, a(u'_j) \rangle_q = w(v'_j, u'_j) = -1$, therefore $\lambda_j a(u'_j) = -v'_j$ and, by the same argument, $\lambda_j a(v'_j) = u'_j$.

Therefore, the plane spanned by (u_j, v_j) and the plane spanned by (u'_j, v'_j) are both the kernel of $a^2 + \lambda_j^2$ (2-dimensional since we made the additional assumption that the λ_i's are pairwise different). Therefore, if ϕ and ϕ' are the endomorphisms which send the canonical basis of $\mathbb{R}^{2n}$ to the basis $(\tilde{u}_1, \ldots, \tilde{u}_n, \tilde{v}_1, \ldots, \tilde{v}_n)$ and the basis $(u'_1, \ldots, u'_n, v'_1, \ldots, v'_n)$ respectively, then the restriction of $\phi^{-1} \circ \phi'$ to any plane spanned by $(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_i})$ (for any $i \in [[1, n]]$) is an orthogonal symplectomorphism from the plane to itself, which is a rotation. $\square$

Let σ be the permutation of $[[1, 2n]]$ defined by:

$$(A.5) \qquad \forall i \in [[1, 2n]], \ \sigma(i) = \begin{cases} 2i - 1 & \text{if } i \leq n, \\ 2(i - n) & \text{if } i \geq n + 1, \end{cases}$$

and let M_σ be the associated permutation matrix (*i.e.* for any $(i, j) \in [[1, 2n]]^2$, $(M_\sigma)_{ij} = \delta_{\sigma(i), j}$).

Now, let us set, for any matrix $S \in \mathcal{M}_{2n}(\mathbb{R})$,

$$(A.6) \qquad S_\sigma = M_\sigma^{-1} S M_\sigma.$$

Let us also denote by $L_{S,i,k}$ for $(i, k) \in [[1, 2n]] \times [[1, n]]$ the vector of $\mathbb{R}^2$ defined by $L_{S,i,k} = \begin{pmatrix} (S_\sigma)_{i,2k-1} \\ (S_\sigma)_{i,2k} \end{pmatrix} \in \mathbb{R}^2$. Then, for $(i, k) \in [[1, n]]^2$, $\mathfrak{s}_{i,k}$ will be the matrix of size 2 whose first line is ${}^t L_{S,2i-1,k}$ and second line is ${}^t L_{S,2i,k}$.

Lemma A.2. *Let A be a positive matrix of size $2n$. Let $\mathcal{S}$ be the (non-empty by Lemma A.1) set of symplectic matrices satisfying*

$$(A.7) \qquad {}^t S A S = \left(\begin{array}{c|c} D_\lambda & 0 \\ \hline 0 & D_\lambda \end{array} \right),$$

where D_λ is the diagonal matrix with $(\lambda_1, \ldots, \lambda_n)$ an n-tuple of positive diagonal elements, which we assume pairwise different. Then:

(1) *The family $(\langle L_{S,i,k}; L_{S,j,k} \rangle)_{(i,j) \in [[1,2n]]^2, k \in [[1,n]]}$ is independent of matrix $S \in \mathcal{S}$.*

(2) *Once the preceding invariants of $\mathcal{S}$ are given, one can construct explicitly a particular matrix of $\mathcal{S}$ (hence all of them by Lemma A.1).*

Proof of Lemma A.2. Let us first prove the first point. Let $(S, T) \in \mathcal{S}^2$. By Lemma A.1, there exist n matrices belonging to $SO_2(\mathbb{R})$ and denoted by $O_1, \ldots, O_n$, such that:

$$(A.8) \qquad T_\sigma = S_\sigma \begin{pmatrix} O_1 & & \\ & \ddots & \\ & & O_n \end{pmatrix} = \begin{pmatrix} \mathfrak{s}_{1,1} O_1 & \cdots & \mathfrak{s}_{1,n} O_n \\ \vdots & & \vdots \\ \mathfrak{s}_{n,1} O_1 & \cdots & \mathfrak{s}_{n,n} O_n \end{pmatrix}$$

and (A.8) is equivalent to:

$$(A.9) \qquad \forall (i, k) \in [[1, 2n]] \times [[1, n]], L_{T,i,k} = {}^t O_k L_{S,i,k}.$$

Hence $(\langle L_{S,i,k}; L_{S,j,k}\rangle)_{(i,j)\in[[1,2n]]^2, k\in[[1,n]]}$ does not depend on the matrix $S \in \mathcal{S}$ and the first point of Lemma A.2 is proven.

Now, let $S \in \mathcal{S}$, and let $(a_{ijk})_{(i,j)\in[[1,2n]]^2, k\in[[1,n]]}$ be the family defined by

$$(A.10) \qquad \forall(i,j) \in [[1,2n]]^2, \ \forall k \in [[1,n]], a_{ijk} = \langle L_{S,i,k}; L_{S,j,k}\rangle.$$

Let us assume that this family is given. Two vectors u and v of $\mathbb{R}^2$ are independent if and only if: $\langle u; v\rangle^2 < \langle u; u\rangle\langle v; v\rangle$. Since the matrix S is invertible, one can choose, for any $k \in [[1,n]]$, a couple of indices $(i_k, j_k) \in [[1,2n]]^2$ such that

$$(A.11) \qquad a_{i_k j_k k}^2 < a_{i_k i_k k} a_{j_k j_k k}.$$

Let $k \in [[1,n]]$ and let us choose a vector $v_{i_k k}$, whose norm is $\sqrt{a_{i_k i_k k}} > 0$. The system of equations with unknown $v \in \mathbb{R}^2$,

$$(A.12) \qquad \begin{cases} \langle v_{i_k k}; v\rangle & = a_{i_k j_k k}, \\ \langle v; v\rangle & = a_{j_k j_k k}, \end{cases}$$

admits exactly two solutions (by (A.11)), denoted by $v_{j_k k}^+$ and $v_{j_k k}^-$ obtained from one another by orthogonal symmetry R_k of axis the line spanned by $v_{i_k k}$.

Let us set $v_{i_k k}^- = v_{i_k k}^+ = v_{i_k k}$. Since the families $(v_{i_k k}^+, v_{j_k k}^+)$ and $(v_{i_k k}^-, v_{j_k k}^-)$ are two bases of $\mathbb{R}^2$, for any $i \in [[1,2n]] \setminus \{i_k, j_k\}$, each one of the two systems

$$(A.13) \qquad \begin{cases} \langle v_{i_k k}; v\rangle & = a_{i_k i k}, \\ \langle v_{j_k k}^+; v\rangle & = a_{j_k i k}, \end{cases} \quad \text{and} \quad \begin{cases} \langle v_{i_k k}; v\rangle & = a_{i_k i k}, \\ \langle v_{j_k k}^-; v\rangle & = a_{j_k i k}, \end{cases}$$

admits exactly one solution denoted respectively by v_{ik}^+ and v_{ik}^-, and satisfying relation $v_{ik}^- = R_k v_{ik}^+$.

We are now able to construct 2^n matrices $(T_A)_{A \in \mathcal{P}([[1,n]])}$ defined, for $A \in \mathcal{P}([[1,n]])$, by

$$(A.14) \qquad \forall(i,k) \in [[1,2n]] \times [[1,n]], L_{T_A,i,k} = \begin{cases} v_{ik}^+ & \text{if } k \in A, \\ v_{ik}^- & \text{if else.} \end{cases}$$

In order to prove the second point of Lemma A.2, it is sufficient to prove the following assertions:

(1) There exists at least one set $A \in \mathcal{P}([[1,n]])$, such that $T_A \in \mathcal{S}$.
(2) There is at most one set $A \in \mathcal{P}([[1,n]])$, such that T_A is symplectic.

Indeed, once those two assertions are proved, there will be exactly one set $A \in \mathcal{P}([[1,n]])$ such that T_A is symplectic, and it will be an element of $\mathcal{S}$, constructed from the values of the family $(a_{ijk})_{(i,j)\in[[1,2n]]^2, k\in[[1,n]]}$ only.

Let us prove the first assertion. Let O_k be the unique element of $SO_2(\mathbb{R})$ for any $k \in [[1,n]]$, such that $L_{S,i_k,k} = O_k v_{i_k k}$ (where S is a particular matrix of $\mathcal{S}$).

The system (A.12) is equivalent to

$$(A.15) \qquad \begin{cases} \langle L_{S,i_k,k}; O_k v\rangle & = a_{i_k j_k k}, \\ \langle O_k v; O_k v\rangle & = a_{j_k j_k k}, \end{cases}$$

which admits exactly two solutions: $v_{j_k k}^+$ and $v_{j_k k}^-$. Hence, for any $k \in [[1,n]]$:

$$(A.16) \qquad L_{S,j_k,k} = O_k v_{j_k k}^+ \ \text{ or } \ L_{S,j_k,k} = O_k v_{j_k k}^-.$$

Let us define the set A by

$$(A.17) \qquad A = \{k \in \mathbb{N} \,|\, L_{S,j_k,k} = O_k v_{j_k k}^+\}.$$

Since each system (A.13) admits a unique solution, we obtain

$$\text{(A.18)} \quad \forall (i,k) \in [[1, 2n]] \times [[1, n]], L_{S,i,k} = \begin{cases} O_k v_{ik}^+ & \text{if } k \in A, \\ O_k v_{ik}^- & \text{if else}, \end{cases}$$
$$= O_k L_{T_A, i, k},$$

that is,

$$\text{(A.19)} \quad T_{A,\sigma} = S_\sigma \begin{pmatrix} O_1 & & \\ & \ddots & \\ & & O_n \end{pmatrix}$$

and $T_A \in \mathcal{S}$ by Lemma A.1.

In order to prove the second assertion, let us use the following lemma.

Lemma A.3. *For any symplectic matrix B of size $2n$, we have*

$$\text{(A.20)} \quad \forall k \in [[1, n]], \ \sum_{i=1}^{n} \det(\mathfrak{b}_{i,k}) = 1.$$

If A_1 and A_2 are two parts of $[[1, n]]$, we get from (A.14) and relation $v_{ik}^- = R_k v_{ik}^+$ that

$$\text{(A.21)} \quad \forall (i,k) \in [[1, 2n]] \times [[1, n]], L_{T_{A_2}, i, k} = \begin{cases} R_k L_{T_{A_1}, i, k} & \text{if } k \in A_1 \Delta A_2, \\ L_{T_{A_1}, i, k} & \text{if else}, \end{cases}$$

where $A_1 \Delta A_2$ is the symmetric difference of A_1 and A_2: $A_1 \Delta A_2 = (A_1 \setminus A_2) \cup (A_2 \setminus A_1)$. Hence,

$$\text{(A.22)} \quad \forall k \in [[1, n]], \ \sum_{i=1}^{n} \det \left((\mathfrak{t}_{A_2})_{i,k} \right) = \epsilon_k \sum_{i=1}^{n} \det \left((\mathfrak{t}_{A_1})_{i,k} \right),$$

where, for $k \in [[1, n]]$, $\epsilon_k = -1$ if $k \in A_1 \Delta A_2$, $\epsilon_k = 1$ if else. Since $A_1 \Delta A_2 = \emptyset$ if and only if $A_1 = A_2$, there exists at most one part A of $[[1, n]]$ such T_A is symplectic. The second assertion, hence the second point of Lemma A.3, is proven. $\square$

Proof of Lemma A.3. Since B is a symplectic matrix, the matrix B_σ satisfies

$$\text{(A.23)} \quad {}^t B_\sigma J_\sigma B_\sigma = J_\sigma.$$

It is sufficient, for $k \in [[1, n]]$, to read equality (A.23) at line $2k$ and column $2k - 1$ to obtain:

$$\text{(A.24)} \quad \sum_{i=1}^{n} \det(\mathfrak{b}_{i,k}) = 1.$$

$\square$

Lemma A.4. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a positive matrix whose eigenvalues are pairwise different. Let D be a diagonal matrix, similar to A. Then there exist exactly 2^n orthogonal matrices conjugating A to D, and they are obtained one from another by a possible change of the sign of each column.*

Proof of Lemma A.4. As A is positive, there exists an orthogonal matrix Q_1 such that

$$\text{(A.25)} \quad Q_1^{-1} A Q_1 = {}^t Q_1 A Q_1 = D.$$

Let $Q_2 \in GL_n(\mathbb{R})$. Then Q_2 is orthogonal and satisfies: $Q_2^{-1}AQ_2 = D$ if and only if $Q_2^{-1}Q_1$ is an orthogonal matrix which commutes to D, that is, because the diagonal elements of D are pairwise different, if and only if $Q_2^{-1}Q_1$ is an orthogonal diagonal matrix. Finally, Q_2 is orthogonal and satisfies: $Q_2^{-1}AQ_2 = D$ if and only if $Q_2^{-1}Q_1$ is diagonal and its elements belong to $\{-1, 1\}$, that is, if Q_2 is obtained from Q_1 by a possible change of the sign of each column. $\square$

APPENDIX B. REALIZING THE POINCARÉ ANGLES

B.1. **The periodic trajectory case.** In this section we indicate how different systems of Fermi coordinates and different Birkhoff normal forms exist for any realization of the Poincaré angles as real numbers and we show how those normal forms are linked to each other. Thus, our results are independent of this ambiguity.

Proposition B.1. *Under the hypothesis of Theorem* 1.4, *the knowledge of the coefficients of the trace formula determines the quantities* $e^{i\theta_i}, i = 1, \ldots, n$. *Moreover, let us denote by* $\mathcal{B}_{\theta_1, \ldots, \theta_n}(x, \xi, \tau) = E + \sum_{i=1}^{n} \theta_i \frac{x_i^2 + \xi_i^2}{2} + \tau + O((x^2 + \xi^2 + |\tau|)^2)$ *the Birkhoff normal form of* H_p *associated to a given choice of angles* θ_i. *For* $k \in \mathbb{Z}^n$, *let* $h_k(x, \xi) = \sum \pi k_i(x_i^2 + \xi_i^2)$ *and let* Φ_k *be the symplectomorphism defined, with the notation of* (2.93), *by*

$$(\text{B.1}) \qquad \Phi_k(x, \xi, t, \tau) = \left(\exp(t\chi_{h_k})(x, \xi), t, \tau + \pi \sum k_i(x_i^2 + \xi_i^2) \right).$$

Then $\mathcal{B}_{\theta_1, \ldots, \theta_n} \circ \Phi_k = \mathcal{B}_{\theta_1 + 2k_1\pi, \ldots, \theta_n + 2k_n\pi}$.

Proof. The first part of the assertion belongs to Fried [6]. The fact that Φ_k is a symplectomorphism can be checked directly. Moreover one sees immediatly that it conjugates the quadratic in (x, ξ)/linear in the τ part of $\mathcal{B}_{\theta_1, \ldots, \theta_n}$ to the one of $\mathcal{B}_{\theta_1 + 2k_1\pi, \ldots, \theta_n + 2k_n\pi}$. Moreover $\mathcal{B}_{\theta_1, \ldots, \theta_n} \circ \Phi_k$ is a function of τ and $x_i^2 + \xi_i^2$ only and it is easy to verify that the algorithmic constructions of the two normal forms are covariantly conjugated by Φ_k. Therefore, it is equal to $\mathcal{B}_{\theta_1 + 2k_1\pi, \ldots, \theta_n + 2k_n\pi}$. $\square$

B.2. **The "bottom of the well" case.**

Proposition B.2. *Under the hypotheses of Theorems* 1.10 *and* 1.14, *the knowledge of the spectrum of* $H(x, \hbar D_x)$ *in* $[H_p(z_0), H_p(z_0) + \epsilon]$, $\hbar = o(\epsilon)$, *determines the* θ_i's *up to permutation.*

Proof. By the quantum normal Birkhoff form construction we know that the bottom part of the spectrum is $\{H_p(z_0) + \sum_{i=1}^{n} \theta_i(\mu_i + 1/2)\hbar + O(\hbar^2), |\mu\hbar| = O(\epsilon)\}$. Therefore, the bottom part of the spectrum determines the set $\Lambda = \{\sum_{i=1}^{n} \theta_i(\mu_i + 1/2), \mu \in \mathbb{N}^n\}$. Let us now assume that the θ_i's are arranged in increasing order. $\theta_1/2$ is then equal to the minimum of Λ. By induction, if $\theta_1, \ldots, \theta_k$ are known for some k, $1 \leq k < n$, let us define $\Lambda_k = \{\sum_{i=1}^{k} \theta_i(\mu_i + 1/2)\}$. Let us set $\lambda_{k+1} := \min \Lambda \cap \Lambda_k^c$. We easily see that $\theta_{k+1} = 2\lambda_{k+1} - \sum_{i=1}^{k} \theta_i$, which concludes the proof. $\square$

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