Left, right and weak model categories

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Definition (Quillen, 1967)

A Quillen model category is a category with all limits and colimits, and three classes of maps called cofibrations, fibrations and weak equivalences such that:

1. Weak equivalences contain isomorphisms and satisfy 2-out-of-3.
2. (trivial cofibration,fibration) is a weak factorization system.
3. (cofibration, trivial fibration) is a weak factorization system.

- As usual *trivial cofibration* means “cofibration and weak equivalence” and *trivial fibration* means fibration and weak equivalence.
- “All limits and colimits” can be replaced by “there is an initial object, a terminal object, fibrations have pullbacks along arrow between fibrant objects and cofibration have pushouts along arrows between cofibrant objects”.

Definition (Quillen, 1967)

A Quillen model category is a category with limits and colimits, and three classes of maps called cofibrations, fibrations and weak equivalences such that:

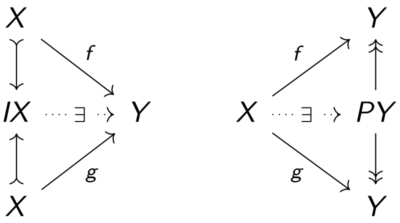
1. Weak equivalences contain isomorphisms and satisfy 2-out-of-3.
2. Every trivial cofibration has the left lifting property against fibrations.
3. Every arrow can be factored as a trivial cofibration followed by a fibration.
4. Every cofibration has the left lifting property against trivial fibrations.
5. Every arrow can be factored as a cofibration followed by a trivial fibration.
6. Cofibrations, fibrations and weak equivalences are closed under retract.

What is it good for ?

Given an object X, we call a cylinder object IX or a path object PX objects that fits in factorizations:

$$X \coprod X \rightarrowtail IX \overset{\sim}{\twoheadrightarrow} X \qquad X \overset{\sim}{\rightarrowtail} PX \twoheadrightarrow X \times X$$

If $f, g : X \to Y$ are two maps one defines the left and right homotopy relation $f \sim_l g$ and $f \sim_r g$ as the existence of dotted maps respectively in



Using the lifting properties (in a clever way), one shows

Proposition

If X is cofibrant and Y fibrant, then the left and right homotopy relation on maps from X to Y coincide and form an equivalence relation compatible to composition.

Proposition

The category $Ho(\mathcal{C})$ of bifibrant objects of $\mathcal{C}$ with homotopy class of maps between them is equivalent to the formal localization $\mathcal{C}[\mathcal{W}^{-1}]$.
The class $\mathcal{W}$ of weak equivalence is strongly saturated, i.e. is exactly the class of maps inverted by $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$.

But one can do better: homotopies between two maps $X \rightrightarrows Y$, are maps $IX \to Y$ (or $X \to PY$), so one can talk about "homotopies between homotopies". Up to some technical details, this allows to construct an ∞-category

$$h_\infty \mathcal{C}$$

where 0-cell are bifibrant objects, 1-cells are maps between them, 2-cells are homotopy between maps, 3-cells are homotopies between homotopies (compatible to the boundary), etc...

Proposition

$h_\infty \mathcal{C}$ is equivalent to the ∞-categorical (Dwyer-Kan) localization $\mathcal{C}[\mathcal{W}^{-1}]$.

Proposition

$h_\infty \mathcal{C}$ has (finite) limits and colimits, with explicit way to compute them.

At some point in the 90's some people were interested in

Question

If $\mathcal{C}$ is a combinatorial monoidal model category, can we construct a transfered model strucutre on the category $Mon(\mathcal{C})$ of monoid in $\mathcal{C}$? (i.e. such that a map between monoid is an equivalence or a fibration if and only if it is one in $\mathcal{C}$).

Theorem (Schwede - Shipley, 1997)

We can if $\mathcal{C}$ satisfies the "Monoid axiom": The arrow of the form $A \otimes j$ where $A \in \mathcal{C}$ and j a trivial cofibration are weak equivalences, as well as all transfinite composite of pushouts of such arrows.
This is the case for example if all objects of $\mathcal{C}$ are cofibrant.

A little latter (1998) Hovey observed that if $\mathcal{C}$ is a combinatorial monoidal model category, Mon($\mathcal{C}$) is "almost" a Quillen model category:

Proposition (Hovey, 1998)

In Mon($\mathcal{C}$) we have

1. *Weak equivalences contain isomorphisms and satisfy 2-out-of-3.*
2. *Trivial cofibrations with a cofibrant domain have the left lifting property against fibrations.*
3. *Every arrow with a cofibrant domain can be factored as a trivial cofibration followed by a fibration.*
4. *Cofibration have left the lifting property against trivial fibration.*
5. *Every arrow can be factored as a cofibration followed by a trivial fibration.*
6. *Cofibrations, fibrations and weak equivalences are closed under retract.*

And these are enough for all the claim about model categories we made above.

Definition (Spitzweck, 2001)

A (Spitzweck) left semi-model category is a category $\mathcal{C}$ with limits and colimits endowed with three classes of maps cofibrations, fibrations and weak equivalences such that*

1. Weak equivalences contain isomorphisms and satisfy 2-out-of-3.
2. Trivial cofibrations with a cofibrant domain have the left lifting property against fibrations.
3. Every arrow with a cofibrant domain can be factored as a trivial cofibration followed by a fibration.
4. Cofibration have left the lifting property against trivial fibration.
5. Every arrow can be factored as a cofibration followed by a trivial fibration.
6. Cofibrations, fibrations and weak equivalences are closed under retract.

Spitzweck called them "J-semi-model categories".

Remark

$*$: *To the previous definition one needs to add the following axiom.*

7. *A pullback of a fibration is a fibration.*

And if one wants infinite homotopy limits to be well behaved as well:

8. *A transfinite composition of fibrations is a fibration.*

In what follows, this type of axiom (as well as its dual) will be ommited from the definitions. But, at least the first one should be added if it cannot be deduced from the rest of the definition.

In the combinatorial/premodel setting we will move to latter, fibration will be be the right class of a weak factorization system, so these will be automatically satisfied.

Example (Spitzweck, 2001)

The category of P-algebra, for P a Σ-cofibrant operads in a monoidal closed combinatorial monoidal category, has a (transfered) left semi-model structure.

Example (Spitzweck, 2001)

The category of operads in a monodial closed combinatorial model category has a left semi-model structures, (transfered from the projective model strucutre on collection).

In both case, some stronger assumptions (like the monoid axiom) allows to get Quillen model categories.

Example (Kapulkin-Lumsdain, Isaev ?, 2016)

The category of contextual categories with Id and Σ-types (and possibly Π-types) carries a left semi-model structures.

Examples of (combinatorial) left semi-model category that are provably not Quillen model category are rare: in general, one just do not know how to prove the axiom of Quillen, this is very frequent for transfered model structure on categories of algebra for a monad or an operad. But there are some:

Example (From David White)

The category of (non-reduced) symmetric operads in $\mathrm{Ch}(\mathbb{F}_2)$ has a left semi-model structure transfered from the projective model structure on $\mathrm{Ch}(\mathbb{F}_2)$ which is not a Quillen model structure.

Another source of left semi-model categories

Proposition (Barwick 2007, Batanin - White)

Left Bousfield localization of combinatorial left semi-model category exists as left semi-model categories.

By comparison:

Proposition

Left Bousfield localization of left proper combinatorial Quillen model categories exists and are combinatorial left proper Quillen model categories.

Overall, everthing works the same in a left semi-model category. Here are the most notable exception:

- If you start from a general object, you can't directly construct a fibrant replacement $X \overset{\sim}{\rightarrowtail} X^{\text{fib}}$. Indeed for this you need X to be cofibrant. So, always take cofibrant replacement before fibrant replacement.
- The category of fibrant object, is *not* a category of fibrant object in the sense of Brown. Indeed, the "weak equivalence, fibration" of a map, usually obtained using a trivial cofibration, only works if the domain is cofibrant.
- Fibrant object that are not cofibrant might not have "path object" $X \overset{?}{\to} PX \twoheadrightarrow X \times X$.

Instead they have "Weak path objects":

$$\begin{array}{ccc} EX & \overset{\sim}{\longrightarrow} & PX \\ {\scriptstyle\sim}\downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ X & \overset{\Delta}{\longrightarrow} & X \times X \end{array}$$

One can dualize the definition (first considered by Barwick in 2007).

Definition

A (Spitzweck) right semi-model category is a category $\mathcal{C}$ with limits and colimits endowed with three classes of maps cofibrations, fibrations and weak equivalences such that

1. Weak equivalences contain isomorphisms and satisfy 2-out-of-3.
2. Trivial cofibrations have the left lifting property against fibrations.
3. Every arrow can be factored as a trivial cofibration followed by a fibration.
4. Cofibration have left the lifting property against trivial fibration with a fibrant target.
5. Every arrow with a fibrant target can be factored as a cofibration followed by a trivial fibration.
6. Cofibrations, fibrations and weak equivalences are closed under retract.

Contrary to left semi-model category, there are plenty of example of (combinatorial) right semi-model categories that are clearly not Quillen model categories, to just mention one

Example

The category of semi-simplicial sets has a combinatorial right semi-model structure with: cofibrations = monomorphisms, equivalences = homotopy equivalence on geometric realization. The fibrant objects are the Kan complex, and the fibration between fibrant objects are the Kan fibrations.

Example (H., 2018)

The category of non-unital polygraphs (computads) and regular polygraphs carries right semi-model categories. The second one is Quillen equivalent to the category of spaces (this is the "regular simpson conjecture"). The first one is conjectured to be equivalent to spaces (equivalent to the general Simpson conjecture).

Proposition (Barwick 2007)

Right Bousfield localization of combinatorial right semi-model category exists as right semi-model categories.

Proposition

Right Bousfield localization of right proper combinatorial Quillen model categories exists and are combinatorial right proper Quillen model categories.

Definition (Fresse, 2009)

A Fressse left semi-model category is a category $\mathcal{C}$ with limits and colimits endowed with three classes of maps cofibrations, fibrations and weak equivalences such that

1. Weak equivalences contain isomorphisms and satisfy 2-out-of-3.
2. Trivial cofibrations with a cofibrant domain have the left lifting property against fibrations.
3. Every arrow with a cofibrant domain can be factored as a trivial cofibration followed by a fibration.
4. Cofibration with a cofibrant domain have left the lifting property against trivial fibration.
5. Every arrow with a cofibrant domain can be factored as a cofibration followed by a trivial fibration.
6. Cofibrations, fibrations and weak equivalences are closed under retract.

Spitzweck actually name them "(I, J)-semi-model categories" in 2001.

Which is dualized in:

Definition

A Fresse right semi-model category is a category $\mathcal{C}$ with limits and colimits endowed with three classes of maps cofibrations, fibrations and weak equivalences such that

1. Weak equivalences contain isomorphisms and satisfy 2-out-of-3.
2. Trivial cofibrations have the left lifting property against fibrations with a fibrant target.
3. Every arrow with a fibrant target can be factored as a trivial cofibration followed by a fibration.
4. Cofibration have left the lifting property against trivial fibration with a fibrant target.
5. Every arrow with a fibrant target can be factored as a cofibration followed by a trivial fibration.
6. Cofibrations, fibrations and weak equivalences are closed under retract.

Remark

If $\mathcal{C}$ is a Fresse left semi-model category, then $\mathcal{C}$ with the same equivalences and fibration, but with:

$$\text{cof}' = \{\textit{Cofibration with cofibrant domain}\}$$

is still a Fresse left semi-model category. It has the same homotopy theoretic properties than the original ones.

Definition

We will call "core cofibrations" the cofibration with cofibrant domain (i.e. between cofibrant objects) and "core fibrations" the fibrations with fibrant target.

Examples

Semi-simplicial sets with:

- cofibrations = monomorphisms,
- equivalences = realization equivalences,
- fibrations = Kan fibrations,

is a Fresse right semi-model category which is not a Spitzweck right semi-model category.

We will see latter that all *combinatorial* Fresse right (/left) model category can easily be modified into a combinatorial Spitzweck semi-model categories by changing its (co)fibration without changing its core (co)fibration.

It seem however that outside the combinatorial world, Fresse semi-model structure are mon common though, but not many examples have been studied yet.

Definition (H. 2018)

A weak model category is a category $\mathcal{C}$ with limits and colimits endowed with three classes of maps cofibrations, fibrations and weak equivalences such that

1. Weak equivalences contain isomorphisms and satisfy 2-out-of-3.
2. Trivial core cofibrations have the left lifting property against core fibrations.
3. Every arrow from a cofibrant to a fibrant object can be factored as a trivial cofibration followed by a fibration.
4. Core cofibration have left the lifting property against trivial core fibrations.
5. Every arrow from a cofibrant to a fibrant object can be factored as a cofibration followed by a trivial fibration.
6. Cofibrations, fibrations and weak equivalences are closed under retract.

As for Fresse semi-model categories, only core cofibration and core fibration matter to the structure of weak model categories.

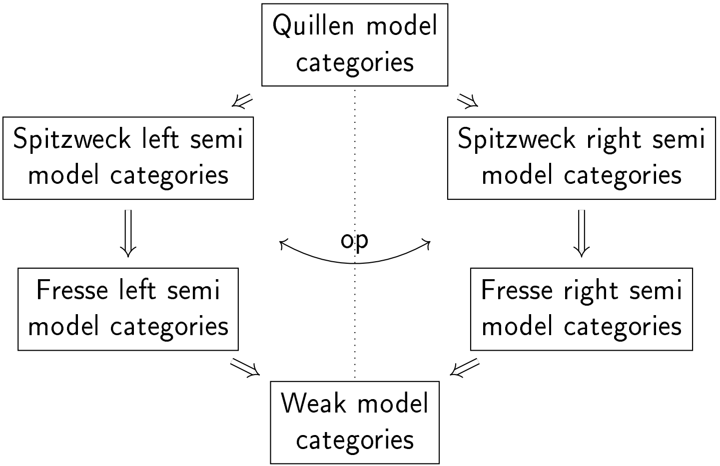
But there is also something similar for weak equivalences: if you start from an object which is neither fibrant nor cofibrant you have no way of taking a fibrant or cofibrant replacement.

Because of this, objects that are neither fibrant nor cofibrant should be ignored from the homotopy theory. Infact, if we replace the class of weak equivalences by:

$$\mathcal{W}' = \left\{ \begin{array}{c} \text{Weak equivalences between} \\ \text{fibrant or cofibrant objects} \end{array} \right\} \cup \{\text{Isomorphisms}\}$$

we still have a weak model category.
We have

$$Ho(\mathcal{C}) \simeq \mathcal{C}^{\text{cof}\vee\text{fib}}[\mathcal{W}^{-1}]$$



Remark

To be honest, I do not have example of weak model category for which I know they are neither left or right semi-model categories.

Let's come back to the example of the category $\mathrm{Mon}(\mathcal{C})$ for $\mathcal{C}$ a combinatorial monoidal closed model category $\mathcal{C}$. We call $F : \mathcal{C} \to \mathrm{Mon}(\mathcal{C})$ the free monoid functor.

Even if $\mathrm{Mon}(\mathcal{C})$ is only a left semi-model category comes with two weak factoirzation system: cofibrantly generated by $F(J)$ and $F(I)$, with I and J the generating (trivial) cofibration of $\mathcal{C}$.

The right class are fibrations, and trivial fibrations. The left class generated by $F(I)$ are the cofibrations. But the left class generated by $F(J)$ are not quite the "trivial cofibrations".

We call "anodyne cofibration" the left class generated by $F(J)$.

We only have that:

$$\{\text{Core anodyne cofibrations}\} = \{\text{Core trivial cofibrations}\}$$

Definition (Barton)

A *premodel category* is a complete and co-complete category with two weak factorization systems called *(cofibrations, anodyne fibrations)* and *(anodyne cofibrations,fibrations)* such that anodyne cofibration are cofibration, or equivalently anodyne fibrations are fibrations.

Definition

A premodel category is said to be a left semi/right semi/weak/Quillen model category if it admits a class of weak equivalence making it as such. Such a class of equivalence is unique when it exists.

Definition

A premodel category is said to be combinatorial (resp. accessible) if it is locally presentable and both weak factorization are cofibrantly generated (resp. accessible). A model category is said to be combinatorial (resp. accessible) if it comes from a combinatorial (resp. accessible) premodel category.

Definition

In a premodel category, an *acyclic cofibration* is a cofibration which has the left lifting property against core fibrations. An *acyclic fibration* is a fibration which has the right lifting property against core cofibration.

Quick summary of the relations.

- In a Quillen model category: acyclic = anodyne = trivial.
- In general, anodyne $\Rightarrow$ acyclic.
- In a weak model category, core acyclic = core trivial.
- In a left semi-model category acyclic fibration = trivial fibrations.

Informally, "acyclic" is a good approximation to "trivial" definable from the premodel structure directly.

Definition

In a premodel category, a “strong cylinder object” for a cofibrant object X is a factorization

$$X \coprod X \rightarrowtail IX \to X$$

where the first map $X \rightarrowtail IX$ is an acyclic cofibration.

Definition

In a premodel category, a “strong path object” for a fibrant object X is a factorization

$$X \to PX \twoheadrightarrow X \times X$$

where the first projection $PX \twoheadrightarrow X$ is an acyclic fibration.

Definition

In a premodel category, a "relative strong cylinder object" for a cofibration $A \rightarrowtail X$ is a factorization

$$X \coprod_A X \rightarrowtail I_A X \to X$$

where the first map $X \rightarrowtail I_A X$ is an acyclic cofibration.

Definition

In a premodel category, a "relative strong path object" for a fibration $X \twoheadrightarrow B$ is a factorization

$$X \to P_B X \twoheadrightarrow X \times_B X$$

where the first projection $P_B X \twoheadrightarrow X$ is an acyclic fibration.

Theorem (H. 2018)

A premodel category is a weak model category if and only if:

- *Every cofibration from a cofibrant objects to a fibrant object admits a relative strong cylinder object.*
- *Every fibration from a cofibrant to a fibrant object admits a relative strong path object.*

sketch of proof.

- These assumptions are exactly what we need to define the homotopy relation and construct the homotopy category.
- One then show that the homotopy category is the localization of the category of fibrant or cofibrant objects at core anodyne (co)fibration.
- One defines the weak equivalences as the arrow inverted by this localization.
- One shows that a core cofibration is acyclic if and only if it is invertible in the homotopy category.

Theorem (H. 2018)

A premodel category is a weak model category if and only if:

- *Every cofibration from a cofibrant objects to a fibrant object admits a relative strong cylinder object.*
- *Every bifibrant object admits a strong path object.*

Theorem (H. 2018)

A premodel category is a weak model category if and only if:

- *Every bifibrant object admits a strong cylinder object.*
- *Every fibration from a cofibrant to a fibrant object admits a relative strong path object.*

Definition

A "relative weak cylinder object" for a cofibration $A \rightarrowtail X$ is a diagram

$$\begin{array}{ccc} X \coprod_A X & \longrightarrow & X \\ \downarrow & & \downarrow \sim \\ I_A X & \longrightarrow & D_A X \end{array}$$

where the first map $X \rightarrowtail I_A X$ is an acyclic cofibration.

Definition

A "relative weak path object" for a fibration $X \twoheadrightarrow B$ is a diagram

$$\begin{array}{ccc} E_B X & \longrightarrow & P_B X \\ \downarrow \sim & & \downarrow \\ X & \xrightarrow{\Delta} & X \times_B X \end{array}$$

where the first projection $P_B X \twoheadrightarrow X$ is an acyclic fibration.

Proposition

In a premodel category, the following conditions are equivalents:

- *Every cofibration from a cofibrant to a fibrant object admits a relative strong cylinder object.*
- *Every cofibration from a cofibrant to a fibrant object admits a relative weak cylinder object.*
- *Every core cofibration admits a relative weak cylinder object.*

And they can be deduced from:

- *For every cofibration $A \rightarrowtail X$ from a cofibrant object to a fibrant object, there exists a cofibration*

$$X \coprod_A X \rightarrowtail J_A X$$

such that both maps $X \rightarrowtail J_A X$ are acyclic cofibrations.

One can also formulate the criterion above in a way that only involves cofibrations and acyclic cofibrations:

Theorem (H. 2018)

A premodel category is a weak model category if and only if

- *Every core cofibrations admits a relative weak cylinder object.*
- *If i, j are composable core cofibrations and $i \circ j$ and j are acyclic, then i is acyclic.*

Example

The category of semi-simplicial sets $\widehat{\Delta_+}$.

It comes with a combinatorial premodel structure, cofibrantly generated by

$$I = \{\partial \Delta_+[n] \rightarrowtail \Delta_+[n]\} \quad J = \{\Lambda^k_+[n] \rightarrowtail \Delta_+[n]\}$$

$\widehat{\Delta_+}$ also has a monoidal closed structure (the geometric product) that makes it a "monoidal premodel category".
The unit is $\Delta_+[0]$, and the cofibration $\Delta_+[0] \coprod \Delta_+[0] \rightarrowtail \Delta_+[1]$ allows to construct weak cylinder and weak path object by tensor and cotensor. For example, for $A \rightarrowtail B$, we construct

$$B \coprod_A B = (B \coprod B) \coprod_{A \coprod A} A \rightarrowtail (B \otimes \Delta_+[1]) \coprod_{A \otimes \Delta_+[1]} A$$

Hence, $\widehat{\Delta_+}$ has a weak model structure. Every object is cofibrant, it is a Fresse right semi-model structure.

More generally:

Example

Let $\mathcal{C}$ be a monoidal closed premodel structure (including the usual compatibility condition between the tensor product and (anodyne) cofibration). Assume there is a cofibration $e \coprod e \rightarrowtail I$, where e is the monoidal unit, such that both maps $e \rightarrowtail I$ are acyclic cofibration, then $\mathcal{C}$ is a (monoidal) weak model category.

Example

Any premodel category enriched (with tensor and cotensor) in a weak model category is a weak model category.

Next step: how to get left/right semi-model categories ?

Theorem

A premodel category is a Fresse left semi-model category if and only if

- *It is a weak model category.*
- *Every cofibrant object admits a strong cylinder object.*
- *Every core acyclic cofibration is anodyne.*

It is a Spitzweck left semi-model structure if and only if it further satisfies

- *Every acyclic fibration is anodyne.*

Sketch of proof.

Using the second assumption one shows that acyclic fibration between cofibrant objects are weak equivalence: they have homotopy inverse. This allows to shows that any two cofibrant replacements of an arbitrary object are equivalents. The last two assumptions will be needed to show some of the lifting properties at the end.

□

The assumption of the kind acyclic = anodyne, or core acyclic = core anodyne are called saturation assumptions. They are actually easy to enforce:

Theorem

If $\mathcal{C}$ is a combinatorial premodel category. Then there is a combinatorial premodel structure $\mathbb{L}\mathcal{C}$ on $\mathcal{C}$, called the left saturation on $\mathcal{C}$, such that:

- *$\mathbb{L}\mathcal{C}$ has the same cofibrations and anodyne fibrations as $\mathcal{C}$.*
- *The anodyne cofibration of $\mathbb{L}\mathcal{C}$ are the acyclic cofibration of $\mathcal{C}$.*
- *$\mathbb{L}\mathcal{C}$ is "left saturated", i.e. its acyclic cofibration are anodyne.*
- *It has the same "core" as $\mathcal{C}$.*

Remark

This construction essentially appear in Cisinski's work and, almost at this level of generality in its generalization by Olschok.

The dual construction also works

Theorem

If $\mathcal{C}$ is a combinatorial premodel category. Then there is a combinatorial premodel structure $\mathbb{R}\mathcal{C}$ on $\mathcal{C}$, called the right saturation of $\mathcal{C}$, such that:

- *$\mathbb{R}\mathcal{C}$ has the same anodyne cofibrations and fibrations as $\mathcal{C}$.*
- *The anodyne fibration of $\mathbb{R}\mathcal{C}$ are the acyclic fibration of $\mathcal{C}$.*
- *$\mathbb{R}\mathcal{C}$ is "right saturated", i.e. its acyclic fibration are anodyne.*
- *It has the same "core" as $\mathcal{C}$, i.e. same core (co)fibrations.*

Remark

$\mathbb{L}\mathbb{R}\mathcal{C}$ and $\mathbb{R}\mathbb{L}\mathcal{C}$ can be different, but they are both left and right saturated.

Remark

These constructions also works if $\mathcal{C}$ is only an accessible premodel category (i.e. is locally presentable with accessible w.f.s.).

Example

If $\mathcal{C}$ is a combinatorial Fresse left semi-model category, then $\mathbb{R}\mathcal{C}$ is a Spitzweck left semi-model category. It has the same core, hence is Quillen equivalent.

Example

if $\mathcal{C}$ is a monoidal combinatorial (or accessible) premodel category and the unit has a strong cylinder object, then $\mathbb{L}\mathcal{C}$ is a Fresse left semi-model category.

Example

If $\mathcal{C}$ is a combinatorial (or accessible) premodel category enriched in a left semi-model category then $\mathbb{L}\mathcal{C}$ is a Fresse left-semi model category.

Our recognition theorem can be dualized:

Theorem (H. 2020)

A premodel category is a Fresse right semi-model category if and only if

- *It is a weak model category.*
- *Every fibrant objects admits a strong path object.*
- *Every core acyclic fibration is anodyne.*

It is a Spitzweck right semi-model structure if and only if it further satisfies

- *Every acyclic cofibration is anodyne.*

Very often, both version of the theorem applies simultaneously, for example, if $\mathcal{C}$ is saturated and enriched (with tensor and cotensor) in a left semi-model category.

What happen then ?

Definition

A premodel category is said to be a two-sided model category if:

- It is a weak model category.
- Every fibrant objects admits a strong path object, every cofibrant objects admits a strong cylinder objects.
- It is bisaturated, i.e. all acyclic cofirations and fibrations are anodyne.

Example

If $\mathcal{C}$ is enriched over a left semi-model category, then $\mathbb{RL}\mathcal{C}$ and $\mathbb{LR}\mathcal{C}$ are both two-sided model category.

Remark

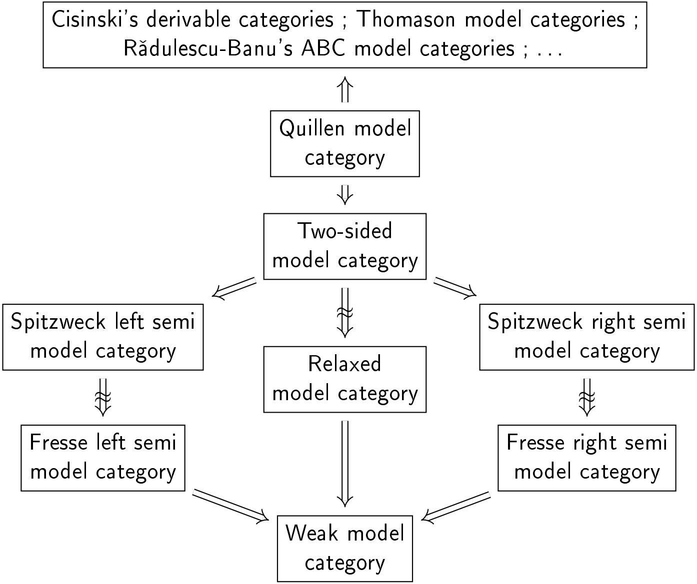
Up to question of saturation, these are the same as R.Barton "relaxed premodel structure".

A two-sided model category has two classes of equivalences:

- A class of "left equivalence" that makes it into a left semi-model structure,
- A class of "right equivalence" that makes it into a right semi-model structure.

They coincide for arrows between object that are either fibrant or cofibrant, but not always in general.

The localization at left and right equivalence are equivalent, but with two different functors $\mathcal{C} \to Ho(\mathcal{C})$ (they agree on fibrant or cofibrant objects). If $\mathcal{W}_L \subset \mathcal{W}_R$, or $\mathcal{W}_R \subset \mathcal{W}_L$, or the two localization functors coincide, or every object is fibrant or cofibrant, then we have a Quillen model category.


Cisinski's derivable categories ; Thomason model categories ;
Rădulescu-Banu's ABC model categories ; . . .
Quillen model category
Two-sided model category
Spitzweck left semi model category
Relaxed model category
Spitzweck right semi model category
Fresse left semi model category
Fresse right semi model category
Weak model category

Some applications:

- Left and right Bousfield localization of combinatorial weak model categories exists as weak model categories.
- Left and right Bousfield localization of combinatorial left semi-model categories exists as left semi-model categories.
- Left and right Bousfield localization of combinatorial right semi-model categories exists as right model categories.
- A left or right Bousfield localization of combinatorial Quillen model categories exists as two-sided model categories.
- One can replace combinatorial by accessible everywhere.

A last application :

It was famoulsy showed by Nikolaus (generalized by Bourke) that if $\mathcal{C}$ is a combinatorial Quillen model category, then you can can consider a fibrant replacement monad T, and the category of T-alg carries a Quillen equivalent transfered model structure where every object is fibrant.

Ching and Riehl, obtained a kind of "dual result": if T is a *simplicial* Quillen model category, and C is *simplicial* cofibrant replacement comonad, then the category of C-coalgebra carries a Quillen equivalence transfered model structure where every object is cofibrant.

Using the result presented in this talk, we obtained a symetric version of these constructions with John Bourke:

Theorem (B.-H., 2020)

Let $\mathcal{C}$ be a combinatorial (or accessible) weak model category. Let C be a cofibrant replacement comonad and T a fibrant replacement monad, then there are Quillen equivalences

$$C\text{-}Coalg \leftrightarrows \mathcal{C} \leftrightarrows T\text{-}Alg$$

Where C-Coalg is a right semi-model structure where every object is cofibrant and T-algebra is a left semi-model structure where every object is fibrant.

One recovers Ching and Riehl result from this: If $\mathcal{C}$ is simplicially enriched and C is simplicial, then C-Coalg is simplically enrihed as well. So it automatically has a two-sided model structure (up to saturation questions). As every object is cofibrant, it is a Quillen model structure.

But that does not quite recover Nikolaus result
And the only assymetrical result of this talk:

Theorem (B.-H., 2020)

If $\mathcal{C}$ is a combinatorial right semi-model category then T-Alg is a Quillen model category.

This only works if T is the fibrant replacement monad obtained from Garner small object argument, applied to a discrete set of arrow.

In particular there is no analogue for accessible weak model categories.

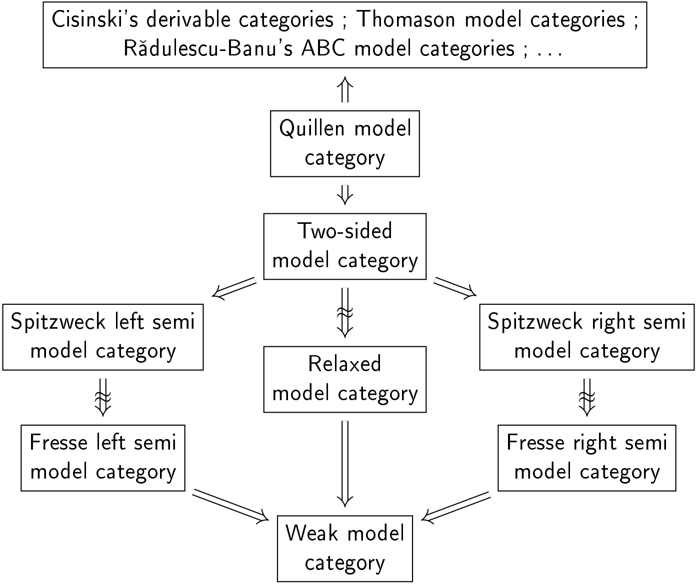
Theorem (B.-H., 2020)

Every combinatorial weak model category is Quillen equivalent to a Quillen model category where every object is fibrant.

We do not know if a similar result holds for accessible weak model categories.

Thank You !

- Combinatorial and accessible weak model categories, H., 2020, ArXiv:2005.02360
- Algebraically cofibrant and fibrant objects revisited, Bourke, H., 2020, Arxiv:2005.05384.
- Weak model categories in classical and constructive mathematics, H., 2018, TAC or ArXiv 1807.02650.


Cisinski's derivable categories ; Thomason model categories ;
Rădulescu-Banu's ABC model categories ; . . .
Quillen model category
Two-sided model category
Spitzweck left semi model category
Relaxed model category
Spitzweck right semi model category
Fresse left semi model category
Fresse right semi model category
Weak model category