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# Des topos à la géométrie non commutative par l'étude des espaces de Hilbert internes

dirigée par Alain Connes

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**Des topos à la géométrie non commutative par l'étude des espaces de Hilbert internes**

*Résumé :* Cette thèse est consacrée à l'étude de relations entre la géométrie non commutative et la théorie des topos, comme deux modèles de topologie généralisée. L'outil principal que nous utilisons est l'étude des champs continus d'espaces de Hilbert sur un topos, définis par l'utilisation de la logique interne du topos. En considérant les algèbres d'opérateurs bornés sur de tels champs on obtient des $\mathcal{C}^*$-algèbres associées au topos.
Dans le chapitre 1 nous étudions cette relation par l'intermédiaire des quantales et dans le cas des topos atomiques. Dans ce cas, la relation avec les algèbres d'opérateurs peut-être décrite explicitement et cela procure un modèle simple des phénomènes qui apparaissent.
Le chapitre 2 définit une notion de théorie de la mesure sur les topos et la relie à la théorie des $W^*$-algèbres, c'est à dire à la théorie de la mesure non commutative. Inspirés par les résultats du chapitre 1 nous définissons une notion de mesure invariante qui apparait comme analogue à la notion de trace. La classification de ces mesures fait apparaitre un $\mathbb{R}^{>0}$-fibré principal canonique sur tout topos booléen intégrable localement séparé, qui est l'analogue de l'évolution temporelle des $W^*$-algèbres (ceci est précisé à la fin du chapitre 2).
Dans le chapitre 3, nous définissons et étudions les notions d'espaces métriques et d'espaces de Banach "localiques". Notre motivation est de pouvoir généraliser les techniques que nous utilisons pour les topos à des groupoides topologiques ou localiques, ainsi que d'obtenir une extension de la dualitée de Gelfand constructive conjecturée par C.J.Mulvey et B.Banachewski. Nous prouvons aussi que dans un topos satisfaisant une certaine condition généralisant la paracompacité, la notion d'espace de Banach localique est équivalente à la notion usuelle d'espace de Banach.

**From toposes to non-commutative geometry through the study of internal Hilbert spaces**

*Abstract:* The goal of this thesis is to study some relations between non-commutative geometry and topos theory, both being seen as generalizations of topological spaces. The main tool we are using is the study of continuous bundles of Hilbert spaces over a topos which are defined as Hilbert spaces in the internal logic of the topos. By looking at the algebras of bounded operators over such Hilbert spaces one can associate $\mathcal{C}^*$-algebras to a topos.
In chapter 1 we study this relation through the use of quantales, and in the case of atomic toposes. For such toposes the relation with operator algebras can be described explicitely, and this provides an interesting toy-model for the case of more general toposes.
In chapter 2 we focus on measure theoretic aspects. We define a notion of generalized measure class over a topos, and this notion appears to be closely related to the theory of $W^*$-algebras. Inspired by the results of chapter 1 we define a notion of invariant measure, which appears to be analogous to the notion of trace on a $W^*$-algebra. The classification of such measures gives rise to a canonical $\mathbb{R}^{>0}$-principal bundle on every integrable locally separated boolean topos, which is the analogue of the modular time evolution of $W^*$-algebras (this analogy is made precise at the end of the chapter).
In chapter 3, we define and study a notion of localic metric spaces and localic Banach spaces. The motivations for such notions are that they allow one to generalize the techniques used on toposes in this thesis to topological and localic groupoids, and that they allows to obtain an extension of the constructive Gelfand duality as conjectured by C.J.Mulvey and B.Banachewski. We also prove that over a topos satisfying a certain technical condition analogous to paracompactness, the notion of localic Banach space is equivalent to the usual notion of Banach space.

# Contents

# Introduction

Les topos et les $C^*$-algèbres sont deux sortes d'objets qui généralisent la notion d'espaces topologiques à des situations où les idées classiques de la topologie ne s'appliquent plus. Il s'avère que ces deux notions affichent certaines similarités assez profondes. Tout d'abord, il existe une longue liste d'exemples d'objets géométriques généralisés auxquels on sait associer aussi bien un topos qu'une $C^*$-algèbre, notamment : tous les groupes discrets, certains groupes localement compacts, les tores non commutatifs et plus généralement l'espace des feuilles d'un feuilletage, les graphes, l'espace des pavages de Penrose et plus généralement une large classe de groupoïdes topologiques.

En plus de ces nombreux exemples, où l'on dispose explicitement d'un topos et d'une algèbre d'opérateurs décrivant tous deux une même situation géométrique, il existe des exemples de problèmes pour lesquels à la fois les topos et les algèbres d'opérateurs jouent un rôle, mais de deux façons qui ne semblent pas faciles à relier. Le premier exemple auquel nous pensons est celui de la physique théorique et plus précisément le problème de la gravitation quantique. Les algèbres d'opérateurs sont clairement des objets pertinents pour étudier la physique quantique et ses généralisations, il existe même un modèle de gravitation quantique complètement basé sur les algèbres d'opérateurs (voir [13], [14], voir aussi la première partie de [18]). D'un autre côté, les topos ont aussi été proposés comme des objets dignes d'intérêt pour la physique théorique moderne et la gravitation quantique (voir [31], [24]).
Le deuxième exemple auquel nous pensons est celui de la théorie des nombres. Certaines algèbres d'opérateurs appelées systèmes de Bost-Connes (introduit dans [9] et généralisé dans [30]) s'avèrent être connectées à des problèmes profonds de théorie des nombres, notamment l'hypothèse de Riemann (voir [17]) ainsi que la théorie du corps de classe explicite (voir aussi les chaptire 2,3 et 4 de [18]). De leurs cotés, les topos ont originellement été introduits dans le but explicite de prouver les conjectures de Weil, qui incluent la version en caractéristique $p$ de l'hypothèse de Riemann. Dans les deux cas, il serait très profitable de mieux comprendre la relation entre l'approche par les algèbres d'opérateurs et celle par la théorie des topos.

Un autre point commun entre ces deux théories est qu'elles ont toutes les deux affiché la capacité à réconcilier le discret et le continu. En effet, dans la conception usuelle qu'une variable réelle est une fonction d'un ensemble $X$ à valeurs dans $\mathbb{R}$ vérifiant certaines conditions de régularité, il est impossible d'avoir de façon intéressante une variable discrète et une variable continue définies simultanément (c'est à dire sur le même ensemble $X$). Maintenant, du point de vue de la physique quantique et de la géométrie non commutative une variable réelle est un opérateur autoadjoint sur un espace de Hilbert. Dans ce contexte il est tout à fait possible d'avoir sur un même espace de Hilbert une variable discrète et une variable continue, même si, en général, elles ne vont pas commuter. Par exemple, sur l'espace de Hilbert $\mathcal{H} = L^2(\mathbb{R}/\mathbb{Z})$ les opérateurs de multiplication par une fonction continue sur $\mathbb{R}/\mathbb{Z}$ donnent lieu à des opérateurs ≪ continus ≫ , alors que l'opérateur (non borné) $d^2/dx^2$ a un spectre discret, et peut (dans la base des fonctions $\exp(inx)$) s'écrire comme un opérateur de multiplication par une fonction à valeurs discrètes. D'une façon similaire, la théorie des topos a aussi cette capacité de réunir des objets discrets et continus. En effet, la théorie des topos permet de mettre simultanément dans une même catégorie naturelle des objets continus comme les espaces topologiques et des objets fondamentalement discrets comme des petites catégories ou des variétés algébriques sur des corps finis. Pour citer Grothendieck (de [29, 2.13]) :

> *Ce "lit à deux places" est apparu (comme par un coup de baguette magique. . . ) avec l'idée du topos. Cette idée englobe, dans une intuition topologique commune, aussi bien les traditionnels espaces (topologiques), incarnant le monde de la grandeur continue, que les (soi-disant) "espaces" (ou "variétés") des géomètres algébristes abstraits impénitents, ainsi que d'innombrables autres types de structures, qui jusque là avaient semblé rivées irrémédiablement au "monde arithmétique" des agrégats "discontinus" ou "discrets".*

D'un point de vue très optimiste, on peut espérer obtenir une forme de dualité de Gelfand non-commutative entre une classe de topos et certaines algèbres d'opérateurs éventuellement équipées de structures supplémentaires. Même sans une telle dualité il y a de très nombreux outils et de nombreuses constructions disponibles dans chacune des deux théories qui pourraient être extrêmement profitables à l'autre si l'on savait comment les transporter. D'un coté les topos disposent d'excellentes théories (co)homologiques et homotopiques et sont connectés de façon très profonde à la logique aussi bien par leur logique interne que par les théories géométriques qu'ils classifient. Et de l'autre coté, les algèbres d'opérateurs sont reliées à l'analyse fonctionnelle et au formalisme de la physique quantique et l'on dispose de nombreux outils puissants comme la $K$-théorie, la (co)homologie cyclique et de Hochschild, la théorie modulaire des algèbres de von Neumann, etc... Enfin même si topos et $C^*$-algèbres peuvent tous les deux être construits à partir de groupoïdes, il existe d'autres procédés qui permettent de les obtenir et qui eux ne sont pas reliés : par exemple un topos peut être obtenu à partir de n'importe quelle petite catégorie, ou comme un topos classifiant et les $C^*$-algèbres peuvent s'obtenir par extension ou déformation d'algèbres d'opérateurs déjà connues.

Il nous faut enfin observer qu'une relation entre ces deux formes de géométrie est nécessairement subtile. En effet, considérons quelques instants le cas de la dualité de Fourier, ou plus généralement de la dualité de Pontryagin. Du point de vue des algèbres d'opérateurs, si $G$ est un groupe abélien discret (ou plus généralement localement compact) alors la $C^*$-algèbre du groupe $G$ est isomorphe à l'algèbre des fonctions continues sur l'espace $X$ sous-jacent au groupe dual de $G$ et l'algèbre de von Neumann du groupe est isomorphe à l'algèbre $L^\infty(X, \mu)$ où $\mu$ est la mesure de Haar de $X$.

Cela signifie que, du point de vue des algèbres d'opérateurs, on identifie le groupe $G$ (vu comme groupoïde à un seul objet) avec l'espace topologique $X$ des caractères de $G$. Du point de vue de la théorie des topos, ces deux objets sont différents, et même assez difficiles à relier. Ainsi, d'une certaine façon, la théorie des algèbres d'opérateurs prise seule semble oublier certaines informations sur la géométrie à cause de ces isomorphismes de Fourier. Mais d'un autre coté, ces isomorphismes sont des outils très efficaces pour étudier ces algèbres de groupe d'un point de vue plus analytique.
La philosophie générale que nous voulons retirer de cet exemple est que la théorie des topos reste plus proche de la géométrie et est plus descriptive, alors que les $C^*$-algèbres sont plus proches de l'analyse et plus efficaces pour produire des invariants non triviaux, et plus spécifiquement des invariants numériques.

Enfin, le meilleur outil que nous ayons pour comprendre la relation qui relie ces deux théories est le fait qu'il est possible de définir (en utilisant la logique interne) une bonne notion de champ continu d'espaces de Hilbert ou de $C^*$-algèbres, ou de modules hilbertiens sur un topos. De plus si $H$ est un champ continu d'espaces de Hilbert sur un topos (ou plus généralement un champ de modules hilbertiens sur un champ de $C^*$-algèbres) alors l'algèbre des opérateurs globalement bornés sur $H$ est une $C^*$-algèbre qui contient beaucoup d'informations sur le topos et qui respectivement peut être étudiée en utilisant la géométrie et la logique du topos. C'est pourquoi l'enjeu principal de cette thèse sera l'étude de ces champs continus sous diverses hypothèses géométriques sur les topos concernés. Enfin, comme les champs continus sont parfois trop restrictifs, on définira aussi une notion de champ mesurable que l'on étudiera.

## Contexte et notion de base

### 1) Topos

Il nous faudra distinguer deux sortes de topos : les topos élémentaires (ou topos de Lawvere) et les topos de Grothendieck, les topos de Grothendieck étant un cas particulier de topos élémentaires.

Un topos élémentaire est par définition une catégorie qui admet des limites finies et des ≪ objets des parties ≫ , c'est à dire une construction universelle qui correspond à la construction de l'ensemble $\mathcal{P}(X)$ des sous-ensembles d'un ensemble $X$ dans la catégorie des ensembles. Cette notion a été introduite par Lawvere et Tierney dans [48] et [69]. La définition que nous avons donnée ici est une forme simplifiée (plus moderne) de la définition originale de Lawvere.

Il est démontré que, dans un certain sens, tout topos élémentaire peut se penser comme un modèle de la théorie des ensembles sans l'axiome du choix, ni l'axiome de l'infini (assurant l'existence d'un ensemble des entiers naturels), et dans lequel le principe du tiers exclu (qui assure que toute proposition est soit vraie soit fausse) peut ne pas être valide. Cette interprétation de la théorie des ensembles dans un topos est appelée la logique interne du topos et est utilisée constamment dans cette thèse. En fait nous travaillerons toujours implicitement dans un topos élémentaire $\mathcal{S}$ dont les objets sont appelés ≪ ensembles ≫ . Nous supposerons toujours que $\mathcal{S}$ a un objet des entiers naturels (c'est à dire satisfait l'axiome de l'infini) mais nous ne supposerons en général pas que sa logique satisfait l'axiome du choix ou le principe du tiers exclu sauf de façon occasionnelle et mentionnée précisément (par exemple, on utilisera l'axiome des choix dépendants pour la preuve du théorème principal de la section 5 du chapitre 3 et on utilisera constamment le principe du tiers exclu dans le chapitre 2). En particulier nous travaillons dans le cadre des mathématiques intuitionnistes [1]. Pour une brève introduction au fonctionnement de cette logique interne, on pourra se référer à [6], le livre [50], donne aussi une présentation beaucoup plus détaillée avec de nombreux exemples et applications.

Les topos de Grothendieck sont un type particulier de topos élémentaires définis et étudiés par Grothendieck et ses collaborateurs en vue d'applications à la géométrie algébrique avant l'invention des topos élémentaires. Ils peuvent être définis comme les topos élémentaires qui sont des catégories présentables (au sens de [7, 5.2]) ou de façon équivalente qui ont toutes les limites inductives et un ensemble de générateurs. Ce sont aussi les catégories équivalentes à la catégorie des faisceaux sur un site (il s'agit de leur définition orignal dans [21]). En particulier, les topos de Grothendieck satisfont toujours l'axiome de l'infini.

La bonne notion de morphisme entre topos de Grothendieck est la notion de ≪ morphisme géométrique ≫ . Un morphisme géométrique entre deux topos correspond à une paire de foncteurs adjoints $(f^*, f_*)$ avec $f_*$ dans le sens du morphisme et $f^*$ qui commute aux limites projectives finies (en plus des limites inductives quelconques). Cette notion fait des topos (de Grothendieck) une 2-catégorie, les 2-morphismes étant les transformations naturelles entre les foncteurs $f^*$. Le foncteur qui envoie un espace topologique sur sa catégorie de faisceaux d'ensembles, et les applications continues sur les foncteurs image inverse et image directe de faisceaux, plonge quasiment [2] pleinement fidèlement la catégorie des espaces topologiques dans la catégorie des topos. C'est ainsi qu'on voit les topos comme une généralisation de la notion d'espace topologique.

Une application très importante de la logique interne est que si l'on construit la théorie des topos de façon constructivement valide (ce qui ne pose pas de problème), on peut parler de topos de Grothendieck à l'intérieur d'un topos de Grothendieck fixé $\mathcal{T}$. Il s'avère que la notion de topos de Grothendieck à l'intérieur de $\mathcal{T}$ (on dira un $\mathcal{T}$-topos) est équivalente à la notion de topos de Grothendieck équipé d'un morphisme géométrique vers $\mathcal{T}$. En particulier, à chaque fois que l'on veut étudier un morphisme géométrique $f : \mathcal{E} \to \mathcal{T}$, on peut se ramener à l'étude d'un $\mathcal{T}$-topos $\mathcal{E}$ en utilisant la logique de $\mathcal{T}$. En particulier, n'importe quelle propriété raisonnable d'un topos donnera immédiatement une propriété relative pour les morphismes géométriques (par exemple, la notion ≪ compact ≫ pour les topos donne la notion ≪ propre ≫ pour les morphismes).

### 2) Algèbre d'opérateurs

1. ou constructives. Pour ce qui nous concerne, constructif et intuitionniste sont synonymes et signifient ≪ valide dans un topos élémentaire ≫ .

2. Il faut se restreindre à une certaines classe d'espace topologique, appelé les espaces topologiques sobres, qui inclue tous les espaces topologiques séparés, mais aussi par exemple tout les espaces topologiques sous jacents à des schémas.

Une $C^*$-algèbre (complexe) est une $\mathbb{C}$-algèbre de Banach équipée d'une involution anti-linéaire $*$ telle que $(ab)^* = b^*a^*$ et $\|x^*x\| = \|x\|^2$. Les deux résultats principaux pour comprendre l'importance de cette définition sont les suivants :

- Les $C^*$-algèbres sont exactement les sous algèbres fermées et stables par adjonction de l'algèbre $B(\mathcal{H})$. Où $B(\mathcal{H})$ désigne l'algèbre des opérateurs bornés d'un espace de Hilbert $\mathcal{H}$.
- Les $C^*$-algèbres commutatives sont exactement les algèbres $C_0(X)$ des fonctions continues qui tendent vers 0 à l'infini sur un espace topologique localement compact $X$. De plus il y a une correspondance entre les morphismes de $C^*$-algèbres commutatives et les applications propres entre les espaces localement compacts correspondants. Cette équivalence de catégorie est appelée la dualité de Gelfand.

Grâce à cette dualité de Gelfand, on peut penser à une $C^*$-algèbre générale comme à un espace topologique localement compact ≪ non-commutatif ≫ . On peut consulter [23] pour une introduction à la théorie des $C^*$-algèbres. Comme nous l'avons mentionné plus haut, de nombreux outils de la topologie classique ont pu être généralisés à cette topologie non commutative, de nombreux exemples sont présentés dans [16].

Les algèbres de von Neumann (voir [22]) sont des $C^*$-algèbres particulières qui ne sont pas juste fermées en norme dans $B(\mathcal{H})$ mais aussi fermées pour la topologie de la convergence faible. Elle peuvent aussi être caractérisées comme les $C^*$-algèbres qui admettent un pré-dual en tant qu'espace de Banach (voir [64]), ou encore comme les $C^*$-algèbres telles que l'ensemble des éléments positifs admet tous les supremums filtrants et qui possède suffisamment de formes linéaires positives qui préservent ces supremums (voir [66, III.3.16]).

Les algèbres de von Neumann abéliennes sont de la forme $L^\infty(X)$, c'est à dire l'algèbre des fonctions mesurables bornées presque partout sur un espace mesuré $X$, modulo les fonctions nulles presque partout. De ce point de vue, la théorie des algèbres de von Neumann s'interprète comme de la théorie de la mesure non-commutative. Comme dans le cas topologique, de nombreux outils de la théorie de la mesure classique ont été généralisés au cas non commutatif, mais la théorie non-commutative possède une particularité supplémentaire : quand on fixe un poids normal semi-fini sur une algèbre de von Neumann $A$ (l'analogue d'une mesure localement finie) il apparaît de façon canonique une évolution temporelle de $A$, c'est à dire un morphisme de $\mathbb{R}$ dans le groupe des automorphismes de $A$. Cette construction est due à Tomita, et a été popularisée par Takesaki (dans [65]), enfin A.Connes a démontré que, modulo les automorphismes intérieurs de l'algèbre, l'évolution temporelle ne dépend pas du choix du poids semi-fini et est complètement déterminée par l'algèbre elle même (voir [15]). Bien sûr, pour une algèbre commutative cette évolution est triviale, et pour l'algèbre $B(\mathcal{H})$ elle est intérieure (et donc triviale au sens de la propriété d'unicité), mais il existe une grande classe d'algèbres (les facteurs de type III) pour lesquelles cette évolution est réellement non triviale et a donné lieu à des invariants fondamentaux de ces algèbres. Une introduction à la théorie de cette évolution temporelle peut-être trouvée dans [67].

**3) Théorie constructive des algèbres d'opérateurs**

Une partie de l'analyse fonctionnelle et de la théorie des algèbres d'opérateurs a été développée constructivement (donc sur des topos) par Mulvey, Banachewski et plusieurs autres mathématiciens dans une série d'articles incluant [12] [57], [60],[3],[4].
Avant de donner une définition constructive de ce qu'est une $C^*$-algèbre qui peut être intéressante à utiliser dans un topos, il nous faut donner la définition constructive de ce qu'est un nombre réel ou complexe. En mathématique classique, il existe de très nombreuses définitions du corps des nombres réels qui sont toutes équivalentes, mais quand on passe aux mathématiques intuitionnistes ces définitions ne sont plus équivalentes. L'ingrédient clé dont nous aurons de toute façon besoin est l'axiome de l'infini pour pouvoir parler de l'ensemble $\mathbb{N}$ des nombres entiers, il est ensuite très simple de construire $\mathbb{Q}$ à partir de $\mathbb{N}$. Il s'avère que, même sans supposer le tiers exclu, les relations usuelles sur $\mathbb{Q}$ sont décidables, c'est à dire par exemple que pour tout couple de nombres rationnels $q$ et $q'$ on a $q < q'$ ou $q = q'$ ou $q > q'$. Dans un topos de Grothendieck les objets correspondants à l'interprétation de $\mathbb{N}$ et $\mathbb{Q}$ sont simplement les faisceaux localement constants égaux à $\mathbb{N}$ et $\mathbb{Q}$.
La définition de $\mathbb{R}$ basée sur les suites de Cauchy est à éviter en l'absence de l'axiome du choix dénombrable. La construction basée sur les coupures de Dedekind marche relativement bien au niveau des propriétés d'ensemble ordonné mais a de nombreux défauts au niveau de sa structure algébrique : les constructions par coupures supérieures et inférieures donnent des ensembles ordonnés différents appelés respectivement réels semi-continus supérieurement et inférieurement. Dans les deux cas, on ne peut pas définir convenablement l'opposé d'un nombre réel ni multiplier des nombres qui ne sont pas positifs. Ainsi les réels semi-continus (supérieurement ou inférieurement) ne forment pas des anneaux ni même des groupes additifs. Ils forment des monoïdes additifs et l'on peut multiplier les éléments positifs entre eux. Malgré cela ils ont leur intérêt et leur interprétation est relativement facile à comprendre, par exemple sur un espace topologique il s'agit des faisceaux des fonctions semi-continues supérieurement et inférieurement.
La bonne définition des nombres réels est celle basée sur les coupures de Dedekind bilatères. Il y a en réalité (au moins) deux façons de définir ce qu'est une coupure bilatère qui ne sont pas constructivement équivalentes et qui donnent à nouveau deux définitions possibles des nombres réels : les réels de Mac-Neville et les réels de Dedekind (ou encore les réel continus). Nous n'utiliserons que les réels de Dedekind (ou réel continus) et nous les appellerons simplement « réels » . L'ensemble des nombres réels ainsi défini n'est pas un corps au sens où l'on n'a pas « $x = 0$ » ou « $x$ est inversible » en général, mais c'est un anneau local et la négation de « $x$ est inversible » est bien « $x = 0$ » (le contraire n'étant en général pas valide). Dans le topos des faisceaux sur un espace topologique, le faisceau des réels de Dedekind est le faisceau des fonctions continues sur $X$ (à valeurs dans $\mathbb{R}$). L'ensemble des nombres réels est inclu à la fois dans l'ensemble des réels semi-continus supérieurement et inférieurement et correspond à chaque fois exactement à l'ensemble des éléments qui admettent un opposé. Il est important de noter que contrairement aux ensembles des réels semi-continus il n'est pas vrai en général que tout sous ensemble borné des nombres réels admet une borne supérieure (en effet, un supremum d'une famille de fonctions continues peut ne pas être continu).

Il est aussi possible de définir les nombres réels continus comme étant la complétion (métrique) de $\mathbb{Q}$, mais il faut utiliser une complétion par filtres de Cauchy plutôt que par suites de Cauchy (comme dans [10, II.3]).

Une fois que les nombres réels sont définis, la définition des nombres complexes ne pose aucun problème : $\mathbb{C}$ est simplement défini comme $\mathbb{R}[X]/(X^2+1)$. Il n'est pas tout à fait exact que $\mathbb{C}$ est algébriquement clos, mais la raison est essentiellement que $\mathbb{C}$ n'est pas un corps, en effet, $\mathbb{C}$ est toujours séparablement clos au sens où tout polynôme qui a un discriminant inversible peut se factoriser en produit de polynômes de degré un.

Il y a deux définitions possibles des espaces de Banach : on peut soit demander que la norme d'un élément soit un nombre réel continu, soit que ce soit un nombre réel semi continu supérieurement. L'idée sous-jacente à cette deuxième version est que dans la plupart des mathématiques il n'est pas nécessaire de minorer la norme d'un élément mais plutôt de la majorer. La deuxième version étant plus générale c'est celle que nous choisirons, et un espace de Banach dont la norme est continue sera dit *continu*. Par exemple l'algèbre de tous les opérateurs sur un espace de Hilbert a en général une norme semi-continue (car il faut prendre une borne supérieure pour définir la norme et que celle ci peut ne pas exister si on reste parmi les réels continus) alors que la sous-algèbre fermée des opérateurs compacts possède une norme continue (car les opérateurs de rang fini ont une norme continue et qu'ils sont denses parmi les opérateurs compacts). La complétude ne doit bien sûr pas être définie par les suites de Cauchy, mais peut être définie de façon équivalente par filtres de Cauchy ou approximations de Cauchy.

Il s'avère que la plus grande partie de la théorie des $C^*$-algèbres se développe relativement bien dans ce cadre constructif, on peut même obtenir une version constructive de la dualité de Gelfand (il faut pour cela remplacer la notion d'espace topologique compact par la notion de locale compacte complètement régulière). Ce résultat est dû à C.J.Mulvey et B.Banachewski ([4]) das le cas des topos de Grothendieck et ensuite adapté par T.Coquand ([20]) dans un cadre complètement constructif (et donc valide dans un topos élémentaire). Nous avons rassemblé ici quelques exemples de différences entre la théorie des algèbres d'opérateurs classique et la théorie constructive :

- Comme mentionné plus haut, $\mathbb{C}$ n'est plus un corps en mathématique constructive. En fait la plupart des propriétés de $\mathbb{C}$ qui ne sont pas des propriétés générales des $C^*$-algèbres commutatives ne sont plus valides constructivement. Par exemple il n'y a en général pas de projection orthogonale sur un sous espace fermé d'un espace de Hilbert.
- Dans le même esprit, les endomorphismes bornés d'un espace de Hilbert n'ont en général pas d'adjoint. On appelle *opérateurs* les endomorphismes qui ont un adjoint. Bien entendu, il est suffisant de supposer le principe du tiers exclu pour avoir l'existence d'adjoint.
- Il n'est pas non plus raisonnable d'espérer pouvoir représenter n'importe quelle $C^*$-algèbre comme une sous algèbre fermée de $B(\mathcal{H})$ car ce résultat utilise le théorème de Hahn-Banach de façon essentielle. En revanche il est toujours possible de représenter n'importe quelle $C^*$-algèbre comme une $C^*$-algèbre fermée dans l'algèbre des opérateurs sur un $C$-module

hilbertien pour $C$ une $C^*$-algèbre commutative. Ce résultat est aussi basé sur la construction GNS, mais appliquée à l'état universel (à valeurs dans $C(X)$ ou $X$ est l'espace[3] des états) plutôt qu'à un état construit par le théorème de Hahn Banach.

- Il ne semble pas possible de donner une bonne définition constructive d'algèbre de von Neumann, ou plus précisément, toute les définitions classiquement équivalentes ne sont plus équivalentes en mathématique constructive et aucune ne se détache réellement. La raison derrière cela est que toute la théorie des algèbres de von Neumann repose sur la construction de projections et que sans le tiers exclu il est très difficile, voire impossible de construire suffisamment de projections.

Quand ces définitions sont interprétées dans la logique du topos des faisceaux sur un espace topologique raisonnable (paracompact) on retrouve les notions classiques de champs continus. Si l'on considère des espaces de Banach avec une norme continue on trouve des champs comme ceux de [23, 10.1.2(iii)], si en revanche on s'autorise des espaces de Banach avec une norme semi-continue, on obtient la notion plus générale de champs de [36]. De même les $C^*$-algèbres internes et les espaces de Hilbert internes sont la même chose que des champs continus de $C^*$-algèbres et d'espaces de Hilbert ( Voir [12] and [58]). De plus si $X$ est un espace topologique localement compact, un espace de Hilbert du topos des faisceaux sur $X$ est la même chose qu'un $C_0(X)$-module hilbertien.

Enfin si $A$ est une algèbre de von Neumann abélienne, alors on peut considérer le topos $\mathcal{T}$ des faisceaux sur l'algèbre Booléenne complète des projections de $A$. Les espaces de Hilbert de $\mathcal{T}$ correspondent exactement aux $W^*$ modules sur $A$, dont on sait qu'ils correspondent aux représentations normales de $A$, et, sous des conditions de dénombrabilité, aux champs mesurables d'espaces de Hilbert sur l'espace mesuré correspondant.

Si $\mathcal{T}$ est un topos et $\mathcal{H}$ un espace de Hilbert dans la logique de $\mathcal{T}$ (on plus généralement un module hilbertien sur une $C^*$-algèbre de $\mathcal{T}$), alors l'algèbre des opérateurs globalement bornés de $\mathcal{H}$ est une $C^*$-algèbre au sens usuel. Cette construction donne une façon naturelle d'attacher des $C^*$-algèbres, ou plus généralement des $C^*$-catégories, à un topos. Malheureusement, ces algèbres sont en général trop grosses pour être intéressantes et toute la difficulté consiste à sélectionner une sous-algèbre plus intéressante (et parfois aussi à choisir un espace de Hilbert intéressant dans le topos).

### 4) Locales et groupoïdes localiques

Un topos de Grothendieck est dit *localique* si les sous objets de l'objet terminal forment une famille génératrice, ou de façon équivalente si il peut être définie par un site dont la petite catégorie sous-jacente est équivalente à un ensemble ordonné. L'exemple principale de topos localique est le topos des faisceaux sur un espace topologique, en fait tout topos localique qui admet suffisamment de points (c'est à dire de morphismes géométriques ayant pour source le topos des ensembles) est de cette forme. Plus généralement, il existe une notion ≪ d'espace

3. ≪ espace ≫ voulant bien sûr dire ≪ locale ≫

topologique (éventuellement) sans point ≫ appelé *locale* telle que la catégorie des topos localiques est équivalente à la catégorie des locales. Les locales sont définies par le treillis de leurs ouverts, qui est un ensemble ordonné vérifiant certaines propriétés faisant de lui un *cadre*[4]. Une locale est un objet extrêmement proche d'un espace topologique : une locale ayant assez de points est la même chose qu'un espace topologique sobre, et même les locales sans points gardent un comportement très géométrique et méritent toujours le nom d'espace. En revanche d'un point de vue plus global, la catégorie des locales a un comportement assez différent de la catégorie des espaces topologiques, notamment le produit de deux espaces topologiques calculé dans la catégorie des espaces topologique et celui calculé dans la catégorie des locales peuvent être distincts, en particulier un groupe topologique n'est pas forcement encore un groupe dans la catégorie des locales (si le groupe est localement compact il n'y a aucun problème) et par exemple un sous groupe localique d'un groupe localique est toujours fermé, ainsi $\mathbb{Q}$ muni de la topologie induite par celle de $\mathbb{R}$ ne peut pas être un groupe localique.

On peut trouver une introduction à la théorie des locales et des cadres dans les deux premiers chapitres de [8], ou dans la partie $C$ de [44]. On peut aussi consulter le livre récent [62] entièrement consacré à leur étude (mais dans un contexte non constructif) ainsi que les deux excellents articles d'introduction de P.T.Johnstone.

Pour un topos (de Grothendieck) général $\mathcal{T}$, il y a un topos localique universel $\mathcal{L}$, qu'on appelle la *réflexion localique* de $\mathcal{T}$ équipé d'un morphisme géométrique de $\mathcal{T}$ dans $\mathcal{L}$. Le treillis des ouverts de $\mathcal{L}$ est exactement le treillis des sous objets de l'objet terminal de $\mathcal{T}$.

À un groupoïde topologique ou localique on peut associer la catégorie des faisceaux équivariants, c'est à dire des faisceaux sur l'espace des unités munis d'une action (continue) des morphismes. Il s'avère que cette catégorie est un topos de Grothendieck. A.Joyal et M.Tierney ont démontré dans [45] que tout topos de Grothendieck peut se réaliser comme le topos des faisceaux équivariants sur un groupoïde localique ouvert (c'est à dire un groupoïde localique dont les applications structurales sont des applications ouvertes). De plus, dans [52], [53] et [11], I.Moerdijk et M.Bunge ont démontré que si on définit les morphismes entre groupoïdes localiques ouverts comme étant les bi-fibrés principaux (pour la topologie des surjections ouvertes) alors :

- La construction de la catégorie des faisceaux équivariants définit un foncteur de la catégorie des groupoïdes vers la catégorie des topos de Grothendieck.
- La construction de Joyal et Tierney définit un unique groupoïde à isomorphisme près dans cette catégorie de groupoïdes, c'est à dire à une notion d'équivalence faible de groupoïdes près.
- Cette construction définit un foncteur des topos vers les groupoïdes qui est l'adjoint à droite du foncteur topos des faisceaux équivariants.
- Ces foncteurs identifient les topos de Grothendieck avec une sous catégorie (pleine) réflexive de la catégorie des groupoïdes localiques ouverts.

4. ≪ frame ≫ en anglais, la traduction par cadre n'étant pas particulièrement standard dans ce contexte.

De plus cette catégorie des groupoïdes (avec les bi-fibrés principaux comme morphismes) peut-être vue comme une sous catégorie (pleine) de la catégorie des champs sur la catégorie des locales équipée de la topologie dont les recouvrements sont les surjections ouvertes (voir aussi [11]).

Comme les $C^*$-algèbres peuvent aussi être construites à partir de groupoïdes topologiques, ces résultats montrent un autre moyen de relier les $C^*$-algèbres et les topos. Mais ce lien n'est pas vraiment disjoint du précédent, si un topos $\mathcal{T}$ correspond à un groupoïde localement compact, il est en général possible de voir l'algèbre réduite (voir l'algèbre maximale du groupoïde) comme des sous-algèbres naturelles d'opérateurs de certains espaces de Hilbert de $\mathcal{T}$. La question de choisir et de construire un système de Haar sur le groupoïde est essentiellement traduite en la question de construire un espace de Hilbert sur le topos, cette dernière question étant souvent plus simple si on la traite du point de vue de la logique interne du topos.

## Principaux résultats de cette thèse

Cette thèse ce décompose en trois développements relativement indépendants (bien que le second soit très fortement inspiré des résultats du premier).

### I) Toposes, quantales and $C^*$-algebras in the atomic case

Dans ce chapitre, nous avons commencé par présenter une équivalence (déjà bien connue) entre la notion de topos de Grothendieck et les objets appelés quantales de Grothendieck (qui sont des cas particuliers de quantales modulaires). Nous avons amélioré certain aspects de cette équivalence, notamment avec le théorème 3.6.3 et son corollaire 3.6.9. Les quantales de Grothendieck sont intéressantes pour nous pour au moins deux raisons : tout d'abord, elle sont essentiellement ce que nous avons envie d'appeler des « algèbres d'opérateurs en caractéristique 1 » et donc sont formellement assez proches des algèbres d'opérateurs classiques, et ensuite il est très naturel (à la vue de certains exemples) de penser qu'on pourra associer une $C^*$-algèbre à un topos comme algèbre de convolution de fonctions « sur la quantale » . (voir 3.8 pour plus de détails sur ce deuxième point).

Dans la dernière section de l'article nous nous concentrons complètement sur les cas des topos atomiques et des quantales atomiques qui leur correspondent, cela correspond essentiellement au cas où il n'y a plus aucune topologie non discrète qui intervient, mais uniquement des problèmes de nature combinatoire. Dans cette situation, nous somme capables de produire une relativement bonne description du lien entre topos et algèbres d'opérateurs par l'intermédiaire des quantales. Nous montrons tout d'abord que les quantales modulaires correspondent naturellement à une notion très naturelle d'hypergroupoïde et que les topos de Grothendieck correspondent à un cas particulier d'hypergroupoïdes que nous qualifions de « semi-simples » .

Nous montrons ensuite que la convolution de fonctions sur le quantale est bien définie si et seulement si certaines conditions de finitude (formulées aussi bien en terme du topos que de l'hypergroupoïde) sont satisfaites, et que sous ces

conditions on obtient une bonne $C^*$-algèbre avec une sous algèbre arithmétique et une évolution temporelle modulaire décrite explicitement.
Nous montrons finalement que ces conditions de finitudes sont équivalentes à une condition géométrique sur le topos : le fait que le topos est localement séparé. De plus, sous cette condition l'évolution temporelle d'une algèbre de von Neumann correspondante est décrite par un $\mathbb{Q}_+^*$ fibré principal sur le topos défini de façon complètement canonique.

L'exemple principal de cette situation est le topos des actions continues d'un groupe localique pro-discret (ou plus généralement une somme disjointe de tels exemples). Les conditions de finitude sont alors équivalentes à la locale compacité du groupe et l'algèbre qui apparaît est simplement une algèbre de Hecke de doubles classes. Il s'avère que, si l'on suppose l'axiome du choix, tout exemple de topos satisfaisant ces conditions est de cette forme. Cela dit, le travail que nous avons fait ici est valide constructivement et peut donc être appliqué dans des topos sans l'axiome du choix et donc à d'autres formes d'exemples. Enfin les méthodes utilisées ici pour traiter ces exemples (n'utilisant pas le groupe localement compact sous-jacent) devraient se généraliser à d'autres sortes de topos.

Nous pensons à ce travail sur les topos atomiques comme un exemple très simple qui servira à nous guider vers une compréhension de phénomènes plus généraux, comme ceux traités dans la partie suivante.

**II) Measure theory over toposes**

Cet article développe essentiellement deux idées :

La première idée est que, par analogie avec la théorie de la mesure classique et plus précisément la théorie de la mesure sur les locales, on peut définir un champ mesurable (par exemple un champ mesurable d'espaces de Hilbert) sur un topos $\mathcal{T}$ comme étant un objet (un espace de Hilbert) vivant dans un autre topos $\mathcal{B}$ qui est Booléen (sa logique satisfait le principe du tiers exclu) et équipé d'un morphisme géométrique injectif vers $\mathcal{T}$. Par exemple, si l'on dispose d'un champ continu sur $\mathcal{T}$ (c'est à dire d'un objet vivant dans $\mathcal{T}$ ) on peut prendre son image inverse qui est un objet de $\mathcal{B}$ et cela revient à oublier la structure de champ continu pour ne garder que celle de champ mesurable. Un tel topos $\mathcal{B}$ est appelé une classe de mesure généralisée sur $\mathcal{T}$.

Nous montrons qu'une classe de mesure généralisée admet une image directe par un morphisme géométrique (par exemple on peut définir la masse de Dirac en un point du topos de cette façon et cela correspondra au topos des actions continues du groupe localique des automorphismes de ce point), et que si le topos Booléen satisfait certaines conditions d'existence de mesure (on dit qu'il est mesurable, ou encore que la classe de mesure correspondante est effective) alors la catégorie des espaces de Hilbert sur ce topos est une $W^*$ catégorie avec un ensemble de générateurs et donc la catégorie des représentations normales d'une certaine algèbre de von Neumann attachée à ce topos (définie à équivalence de Morita près).

Une classe de mesure généralisée sur un espace topologique localement compact $X$ est la même chose qu'une $C^*$-algèbre monotone fermée $A$ qui contient $\mathcal{C}_0(X)$ comme une sous algèbre dense au sens où $\mathcal{C}_0(X)$ n'est contenu dans aucune sous-algèbre monotone fermée stricte de $A$. La classe de mesure généralisée correspondante est alors effective si et seulement si $A$ est en fait une algèbre de von Neumann. En particulier pour tout espace topologique localement compact il existe une plus grande classe de mesure généralisée effective (correspondant à l'algèbre de von Neumann enveloppante) mais les classes de mesure généralisée quelconques ne forment pas forcement un ensemble et peuvent ne pas avoir d'élément maximal. La situation est la même quand $X$ est une locale, (il existe une plus grande classe de mesure généralisée effective) mais nous ignorons si c'est encore le cas pour un topos de Grothendieck arbitraire.

La deuxième idée développée ici est que la construction du $\mathbb{Q}_+^*$ fibré principal faite pour les topos atomiques localement séparés du premier développement peut être adaptée à un topos Booléen (donc à une classe de mesure généralisée quelconque sur un topos), la condition pertinente étant encore que le topos (booléen) soit localement séparé (et mesurable). Dans cette situation, cette construction produit un $\mathbb{R}_+^*$ fibré principal qui peut s'interpréter comme le fibré des mesures localement finies à support maximal (qui est un fibré principal à cause du théorème de Radon-Nikodym). À partir de là il y a trois possibilités :

- Soit le topos est séparé, dans ce cas le fibré est trivial pour des raisons complètement explicites. C'est l'analogue [5] des algèbres de von Neumann de type I .
- Soit le topos est juste localement séparé, mais le fibré est tout de même trivial, dans ce cas une section globale du fibré donne lieu à une mesure « invariante » qui permet de définir la masse de n'importe quel objet du topos (qui est un nombre réel) d'une façon naturelle, ou encore de construire une trace sur l'algèbre de von Neumann des endomorphismes de certains espaces de Hilbert du topos.
- Soit le fibré principal est non trivial, et dans ce cas il donne une évolution temporelle modulaire des espace de Hilbert du topos qui est explicitement reliée à l'évolution temporelle modulaire des algèbres de von Neumann.

Cette décomposition est très similaire à la décomposition en type des facteurs de von Neumann, et la conclusion générale est que, comme les espaces non commutatifs, les topos aussi ont leur propre dynamique canonique au niveau de la théorie de mesure.

### III) Localic Banach spaces

Comme nous l'avons mentionné plus haut, les topos sont des cas particuliers de groupoïdes localiques, on appelle ces groupoïdes des groupoïdes étales complets [6]. Le groupoïde obtenu par l'action d'un groupe localique sur un point est étale complet essentiellement si le groupe est pro-discret (ce n'est pas tout à fait vrai, ou au moins il faut être plus précis sur la définition de pro-discret, mai c'est une bonne approximation à avoir en tête). En particulier, les groupes

5. Il s'avère, même si cela n'a pas été inclus dans l'article que dans ce cas toute algèbre de von Neumann endomorphisme d'un espace de Hilbert sur le topos est de type I.

6. Tout les groupoïdes étales sont étales complets, mais la réciproque n'est pas vraie.

connexes localement compacts (qui sont alors aussi des groupes localiques) ne correspondent pas à des topos.

Du point de vue de la théorie des algèbres d'opérateurs, ceci est une faiblesse de la théorie des topos. En effet, ces groupes, ou plus précisément leurs $C^*$-algèbres de groupe, ont des propriétés aussi bonnes que celle des groupes discrets, et on aimerait pouvoir étudier aussi de tels exemples dans le formalisme de la théorie des topos. Plus généralement on aimerait pouvoir parler d'espace de Hilbert et de $C^*$-algèbre pas seulement au dessus d'un topos mais aussi au-dessus d'un groupoïde localique.

Le problème pour faire cela est que si on l'on veut une bonne définition (notamment qui soit invariante par équivalence de groupoïdes) de tels objets il faut et il suffit que la notion d'espace de Hilbert qu'on utilise ait la propriété de descente par les surjections ouvertes, et il s'avère que ce n'est pas le cas. Pour remédier à cette difficulté nous définissons une généralisation de la notion d'espace de Banach : les espaces de Banach localiques, dont l'espace vectoriel sous-jacent n'est plus un ensemble mais une locale. Nous développons la théorie de tels objets de façon systématique et nous prouvons qu'ils sont effectivement la solution optimale au problème de descente mentionné.

Nous prouvons aussi, que la dualité de Gelfand constructive peut s'entendre en une dualité entre $C^*$-algèbres localiques et locales compacts régulières comme cela avait été conjecturé par Mulvey et Banachewski dans [4].

Pour finir, nous prouvons un théorème de spatialité montrant que [7] au dessus d'un espace topologique paracompact, ou plus généralement d'une classe de topos satisfaisant une condition technique qui généralise la paracompacité (plus précisément l'existence de partitions de l'unité), il n'y a aucune différence entre espaces de Banach localiques et espaces de Banach ordinaires. En particulier, au dessus d'un groupoïde localique paracompact (et en supposant l'axiome du choix dans le topos de base) les champs d'espaces de Banach classiques sont bien définis sans difficulté supplémentaire.

## Autres développements possibles

### Vers une dualité de Gelfand non abélienne ?

À n'importe quel topos $\mathcal{T}$ on peut associer sa catégorie de $\mathcal{T}$-espaces de Hilbert. C'est une $C^*$-catégorie munie d'une structure monoïdale symétrique. Si $p$ est un point de $\mathcal{T}$, alors $p^*$ induit une représentation de cette $C^*$-catégorie compatible à la structure monoïdale et vérifiant une condition de ≪ normalité ≫ . Il semble que pour une certaine classe de topos (qui doit encore être déterminée précisément) $\mathcal{T}$ peut se reconstruire comme le topos classifiant de la théorie des représentations de sa $C^*$-catégorie vérifiant ces conditions. Cette classe devrait notamment inclure les topos Booléens localement séparés, ce cas particulier précis étant un travail en cours.

### Le cas des topos séparés compacts.

7. En supposant l'axiome du choix dans le topos de base.

Dans les deux premiers chapitres, les conditions de séparation et de séparation locale ont joué un rôle essentiel. Des idées similaires montrent que ces conditions devraient aussi pouvoir être utilisées pour des topos non booléens. Une application interne de construction similaire à celle faite dans le premier développement combinée avec le bon comportement des champs continus sur des locales séparés compacts devrait permettre d'obtenir une très bonne description de la catégorie des $C^*$-algèbres et des modules hilbertiens au dessus d'un topos séparé compact localement décidable (les conditions précises à mettre sur le topos n'étant pas encore complètement bien comprises). En particulier, nous essayons de montrer que pour un tel topos $\mathcal{T}$, la catégorie des $C$-modules hilbertiens de $\mathcal{T}$ (pour n'importe quelle $C^*$-algèbre $C$ du topos) est équivalente à la catégorie des modules hilbertiens sur une $C^*$-algèbre $C \rtimes \mathcal{T}$. Si nous parvenons à faire cela, on pourra ensuite essayer d'appliquer cette construction pour définir $C \rtimes \mathcal{T}$ quand $\mathcal{T}$ est juste localement de la forme précédente (en particulier, juste localement séparé). Ceci est aussi un travail en cours.

**Thermodynamique des topos.**

Si on en reste au cas des topos booléens dans l'étude de l'évolution temporelle il est impossible d'aller plus loin et d'étudier par exemple les états KMS quand la température varie, car ceux ci correspondent à différentes classes de mesure généralisées et donc à différents topos booléens. Ainsi il serait intéressant d'étendre la définition de mesure invariante à des topos non booléens et d'essayer d'obtenir une bonne théorie des états KMS pour les topos. Dans ce but, l'étude du cas des topos de pré-faisceaux sur une petite catégorie simplifiable à gauche devrait être très intéressante. En effet, ces topos sont relativement simples du point de vue géométrique (ce sont des groupoïdes étales avec un espace d'unités qui est le spectre d'un ensemble ordonné), leurs classes de mesures généralisées sont toujours localement séparées, et ils contiennent de nombreux exemples dont la thermodynamique a été beaucoup étudiée (les algèbres de graphes, le système BC etc...).

**$K$-théorie relative à un topos et conjecture de Baum-Connes.**

Il semble possible d'étendre la définition de Kasparov de KK-theorie équivariante d'une paire de $C^*$-algèbres munie d'une action d'un groupe $G$ à une paire de $C^*$-algèbres vivant dans un même topos [8] $\mathcal{T}$, pour définir un groupe $KK_{\mathcal{T}}(A, B)$ qui satisferait un analogue de la propriété universelle de la KK-théorie équivariante (énoncée et démontrée pour les groupes localement compacts dans [68]).
De plus, en observant qu'un espace topologique $X$ équipé d'une action d'un groupe $G$ est un $G$ espace propre et compact si et seulement si le topos $X/G$ des faisceaux $G$-équivariants sur $X$ est séparés et compacts on devrait pouvoir utiliser les résultats sur les topos séparé compact pour donner une interprétation dans ce contexte de la conjecture de Baum-Connes.
La version la plus générale de cette conjecture que nous pourrions énoncer dans ce cadre est fausse (les contre exemples donnés par N.Higson, V.Lafforgue et G.Skandalis dans [35] sont aussi valides dans ce cadre), mais de nombreux cas sont déjà démontrés : par exemple le cas des groupes avec la propriété de Haagerup est dû à Higson et Kasparov dans [34], et un grand nombre d'autres cas

8. Ou peut-être plus généralement, au dessus d'un même groupe localique.

incluant tous les groupes de Lie simples et les groupes hyperboliques ont été obtenue par V.Lafforgue (voir [47]).
Nous espérons que les preuves de ces cas particuliers pourront s'interpréter dans le formalisme de la théorie des topos.

# Introduction

Toposes and $C^*$-algebras are two notions which generalize the concept of topological space to more general situations where usual topology is no longer of help. They appear to have several deep similarities. First of all, there is a long list of examples of "generalized geometric objects" to which one can associate both a $C^*$-algebra and a topos. This list includes discrete groups, some locally compact groups, the non commutative tori, the space of leaves of foliations, graphs, the space of Penrose tilings, and more generally a large class of topological groupoids.

In addition to this list of examples where one has each time both a $C^*$-algebra and a topos describing a same geometric situation, there are examples of problems where both toposes and $C^*$-algebras play a role but in ways that are not easy to relate. In theoretical physics for example, and more precisely on the issue of quantum gravity, operator algebras are clearly relevant objects to study quantum theory and its generalizations, there is even a model of quantum gravity completely based on operator algebras (see [13], [14], see also the first part of [18]). Toposes have also been proposed more recently as possible object of interest in the formulation of quantum gravity (see [31] and [24]).
Also, in number theory, the so called Bost-Connes systems (first introduced in [9] and generalized in [30]) are $C^*$-algebras which appear to be connected to deep number theoretic problems, notably, the Riemann hypothesis (see [17]) and explicit class field theory (see also chapter 2,3 and 4 of [18]). On the other hand, toposes have been introduced explicitly in order to prove the Weil conjectures, including the analogue in characteristic $p$ of the Riemann Hypothesis. In both cases it would be interesting to have a better understanding of the relation between topos theoretic and $C^*$-algebraic approaches.

Also the two theories have independently shown their ability to reconcile the continuum and the discrete. Indeed, in the purely classical conception that a real "variable" should be a function from a set $X$ to $\mathbb{R}$ with some regularity conditions, it is impossible to have, in an interesting way, a discrete and a continuous variable defined simultaneously (that is one the same set $X$). Now, from the point of view of quantum mechanics and of non commutative geometry, a variable is an operator $A$ acting on some Hilbert space $\mathcal{H}$ and satisfying $A = A^*$ if one deals with a real variable. In this context one can have on the same Hilbert space continuous and discrete variables (although they will generally not commute). For example on the space $\mathcal{H} = L^2(\mathbb{R}/\mathbb{Z})$ multiplication by continuous functions on the circle give operators on $\mathcal{H}$ corresponding to continuous variables, whereas the (unbounded) operator $d^2/dx^2$ has a discrete spectrum, and can be written (in the basis given by functions $\exp(inx)$ ) as multiplication by a function with discrete values. Similarly, toposes also have this ability to unite continuous and discrete objects. They put together in the same natural category continuous objects like topological spaces, and extremely discrete objects like algebraic varieties over finite fields or small categories. To quote Grothendieck (from [29, 2.13]):

> *Ce "lit à deux places" est apparu (comme par un coup de baguette magique. . . ) avec l'idée du topos. Cette idée englobe, dans une intuition topologique commune, aussi bien les traditionnels espaces (topologiques), incarnant le monde de la grandeur continue, que les (soi-disant) "espaces" (ou "variétés") des géomètres algébristes abstraits impénitents, ainsi que d'innombrables autres types de structures, qui jusque là avaient semblé rivées irrémédiablement au "monde arithmétique" des agrégats "discontinus" ou "discrets".*

From a very optimistic point of view, one can hope for a form of non-commutative Gelfand duality between a class of toposes and certain $C^*$-algebras eventually equipped with additional structures. Even without such a duality there are a lot of tools and constructions on each side that could benefit the other if one were able to transport them. On one hand, toposes have a really nice (co)homology theory and homotopy theory, and are deeply connected to logic both through their internal logic and through the geometric theory they classify. On the other hand, $C^*$-algebras are connected to functional analysis and to the formalism of quantum physics and one also disposes of a lot of powerful tools to study them, like K-theory, cyclic and Hochschild (co)homology, the modular theory of von Neumann algebras, etc... Moreover, even if $C^*$-algebras and toposes can both be constructed out of groupoids, there are also other processes that can be used to construct them, for example a topos can be obtained from any small category or as a classifying topos, and $C^*$-algebras can be constructed by deformations or by extensions of previously known $C^*$-algebras.

We also need to observe that a relation between these two forms of geometry is necessarily subtle. Specifically, consider the case of the Fourier isomorphisms and more generally of Pontrjagin duality. If $G$ is a discrete abelian group (or more generally an abelian locally compact group) then from the point of view of operator algebras, the convolution algebra of $G$ is isomorphic to the algebra of continuous functions on the dual group $X$ of $G$, and similarly the von Neumann algebra of the group is isomorphic to the von Neumann algebra $L^\infty(X, \mu)$ where $\mu$ is the Haar measure of $X$. This means that from the point of view of operator algebras one identifies the group $G$ (acting on a one point space) and the topological space $X$ of characters of $G$. From the topos perspective these two objects are different, and relatively unrelated. Hence, in some sense, $C^*$-algebras alone seem to forget some information about the geometry because of this Fourier isomorphism. But one the other hand, the existence of this isomorphism is an extremely powerful tool in order to study these objects from a more analytic point of view. The general philosophy that we will remember from this case, is that topos theory seems to stay closer to geometry and be more "descriptive" than $C^*$-algebras whereas $C^*$-algebra theory is closer to analysis and more efficient to provide non trivial invariants and especially numerical ones.

Finally, one of the best tools we have at our disposal to understand the relation between toposes and $C^*$-algebras, is the fact that it is possible (using internal logic) to define a good notion of continuous bundle of Hilbert spaces or $C^*$-algebras or Hilbert modules over a topos. Moreover, if $H$ is a continuous bundle of Hilbert spaces (or of Hilbert modules over some continuous bundle of $C^*$-algebras) then the algebra of bounded operators on $H$ is a $C^*$-algebra which contains a lot of information about the topos, and can be studied using the geometric and logical structure of the topos. This is why the major concern of this thesis is the study of some properties of these continuous fields of operator algebraic structures over toposes, under various geometric assumptions on the concerned toposes. Also, as continuous fields may fail to exist on certain toposes, or maybe too rigid we will define and study a notion of measurable field.

## Context and basic notion

### 1) Toposes

We need to distinguish two sorts of toposes, the elementary toposes (or Lawvere toposes) and the Grothendieck toposes, Grothendieck toposes being a special case of elementary toposes.

An elementary topos is a category which admits finite limits and power-objects, i.e. a universal construction corresponding to the construction of the set $\mathcal{P}(X)$ of subsets of a given set $X$. This notion has been introduced by Lawvere and Tierney, see [48] and [69], the definition has been simplified afterwards.

It has been proved that, in some sense, any elementary topos can be considered as a model of usual set theory, without the axiom of choice, the axiom of infinity (asserting that the set of all natural numbers exists), and in which the law of excluded middle (the assertion that for any proposition $P$ one has either $P$ or the negation of $P$) may not hold. This interpretation of set theoretical constructions is called the "internal logic" of the topos and is used constantly in all our work, in fact we are always implicitly assuming that we are are working internally in some elementary topos $\mathcal{S}$ whose objects are called "sets". We will generally assume that $\mathcal{S}$ has a natural number object (i.e. satisfies the axiom of infinity), but only occasionally assume that the base topos satisfies additional axioms, like the law of excluded middle in chapter 3 or the axiom of dependent choice in section 5 of chapter 4. In particular, unless explicitly stated otherwise we generally work in the framework of intuitionist [9] mathematics. For a brief introduction of how internal logic works, one can consult [6] and for a more detailed one with a lot of examples see [50].

Grothendieck toposes form a particular class of well behaved elementary toposes, which has been defined and studied prior to elementary toposes by Grothendieck and his collaborators for the purpose of algebraic geometry. They can be defined as elementary toposes which are presentable categories (as in [7, 5.2]), or which have all inductive limits and a small set of generators. They can equivalently be defined as categories equivalent to a category of sheaves over a site (this is their

9. Or "constructive", for our concern, intuitionist and constructive will be synonym and will mean "topos valid"

original definition in [21]). In particular, Grothendieck toposes always satisfy the internal axiom of infinity.

The good notion of morphism between Grothendieck toposes is the notion of "geometric morphism". A geometric morphism between two toposes corresponds to a pair of adjoint functors ($f^*$,$f_*$) with $f_*$ in the same direction as the morphism, and $f^*$ which commutes to finite limits. This turns toposes into a 2-category, with the 2-morphisms being the natural transformations between the $f^*$ functors. The functor which sends a topological space to its topos of sheaves, and continuous maps to the functor of pullback and pushforward of sheaves, embeds almost [10] fully faithfully the category of topological spaces into the category of toposes. It is in this sense that toposes are a generalization of topological spaces.

An extremely important application of internal logic, is that if one studies topos theory constructively (which is not a problem) one can talk about Grothendieck toposes internally in some fixed topos $\mathcal{T}$. It appears, that these are exactly equivalent to Grothendieck toposes endowed with a geometric morphism $\mathcal{E} \to \mathcal{T}$. Hence every time one has such a geometric morphism, one can bring its study to the study of $\mathcal{E}$ "internally" in $\mathcal{T}$. In particular, any reasonable property of toposes automatically has a relative version for geometric morphisms (for example, the notion "compact" for toposes gives the relative notion "proper" for geometric morphisms).

**2) Operator algebras**

A (complex) $C^*$-algebra is a Banach algebra over $\mathbb{C}$, endowed with an anti-linear involution $*$ such that $(ab)^* = b^*a^*$ and $\|x^*x\| = \|x\|^2$. The two main results to understand the importance of this definition, are:

- $C^*$-algebras are exactly the closed sub-algebras of the algebra $B(\mathcal{H})$ which are stable by adjunction (the adjunction being the $*$ operation) where $B(\mathcal{H})$ stands for the algebra of bounded endomorphisms of a Hilbert space $\mathcal{H}$.
- Commutative $C^*$-algebras are exactly the algebras $\mathcal{C}_0(X)$ of continuous functions which tend to zero at infinity over a (Hausdorff) locally compact topological space $X$, with the involution given by the complex conjugation. Moreover morphisms of involutive algebras correspond exactly to pre composition by continuous proper maps. This result is known as the Gelfand duality.

Because of the Gelfand duality, one can think of general $C^*$-algebras as generalized "non-commutative" locally compact topological spaces. One can consult [23] for an introduction to the theory of $C^*$-algebra. As mentioned earlier, a lot of tools from topology and geometry have been generalised to this non-commutative context, a lot of examples of this can be found in [16].

Von Neumann algebras (see [22]), are particular $C^*$-algebras, which are not just closed in $B(\mathcal{H})$ for the norm but also for one of the weaker topology on $B(\mathcal{H})$

10. One has to restrict to a certain class of topological spaces, called "sober" topological spaces which include all Hausdorff topological spaces, and all underlying topological spaces of schemes.

(for example, for the topology of simple convergence of operators). It has been shown that von Neumann algebras can be characterized as $C^*$-algebras which are, as Banach spaces, the dual of some other Banach space (see [64]), or as those which have arbitrary directed supremum of positive elements and enough positive linear functionals which commute to directed supremum of positive elements (see [66, III.3.16]).

Abelian von Neumann algebras are of the form $L^\infty(X)$, i.e. the algebra of all bounded measurable functions on a measured space $X$ modulo functions which are zero almost every where. Hence the theory of general von Neumann algebras can be thought of as non commutative measure theory. As in the topological case, a lot of usual constructions and tools of usual measure theory can be extended to the non-commutative case. But non-commutative measure theory comes with an additional feature: when one puts a semi-finite weight (the analogue of a locally finite measure) on a von Neumann algebra $A$, there is, attached to it, a canonical "time evolution of $A$", i.e. a morphism from $\mathbb{R}$ to the group of automorphism of $A$. This construction is originally due to Tomita, has been popularized by Takesaki (in [65]), and Connes proved that in fact, up to inner automorphisms the time evolution does not depend on the weight and is completely canonically attached to the algebra (see [15]). Of course for a commutative algebra this evolution is trivial and for $B(\mathcal{H})$ it is inner (hence trivial in the sense of the uniqueness property). But there is a large class of algebras (type III von Neumann algebra) for which this time evolution is really non trivial and has lead to interesting invariants of these algebras. Another introduction to the theory of this time evolution can be found in [67].

**3) Constructive operator algebra theory**

A part of functional analysis and of the theory of operator algebras has been developed in a constructive context (hence over toposes) by Mulvey, Banachewski and other people in a series of papers including [12] [57], [60],[3],[4].
Before giving a constructive definition of $C^*$-algebras that could be interpreted in a topos, one needs to define constructively what is a real (or complex) number. In classical mathematics, there are a lot of equivalent ways to define the field of real numbers, but these different methods are no longer equivalent in an intuitionist context. First of all one needs to have the axiom of infinity in order to define $\mathbb{N}$ and $\mathbb{Q}$. It appears that even if one does not assume the law of excluded middle, the usual relations on $\mathbb{N}$ and $\mathbb{Q}$ are all decidable, this means that for example, for all pairs $(q, q')$ of rationals one has either $q < q'$,$q = q'$ or $q > q'$. In a Grothendieck topos, the objects corresponding to the interpretation of $\mathbb{N}$ and $\mathbb{Q}$ are simply the locally constant sheaves with values $\mathbb{N}$ and $\mathbb{Q}$.
The definition of $\mathbb{R}$ involving Cauchy sequences has to be avoided in the absence of the axiom of (dependent) choice. The construction involving usual Dedekind cuts works relatively well at the level of ordered sets, but has several hiccups: the constructions by upper and lower Dedekind cuts give two different (non isomorphic as ordered sets) sets of reals called respectively the upper and lower semi-continuous real numbers. In both cases one has trouble with defining the opposite of a number, or more generally multiplication by non positive elements. Lower or upper semi-continuous real numbers do not form rings, or even additive groups. They form ordered additive monoids, and one can define

the multiplication of two positive elements. Despite of this they will be of interest, and they are easy to understand: for example on a topological space, they correspond to the sheaves of upper and lower semi continuous functions with value in $\mathbb{R}$.
The good definition of real number is done using two sided Dedekind cuts. There are actually (at least) two ways of giving the definition of a two sided Dedekind cut which are not constructively equivalent and which give again two different notions of real numbers called Macneille real numbers and Dedekind (or continuous) real numbers. We will use only Dedekind/continuous real numbers and call them simply "real numbers". The set of Dedekind real numbers is not a field in the sense that we do not have "$x = 0$ or $x$ is invertible" in general, but it is a (local) ring, and the negation of "$x$ is invertible" is "$x = 0$" (the opposite does not necessary hold). On a topological space, the sheaf of Dedekind real numbers corresponds to the sheaf of continuous real valued functions. The set of real numbers is included both in the set of lower and upper real numbers, and in both cases corresponds exactly to the set of elements which have an opposite. One should note that, contrary to the case of upper and lower real numbers it is not true in the set of (continuous) real numbers that every bounded set has a supremum. For more information about this various definition of real numbers see [44, D4.7].

It is also possible to define the set of Dedekind real numbers as a completion of $\mathbb{Q}$, but one has to use a completion by Cauchy filters instead of Cauchy sequences (as in [10, II.3]).

Once we define real numbers, the definition of complex numbers does not pose any problem: $\mathbb{C}$ is simply defined as $\mathbb{R}[X]/(X^2+1)$. It is not completely exact that $\mathbb{C}$ is still algebraically closed, but the problem is essentially that it is not a field, and it is still true that $\mathbb{C}$ is separably closed, that is if a polynomial with coefficient in $\mathbb{C}$ has an invertible discriminant then it can be factored.

There are two possible definitions of Banach spaces: one can either ask the norm of an element to be a continuous real number, or an upper semi-continuous real number, corresponding to the idea that in most of mathematics one only needs to give upper bounds on norms of certain elements (and we do not need lower bound). The second case is more general, and Banach spaces are in general assumed to have a norm with value in upper semi-continuous real numbers, Banach spaces whose norm takes value in continuous real numbers are generally called continuous Banach spaces. For example the set of all bounded operators over a Hilbert space is in general not a continuous Banach space, but the closed sub algebra of compact operators is a continuous Banach space (because it is defined as the closure of the set of finite rank operators and finite rank operators have continuous norm). Completeness should not be defined by Cauchy sequences, but can be defined equivalently by Cauchy filters or Cauchy approximations.

It appears that most of the theory of $C^*$-algebras can be developed relatively well, one has even been able to give a constructive version of the Gelfand duality (one has to replace compact topological spaces by compact completely regular locales). It is due in the first place to Mulvey and Banachewski [4] in the case of Grothendieck toposes, and to T.Coquand in [20] in the more general framework of constructive mathematics (hence also inside elementary toposes). Some examples of differences between constructive and classical operator algebras are listed here:

- As mentioned earlier, $\mathbb{C}$ is no longer a field, in fact most of the properties of $\mathbb{C}$ which are not true for general commutative algebras no longer hold. For example it is not true that there is an orthogonal projection over a closed subspace of an Hilbert space. This should be related to the fact that there is in general no orthogonal projection over a closed sub module of a $C(X)$-Hilbert module.
- In the same spirit, bounded endomorphisms of Hilbert spaces have in general no adjoint. One calls *operators* the endomorphisms which have an adjoint. It is enough to assume the law of excluded middle to avoid this issue.
- One cannot hope to represent an arbitrary $C^*$-algebra as a sub-algebra of $B(\mathcal{H})$ for $\mathcal{H}$ a Hilbert space, because the proof of this result uses the Hahn Banach theorem in an essential way. But it is still possible to represent any $C^*$-algebra as a closed involutive sub $C^*$-algebra of $B(\mathcal{H})$ for $\mathcal{H}$ an Hilbert module over a commutative $C^*$-algebra $C(X)$. This also uses the GNS construction, but applied to the "universal state" (which takes value in $C(X)$, for $X$ the space [11] of states of the algebra) instead of a particular well chosen state.
- It does not seem possible to give a good definition of von Neumann algebras without the law of excluded middle (or more precisely, all the classically equivalent definitions will no longer be equivalent). The reason for this is that without the law of excluded middle there is no way of constructing projections in general, and that projections are one the most essential tools of basic von Neumann algebra theory.

Finally, this internal theory of operator algebras corresponds to well known objects, at least when interpreted over a paracompact topological space: an internal Banach space corresponds to a Banach bundle whose sections are assumed to have a continuous norm as in [23, 10.1.2(iii)], or a semi-continuous norm as in [36] depending on whether the internal Banach space is continuous or not. Similarly internal $C^*$-algebras and Hilbert spaces correspond to continuous fields of $C^*$-algebras and Hilbert spaces (See [12] and [58]). Also, if $X$ is a locally compact topological space, Hilbert spaces in $\mathsf{Sh}(X)$ are equivalent to Hilbert modules over $C_0(X)$.

Moreover, if $A$ is an abelian von Neumann algebra, one can consider the topos $T$ of sheaves over the complete boolean algebra of projections of $A$. Hilbert spaces in $T$ correspond to $W^*$ modules over $A$, which are known to correspond to normal representations of $A$, and under some countability condition to measurable fields of Hilbert spaces over the corresponding measured space.

Finally, if $\mathcal{T}$ is a topos and $\mathcal{H}$ an Hilbert space in the logic of $\mathcal{T}$ (or more generally an Hilbert module over some $C^*$-algebra in $\mathcal{T}$), then, the algebra of (globally) bounded operators on $\mathcal{H}$ defines a $C^*$-algebra in the base topos. This

11. "Space" has to be interpreted as "locale" here.

construction gives a natural way to associate $C^*$-algebras, or more generally $C^*$-categories to toposes. Unfortunately, these algebras are generally a little too big to be really interesting: all the difficulty lies in selecting interesting (smaller) classes of operators which will give interesting $C^*$-algebras (and sometimes also in constructing a nice Hilbert space over $\mathcal{T}$).

**4) Locales and localic groupoids**

A *localic* topos is a Grothendieck topos that is generated by subobjects of its terminal object, or equivalently that can be defined by a site whose underlying category is a pre ordered set. The key example of localic toposes is the category of sheaves over a topological space. The category of localic toposes is equivalent to the category of "locales". Locales can be defined by certain ordered sets called frames and interpreted as the lattice of their of open subsets (more precisely, the category of locales is the opposite of the category of frames). A locale is essentially the same thing as a topological space but may fail to have "points" (i.e. geometric morphisms from the topos of sets to them). A locale having enough points is the same thing as a sober topological space. Despite this, the category of locales is generally a little better behaved than the category of topological spaces. For example as the product of locales is in general different from the product of topological spaces, the notion of localic group (or groupoid) is different from the notion of topological group (or groupoid) and one has that any localic subgroup of a localic group is closed ! In particular, $\mathbb{Q}$ endowed with the topology induced by the topology of $\mathbb{R}$ is not a localic group. This difference disappears in the case of locally compact groups.

An introduction to the theory of frames and locales can be found in the first two chapters of [8], in the part $C$ of [44]. There is also a recent book ([62]) entirely devoted to them (but in a non-constructive framework). The reader can also consult the two excellent survey papers of P.T.Johnstone on the subject: [41], [40].

For an arbitrary topos $\mathcal{T}$ there is a universal localic topos $\mathcal{L}$, called the *localic reflection* of $\mathcal{T}$ equipped with a geometric morphism from $\mathcal{T}$ to $\mathcal{L}$. The frame of "open subsets" of $\mathcal{L}$ can be obtained as the set of subobjects of the terminal object of $\mathcal{T}$.

To a localic or topological groupoid one can associate a category of equivariant sheaves on it. It appears that this category is a topos. A.Joyal and M.Tierney proved in [45] that every Grothendieck topos can be realized as the category of sheaves over an open localic groupoid (i.e. a localic groupoid whose structure maps are open maps). Moreover, in [52],[53] and [11] I.Moerdijk and M.Bunge showed that if one defines morphisms of open localic groupoids as being principal bi-bundles (for the topology of open surjections), then:

- The construction of the category of equivariant sheaves defines a functor from groupoids to Grothendieck toposes.
- The construction of Joyal and Tierney defines a unique groupoid up to isomorphism in this category of groupoids, i.e. up to some notion of equivalence.

- This defines a functor from toposes to groupoids which is a right adjoint to the functor of equivariant sheaves.
- This turns Grothendieck toposes into a full reflexive sub-category of the category of open localic groupoids.

This category of groupoids with principal bi-bundles as morphisms can also be seen as the full subcategory of the category of stacks on the site of locales with the topology whose covering are open surjections (see also [11]).

As $C^*$-algebras can be constructed out of locally compact groupoids, these results show another path to relate toposes and $C^*$-algebras. It is in fact close to the previous one: If a topos $\mathcal{T}$ corresponds to a locally compact groupoid, it is generally possible to see 'the' reduced $C^*$-algebra attached to this groupoid as a sub algebra of endomorphisms of some Hilbert space over $\mathcal{T}$. The question of choosing and constructing a Haar system on the groupoid in order to define the algebra translates into choosing and constructing a suitable Hilbert space in the topos, which is in general an easier question if one considers it from the point of view of internal logic.

## Main results of this thesis

This thesis decomposes into three relatively independent parts.

**I) Toposes, quantales and $C^*$-algebras in the atomic case**

In this article, we start by reviewing the (already known) equivalence between Grothendieck toposes and ordered theoretic objects called Grothendieck quantales (a particular case of modular quantales). We improved some aspects of this equivalence, especially with theorem 3.6.3 and corollary 3.6.9. Grothendieck quantales are interesting for our concern for two different reasons: first, they look a lot like something that one wants to call "characteristic one operator algebras" hence they should be at least formally closer to operator algebra, and secondly it seems reasonable to think that operator algebras attached to a topos will arise as convolution algebras of functions "on the quantale" (see 3.8 for more detail on this second point).

We then completely focus on the case of "atomic toposes" and "atomic quantales", where essentially, there is no longer any non discrete topology involved. And in this case we are able to provide a relatively complete description of the relation between toposes and $C^*$-algebras through the use quantales. We first show that atomic modular quantales correspond exactly to a natural notion of hypergroupoid, and atomic Grothendieck quantales to a special case of hypergroupoids that we call semi-simple hypergroupoids.

We then show that the convolution of functions on the quantale is well defined if and only if some explicit finiteness conditions (stated both in terms of the topos and in terms of the hypergroupoid) are satisfied and that in this situation one gets a nice $C^*$-algebra with an arithmetic sub-algebra and a modular time evolution explicitly described.

We also show that these finiteness conditions are equivalent to a geometric condition: the fact that the topos is locally separated, and that under this assumption the time evolution of the corresponding von Neumann algebra is defined by a $\mathbb{Q}_+^*$ principal bundle on the topos which is completely canonical.

The main example of this situation is the topos of continuous actions of a pro-discrete localic group (or more generally a disjoint sum of such toposes). The finiteness condition corresponds to the local compactness of the group and the algebra arising is a Hecke algebra of double co-sets with respect to certain open subgroups (corresponding to a base of atoms that we can chose). It appears that, assuming the axiom of choice, every (connected) example satisfying the finiteness condition is of this form. All this work is constructively valid and can hence be applied internally to other toposes where other sorts of examples can exist. Also the methods we have used here are meant to be generalized to other topos theoretic situations.

We think of this case of atomic toposes as a "toy model" that will lead us towards the understanding of more general phenomena.

**II) Measure theory over toposes**

This paper is devoted to two main ideas:

The first idea is that, by analogy with usual measure theory, one can define a measurable field of something (like a measurable field of Hilbert spaces) over a topos $\mathcal{T}$ as being an object (an Hilbert space) living in a boolean topos $\mathcal{B}$ endowed with an injective geometric morphism to $\mathcal{T}$. For example, if one has a continuous field on $\mathcal{T}$ one can "forget" that it is a continuous field and keep only the structure of measurable field by pulling it back to $\mathcal{B}$. Such a topos $\mathcal{B}$ is called a generalized measure class on $\mathcal{T}$.

We show that generalized measure classes can be pushed-forward (in particular, to any point $p$ of the topos one can attach the Dirac mass at this point, which corresponds to the topos of continuous actions of the localic group of automorphisms of $p$). And that if the boolean topos satisfies some condition of existence of measures (one says that $\mathcal{B}$ is measurable, or that the generalized measure class is effective) then the category of measurable fields of Hilbert spaces is a complete $W^*$ category with a small set of generators, hence the category of modules over a von Neumann algebra.

A generalized measure class over a locally compact topological space $X$ is the same thing as a monotone complete $C^*$-algebra $A$ which contains $\mathcal{C}_0(X)$ as a dense sub-algebra in the sense that there is no proper monotone complete sub-algebra of $A$ which contains $\mathcal{C}_0(X)$. The generalized measure class is effective if and only if $A$ is a von Neumann algebra. In particular, for each locally compact topological space there is a largest effective generalized measure class on $X$ corresponding to the enveloping von Neumann algebra, but the family of all generalized measure class on $X$ may not be a proper set, and may not have a maximal element. The situation is the same if $X$ is any locale (there is largest effective generalized measure class) but we do not know if it is also the case when $X$ is a general topos.

The second idea of this paper, is that the construction of the canonical $\mathbb{Q}_+^*$ bundle over locally separated atomic toposes done in the first article can be carried out in boolean toposes, again under the assumption that the topos is locally separated (and satisfying the measurability condition). This produces a canonical $\mathbb{R}_+^*$ bundle which can be interpreted as the bundle of locally finite well supported measures, and is a principal bundle exactly because of the Radon-Nikodym theorem. One then has three possible situations:

- If the topos is not just locally separated, but actually separated, the bundle is trivial for completely explicit reasons. This is analogue of the case of type I von Neumann algebras [12].
- If the topos is not separated, but the bundle is trivial, then a global section of the bundle gives an "Invariant measure" that allows one to measure the "mass" of any object of the topos (which is a real number) in a natural way, and to define a trace on the von Neumann algebra of endomorphisms of certain internal Hilbert spaces.
- If the bundle is non trivial then it gives a natural "modular time evolution" of Hilbert spaces over the (boolean) topos, related to the usual modular time evolution of von Neumann algebras.

This is really similar to the type decomposition of von Neumann factors. The general conclusion is that as for non-commutative spaces, measure theory over toposes has its own canonical dynamic.

**III) Localic Banach spaces**

As mentioned earlier, toposes correspond to particular localic groupoids, called etale complete groupoids. The groupoid obtained from by the action of a localic group $G$ over a point is etale complete essentially if the group is "pro-discrete" (this is not exactly true, or at least one has to be careful with the definition of pro-discrete, but this is still a good picture to keep in mind). In particular, connected locally compact topological groups (they are also localic groups) do not corresponds to toposes.

From the point of view of operator algebras this is a clear weakness of topos theory. Indeed, these groups are known to have as good operator algebraic properties as discrete groups, and one would like to be able to handle them in the topos theoretic formalism. More generally one would like to be able to talk about Hilbert spaces and $C^*$-algebras not only over toposes but also over (open) localic groupoids.

There is one problem in order to have a nice definition (invariant by equivalence of groupoids) of such objects. It is essentially the notion of Banach space (as well a Hilbert space, and $C^*$-algebra) does not descend along open surjections (see the section 2.6 of this chapter). In order to fix this problem we define a generalization: the notion of localic Banach spaces, which are Banach spaces whose underlying vector spaces are no longer sets but are locales. We develop the theory of such objects in a systematical way, and we prove that they indeed provide the optimal solution to the problem of descent mentioned earlier, and

12. Although it has not been included in the paper, it is true that an algebra of endomorphisms of an Hilbert space over a separated boolean topos is always a type I $AW^*$ algebra.

hence give a good definition of fields of Banach spaces over an open localic groupoid.

We also prove that one has an extended Gelfand duality between localic $C^*$-algebras and compact regular locales as conjectured by Mulvey and Banachewski in [4].

Finally, we prove a spatiality theorem showing that [13], over a paracompact topological space, or more generally over a class of toposes satisfying some technical condition generalizing paracompactness, there is no difference between localic Banach spaces and usual Banach spaces. Hence for "paracompact" localic groupoids (and assuming the axiom of dependent choice in the base topos) usual fields of Banach spaces will be well defined without any additional difficulties.

This last result is inspired from a theorem of Douady and Dal Soglio-Hérault (which can be found in the appendix of [25]) which states that over a paracompact topological space every Banach bundle has enough continuous sections, where "Banach Bundle" is a notion which is extremly close to our notion of localic Banach space interpreted in the topos of sheaves over a topological space. The proof by Douady and Dal Soglio-Hérault is based on the construction of "$\epsilon$-continuous section", which rougly are discontinuous sections whose discontinuities are $\epsilon$-small. Such sections do not make sense in a purely topos theoretic context (they are discontinuous) but we have been able to replace them by the notion of section of the sheaf of positive $\epsilon$-small open sublocales, such a section corresponding to a tubular neighborhood of a $\epsilon$-continuous section.

## Future possible developments

### A Non abelian monoidal Gelfand duality ?

To any topos $\mathcal{T}$ one can associate its category of internal Hilbert spaces. It is a $C^*$-category endowed with a symmetric monoidal structure. If $p$ is any point of $\mathcal{T}$, then $p$ induces a monoidal representation of this category satisfying some normality condition. It seems that for a large class of locally separated toposes it is possible to reconstruct the topos from its $C^*$-category as a classifying topos of "normal" monoidal representations of the $C^*$-category (the term normal has to be defined properly). This large class includes boolean locally separated toposes, and this precise case is a work in progress.

### Locally compact separated toposes

In chapter I and II, the condition of separation and local separation appears to play a major role. A similar idea seems to show that these properties can also be used outside the realm of boolean toposes. An internal application of results similar to those of chapter I, combined with the good behavior of continuous fields over locally compact topological spaces, should give an extremely good description of the category of $C^*$-algebras and their Hilbert modules over an arbitrary separated locally compact locally decidable topos $\mathcal{T}$. In particular, we

13. assuming the axiom of dependant choice in the base topos

are currently trying to prove that for any $C^*$-algebra $C$ over such a topos, there exists a $C^*$-algebra $C \rtimes \mathcal{T}$ in the base topos such that the category of $C$-Hilbert module in $\mathcal{T}$ is equivalent to the category of $C \rtimes \mathcal{T}$ Hilbert modules. Once this is achieved, one can try to apply this to define $C \rtimes \mathcal{T}$ in the case where $\mathcal{T}$ is just locally of the previous form. One can also try to extend the non-abelian Gelfand duality of the previous paragraph to toposes of this form. This is a work in progress .

**Thermodynamics of toposes.**

If we stick to boolean toposes in the study of the time evolution it is not possible to go further and to study KMS states at different temperatures, because these correspond to different generalized measure classes, hence to different boolean toposes. Hence it should be interesting to try to extend the notion of invariant measure to a non boolean topos and to try to obtain a nice theory of KMS states for toposes. In order to do so a good example to study would be the case of toposes of pre-sheaves over a left cancellative small category. These have the advantage to be relatively easy to describe at a geometric level (they correspond to etale groupoids, and their space of units is the spectrum of an ordered set). They include all graph algebras, and the $BC$-systems whose thermodynamics has been extensively studied. Also, all their generalized measure classes satisfy the local separability conditions.

**$K$-theory relative to a topos and the Baum-Connes conjecture.**

It seems possible to extend the definition of Kasparov equivariant KK-theory of a pair of $C^*$-algebras with an action of a group $G$, to a pair of $C^*$-algebras living in a same topos [14] $\mathcal{T}$, defining a group $KK_{\mathcal{T}}(A, B)$ following an analogue of the universal property of equivariant $KK$-theory (stated and proved for locally compact groups in [68]).
Also, observing that a topological space $X$ endowed with an action of a group $G$ is a proper compact $G$ set if and only if the topos $X/G$ is compact and separated, one should be able to use the results about locally compact separated toposes (of the second possible development) to generalize the definition of the Baum-Connes assembly map to this topos theoretic context, and formulate a topos theoretic analogue of the Baum-Connes conjecture.
The most general form of the Baum-Connes conjecture we can state is known to be false (counter example has been given by Higson, Laforgue and Skandalis in [35]), hence one cannot hope that this version of the conjecture will be true in general, but a lot of cases of the conjecture are known to be true, for example the Baum-Connes conjectures with coefficient has been proved for group with the Haagerup property by Higson and Kasparov in [34], and a large number of cases (all simple Lie group, Hyperbolic groups etc...) have been obtained by V.Lafforgue see [47]. Our hope is that the proof of these particular cases might be interpretable in the topos theoretic context.

14. Or maybe more generally of a couple of localic $C^*$-algebras over a same open localic groupoids.

# Notations, conventions and preliminaries

In all this thesis, we work internally in a base topos $\mathcal{S}$ which is an elementary topos with a natural number object $\mathbb{N}$. Objects of $\mathcal{S}$ are called sets. All other toposes we will be toposes endowed with a geometric morphisms to $\mathcal{S}$ and we will assume that they are bounded over $\mathcal{S}$, or equivalently that they are Grothendieck topos over $\mathcal{S}$, i.e. categories of $\mathcal{S}$ valued sheaves on a Grothendieck site internal to $\mathcal{S}$. We will not assume that the logic of $\mathcal{S}$ satisfies the axiom of choice or the law of excluded middle, except in chapter 2 where the base topos $\mathcal{S}$ is assumed to be boolean and in the last section of chapter 3 where we will at some point assume that it satisfies the axiom of dependant choice (it will be indicated clearly).
Most of the basic notions of topos theory can be found in [49], for the others we will give precise references in [44].
In all this thesis we will use the following notations and conventions:

- If $\mathcal{C}$ is a category (or a topos) then $|\mathcal{C}|$ denotes the set (or the class) of objects of $\mathcal{C}$. The symbol $\mathcal{C}$ denotes the set (or class) of all maps, and we will equivalently use the notation $\mathcal{C}(a,b)$, $\hom(a,b)$ or $\hom_{\mathcal{C}}(a,b)$ for the set of morphisms from $a$ to $b$.
- The letters $\mathcal{T}$ and $\mathcal{E}$ will always denotes a topos over $\mathcal{S}$.
- $\Omega_{\mathcal{T}}$ denotes the sub-object classifier of the topos $\mathcal{T}$, $1_{\mathcal{T}}$ denotes its terminal object and if $X \in |\mathcal{T}|$ then $\mathcal{P}(X)$ stand for the power object of $X$ (isomorphic to ${\Omega_{\mathcal{T}}}^X$), and $\mathsf{Sub}(X)$ for the set of sub-objects of $X$, i.e. the set of global sections of $\mathcal{P}(X)$.
- $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ denote respectively the set of non negative integers, integers and rational numbers of the base topos. We recall that even if we do not assume the law of excluded middle in the base topos it is true that for all pair of rational number $a$ and $b$ one has $a = b$ or $a < b$ or $a > b$.
- $\mathbb{N}_{\mathcal{T}}$, $\mathbb{Z}_{\mathcal{T}}$ and $\mathbb{Q}_{\mathcal{T}}$ denote the sheaves of natural numbers, integers and rational numbers in the topos $\mathcal{T}$. As we restrict ourselves to toposes endowed with a geometric topos to the base topos they are simply $p^*(\mathbb{N})$,$p^*(\mathbb{Z})$ and $p^*(\mathbb{Q})$ where $p$ is the canonical geometric morphism from $\mathcal{T}$ to the base topos $\mathcal{S}$.
- $\mathbb{R}_{\mathcal{T}}$ denotes the sheaf of continuous real numbers, i.e. two sided Dedekind cuts (see [44, D4.7]). In a Grothendieck topos, it can be described externally by the following properties: for any $X \in |\mathcal{T}|$, $\hom(X, \mathbb{R}_{\mathcal{T}})$ is the set of continuous functions from the underlying locale of $X$( whose frame

of opens is $\mathsf{Sub}(X)$) to the space of real numbers [15]. We also denote by $\mathbb{R}$ the object of coninuous real number of the base topos.

- $\mathbb{C}_{\mathcal{T}}$ denotes the object of $\mathcal{T}$ of (continuous) complex number, i.e. $\mathbb{R}_i$.
- $\mathbb{R}$ and $\mathbb{C}$ denotes the object of continuous real and complex number of the base topos. In chapter 3 it will also denote the formal locales of real and complex numbers in the base topos.
- $\overrightarrow{\mathbb{R}_+^\infty}$ denotes the set of positive lower semi-continuous real numbers (possibly infinite). In presence of the law of excluded middle it is the set $\mathbb{R}_+ \cup \{\infty\}$. In a topos it is the sheaf defined by the fact that $\hom(X, \overrightarrow{\mathbb{R}_+^\infty})$ is the set of functions from the locale $\mathsf{Sub}(X)$ to $\mathbb{R}_+ \cup \{\infty\}$ [16] endowed with the topology where the $(a, \infty]$ are a basis of open sets, i.e. it is the set of lower semi-continuous functions (possibly infinite) on the locale $\mathsf{Sub}(X)$. Internally, $\overrightarrow{\mathbb{R}_+^\infty}$ is defined as the set of $P \subset \mathbb{Q}_{\mathcal{T}}$ such that if $q < 0$ then $q \in P$, and $q \in P \Leftrightarrow \exists q' \in P, q < q'$. See [44, D4.7].
- A proposition (internal to a topos) is said to be *decidable* if it is complemented (i.e. such that $P \vee \neg P$ holds). An object is said to have decidable equality, or to be *decidable*, if its diagonal embedding $X \to X \times X$ is complemented.
- A set (or an object $X \in |\mathcal{T}|$) is said to be *inhabited* if it satisfies (internally) $\exists x \in X$ (which in constructive mathematics is stronger than the assertion that $X$ is not empty). For an object of a topos it is equivalent to the fact that the canonical map $X \to 1_{\mathcal{T}}$ is an epimorphism.
- A set $X$ (or an object of a topos $\mathcal{T}$) will be said to be *finite* if it is (internally) Kuratowski finite, i.e. if internally $\exists n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ such that for all $x \in X$ $\exists i, x = x_i$. On can consult [44, D4.5] for the theory of Kuratowski finite sets.
  Roughly, a quotient of a finite set is finite, but the proof that a subset of a finite set is finite requires the subset to be complemented and may fail in full generality. If a set $X$ is finite and has decidable equality, then there exists $n \in \mathbb{N}$ such that $X$ is isomorphic (internally [17]) to $\{1, \dots, n\}$, and a subset of $X$ is finite if and only if it is complemented.
- If $X \in |\mathcal{T}|$ is an object of a topos, we denote by $\mathcal{T}_{/X}$ the slice category whose objects are object of $\mathcal{T}$ endowed with an arrow to $X$ and whose arrows are commutative triangles. It is a topos, and it play the role of the etale space of $X$.
- A *sub-quotient* of an object $X \in |\mathcal{T}|$ is a quotient of a sub-object of $X$ (or equivalently, but less naturally, a sub-object of a quotient).
- An object $B \in |\mathcal{T}|$ is said to be a *bound* of $\mathcal{T}$ if any object of $\mathcal{T}$ can be written as a sub-quotient of an arbitrary co-product of copies of $B$ (see [44, B3.1.7]). Equivalently $B$ is a bound of $\mathcal{T}$ if $\mathsf{Sub}(B)$ is a generating family of $\mathcal{T}$, i.e. $\mathsf{Sub}(B)$, seen as a full subcategory of $\mathcal{T}$ and endowed with the restriction of the canonical topology of $\mathcal{T}$, forms a site of definition for $\mathcal{T}$. This means essentially that $B$ is big enough to generate $\mathcal{T}$, for example: in the topos $G-\mathrm{Set}$ of sets endowed with an action of a group

15. In a non-boolean context, the "space of or real numbers" has to be interpreted as "the formal locale of real numbers".

16. Here we have assumed the law of excluded middle in the topos of sets in order to simplify the notation.

17. As the isomorphism is not canonical, it might not lift to a global isomorphism if we are working internally in a topos.

$G$, an object $X$ is a bound if and only if the map $G \to Aut(X)$ is injective. A topos is the topos of sheaves over a locale if and only if $1_{\mathcal{T}}$ is a bound (see [44, definition A4.6.1 and theorem C1.4.6]). When a topos is given by a site, the simplest way to obtain a bound is to choose an object which contains a copy of each representable objects (for example, the direct sum of all the representable objects, see [44, B3.1.8(b)]). Existence of a bound, together with the existence of enough (co)limits, characterize Grothendieck toposes among elementary toposes (see [44, C2.2.8]).

- When one has a product $E_1 \times \cdots \times E_n$ of objects of any kind (generally locales) we will denote by $\pi_i$ the projection onto $E_i$, by $\pi_{i,j}$ the projection onto $E_i \times E_j$, etc... We generally do not specify the domain of definition and we hope that it will be clear from the context. For example one has: $\pi_1 \circ \pi_{i,j} = \pi_i$ and $\pi_2 \circ \pi_{i,j} = \pi_j$ because in these formulas $\pi_1$ and $\pi_2$ denote the two projections from $E_i \times E_j$ to $E_i$ and $E_j$ respectively. This will be used essentially in chapter 3.

# Chapter 1

# Toposes, quantales and $C^*$-algebras, the atomic case.

## 1 Introduction

In some sense, quantales are a third kind of generalized spaces which appears to be related both to operator algebras and to toposes. In operator algebra they have been introduced by C.J.Mulvey (or more precisely named, see [59] and [38]) in an attempt to formalize the notion of "quantum topology" studied by R.Giles and H.Kummer in [28] and C.A.Akemann in [1]. In topos theory they arise in the description of the category of sup-lattices of a given topos studied in [45] and, because of the results in [33] (see also the first part of the present chapter of this thesis) they completely describe a topos in the sense that a topos endowed with a bound is essentially the same thing as a special kind of quantale, called a "Grothendieck quantale".

The types of quantales appearing in operator algebra and in topos theory have extremely different properties: the Grothendieck quantales are quantales of relations on a bound of a topos, and behave like the quantale of relations on a set, in particular they are distributive and modular. On the other side, quantales appearing in operator algebra are in general not modular but correspond to particular subquantales of the quantale of projections in a Hilbert space, hence deserving the name "quantum" in a more precise manner. These differences exclude a straightforward comparison of the theories of Grothendieck toposes and of operator algebras through the associated quantales, and show that the relation between the two theories is necessarily more involved. We nevertheless use the Grothendieck quantale associated to a topos as a starting point and show that under suitable hypothesis a Grothendieck quantale can be used to construct a convolution $C^*$-algebra attached to a topos.

We would also like to stress out that modular quantales and Grothendieck quantales are extremely good candidates to be thought of as characteristic one operator algebras. First there are several formal similitudes: the fact that sup-lattices enriched categories are a form of "characteristic one additive categories", the presence and the important role of the $*$ involution, and other more specific points like the fact that the initial and terminal support of $a$ are given by $a^*a$ and $aa^*$. Secondly, Grothendieck quantales (and conjecturally modular quantales) are interpreted as quantales of relations on objects of topos (see 2.4.5 and 2.7.1), i.e. as characteristic one matrix algebras. Hence results of sections 3 underline the fact that there is a close relation between topos theory and non-commutative geometry in characteristic one. It might be interesting to make this relation more precise, for example by giving an interpretation of the distributivity (Q3) and the modularity (Q9) conditions of 2.1.1 in term of characteristic one semirings.

In section 2 we focus on the relation between Grothendieck quantales and Grothendieck toposes. Most results of the section 2.1 to 2.5 are already well known and present in [63] or [33]. The only originality of our approach is that we give a direct proof that the category of sup-lattices of a topos is the category of modules over a quantale of relations, and then we use this to describe the objects of the topos in terms of this quantale, the previous approach (mainly [63]) generally works in the other direction. In 2.1 we review the basic theory of sup-lattices internal to a topos. In 2.3 the basic theory of sup-lattices enriched category and we give a characterization of the sup-lattice enriched categories of modules over a unital quantale. In 2.4 we explain the correspondence between Grothendieck toposes and Grothendieck quantales, and give a description of a topos attached to the quantale $Q$ in terms of a notion of $Q$-set completely similar to the notion of $\mathcal{L}$-set when $\mathcal{L}$ is a frame. Section 2.5 produces a description of the topos attached to a quantale as a classifying topos making the previous correspondence functorial.
The main new contribution of section 2 is in 2.6 and consists of a description of the locales internal to a topos in terms of "modular actions" of the corresponding quantale on classical locales, as well as more generally a description of bi-linear maps between sup-lattices in terms of the corresponding $Q$-modules.
In 2.8 we explain why attaching a Grothendieck quantale to a topos is an interesting step towards the construction of $C^*$-algebras.

In section 3 we focus on the case of an atomic topos, showing that in this case the attached Grothendieck quantale corresponds to a "hypergroupoid". Under some reasonable finiteness assumptions there is indeed a "Hypergroupoid $C^*$-algebra" attached to that quantale in the way sketched in 2.8. This $C^*$-algebra comes in two forms: a reduced algebra and a maximal algebra; in both cases it comes with a natural and explicit time evolution attached through Tomita theory to a "regular" representation, and with a generating $\mathbb{Z}$ sub-algebra with interesting combinatorial properties. We also characterize in section 3.7 the atomic toposes for which the construction is possible as the locally decidable locally separated toposes. We also show that in this situation the time evolution is canonical and described by a principal $\mathbb{Q}_+^*$ bundle. The main example of this situation are the well-known double-cosets algebras.

# 2 Toposes, quantales and sup-lattices

## 2.1 Introduction

2.1.1. Let $X$ be any object of a topos $\mathcal{T}$, and denote by $\mathsf{Rel}(X)$ the set of relations on $X$, i.e. the set $\mathsf{Sub}(X \times X)$ of sub-objects of $X \times X$. Then $Q = \mathsf{Rel}(X)$ is endowed with several structures:

(Q1) The inclusion of subobjects gives an order relation on $Q$.

(Q2) $Q$ has arbitrary supremums for this order relation. We will denote the supremum of a family by $\bigvee_{i \in I} a_i$, supremums are also called unions [1]. Moreover the existence of supremum implies the existence of infimums denoted $\bigwedge$ and sometime called intersections.

(Q3) Finite infimums distribute over arbitrary supremums: $a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$.

(Q4) There is an associative composition law on $Q$ defined internally by $RP = \{(x,y) | \exists z \in X, xRz \text{ and } zPy\}$.

(Q5) The composition law is order preserving and distribute over supremum.

(Q6) The diagonal subobject of $X$ provide an element $1 \in Q$ which is a unit for the composition law.

(Q7) There is an order preserving involution: $R \mapsto R^* = \{(y,x)|xRy\}$ of $Q$.

(Q8) For all $P, R \in Q$ one has: $(PR)^* = R^* P^*$.

(Q9) For all $P, R, T \in Q$ one has $P \wedge RT \leqslant R(R^*P \wedge T)$.

If we assume additionally that $X$ is a bound of $\mathcal{T}$, then one has additionally:

(Q10) There exist two families $(v_i)_{i \in I}, (u_i)_{i \in I}$ of elements of $Q$ such that: $\forall i, u_i u_i^* \leqslant 1, v_i v_i^* \leqslant 1$ and $\top = \bigvee_i v_i u_i^*$. where $\top$ denotes the top element of $Q$.

Some of these points deserve a proof and a few comments.

- $(Q9)$ is called the *modular* law. It is easy to prove using internal logic: Let $(a,b) \in (P \wedge RT)$. One has: $(a,b) \in P$ and there exists $c \in X$ such that $(a,c) \in R$ and $(c,b) \in T$. Hence $(a,c) \in R$ and $(c,b) \in (R^*P \wedge T)$ so $(a,b) \in R(R^*P \wedge T)$. As this proof uses only intuitionist logic, it is valid in any topos.
- $(Q3)$ is sometimes also called the modular law, which gives rise to a conflict of terminologies. We will prefer the term *distributivity* law for $(Q3)$.
- $(Q10)$ expresses the fact that, as $X$ is a bound of $\mathcal{T}$, $X \times X$ has to be a sub-quotient of a co-product of an $I$-indexed family of copies of $X$.
  Indeed, in this situation, there is a family $(u_i, v_i)_{i \in I}$ of partial functions from $X$ to $X \times X$. A partial function $f$ from $X$ to $X$ can be represented by its graph: the relation $R$ such that $(yRx)$ if and only if $f(x)$ is defined and $y = f(x)$. A relation $R$ on $X$ is the graph of a partial function if and only if $RR^* \leqslant 1$. So one has two families of relations on $X$, also denoted $(u_i)$ and $(v_i)$, such that for all $i$, $u_i u_i^* \leqslant 1$ and $v_i v_i^* \leqslant 1$. The relation $\bigvee_i v_i u_i^*$ is the union of the image of $X$ in $X \times X$ by all the partial maps $(v_i, u_i)$. So the relation $\bigvee_i v_i u_i^* = \top$ expresses the fact that the corresponding map is onto.

1. because they corresponds to internal union

2.1.2. In all this chapter, $X$ will denote an object of a topos $\mathcal{T}$, and when we assume that $X$ is a bound we will call it $B$ instead in order to be clear on which result assume that $X$ is a bound and which does not.

2.1.3. **Definition :** *A Set satisfying* $(Q1)$ *and* $(Q2)$ *is a sup-lattice. A Set satisfying* $(Q1)$, $(Q2)$, $(Q4)$ *and* $(Q5)$ *is called a quantale, (unital if it also satisfies* $(Q6)$*). We will call a modular quantale, a quantale satisfying all the axioms from* $(Q1)$ *to* $(Q9)$*, and a Grothendieck quantale one satisfying all the axioms from* $(Q1)$ *to* $(Q10)$.

The term quantale is due to C.J. Mulvey in [59]. The name Grothendieck quantale has been given by H.Heymans and I.Stubbe in [32] and [33]. For the term "modular quantale", our terminology differs slightly from previous work (like [33]), where the axiom $(Q3)$ is not included in the definition of a modular quantale. The main reasons for our choice of terminology is simply that we only want to consider quantales that arise as relations on objects in a topos and hence satisfy the axiom $(Q3)$. Also, we think it is more natural to assume a compatibility between intersection and supremum (given by $(Q3)$) as soon as we assume both a compatibility between intersection and the composition law (given by $(Q9)$) and a compatibility between the composition law and supremum (given by $(Q5)$).

2.1.4. The main result relating toposes to quantales (which should probably be attributed to P.J.Freyd and A.Scedrov in [26]), is the fact that if $\mathcal{T}$ is a topos and $B$ is a bound of $\mathcal{T}$ then $\mathcal{T}$ can be completely reconstructed from the Grothendieck quantale $Q = \mathsf{Rel}(B)$, and that every Grothendieck quantale can be written (essentially uniquely) in the form $\mathsf{Rel}(B)$ for a bound $B$ of a topos $\mathcal{T}$.
This result (at least its first part, the second part being a little harder) can actually be proven directly using the following construction:

**Definition :** *If* $Q$ *is a Grothendieck quantale, we will denote by* $Site(Q)$ *the site whose objects are the* $q \in Q$ *such that* $q \leqslant 1$ *and whose morphisms are given by:*

$$\hom(q,q') = \{f \in Q | 1 \wedge f^* f = q \text{ and } f f^* \leqslant q'\}.$$

*The composition is given by* $f \circ g = fg$*. The identity morphism of* $q$ *is* $q$ *itself*[2]*. And a Sieve* $J$ *on an object* $q$ *is covering if:*

$$\bigvee_{\substack{q' \in Q, q' \leqslant 1 \\ f \in J(q')}} f f^* = q$$

The fact that for any Grothendieck quantale $\mathrm{Site}(Q)$ is indeed a site is not straightforward. Apparently [3] it can be checked directly, but this proof is quite

2. The modular law imply that if $q \leqslant 1$ then $q^2 = q$.
3. We checked it, but unfortunately, it does not seems that a proof of this kind had ever been published.

long and is not necessary because one has a more abstract proof, using the following easier proposition, and theorem 2.4.5.

**Proposition :** *If $Q = \mathsf{Rel}(B)$, for a bound $B$ of a topos $\mathcal{T}$, then Site$(Q)$ is the site of subobjects of $B$. In particular, it is a site of definition for $\mathcal{T}$.*

**Proof :**
We use the same kind of argument as the proof that $Q$ satisfies $(Q10)$.
As $1 \in Q = \mathsf{Sub}(B \times B)$ corresponds to the diagonal sub-object of $B \times B$, an element $q \in Q$ such that $q \leqslant 1$ corresponds to a unique sub-object of $B$. Let $q$ and $q'$ be two sub-objects of $B$, and $f \in Q = \mathsf{Rel}(B)$ satisfying the two conditions $1 \wedge f^* f = q$ and $f f^* \leqslant q'$. The first condition asserts (internally) that element $x \in B$ such that $(\exists y \, , y f x)$ are exactly the elements of $q$, and the second condition asserts that if $y f x$ and $y' f x$ then $y = y'$ and $y \in q'$. This is exactly the conditions that characterizes the graph of a function from $q$ to $q'$, hence $\hom_{\mathrm{Site}(Q)}(q, q')$ is indeed isomorphic to $\hom_{\mathcal{T}}(q, q')$ and as the composition of relations extends the composition of functions this correspondence is indeed an equivalence of categories.
It only remains to check that the topology of Site$(Q)$ is indeed the canonical topology of the topos, but for any collection of map $f_i : q_i \rightarrow q$, the sub-object $f_i f_i^* \leqslant q$ is exactly the image of $f_i$ in $q$ and hence the condition that:

$$\bigvee_i f_i f_i^* = q$$

simply asserts that the family is jointly surjective. $\square$

One of our goal is to provide a way to reconstruct $\mathcal{T}$ from $Q$ without using sites.

## 2.2 The category $\mathsf{sl}(\mathcal{T})$ of sup-lattices

In this subsection we recall the definition and basic properties of the categories of sup-lattices of a topos as it is studied in [45]. We will not give any proofs, but most of them are straightforward and they all can be found in [45].

2.2.1. A sup-lattice is an ordered set which admit arbitrary supremum, a morphism of sup-lattices is an order preserving function which preserve supremum.

**Definition :** *We will denote by $\mathsf{sl}(\mathcal{T})$ the category of sup-lattices internal to $\mathcal{T}$. And by $\mathsf{sl}$ the category of sup-lattices internal to the base topos $\mathcal{S}$.*

In all of this sub-section, we will proves result for $\mathsf{sl} = \mathsf{sl}(\mathcal{S})$, but as $\mathcal{S}$ is an arbitrary topos one can apply everything internally to $\mathcal{T}$ and deduce the same result for $\mathsf{sl}(\mathcal{T})$.

2.2.2. Although we use the term "sup"-latices, it is a classical fact of ordered set theory that if every subset admits a supremum then every subset also admits an infimum, and hence a sup-lattice is the same thing as an inf-lattice. The term "sup" is here to denote the fact that we are considering sup-preserving morphisms (which are different from inf-preserving morphisms).
This duality has a consequence on the category $\mathsf{sl}$: it is endowed with an involutive contravariant functor, that we will denote by $(\_)^*$. Indeed if $S$ is a sup-lattice then if we define $S^*$ as being $S$ endowed with the reverse order relation it is again a sup-lattice, and if $f$ is a morphism then we denote by $f^*$ its right adjoint (it always exists because $f$ commutes to supremum) which is a morphism of sup-lattices for the opposite order relations. One has $f^{**} = f$ because of the reversing of the order relations, and hence $*$ is an involutive anti-equivalence of categories.

This involution allows to compute colimits in the category of sup-lattices: indeed one can easily check that the category $\mathsf{sl}$ has all limits and that they are computed at the level of the underlying set. As $(\_)^*$ transforms co-limits into limits, $\mathsf{sl}$ also have all co-limits.

2.2.3. If $X$ is a set, then $\mathcal{P}(X) = {\Omega_\mathcal{T}}^X$ (the power object of $X$) is a free sup-lattice generated by $X$, i.e.:

$$\hom_\mathcal{T}(X, S) = \hom_{\mathsf{sl}}(\mathcal{P}(X), S)$$

This adjunction formula turns $\mathcal{P}$ into a functor from sets to $\mathsf{sl}$ that sends a map $f : X \to Y$ to the direct image map $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$.

2.2.4. Knowing how to construct a free sup-lattice (using $\mathcal{P}$) and a quotient of sup-lattice (using the involution $^*$), one can construct sup-lattices by "generators and relations". More precisely, if $I$ is a set, and $R$ is a family of couples of subsets $(r_1, r_2)$ of $I$, interpreted as relation of the form:

$$\bigvee_{x \in r_1} x \leqslant \bigvee_{y \in r_2} y$$

then the sup-lattice presented by the set of generators $I$ and the set of relations $R$ identifies with:

$$\{V \subseteq I | \forall (r_1, r_2) \in R, (r_2 \subseteq V) \Rightarrow (r_1 \subseteq V)\}$$

2.2.5. If $S$ and $S'$ are sup-lattices, then the set of sup-lattice morphisms between $S$ and $S'$ is again a sup-lattice, for the point-wise ordering, with supremum computed point-wise. This sup-lattice is denoted by $[S, S']$.

These internal hom objects come with a monoidal structure given by the universal property:

$$\hom(M \otimes N, P) \simeq \hom(M, [N, P])$$

Equivalently, the morphisms from $M \otimes N$ to $P$, are the functions from $M \times N$ to $P$ which are morphisms of sup-lattices in each variable (when fixing the other variable). We will call such maps bilinear maps from $M \times N$ to $P$. The explicit construction of the tensor product is conducted exactly as for modules over a ring by a construction by generators (the $(m \otimes n)$ for $m, n \in M \times N$) and relations expressing the notion of bi-linear map.

2.2.6. In addition of being a closed monoidal category endowed with an involution, the category $\mathsf{sl}$ also satisfies the following interesting properties.

$$\Omega_{\mathcal{T}} \otimes N = N$$
$$[\Omega_{\mathcal{T}}, N] = N$$
$$M^* = [M, {\Omega_{\mathcal{T}}}^*]$$
$$(M \otimes N)^* = [M, N^*]$$

In particular, even if we will not use this concept here, this means that $\mathsf{sl}$ (endowed with all these structures) is a $*$-autonomous category in the sense of [5], with ${\Omega_{\mathcal{T}}}^*$ as dualizing object.

2.2.7. Let $\mathcal{T}$ and $\mathcal{E}$ be two toposes, and $f$ a geometric morphism from $\mathcal{T}$ to $\mathcal{E}$. Let also $S$ be a sup-lattice in $\mathcal{T}$, then $f_*(S)$ is a sup-lattice in $\mathcal{E}$: indeed (working internally in $\mathcal{E}$) if $P$ is a subset of $f_*(S)$ then by adjunction there is a map from $f^*(P)$ to $S$, we can consider the supremum $s$ of the image of this map. As $s$ is a uniquely defined element, it is a global section of $S$, i.e. an element of $f_*(S)$. From here one can check that $s$ is also a supremum for $P$. This defines a functor $f_* : \mathsf{sl}(\mathcal{T}) \to \mathsf{sl}(\mathcal{E})$. We also note that $f_*$ preserves bi-linear maps between sup-lattices.

In the other direction, if $S$ is a sup-lattice in $\mathcal{E}$ then $f^*(S)$ is in general just a pre-ordered set in $\mathcal{T}$, but one can construct a completion, denoted by $f^{\#}(S)$. In order to do so, we chose any presentation by generators and relations of $S$ (for example, taking all elements and all relations), and then we define $f^{\#}(S)$ by generators and relations using the pullback of the system of generators and relations chosen for $S$. At first sight, it is not clear that this definition of $f^{\#}(S)$ does not depend of the presentation of $S$, but one can prove an adjunction formula:

$$\hom_{\mathsf{sl}(\mathcal{T})}(f^{\#}(S), T) \simeq \hom_{\mathsf{sl}(\mathcal{E})}(S, f_*(T))$$

which is natural in $T$. This implies that $f^{\#}(S)$ does not depend on the presentation of $S$, that it is functorial in $S$ and that $f^{\#}$ is a left adjoint of $f_*$. We will use the same technique in the proof of the third point 2 of proposition 2.3.3. This result can actually be seen as a special case of the first two points of proposition 2.3.3 applied internally in $\mathcal{E}$ to the category $\mathcal{C} = \mathsf{sl}(\mathcal{T})$ with $B = \Omega_{\mathcal{T}}$.

## 2.3 Categories enriched over $\mathsf{sl}$

2.3.1. Thanks to the monoidal structure on $\mathsf{sl}$ one can talk about $\mathsf{sl}$-enriched categories. Precisely, a $\mathsf{sl}$-enriched category $\mathcal{C}$ is a category such that morphism sets are endowed with an order relation which turns them into sup-lattices and composition into a bi-linear map.

Here are the two main examples of $\mathsf{sl}$-enriched categories we want to consider:

**Proposition :** *Let $\mathcal{T}$ be a Grothendieck topos, then $\mathsf{sl}(\mathcal{T})$ is a $\mathsf{sl}$ enriched category.*

**Proof :**
If $S$ and $S'$ are two objects of $\mathsf{sl}(\mathcal{T})$ and $p$ denotes the structural geometric morphism from $\mathcal{T}$ to the topos of sets, then

$$\hom(S, S') = p_*([S, S'])$$

which is a sup-lattice thanks to 2.2.7. The composition is a bi-linear map because it is given (through an application of $p_*$) by an internal bi-linear map:

$$[S, S'] \times [S', S''] \to [S, S''].$$

$\square$

A (unital) quantale, as defined in 2.1.3, is exactly a monoid object of $\mathsf{sl}$, i.e. it is a sup-lattice endowed with the structure of a (unital) monoid such that the composition law is bi-linear. A right (or left) module over a unital quantale $Q$, is a sup-lattice $S$ endowed with a right (or left) action of the underlying monoid of $Q$ such that the corresponding map $S \times Q \to S$ is bi-linear.
The category of right modules over $Q$ (with $Q$-linear morphisms) is denoted by $\mathsf{Mod}_Q$, this is the other important example of $\mathsf{sl}$-enriched category we will consider.

If one thinks of the supremum of a family of elements as a form of addition, a sup-lattice enriched category is really close to being an additive category (maybe something we would like to call a "locally complete characteristic one additive category" as our addition is characterized by the property that $x + x = x$ ). The following two results are in this spirit.

2.3.2. From the technique of computation of co-limits in $\mathsf{sl}$ explained in 2.2.2 one can see that the co-product of a family of objects in $\mathsf{sl}$ is isomorphic to the product of the same family. This is actually a general well known [4] result:

**Proposition :** *Let $\mathcal{C}$ be a $\mathsf{sl}$-enriched category, let $(A_i)_{i\in I}$ be a family of objects of $\mathcal{C}$ and $A$ be an object of $\mathcal{C}$, then the following three conditions are equivalent:*

1. *$A$ is the co-product of the family $(A_i)$.*

4. it appears, for example, under a slightly different form in [26, 2.214 and 2.223].

2. *$A$ is the product of the family $(A_i)$.*
3. *There are two families of morphisms $f_i : A_i \to A$ and $p_i : A \to A_i$ such that $\sup_i f_i \circ p_i = Id_A$ and for all $i, j \in I$:*

$$p_j \circ f_i = \sup\{f : A_i \to A_j | i = j \text{ and } f = Id_{A_i}\}\ [5]$$

*Moreover, in this situation, the morphisms $f_i$ and $p_i$ given in* 3. *are the natural morphisms asserting that $A$ is the (co)-product of the $A_i$.*

**Proof :**
Passing from $\mathcal{C}$ to $\mathcal{C}^{op}$ preserves property 3. and exchanges properties 1. and 2., hence it is enough to show that 2. and 3. are equivalent.
We will start by showing that 3. $\Rightarrow$ 2..
We assume 3. holds, in particular $A$ is already endowed with maps $(p_i)$ from $A$ to $A_i$ for each $i$, we have to show that $A$ and the $(p_i)$ are universal for this property.
Let $X \in \mathcal{C}$ be any object and assume we have a collection of map $h_i : X \to A_i$.
Let $h = \sup_i (f_i \circ h_i) : X \to A$. Then for every $i$:

$$p_i \circ h = \sup_j p_i \circ f_j \circ h_j = \sup_j \sup\{f \circ h_j | i = j \text{ and } f = Id_{M_i}\} = h_i.$$

We also have to show that this map is unique: let $h'$ be any other map from $X$ to $A$ such that for every $i$, $p_i \circ h' = h_i$. Then:

$$h = \sup f_i \circ h_i = (\sup f_i \circ p_i) \circ h' = h'.$$

Assume now that $A$ is the product of the $A_i$. The maps $p_i$ are the structural maps, the maps $f_i$ are uniquely defined morphisms (using the universal property of the product) by the formula given for $p_j \circ f_i$. Hence the formula for $p_j \circ f_i$ holds by definition, and the equality $\sup_i f_i \circ p_i = Id_A$ because of the relation

$$p_j \circ \sup_i f_i \circ p_i = p_j$$

(obtained by the same computation as in the first part of the proof) and the uniqueness in the universal property of the product.
□

This proposition has interesting consequences: First, any $\mathsf{sl}$-enriched functor will automatically preserve each product and each co-product (because 3. is clearly preserved by any $\mathsf{sl}$-enriched functor).
Additionally, one can describe the morphisms between two co-products (or products) by something which looks like (infinite) matrix calculus. More precisely, a morphism $f$ from $\coprod_{j \in J} A_j$ to $\coprod_{i \in I} B_i$ is the same thing as a morphism from $\coprod_{j \in J} A_j$ to $\prod_{i \in I} B_i$, hence it is given by the datum of a morphism $f_{i,j} : A_j \to B_i$ for each $i$ and each $j$.
The composition with a $g : \coprod_{i \in I} B_i \to \coprod_{k \in K} C_k$, is:

5. If we assume the law of excluded middle, or more specifically that the set of indices $I$ has a decidable equality, then this formula reduces to the more classical: $p_i \circ f_i = Id_{A_i}$ and $p_j \circ f_i = 0$ if $i \neq j$

$$g \circ f = \bigvee_{i \in I} g_{k,i} \circ fi, j$$

In the special case where all the $A_j$ and $B_i$ are isomorphic to a same object $A$, then $\hom(A, A) = Q$ is a quantale and we will denote by $M_{I,J}(Q)$ the set of morphisms from $A^{(J)}$ to $A^{(I)}$, which can be identified with $Q^{I \times J}$.

2.3.3. The next result can be thought of as a $\mathsf{sl}$-enriched form of the Mitchell embedding theorem which asserts that every abelian category is a full sub-category of a category of modules over a ring, but restricted to the case where there are enough "projective" objects.

**Proposition :** *Let $\mathcal{C}$ be a $\mathsf{sl}$-enriched category, $A$ an object of $\mathcal{C}$ and $Q = \hom_{\mathcal{C}}(A, A)$.*

1. *$Q$ is a quantale for composition, and $R_A : X \mapsto \hom_{\mathcal{C}}(A, X)$ induces a functor from $\mathcal{C}$ to $\mathsf{Mod}_Q$.*
2. *If $\mathcal{C}$ has all co-limits, then $R_A$ has a left adjoint denoted $T_A : Y \mapsto Y \otimes_Q A$.*
3. *If in addition $R_A$ preserves co-equaliser, then $T_A$ is fully faithful.*
4. *If in addition $\mathcal{C}(A, \_)$ reflects isomorphisms (i.e. if $f$ is a map in $\mathcal{C}$ such that $\mathcal{C}(A, f)$ is an isomorphism then $f$ is an isomorphism), then $R_A$ and $T_A$ realize an equivalence of categories between $\mathcal{C}$ and $\mathsf{Mod}_Q$.*

**Proof :**

1. As $\mathcal{C}$ is an $\mathsf{sl}$ enriched category, composition are bilinear, hence $Q = \hom_{\mathcal{C}}(A, A)$ is a quantale for composition, the action of $Q$ on $\hom_{\mathcal{C}}(A, X)$ by pre-composition is also bi-linear, and for any morphism $f : X \to Y$ the induced morphism from $\hom_{\mathcal{C}}(A, X)$ to $\hom_{\mathcal{C}}(A, Y)$ is a $Q$-linear morphism of sup-lattices.
2. Let $X$ be a right $Q$ module, then (in $\mathsf{Mod}_Q$) one has a surjection $p : \coprod_{x \in X} Q \twoheadrightarrow X$. Let $f_1, g_1 : K \rightrightarrows \coprod_{x \in X} Q$ be the kernel pair of $p$. Let $p_2 : \coprod_{k \in K} Q \twoheadrightarrow K$, and let $f = f_1 \circ p_2$ and $g = g_1 \circ p_2$.
   $X$ is the co-equaliser of the two $Q$ linear maps (for the right action) $f$ and $g : \coprod_{k \in K} Q \rightrightarrows \coprod_{x \in X} Q$, which correspond to elements of $M_{X,K}(Q)$. Let $A^{(X)} = \coprod_{x \in X} A$ and $A^{(K)} = \coprod_{k \in K} A$. Thanks to a remark done in 2.3.2, maps between $A^{(K)}$ and $A^{(X)}$ can also be identified with elements of $M_{X,K}(Q)$, hence there are two maps corresponding to $f$ and $g$ from $A^{(K)}$ to $A^{(X)}$. We define $T_A(X)$ to be the co-equaliser of these two maps.
   One easily checks that for any $B \in \mathcal{C}$, there is a canonical (functorial in $B$) isomorphism $Hom(T_A(X), B) \simeq Hom(X, R_A(B))$ (they are the same once we develop all the inductive limits involved) which implies both the adjunction between $T_A$ and $R_A$ and the functoriality of $T_A$.
3. As $T_A(X)$ is computed as the co-equalizer of two arrows $f, g : A^K \rightrightarrows A^X$ such that the co-equalizer of $R_A(f), R_A(g)$ is $X$, if $R_A$ commutes to co-equalizer then one can deduce that $R_A(T_A(X)) \simeq X$ which (thanks to the adjunction) means that $T_A$ is fully faithful.

4. We already know that $X \simeq R_A \circ T_A(X)$ (by the unit of the adjunction). Let $c_X : T_A(R_A(X)) \to X$ be the co-unit of the adjunction then, $R_A(c_X) : R_A(T_A(R_A(X))) \to R_A(X)$ is a retraction (by general properties of the unit and co-unit) of the unit of the adjunction valued in $R_A(X)$ (i.e. the canonical map $R_A(X) \to R_A(T_A(R_A(X))))$ but this map is known to be an isomorphism, hence $R_A(c_X)$ is an isomorphism and since $R_A$ detects isomorphism, we proved that $c_X$ is an isomorphism.

□

The following theorem can then be seen as a corollary of the previous proposition.

2.3.4. **Theorem :** *Let $\mathcal{T}$ be a Grothendieck topos, and $B$ a bound of $\mathcal{T}$. Then* $\hom_{\mathcal{T}}(B, \_)$ *induces (one half of) an equivalence of categories from* $\mathsf{sl}(\mathcal{T})$ *to* $\mathit{Mod}_Q$ *where $Q$ is the quantale* $\mathsf{Rel}(B)$*.*

This result is essentially the same as the theorem 5.2 of [63].

**Proof :**
We will prove that with $\mathcal{C} = \mathsf{sl}(\mathcal{T})$ and $A = \mathcal{P}(B)$, all the hypotheses of the four points of the previous proposition are verified, and $Q = \mathsf{Rel}(B)$.
Note that:

$$\hom_{\mathcal{T}}(B, S) = \hom_{\mathsf{sl}(\mathcal{T})}(A, S).$$

- $\mathsf{sl}(\mathcal{T})$ has all co-limits (and also all limits) because they can be computed internally in $\mathcal{T}$.
- $R_A$ commutes to co-equalizer because of the following formula:

$$\begin{aligned} R_A(X) &= \hom_{\mathcal{T}}(B, X) \simeq \hom_{\mathcal{T}}(B, X^*)^* \\ &\simeq \hom_{\mathsf{sl}(\mathcal{T})}(A, X^*)^* \simeq \hom_{\mathsf{sl}(\mathcal{T})}(X, A^*)^* \end{aligned}$$

  And the last term clearly commutes to every inductive limit.
- $Q$ is identified with $\mathsf{Rel}(B)$ through the isomorphism:

$$\hom_{\mathsf{sl}(\mathcal{T})}(\mathcal{P}(B), \mathcal{P}(B)) \simeq \hom_{\mathcal{T}}(B, \mathcal{P}(B)) \simeq \mathsf{Sub}(B \times B)$$

  Internally, this corresponds to the map which sends a morphism $f$ to the relation $y(R_f)x :=$ "$x \in f(\{y\})$". The fact that composition of morphisms coincides with the composition of relations is checked internally:

$$zR_fR_gx = (\exists y, x \in f(\{y\}) \text{ and } y \in g(\{z\})) = xR_{f \circ g}z.$$

- $R_A$ detects isomorphisms:
  Let $f : S \to S'$ such that $R_A(f)$ is an isomorphism. For any sub-object $U \subseteq B$, every map $t : U \to S$ can be extended canonically to a map $\tilde{t} : B \to S$ by the (internal) formula:

$$\tilde{t}(x) = \sup\{y | x \in U \text{ and } y = t(x)\}$$

If $t$ is a map from $B$ to $S$, we can restrict $t$ to $U$ and then extend $t|_U$ into $\tilde{t}$, one then has the formula

$$\tilde{t}(x) = \sup\{y|x \in U \text{ and } y = t(x)\} = t.\delta_U$$

where $\delta_U$ is the element of $Q$ corresponding to the diagonal embedding of $U$ in $B \times B$ and the product is the natural right action of $Q$ on $\hom(B,S)$. Finally, as $\delta_U^2 = \delta_U$, $\hom(U,S)$ is identified with $\hom(B.S).\delta_U$.
As $R_A(f)$ is an isomorphism, all the maps $\hom(U,f)$ for every sub-object $U$ of $B$ are isomorphisms, because they are retractions of the map $R_A(f)$. The object $B$ being a bound of $\mathcal{T}$, the sub-objects of $B$ form a generating family and so $f$ is an isomorphism.

□

## 2.4 Quantale Sets

In the previous section we showed that, for any Grothendieck topos $\mathcal{T}$ endowed with a bound $B$, the quantale $Q = \mathsf{Rel}(B)$ already determines the category $\mathsf{sl}(\mathcal{T})$. We will now show that if we add [6] the operation $(\_)^*$ on $Q$, then we can give a complete description of $\mathcal{T}$ in terms of $Q$.

The theorems 2.4.3 and 2.4.5 are the main (previously known) results relating Grothendieck toposes to Grothendieck quantales, they can be found explicitly in [32] and [33] and under different forms in [26] and [63].

2.4.1. Our starting point will be the following lemmas:

**Lemma :** *Let $X$ be an object of $\mathcal{T}$, and $Y$ be a sub-quotient of $X$, then the relation on $X$ defined by*

$$xRy = \text{“ } x \text{ and } y \text{ both have an image in } Y \text{ and these coincide ”}$$

*is symmetric ($R^* = R$) and transitive ($R^2 \leqslant R$). This induces a correspondence between sub-quotients of $X$ and symmetric transitive relations on $X$ (also called partial equivalence relations).*

**Proof :**
The symmetry and transitivity of the relation are clear. Moreover $Y$ is fully determined by $R$: it is the quotient of $U = \{x|xRx\}$ by $R$ (which is an equivalence relation on $U$). Conversely, let $R$ be any symmetric and transitive relation on $X$. Let $U = \{x|xRx\}$, $R$ induces an equivalence relation on $U$, and we have $xRy \Rightarrow xRx$. Hence, $xRy \Leftrightarrow (x \in U) \wedge (xRy) \wedge (y \in U)$, i.e. $R$ is indeed the relation induced by the sub-quotient $U/R$. □

6. Actually, because we know that $Q$ is of the form $\mathsf{Rel}(B)$, the $^*$ operation is fully determined by the underlying quantale. This comes from the property (Q10) together with this lemma: the condition $f = g^*$ and $gg^* \leqslant 1$ is equivalent to the condition $\exists u \leqslant 1$, $uf = f$, $gu = g$ , $gf \leqslant 1$ and $u \leqslant fg$ . This lemma is proved using internal logic.

2.4.2. **Lemma :** *In the situation of the previous lemma, one actually has $R^2 = R$, and the map which sends a sub-object of $Y$ to its pullback in $X$ identifies $\mathcal{P}(Y)$ with $R(\mathcal{P}(X))$ (where $R$ denotes the endomorphism of sup-lattices of $\mathcal{P}(X)$ corresponding to the relation $R$).*

**Proof :**
Indeed, if $(xRy)$ then $(xRx)$ and $(xRy)$ hence $(xR^2y)$, this proves that $R \leqslant R^2$, and hence $R = R^2$. Let $P$ be a subset of $X$, $R(P) = \{x \in X | \exists z \in P, xRz\}$. So $P = R(P)$ if and only if $P$ is included in $U = \{x | xRx\}$ and saturated for the equivalence relation induced by $R$ on $U$. These are exactly the subsets which are pullbacks of subsets of $Y$. $\square$

2.4.3. **Theorem :** *The category $\mathsf{Rel}(\mathcal{T})$ whose objects are the objects of $\mathcal{T}$ and morphisms from $X$ to $Y$ are sub-objects of $Y \times X$ (the composition being given by the composition of relations) is equivalent to the following category $\mathsf{Proj}(Q)$:*

- *The objects are the couples $(I, P)$ where $I$ is a set, and $P$ is a matrix in $M_{I,I}(Q)$ such that $P^2 = P$ and $P^* = P$ where $(P^*)_{i,j} = (P_{j,i})^*$.*
- *The morphisms from $(J, P')$ to $(I, P)$ are the matrices $M \in M_{I,J}(Q)$ such that $P.M = M$ and $M.P' = M$ (the composition being the product of matrices).*

*Under this equivalence, the opposite of a relation corresponds to the "trans-conjugation" of a matrix: $(M^*)_{i,j} = (M_{j,i})^*$.*

Before proving this theorem we will need one more simple lemma, which is actually the last point of the theorem:

**Lemma :** *Let $R$ be a sub-object of $(\coprod_{i \in I} B) \times (\coprod_{j \in J} B)$ corresponding to a morphism $R : \mathcal{P}(B)^I \to \mathcal{P}(B)^J$ represented by a matrix: $(R_{i,j})_{i \in I, j \in J}$, then the opposite relation corresponds to the trans-conjugate matrix $(R^*)_{j,i} = (R_{i,j})^*$*

**Proof :**
This can be checked internally: Since $R_{i,j}$ corresponds to the intersection of $R$ with the inclusion $(f_i, f_j)$ of $B \times B$ in $(\coprod_{i \in I} B) \times (\coprod_{j \in J} B)$, taking the opposite relation will reverse $R_{i,j}$ and exchange the indices. This concludes the proof of the lemma. $\square$

We now prove theorem 2.4.3:

**Proof :**
In order to prove the equivalence of $\mathsf{Proj}(Q)$ and $\mathsf{Rel}(\mathcal{T})$, we will consider a third category $\mathcal{C}$, the full sub-category of $\mathsf{sl}(\mathcal{T})$ of sup-lattices which are of the form $\mathcal{P}(X)$ for $X$ an object of $\mathcal{T}$, and show that both $\mathsf{Proj}(Q)$ and $\mathsf{Rel}(\mathcal{T})$ are equivalent to $\mathcal{C}$.

The association $X \to \mathcal{P}(X)$ is (one half of) an equivalence from $\mathsf{Rel}(\mathcal{T})$ to $\mathcal{C}$. Indeed, it is essentially surjective by definition of $\mathcal{C}$, and we have already mentioned that morphisms between power objects are the same thing as relations, so it is also fully faithful.

The association $(I,P) \to P(\mathcal{P}(B)^I)$ is (one half) of an equivalence from $\mathsf{Proj}(Q)$ to $\mathcal{C}$.
Indeed, as $B$ is a bound of $\mathcal{T}$, any object $X$ of $\mathcal{T}$ is a sub-quotient of some $\coprod_{i\in I} B$, hence by lemmas 2.4.1 and 2.4.2 there is an endomorphism $F$ of $\mathcal{P}(\coprod_{i\in I} B) = \mathcal{P}(B)^I$ such that $F^2 = F$, $F^* = F$, and $\mathcal{P}(X) = F(\mathcal{P}(B)^I)$. By the equivalence of 2.3.4 (and also by 2.3.2), such an endomorphism corresponds exactly to a matrix $P$ such that $(I,P)$ is indeed an object of our category. So this functor is full and well defined (at least on objects). Now a morphism from $P'(\mathcal{P}(B)^J)$ to $P(\mathcal{P}(B)^I)$ is exactly the data of a matrix $M$ such that $P.M = M$ and $M.P' = M$. This concludes the proof of the equivalences. The last point of the theorem being proved by the lemma.
□

2.4.4. **Corollary :** *The topos $\mathcal{T}$ is equivalent to the (non full) subcategory of $\mathsf{Proj}(Q)$, with all objects and with morphisms from $(J,P')$ to $(I,P)$ only the matrices $M$ which satisfy the additional condition: such that $MM^* \leqslant P$ and $P' \leqslant M^*M$.*

**Proof :**
These two additional conditions indeed characterize functional relations [7] among arbitrary relations, and in a topos functional relations are in correspondence with morphisms. □

2.4.5. **Theorem :** *For every Grothendieck quantale $Q$, there exists a topos $\mathcal{T}$ and a bound $B$ of $\mathcal{T}$ such that $Q = \mathsf{Rel}(B)$.*

Of course, from the previous theorem, such a topos is unique.

**Proof :**
One could use the construction of Site$(Q)$ given in the introduction, but the proof that this is indeed a site and that it gives back $Q = \mathsf{Rel}(B)$ is long and not really illuminating. Instead, we will use results from the theory of allegories (see [26], or [44, A.3]) which is closely connected to what we are doing here:
In the language of [26] a modular quantale $Q$ is a one object locally complete distributive allegory, and $\mathsf{Proj}(Q)$ is the systemic completion of $Q$. The result [26, 2.434] proves that $\mathsf{Proj}(Q)$ is a power allegory and [26, 2.226] proves that it has a unit. So in order to apply [26, 2.414] and conclude that $\mathsf{Proj}(Q)$ is the category of relations on an elementary topos, we need to prove that it is "tabular". Using [26, 2.16(10)] it is enough to prove that [8] for each set $X$ the maximal matrix of $M_{X,X}(Q)$ can be written $FG^*$ for $F,G \in M_{X,Y}(Q)$ with

7. i.e. relations of the form $\{(f(x),x)\}$ for $f$ a (totally defined) function.
8. The reader should note that [26] uses a reverse composition order for morphisms in category, whereas we use the standard composition order. This explains why the formula we give is different from the one given in the reference.

$FF^* \leqslant Id_X$ and $GG^* \leqslant Id_Y$. But $(Q10)$ is exactly the assertion that this is true when $X$ is a singleton, and the general case follows easily from $(Q10)$ by taking $Y = X \times I$.

The elementary topos we obtain in this way has arbitrary co-products and is bounded, hence it is a Grothendieck topos.

Finally, if $B$ is the object of $\mathsf{Proj}(Q)$ represented by the set $X = \{*\}$ and $P = 1$, then $\mathsf{Rel}(B) = Q$ and this concludes the proof. $\square$

2.4.6. In the remainder of this section we just give a simpler description of the category $\mathsf{Proj}(Q)$ in term of $Q$-Set inspired from the notion of $\mathcal{L}$-sets, when $\mathcal{L}$ is a locale (see for example [8, 2.8 and 2.9] or [44, C1.3]). Our aim is both to provide a formalism suitable for computation and to show that $\mathsf{Proj}(Q)$ is exactly a non-commutative generalization of $\mathcal{L}$-set. We do not know if this formulation has already been presented somewhere or not.

**Definition :**

- *A $Q$-Set is a set $X$ endowed with a function $[\_ \approx \_] : X \times X \to Q$ such that:*

$$\begin{array}{ll} (S1) & \forall x, y \in X, [x \approx y] = [y \approx x]^* . \\ (S2) & \forall x, y, z \in X, [x \approx y]\,[y \approx z] \leqslant [x \approx z] . \end{array}$$

- *A $Q$-relation $R$ from $X$ to $Y$ (two $Q$-sets) is a map:*

$$\begin{array}{ccc} Y \times X & \to & Q \\ (y, x) & \mapsto & [yRx] \end{array}$$

*such that:*

$$\begin{array}{lll} (R1) & [y \approx y']\,[y'Rx] \leqslant [yRx] & \text{with equality whenever } y = y' \\ (R2) & [yRx']\,[x' \approx x] \leqslant [yRx] & \text{with equality whenever } x = x'. \end{array}$$

- *A $Q$-function from $X$ to $Y$ is a $Q$-relation:*

$$\begin{array}{ccc} Y \times X & \to & Q \\ (y, x) & \mapsto & [y \approx f(x)] \end{array}$$

*which (in addition to $(R1)$ and $(R2)$) satisfies:*

$$\begin{array}{ll} (F1) & [y \approx f(x)]\,[y' \approx f(x)]^* \leqslant [y \approx y'] \\ (F2) & [x \approx x] \leqslant \bigvee_y [y \approx f(x)]^*\,[y \approx f(x)] \end{array}$$

- *$Q$-relations and $Q$-functions can be composed by the formula:*

$$[zRQx] = \bigvee_y [zRy]\,[yQx]$$

$$[z \approx f \circ g(x)] = \bigvee_y [z \approx f(y)]\,[y \approx g(x)]$$

- *The opposite of a $Q$-relation is given by:*

$$[xR^*y] = [yRx]^*$$

2.4.7. **Proposition :** *Consider the following modification of the axioms:*
*(S2')* $[x \approx y] = \bigvee_t [x \approx t]\,[t \approx y]$.
*(R1')* $\bigvee_{y'} [y \approx y']\,[y'Rx] = [yRx]$
*(R2')* $\bigvee_{x'} [yRx']\,[x' \approx x] = [yRx]$
*(F2')* $[x \approx x'] \leqslant \bigvee_y [y \approx f(x)]^*\,[y \approx f(x')]$
*Then assuming $(S1)$ holds, $(S2)$ and $(S2')$ are equivalent.*
*And assuming $X$ and $Y$ are $Q$-sets, $(R1)$ is equivalent to $(R1')$, $(R2)$ is equivalent to $(R2')$ and assuming additionally $(R2)$ then $(F2)$ is equivalent to $(F2')$.*

*In particular $Q$-Sets are exactly the same as objects of $\mathsf{Proj}(Q)$, and $Q$-relations and $Q$-functions correspond respectively to morphisms in $\mathsf{Proj}(Q)$, and morphisms which are sent to functional relations by the equivalence of 2.4.3.*

We will need the following lemma:

**Lemma :** *In any $Q$-set, one has*

$$[x \approx y]\,[y \approx y] = [x \approx y]$$

$$[x \approx x]\,[x \approx y] = [x \approx y]$$

**Proof :**
Indeed (for the second equality), for any $q \in Q$ element of a modular quantale one has:

$$q \leqslant (1.q \wedge q) \leqslant (1 \wedge qq^*)q \leqslant qq^*q$$

So:

$$[x \approx y] \leqslant [x \approx y]\,[y \approx x]\,[x \approx y] \leqslant [x \approx x]\,[x \approx y]$$

The reverse inequality being a consequence of $(S2)$, one has the desired equality. □

We now prove the proposition:
**Proof :**

- Clearly, $(S2')$ implies $(S2)$. Assume that $(S2)$ and $(S1)$ hold, hence that $X$ is a $Q$-set. One can apply the lemma and one has:

  $$[x \approx y] \leqslant \bigvee [x \approx t]\,[t \approx y]$$

  by taking $t = x$ or $t = y$. The reverse inequality follows from $(S2)$.

- $(R1) \Rightarrow (R1')$ is clear because of the equality case. Assuming $(R1')$ one has immediately $[y \approx y']\,[y'Rx] \leqslant [yRx]$. So we just have to prove that $[y \approx y]\,[yRx] = [yRx]$. But:

$$[y \approx y]\,[yRx] = \bigvee_{y'} [y \approx y]\,[y \approx y']\,[y'Rx] = \bigvee_{y'} [y \approx y']\,[y'Rx] = [yRx]$$

  The equivalence of $(R2)$ and $(R2')$ is proved the same way.
- $(F2)$ is a special case of $(F2')$. Assume $(F2)$ then:

$$\begin{aligned}[x \approx x'] = [x \approx x]\,[x \approx x'] &\leqslant \textstyle\bigvee_y [y \approx f(x)]^*\,[y \approx f(x)]\,[x \approx x'] \\ &\leqslant \textstyle\bigvee_y [y \approx f(x)]^*\,[y \approx f(x')]\end{aligned}$$

The fact that $Q$-Sets are exactly the same as objects of $\mathsf{Proj}(Q)$, and $Q$-relations and $Q$-functions correspond respectively to morphisms in $\mathsf{Proj}(Q)$, and morphisms which are sent to functional relations by the equivalence of 2.4.3 is now immediate: If we replace the original axioms by this modified version, and if we interpret $[x \approx y]$,$[xRy]$ and $[x \approx f(y)]$ as matrix coefficients then the conditions imposed on them are exactly those for being objects and morphisms of $\mathsf{Proj}(Q)$. $\square$

## 2.5 Relational representations of Grothendieck quantales

In the previous section we constructed a topos $Q$-Sets from a Grothendieck quantale $Q$. In this section we describe the theory classified by this topos, that is study the morphisms from an arbitrary topos $\mathcal{T}$ to the topos of $Q$-sets. This also explains in which sense the equivalence between Grothendieck quantales and Grothendieck toposes is functorial.

2.5.1. **Definition :** *A morphism of modular quantales is a function $f : Q \to Q'$ between two modular quantales such that:*

- *$f$ commutes to arbitrary supremum (in particular it preserves the smallest elements)*
- *$f$ commutes to finite intersections (in particular it preserves the top element $\top$).*
- *$f$ is a morphism of unitary monoids (in particular it preserves $1$).*
- *$f$ commutes to the involution.*

*A Relational representation of a modular quantale $Q$ is the datum of an inhabited set $X$ endowed with a modular quantale morphism $\pi$ from $Q$ to $\mathit{Rel}(X)$. A morphism of relational representations is a map from $X$ to $X'$ such that for each $q \in Q$ if $(x,y) \in \pi(q)$ then $(f(x), f(y)) \in \pi'(q)$ .*

2.5.2. **Theorem :** *The topos of $Q$-sets classifies the relational representations of $Q$, the universal representation being given by the action of $Q$ on the bound $B$ (which corresponds to the $Q$-set $\{*\}$ with $[* \approx *] = 1$). In other words, if $\mathcal{E}$ is any topos then there is an equivalence of categories between the geometric morphisms from $\mathcal{E}$ to $Q$-sets, and the relational representations of $Q$ inside $\mathcal{E}$. And this equivalence is given by $f \mapsto f^*(B)$.*

This theorem is essentially the same as theorem 2.9 of [63].

**Proof :**
As a bound, the object $B$ has to be in particular inhabited, hence it is indeed a relational representation. So any geometric morphism from $\mathcal{E}$ to $Q$-sets does yield a relational representation of $Q$ on $f^*(B)$ and any natural transformation gives a morphism of representations. So the functor mentioned in the theorem indeed exists.

If $f$ is a geometric morphism from $\mathcal{E}$ to $Q$-sets, then $f^*$ induces a $\mathsf{sl}$-enriched functor from $\mathsf{Proj}(Q)$ to $\mathsf{Rel}(\mathcal{E})$.
Because $\mathsf{Proj}(Q)$ is generated by $B$ under co-product and splitting of projection, and since by proposition 2.3.2 arbitrary co-products (as well as splitting of projection) are preserved by any $\mathsf{sl}$-enriched functor, any relational representation $(X, \pi)$ of $Q$ in a topos $\mathcal{E}$ extends in a uniquely defined $\mathsf{sl}$-enriched functor from $\mathsf{Proj}(Q)$ to $\mathsf{Rel}(\mathcal{E})$: one has to send the couple $(I, P)$ on $\pi(P) \coprod_{i \in I} X$, and any morphism in $\mathsf{Proj}(Q)$ is a matrix $M$ which one has to send to the matrix "$\pi(M)$" defining a relation in $\mathcal{E}$.
Moreover if $f$ and $g$ are two geometric morphisms from $\mathcal{E}$ to $Q$-sets, then morphisms between the relational representations they induce uniquely extend to natural transformations between $f^*$ and $g^*$.
So we just have to prove that if $(X, \pi)$ is a relational representation of $Q$, then the induced functor $v$ from $\mathsf{Proj}(Q)$ to $\mathsf{Rel}(\mathcal{E})$ comes from a geometric morphism from $\mathcal{E}$ to $Q$-sets.

- As $\pi$ commutes with $*$, so does $v$. Hence $v$ preserves functional relations and induces a functor from $Q$-sets to $\mathcal{E}$.
- The terminal object of $Q$-sets is the quotient of $B$ by its maximal relation, and since $\pi$ preserves the maximal relation, the terminal object of $Q$-sets is sent to the quotient of $X$ by its maximal relation, which is the terminal object of $\mathcal{E}$ because $X$ is inhabited. So $v$ preserves the terminal object.
- Let

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & S \end{tikzcd}$$

  be a pull back diagram in $Q$-sets, then $P$ can be identified with the relation $f^*g$ on $X \times Y$, indeed internally $f^*g$ is the relation $\{(x, y | f(x) = g(y)\}$ hence it is the fiber product $X \times_S Y$. So $v$ preserves pullback. As $v$ preserves the terminal object, it preserves all limits.

- In a topos, a collection of maps $f_i : A_i \to A$ is a covering if and only if $1_A \leqslant \bigvee_{i \in I} f_i f_i^*$ as $v$ preserves all the structures involved, $v$ preserves covering families.

All these properties together imply that $v$ is indeed the $f^*$ functor of a geometric morphism and conclude the proof.
□

## 2.6 Internally bi-linear maps between $Q$-modules

The category $\mathsf{sl}(\mathcal{T})$ is endowed with a tensor product. In the case where $\mathcal{T} = \mathsf{Sh}(\mathcal{L})$ is the topos of sheaves on a frame $\mathcal{L}$, one can see that through the identification of $\mathsf{sl}(\mathcal{T})$ with $\mathsf{Mod}_{\mathcal{L}}$ this tensor product corresponds to the natural tensor product over $\mathcal{L}$, which as in the case of commutative algebras is defined by the universal property: the maps from $M \otimes_{\mathcal{L}} N$ to $P$ are the bi-linear morphisms from $M \times L$ to $P$ such that for all $l \in \mathcal{L}$, $f(m, n.l) = f(m.l, n) = f(m,n).l$.
The main result of this section is that, in the general case, even if the tensor product of two $Q$-modules can be difficult to compute explicitly, the set $\mathsf{Bil}_{\mathcal{T}}(M \times N, P)$ of internal bilinear map from $M \times N$ to $P$ has a simple description in terms of the corresponding right $Q$-modules. This leads in particular to a simple description of the category of internal locales of $\mathcal{T}$ in terms of a Grothendieck quantale representing $\mathcal{T}$. More precisely:

2.6.1. **Definition :** *If $A$, $B$ and $C$ are three right modules over a Grothendieck quantale $Q$, we say that a map $f : A \times B \to C$ is $Q$-bilinear if it is a bi-linear morphism of sup-lattices and if it satisfies the following three conditions:*

1. $f(aq, b) \leqslant f(a, bq^*)q$
2. $f(a, bq) \leqslant f(aq^*, b)q$
3. $f(a,b).q \leqslant f(aq, bq)$.

*We will denote by $\mathsf{Bil}_Q(A \times B, C)$ the set of $Q$-bilinear maps.*

$\mathsf{Bil}_Q(A \times B, C)$ is a sup-lattice for the pointwise ordering (with supremum computed pointwise), and it is an $\mathsf{sl}$-enriched functor in each of the three variables (contravariant in the first two variables) with the functoriality given by composition.
The main result of this section (theorem 2.6.3) is that this functor is isomorphic to the functor of internal bilinear maps.

2.6.2. Let $M$,$N$ and $P$ be internal sup-lattices in $\mathcal{T}$, let $\widetilde{M}$, $\widetilde{N}$ and $\widetilde{P}$ be the corresponding right $Q$-modules (i.e. $\tilde{M} = \hom_{\mathcal{T}}(B, M)$). Let $f$ be a bilinear morphism from $M \times N$ to $P$.
Then one can define a map $\tilde{f}$ from $\widetilde{M} \times \widetilde{N}$ to $\widetilde{P}$ by the (internal) formula:

$$\tilde{f}(m,n) := b \mapsto f(m(b), n(b)) \in P$$

With $m$ and $n$ elements of $\widetilde{M}$ and $\widetilde{N}$, that is, maps from $B$ to $M$ and $N$, then $\tilde{f}(m,n)$ is indeed an element of $\widetilde{P} = \hom_{\mathcal{T}}(B,P)$.

**Proposition :** *The map $\tilde{f}$ is a $Q$-bilinear morphism map in the sense of 2.6.1. Moreover the construction $f \to \tilde{f}$ defines a morphism of $\mathsf{sl}$-enriched functors:*

$$\mu(M,N,P) : \mathit{Bil}_{\mathcal{T}}(M \times N, P) \to \mathit{Bil}_{Q}(\widetilde{M} \times \widetilde{N}, \widetilde{P})$$

**Proof :**
The (sup-lattice) bilinearity is immediate: supremum in $\widetilde{M}, \widetilde{N}$ and $\widetilde{P}$ corresponds to pointwise internal supremum hence the bilinearity of $\tilde{f}$ simply comes from the internal bilinearity of $f$.
Recall that by definition one has internally for any $m \in \widetilde{M}$ and $q \in Q = \mathsf{Rel}(B)$:

$$m.q(b) = \bigvee_{(b',b) \in q} m(b').$$

All three properties defining $Q$-bilinearity are then easily checked internally:

1. $$\tilde{f}(mq,n)(b) = \bigvee_{(b',b) \in q} f(m(b'),n(b))$$

   Whereas:

   $$\tilde{f}(m,nq^*)q(b) = \bigvee_{(b',b) \in q} \tilde{f}(m,nq^*)(b') = \bigvee_{(b',b) \in q, (b',b'') \in q} f(m(b'),n(b''))$$

   So the first term corresponds to the restriction of the union to $b = b''$ of the second and is indeed smaller.
2. Same proof.
3. $$\left[\tilde{f}(m,n)q\right](b) = \bigvee_{(b',b) \in q} f(m(b'),n(b'))$$

   Whereas

   $$\tilde{f}(mq,nq)(b) = f(mq(b),nq(b)) = \bigvee_{(b',b) \in q, (b'',b) \in q} f(m(b''),n(b'))$$

   So the first term corresponds to the restriction of the union to $b' = b''$ of the second and is indeed smaller.

Also $f \mapsto \tilde{f}$ commutes to supremum, because if one takes $f_i$ an (external) family of internal bilinear maps, $m \in \widetilde{M}$ and $n \in \widetilde{N}$ then (internally) for any $b \in B$:

$$\left(\widetilde{\bigvee_i f_i}\right)(m,n)(b) = \bigvee_i f_i(m(b),n(b)) = \left(\bigvee_i \tilde{f}_i(m,n)\right)(b)$$

And the functoriality is immediate: $\tilde{f}(m,g(n)) := b \mapsto f(m(b),g(n(b))$ is indeed the map attached to $f(\_,g(\_))$ and $\tilde{g}(\tilde{f}(m,n) := b \mapsto g(f(m(b),n(b)) = \widetilde{g \circ f}(m,n)$.
□

2.6.3. **Theorem :** *The construction $f \mapsto \tilde{f}$ from 2.6.2, defines an isomorphism of $\mathsf{sl}$ enriched functors:*

$$\mu : \mathsf{Bil}_{\mathcal{T}}(M \times N, P) \simeq \mathsf{Bil}_{Q}(\widetilde{M} \times \widetilde{N}, \widetilde{P}).$$

The functoriality of the association and the fact that it commutes to supremum have already been mentioned, so it only remains to prove that it is a bijection. The proof of this theorem will be completed in 2.6.8 after proving a few lemmas.

2.6.4. **Lemma :** *The association $f \mapsto \tilde{f}$ of 2.6.2 is injective.*

**Proof :**
Let $f$ and $g$ be two internal bi-linear maps from $M \times N$ to $P$ such that $\tilde{f} = \tilde{g}$. This means that for each map $(m,n) : B \to M \times N$ one has internally:

$$\forall b \in B, f(m(b), n(b)) = g(m(b), n(b))$$

i.e.:

$$f \circ (m,n) = g \circ (m,n).$$

But we already explained in the last part of the proof of 2.3.4 that any map from a sub-object $U$ of $B$ to a sup-lattice can be extended (canonically) to a map on all of $B$. As $B$ is a bound, maps from sub-objects of $B$ can cover $M \times N$ and by the extension arguments, maps from $B$ cover $M \times N$, so we can conclude from the previous formula that $f = g$.
□

2.6.5. **Lemma :** *Let $h : Q \times Q \to \tilde{P} \in \mathsf{Bil}_Q(Q \times Q, \tilde{P})$ where $Q$ is endowed with its right action on itself. Then:*

- *Let $c \in \tilde{P}$ and $a, b \in Q$ such that $aa^* \leqslant 1$, $bb^* \leqslant 1$. If one has $c \leqslant h(a,b)$ then for all $x, y \in Q$:*

$$c(a^*x \wedge b^*y) \leqslant h(x,y)$$

- *For all $x, y$ one has:*

$$h(x,y) = \bigvee_{aa^* \leqslant 1, bb^* \leqslant 1} h(a,b)(a^*x \wedge b^*y)$$

**Proof :**
For the first point:
Let $t = (a^*x \wedge b^*y) \in Q$. Then one has:

$$at \leqslant (aa^*x \wedge ab^*y) \leqslant aa^*x \leqslant x$$

$$bt \leqslant (ba^*x \wedge bb^*y) \leqslant bb^*y \leqslant y$$

So:

$$ct \leqslant h(a,b)t \leqslant h(at,bt) \leqslant h(x,y)$$

For the second point, let:

$$h'(x,y) = \bigvee_{aa^* \leqslant 1, bb^* \leqslant 1} h(a,b)(a^*x \wedge b^*y).$$

Clearly, $h'$ is also in $\mathsf{Bil}_Q(Q \times Q, \tilde{P})$.
The first point shows that $h' \leqslant h$. For the reverse inequality we will proceed in several steps:

- If $(xx^* \leqslant 1), (yy^* \leqslant 1)$. Let $D(x) = 1 \wedge x^*x$ and $D(y) = 1 \wedge y^*y$. We note that for elements smaller than 1, the involution is the identity and composition and intersection coincide (these can be proved by applying the modularity law, or by using theorem 2.4.5 as a black box and checking it internally in the corresponding topos). So

  $$xD(x) = x(1 \wedge x^*x) \geqslant (x \wedge x) = x$$

  hence $xD(x) = x$ and $yD(y) = y$ also,

  $$x^*x \wedge y^*y \geqslant D(x) \wedge D(y) = D(x)D(y)$$

  Aslo for any $e \leqslant 1$:

  $$h(xe,y) \leqslant h(x,ye)e \leqslant h(x,y)e \leqslant h(xe,ye) \leqslant h(xe,y)$$

  hence

  $$h(xe,y) = h(x,ye) = h(x,y)e$$

  Finally:

  $$h(x,y)(x^*x \wedge y^*y) \geqslant h(x,y)D(x)D(y) = h(xD(x), yD(y)) = h(x,y)$$

  So $h'(x,y) \geqslant h(x,y)$
- We will now assume that $x$ is arbitrary and $y$ is simple. As $Q$ is a Grothendieck quantale, $x$ can be written as a supremum of elements of the form $uv^*$ with $u$ and $v$ simple. So, by bi-linearity of $h$, it is enough to prove that $h(x,y) \leqslant h'(x,y)$ when $x$ is of the form $uv^*$. In this case:

  $$h(uv^*,y) \leqslant h(u,yv)v^* = h'(u,yv)v^* \leqslant h'(uv^*,yvv^*) \leqslant h'(uv^*,y)$$
- If both $x$ and $y$ are now arbitrary, then the same technique allows ones to conclude.

□

2.6.6. **Corollary :** *Theorem 2.6.3 holds whenever* $M = N = \mathcal{P}B$ *(ie.* $\mu(\mathcal{P}(B), \mathcal{P}(B), \tilde{T})$ *is an isomorphism)*

**Proof :**
Injectivity is known by lemma 2.6.4. So we just have to prove surjectivity. Let $h$ be any bi-linear map from $Q \times Q = \tilde{M} \times \tilde{N} \to \tilde{P}$ satisfying our three conditions. Then, by lemma 2.6.5, one has that:

$$h(x, y) = \bigvee_{aa^* \leqslant 1, bb^* \leqslant 1} h(a, b)(a^* x \wedge b^* y).$$

One can see that maps of the form $(x, y) \mapsto t(ux \wedge vy)$ with $u, v \in Q$ and $t \in \tilde{T}$ correspond to the internal bi-linear map which sends $(p, q) \in \mathcal{P}(B) \times \mathcal{P}(B)$ to $t(u(p) \wedge v(q)$ where $u$ and $v$ are seen as endomorphisms of $\mathcal{P}(B)$ and $t$ as a morphism from $\mathcal{P}(B)$ to $T$. Indeed if $g(p, q) = t(u(p) \wedge v(q))$ then

$$\tilde{g}(x, y) = t(ux(b) \wedge vy(b)) = \bigvee_{(b', b) \in (ux \wedge vy)} t(b') = t(ux \wedge vy)$$

Hence $h$ can be written as a supremum of maps coming from internal bi-linear maps, but as $f \mapsto \tilde{f}$ commutes to arbitrary supremum, this shows that $h$ does also come from a bilinear map. $\square$

2.6.7. At this point, we have two possibilities. We can conclude by an argument of extension by inductive limit, or use the following argument which we found more convincing:

**Proposition :** *Assume that* $\mu(\mathcal{P}(B), M, P)$ *is an isomorphism for some* $M$ *and* $P$ *in* $\mathsf{sl}(\mathcal{T})$*. Then* $\mu(N, M, P)$ *is an isomorphism for any* $N$ *in* $\mathsf{sl}(\mathcal{T})$*.*

**Proof :**
We already know that $\mu$ is injective. Hence it remains to show the surjectivity. Let $M$ and $P$ be such that $\mu(\mathcal{P}(B), M, P)$ is an isomorphism. The internal sup-lattice $[M, P]$ then corresponds to the right $Q$-module:

$$\begin{aligned} \hom_{\mathcal{T}}(B, [M, P]) &= \hom_{\mathsf{sl}(\mathcal{T})}(\mathcal{P}(B), [M, P]) = \mathsf{Bil}_{\mathcal{T}}(\mathcal{P}(B) \times M, P) \\ &= \mathsf{Bil}_Q(Q \times \tilde{M}, \tilde{P}) \end{aligned}$$

where the action of $Q$ on the last term is given by the left action of $Q$ on itself. Now let $g \in \mathsf{Bil}_Q(\tilde{N} \times \tilde{M}, \tilde{P})$. For any $n \in \tilde{N}$, the map:

$$g_n : (q, m) \mapsto g(nq, m)$$

is an element of $\mathsf{Bil}_Q(Q \times \tilde{M}, \tilde{P})$. The map $(n \mapsto g_n)$ is a morphism of right $Q$-modules and hence internally corresponds to a map $f : N \to [M, P]$ which in turn corresponds to a map $f \in \mathsf{Bil}_{\mathcal{T}}(N \times M, P)$. Finally $\tilde{f} = g$ because for any $n \in \tilde{N}$, $(q, m) \mapsto \tilde{f}(nq, m)$ is by construction of $\tilde{f}$ the map $g_n \in \mathsf{Bil}_Q(Q \times \tilde{M}, \tilde{P})$ and hence $\tilde{f}$ agrees with $g$. This concludes the proof.
$\square$

2.6.8. We can now finish the proof of 2.6.3: by 2.6.6 we know that for any $T \in \mathsf{sl}(\mathcal{T})$, $\mu(\mathcal{P}(B), \mathcal{P}(B), T)$ is an isomorphism. Hence by 2.6.7, $\mu(N, \mathcal{P}(B), T)$ is an isomorphism for each $N$ and $T$, as one can freely exchange the first two variables, $\mu(\mathcal{P}(B), N, T)$ is also an isomorphism, and a second application of 2.6.7 allows one to conclude.

2.6.9. **Corollary :** *The category of internal locales of $\mathcal{T}$ (equivalently, the category of toposes which are localic over $\mathcal{T}$) is equivalent to the category of locales $\mathcal{L}$ endowed with a right action of $Q$ such that:*

- *As a sup-lattice $\mathcal{L}$ is a right $Q$-module.*
- *One has the modularity condition:*

$$\forall m, n \in \mathcal{L}, \forall q \in Q, m \wedge nq \leqslant (mq^* \wedge n)q$$

*We will call such an action a modular action (the morphisms of this category being the morphism of locales whose $f^*$ part is $Q$-equivariant).*

Of course, the equivalence is given by the usual functor $\mathcal{L} \mapsto \hom_{\mathcal{T}}(B, \mathcal{L})$.

**Proof :**
Let $\mathcal{L}$ be a locale in $\mathcal{T}$ then $\tilde{\mathcal{L}}$ is indeed endowed with an operation $\tilde{\wedge}$ which is $Q$-bilinear. Also:

$$\tilde{\wedge}(m, n)(b) = m(b) \wedge n(b)$$

is the intersection of $\hom_{\mathcal{T}}(B, \mathcal{L})$, hence $\mathcal{L}$ is indeed a locale and intersection is indeed $Q$-bilinear.

Conversely, if $\mathcal{L}$ is a locale endowed with a modular action of $Q$, then the operation $\wedge$ is $Q$-bilinear (the second axiom comes from the symmetry, and the third axiom because multiplication by $q$ is order preserving). Hence $\mathcal{L}$ corresponds to an internal sup-lattice $L$ equipped with a bi-linear map $m$ coming from $\wedge$. This bi-linear map has to be the intersection map of $L$ because both $m$ and $\wedge_L$ induce the same map when we externalise it by watching the morphisms from $B$ to $L \times L$, and the proof of the injectivity of the externalisation process done in 2.6.4 works without assuming that the map $f$ is bilinear.
□

2.6.10. Also, from the description of $\mathsf{Bil}_{\mathcal{T}}(M \times N, P)$ it is possible to obtain an explicit description of both the tensor product and the internal hom objects in terms of the corresponding $Q$-modules: for the hom-object it has been done in the proof of 2.6.7 and for the tensor product, the description of $\mathsf{Bil}_{\mathcal{T}}(M \times N, P)$ translates into a presentation by generators and relations of $\widetilde{M \otimes N}$. In order to completely handle the monoidal structure of $\mathsf{sl}(\mathcal{T})$ in terms of the category of $Q$-modules it remains to understand what $\widetilde{M^*}$ is.
An element of $\widetilde{M^*}$ is a function from $B$ to $M^*$, hence it is the same thing as a function from $B$ to $M$ (i.e. an element of $\widetilde{M}$) but with the reverse order relation,

hence as a sup-lattice $\widetilde{\widetilde{M^*}} = \widetilde{M^*}$. A simple computation shows that $q \in Q$ acts on $\widetilde{M^*}$ by the adjoint of the action of $q^*$ on $M$.

We also note that the notion of $Q$-bilinear map makes sense when $Q$ is only a modular quantale, and that one can define a "tensor product" $M \otimes_Q N$ universal for $Q$-bilinear maps from $M \times N$. But in general this tensor product fails to be associative.

## 2.7 Representations of modular quantales

2.7.1. Conjecture: Every modular quantale is of the form $\mathsf{Rel}(X)$ with $X$ an object of a topos.

2.7.2. Let $Q$ be any modular quantale, we can try to consider the classifying topos of the theory of relational representations of $Q$ and hope that the quantale of relations on the universal representation is isomorphic to $Q$. Unfortunately this is not true in general as the following example shows:

Let $n$ be any integer $> 2$, and $G = S_n$ be the permutation group and consider the $S_n$-set $X_n = \{1, \dots, n\}$ endowed with its natural action of $S_n$. The only relations on $X_n$ (in $S_n$-sets) are: $\emptyset, 1, \top, \Delta$ where $\Delta$ is the complementary relation of 1. As $n > 2$ one has $\Delta^2 = \top$. Let $Q$ be the quantale of relations on $X_n$ in $S_n$-sets. It appears that $Q$ does not depend on $n$. A relational representation of $Q$ (in a topos) is just an object $S$ which has a decidable equality (the diagonal sub-object is complemented) and at least three distinct elements. The universal model $U$ of this theory has more relations on it than just the four elements of $Q$:
Indeed, let

$$P = \{(x, y) \in U | \exists x_1, \dots, x_4 \in U^4 \text{ pairwise disjoint in U}\}.$$

then the pullback of $P$ in the representation $X_3$ is $\emptyset$ whereas the pullback of $P$ in the representation $X_n$ for $n > 3$ is $\top$ hence $P$ cannot be any of the objects of $Q$ in the universal representation.

## 2.8 Towards a convolution $C^*$-algebra attached to a quantale

In the special case where the topos $\mathcal{T}$ is an étendue, ie the topos of equivariant sheaves over an étale (localic) groupoid $G = (G_0, G_1)$, the quantale associated to the bound $B$ such that the slice topos $\mathcal{T}_{/B}$ is the locale $G_0$ is the set of open subsets of $G_1$, and the composition law is given by the direct image of open subsets in the composition law of $G_1$.

With this fact and the usual construction of a $C^*$-algebra from a groupoid (see [61] ) in mind it is natural to try to construct a $C^*$-algebra from a quantale by defining a convolution product over a subset of continuous functions "over $Q$"

(or over open subspaces of $Q$). Indeed we can view $Q$ as a locale by forgetting its composition law and involution, then use the involution to get an involution on continuous functions and hope [9] that the composition law on $Q$ will allow us to construct a convolution product on continuous functions.

In order to perform this construction the general idea is the following: a continuous function on $Q$ is the same thing as a function from $X \times X$ to the object $\mathbb{R}_{\mathcal{T}}$ of real or $\mathbb{C}_{\mathcal{T}}$ of complex numbers. Hence it can be thought of, internally, as an infinite matrix whose rows and columns are indexed by $X$, and we can use the multiplication of matrices to define the convolution product. If the coefficients are all non-negative and we allow infinite coefficients the product should be always defined. There are two difficulties that arise when we try to define the matrix product internally:

- Matrix multiplication requires a summation indexed by the elements of $X$. It cannot be done if we do not assume that $X$ has a decidable equality. The reason for this is that without this assumption, when we look at partial sums $f(x_1)+\cdots+f(x_n)$, for $(x_1, \cdots, x_n)$ elements of $X$ we cannot say whether the elements $x_i$ are distinct, hence we cannot assert that we have not counted some value of $f$ twice.
- The sum of an infinite number of terms will in full generality be defined as the supremum of all the possible finite sums. In general the object $\mathbb{R}_{\mathcal{T}}$ of continuous real numbers (i.e. two sided Dedekind cut) does not always have supremum. In order to define a supremum we need to replace the usual "continuous" real numbers by the lower semi-continuous real numbers (one-sided Dedekind cut), so the result of the convolution will in general be a lower semi-continuous function.

2.8.1. We now move to the precise definition:

**Proposition :** *Let $X$ be an object of a topos $\mathcal{T}$ with a decidable equality. Let (internally) $f$ and $g$ be functions from $X \times X$ to $\overrightarrow{\mathbb{R}_{+}^{\infty}}$. Then we can define a function $(f*g)$ from $X \times X$ to $\overrightarrow{\mathbb{R}_{+}^{\infty}}$ by the internal formula:*

$$(f*g)(x,x') = \sum_{x'' \in X} f(x,x'')g(x'',x')$$

Of course this $X$ indexed sum has to be interpreted as, for all $q \in \mathbb{Q}_{\mathcal{T}}$:

$$q < (f*g)(x,x')$$

if and only if: $\exists n \in \mathbb{N}_{\mathcal{T}}, x_1, \cdots, x_n \in X$ such that:

$$\forall i \neq j, x_i \neq x_j$$

and

$$q < \sum_{i=1}^{n} f(x,x_i)g(x_i,x')$$

9. This will not be the case in full generality.

where $q < \sum_{i=1}^{n} f(x,x_i)g(x_i,x')$ naturally means:
$\exists u_1, \cdots, u_n, v_1, \cdots v_n \in \mathbb{Q}_{\mathcal{T}}$, such that $u_i < f(x,x_1), v_i < g(x_i,x')$ and $q < \sum_{i=1}^{n} u_i v_i$.
**Proof :**
All we have to do is check that the set $L$ of $q$ such that:

$$\exists n \in \mathbb{N}_{\mathcal{T}}, x_1, \cdots, x_n \in X, \text{ such that } \forall i \neq j, x_i \neq x_j \text{ and } q < \sum_{i=1}^{n} f(x,x_i)g(x_i,x')$$

is indeed a positive one-sided Dedekind cut.
It is positive because, by taking $n = 0$ all negative $q$ are in $L$. If $q' < q$ and $q \in X$ then clearly $q' \in X$. If $q \in L$, then $q < \sum f(x,x_i)g(x_i,x')$ so there exists $q' > q$ such that $q' < \sum f(x,x_i)g(x_i,x')$ hence $q' \in L$. This concludes the proof. $\square$

2.8.2. **Proposition :** *The convolution product defined is associative and the characteristic function of the unit of $Q$ is a unit.*

**Proof :**
This an immediate consequence of the fact that internally, the composition of matrices is associative and that the identity matrix is a unit for the composition of matrices. All we need is a constructive version of Fubini's theorem for sums (indexed by decidable sets) of positive lower semi-continuous real numbers. The usual proof can easily been made constructive, or we can apply the general Fubini's theorem proved in [73]. $\square$

This is interpreted externally as the construction of a convolution product on the set of lower semi-continuous functions on the underlying space of $Q$. This corresponds exactly to the construction of the convolution algebra of an étale groupoid (see for example [61] [10] for this construction).
It should be possible to obtain something similar to the more general construction of the convolution algebra of a locally compact groupoid endowed with a Haar system, by replacing the bound $B$ by an internal space endowed with an internal "measure" (a valuation to be more precise). This would require using internal measure theory as developed in [72] and [73].

Of course, in full generality this kind of construction may fail to give anything interesting: it may not restrict to an operation on continuous functions, or it may even yield an everywhere infinite result. The rest of this chapter will give a more precise picture of the situation in the special case where the underlying topological space of $Q$ is discrete.

10. In this reference, étale groupoids are called r-discrete groupoids.

# 3 Atomic Quantales

## 3.1 Introduction

In this section we focus on a really special case of the theory explained in the previous section: when the underlying space of the quantale $Q$ (that is, the space obtained by forgetting the product on $Q$ and seeing it as a locale) is a discrete topological space. This corresponds to the study of atomic toposes (see [44, C.3.5] for the theory of atomic toposes).
In this situation, the convolution product constructed at the end of the previous section is easier to understand, and among other things we will explain a simple necessary and sufficient condition for this convolution product to be defined and interesting on continuous functions. In this special case two additional features appear: a canonical "time evolution" on the $C^*$-algebra obtained this way, and we will observe that when the convolution product is well defined it restricts into a product on finitely supported integer valued functions, which gives rise to an algebra over $\mathbb{Z}$ with possibly interesting arithmetic and combinatorial properties. This algebra can be interpreted as a subalgebra of the algebra of endomorphisms of the free $\mathbb{Z}_{\mathcal{T}}$ modules of base $B$.

All the constructions will be made constructively, but at some point we will need to take the assumption that the underlying space of $Q$ is decidable.

## 3.2 Atomic quantales and atomic toposes

An object $X$ of a topos is said to be an atom if $\mathsf{Sub}(X) \simeq \Omega$ [11]. A direct image of an atom by a morphism is again an atom. A topos is said to be atomic if it satisfies one of the following equivalent properties:

- The atoms form a generating family.
- For every object $X \in |\mathcal{T}|$, $\mathsf{Sub}(X)$ is an atomic locale (i.e. of the form $\mathcal{P}(S)$ for some set $S$).
- $\mathcal{T}$ is the topos of sheaves over an atomic site (i.e. a site such that the covering sieves are exactly the inhabited sieves).

For details on the theory of atomic toposes one can consult [44, C3.5] [12].

3.2.1. **Proposition :** *Let $\mathcal{T}$ be a topos and $B$ be a bound of $\mathcal{T}$, then the following conditions are equivalent:*

1. *$\mathcal{T}$ is atomic over the base topos $\mathcal{S}$.*
2. *$Q = \mathit{Rel}(B)$ is (as a locale) atomic, i.e. its underlying poset is of the form $\mathcal{P}(X)$ for some set $X$.*
3. *$Z(Q) = \mathit{Sub}(B) = \{q \in Q | \leqslant 1\}$ is atomic.*

11. Assuming classical logic in the base topos, this means that $X$ is non-empty and has no non trivial sub-object.
12. This reference studies atomic geometric morphisms, but a geometric morphism $f : \mathcal{E} \to \mathcal{T}$ is atomic if and only if $\mathcal{E}$ is atomic as a $\mathcal{T}$-topos.

**Proof :**
1. $\Rightarrow$ 2. is clear, because in an atomic topos all the lattices $\mathsf{Sub}(X)$ are atomic. The implication 2. $\Rightarrow$ 3. is also clear. We will prove 3. $\Rightarrow$ 1.: Every object of $\mathcal{T}$ can be covered by subobjects of $B$, so if every subobject of $B$ can be covered by atoms (which is the assumption in 3.) every object of $\mathcal{T}$ can be covered by atoms and hence $\mathcal{T}$ is atomic. This concludes the proof. $\square$

Of course, if $\mathcal{T}$ is an atomic topos and $X$ is any object of $\mathcal{T}$, then $\mathsf{Rel}(X)$ is an atomic modular quantale.

3.2.2. The notion of atomic quantale will be closely related to the notion of Hypergroupoid, which is a natural generalisation of the notion of canonical hypergroup which can be found in [51] or [46].

**Definition :** *A hypergroupoid $G$ is the data of:*

- *A set $E(G)$ of objects.*
- *For each $e, e' \in E(G)$ a set $G(e, e')$ of "arrows" from $e$ to $e'$.*
- *For each $g \in G(e_1, e_2)$ and $h \in G(e_2, e_3)$ an inhabited set $hg \subseteq G(e_1, e_3)$ of "possible compositions".*
- *And for each $x \in G(e, e')$ an element $x^* \in G(e', e)$ called the inverse of $x$.*

*With the following axioms:*

*(HG1)* $\forall e \in E(G), \exists 1_e \in G(e, e),$ *such that* $\forall x \in G(e', e), 1_e x = \{x\}$ *and* $\forall x \in G(e, e'), x1_e = \{x\}$.
*Such a $1_e$ is unique and is also denoted by $e$.*

*(HG2) for all $x, y, z$ three arrows such that the composition $xy$ and $yz$ are defined, one has $(xy)z = x(yz)$ where the product of an element $x$ with a set $S$ is defined by $xS = \bigvee_{s \in S} xs$.*

*(HG3) if $x \in yz$ then $z \in y^* x$ and $y \in xz^*$.*

We will adopt the convention that if $g$ and $g'$ are two non-composable arrows of a hypergroupoid then $gg'$ is defined to be the empty set. Or constructively that for two general arrow $g$ and $g'$ of a hypergroupoid $gg' = \{u | g \text{ and } g' \text{ are composable and } u \in gg'\}$.

3.2.3. **Proposition :** *Let $G$ be a hypergroupoid, and let $X$ be the set of all arrows:*

$$X = \coprod_{e, e' \in E(G)} G(e, e')$$

*Then $\mathcal{P}(X)$ endowed with the following structure:*

$$U^* = \{x^*, x \in U\}$$

$$UV = \{t | \exists u \in U, v \in V, u, v \textit{ are composable and } t \in uv\} = \bigvee_{\substack{u \in U \\ v \in V}} uv$$

*is a (atomic) modular quantale.*

**Proof :**
$\mathcal{P}(X)$ is by definition an atomic locale. We check the remaining axioms:

- The associativity of the product: if $g \in U(VW)$ then $\exists u \in U$, $f \in (VW)$ such that $g \in uf$ but $b \in VW$ means that there exists $v \in V$, $w \in W$ such that $f \in v.w$. hence $g \in u(vw) = (uv)w$. So there exists an $h \in (uv)$ (in particular $h \in UV$) such that $g \in hw$ and, $g \in (UV)W$. The reverse inclusion is exactly the same.
- The composition is clearly bi-linear because it is defined so that:
$$UV = \bigvee_{\substack{u \in U \\ v \in V}} \{u\}.\{v\}$$
- The Set $1 = \{1_e, e \in E(G)\}$ is also clearly a unit for $Q$, because for any $u \in G(e, e')$, $\{u\}.\{1_e\} = \{u\}$ and $\{u\}\{1_{e''}\} = \emptyset$ for any other $e''$, hence $\{u\}1 = \{u\}$, one obtains the general result by bi-linearity (and symmetry for the fact that 1 is also a left unit).
- $(UV)^* = V^*U^*$: Let $x \in V^*U^*$ i.e. $x \in v^*u^*$ for $u$ and $v$ respectively in $U$ and $V$. Then $v^* \in x.u$, $u \in x^*v^*$, and finally $x^* \in uv$, ie $x \in (UV)^*$. This reasoning can be conducted backwards to obtain the reverse inequality.
- The modularity law:
Let $x \in U \wedge VW$ then $x \in U$ and $x \in vw$ with $v \in V$, $w \in W$. We have $v \in xw^*$ hence $v \in (UW^* \wedge V)$ and $x \in (UW^* \wedge V)W$.

$\square$

3.2.4. **Proposition :** *Conversely, any atomic modular quantale is of the form $\mathcal{P}(X)$ where $X$ is the set of arrows of a hypergroupoid.*

**Proof :**
Let $Q$ be an atomic modular quantale. So $Q = \mathcal{P}(X)$ for some set $X$.
Let $E = \{x \in X | x \in 1_Q\}$. In order to simplify notations we will identify an element of $X$ with the corresponding singleton element in $Q$.
For any $q \in X$, as $1_Q q = q$ there exists $e \in E$ such that $ex = x$. Such an $e$ is unique because if $e'x = x$ then $e'ex = x$. But as $e'$ and $e$ are subobjects of 1 in a modular quantale, one has $ee' = e \wedge e'$ and in particular $x = (e' \wedge e)x$. Hence $e' \wedge e$ is inhabited and finally $e = e'$. Similarly, for each $x \in Q$ there is a unique $e' \in E$ such that $xe' = x$.
Let:
$$G(e, e') = \{x \in X | xe = x \text{ and } e'x = x\}$$
We just show that $X$ is the disjoint union of all the $G(e, e')$. If $x \in G(e, e')$ then $x^* \in G(e', e)$. Also, if $a \in G(e, e')$ and $b \in G(e', e'')$ then $ba \subseteq G(e, e'')$, indeed, if $c \in ba$ then there exists a unit $f \in E$ such that $fc = c$. In particular $c \in fba = fe''ba = (f \wedge e'')ba$. Hence $e'' = f$. The proof work exactly the same for the other side.
We will prove that this is indeed a hypergroupoid structure:

- Let $a \in G(e,e')$ and $b \in G(e',e'')$. One has $e'a = a$ and $be' = b$. hence $e' \in aa^*$ and $b \in baa^*$ so $ba$ is inhabited.
- if $a \in G(e',e)$ then $ea = a$ and $ae' = a$ by definition.
- the associativity of the product comes from the associativity of the product of the quantale and the fact that if there exists $a \in uv$ for $u, v \in X$ then $u$ and $v$ are composable elements: this assert that the product of the hypergroupoid is exactly the product of the quantale, and its associativity follows by restriction to composable pairs of morphisms.
- If we assume that $x \in yz$ then $x = x \wedge yz \leqslant (xz^* \wedge y)z$ hence $y \wedge xz^*$ is inhabited so $y \in xz^*$.

Finally as we have proved already that multiplication in $Q$ and in $X$ are essentially the same it is a routine check to prove that $Q$ will be isomorphic to $\mathcal{P}(X)$ as a modular quantale.
□

3.2.5. We have essentially proved that an atomic modular quantale is the same thing as a hypergroupoid. If we define a morphism of hypergroupoid from $G$ to $G'$ to be a function $f$ from $E(G)$ to $E(G')$ and a collection of maps (all called $f$) $f : G(e,e') \to G'(f(e), f(e'))$ such that $f(1_e) = 1_{f(e)}$ and $f(xy) \subseteq f(x)f(y)$ [13], then one has:

**Theorem :** *There is an anti-equivalence of categories between the category of atomic modular quantales (with weakly unital morphisms) and the category of hypergroupoids.*

Here "weakly unital morphisms" of (atomic) modular quantales are those defined in 2.5.1, except that we no longer assume them to preserve the unit, but only to satisfy the weaker conditions $1 \leqslant f(1)$.

**Proof :**
Let $Q$ and $Q'$ be two atomic modular quantales and $g : Q' \to Q$ a morphism of modular quantales. By 3.2.4, $Q$ and $Q'$ can be written $Q' = \mathcal{P}(X')$ and $Q = \mathcal{P}(X)$ where $X'$ and $X$ are the sets of all arrows of two hypergroupoids $G$ and $G'$. As $g$ is in particular a morphism of locale, it induces a map $f : X \to X'$ characterized by the fact that for all $x \in X$, $f(x)$ is the unique element of $X'$ such that $x \in g(\{f(x)\})$.
In particular, let $c \in ab$ then $ab \subseteq g(f(a))g(f(b))$ hence $c \in g(f(a)f(b))$ ie $f(c) \in f(a)f(b)$. This proves that $f(ab) \subseteq f(a)f(b)$. As $1 \leqslant g(1)$, and $1 \in G$ corresponds to $E(G) \subseteq X$, the map $f$ acts on the unit set and preserves the identity element. One also has $x^* \in g(f(x))^* = g(f(x)^*)$ hence $f(x)^* = f(x^*)$ and finally, if $g \in G(e,e')$ then $e \in g^*g$, $e' \in gg^*$ and $f(e) \in f(g)^*f(g)$, $f(e') \in f(g)f(g)^*$ which proves that $f(g)$ is an element of $G(f(e), f(e'))$ which concludes the proof that $f$ is a morphism of hypergroupoids.
Conversely, if $f$ is a morphism of hypergroupoids, then as $f(1_e) \in f(g)f(g^*)$ one can conclude that $f(g^*) \in f(g)^*f(1_e) = f(g)^*$ hence $f(g^*) = f(g)^*$. One can then define $g = f^{-1}$ which is a frame homomorphism and compatible with multiplication and involution, and $1 \leqslant g(1)$ because each unit is sent by $f$ to

13. Of course $f(xy)$ denote the direct image by $f$ of the set $xy$.

a unit, hence it is a morphism of modular quantales. These two constructions are clearly compatible with compositions and inverse from each other, hence, together with propositions 3.2.3 and 3.2.4, this concludes the proof of the theorem.
□

3.2.6. Finally we investigate the case of Grothendieck quantale:

**Proposition :** *Let $Q$ be an atomic modular quantale, and $G$ be the corresponding hypergroup. The following conditions are equivalent:*

1. *$Q$ is a Grothendieck quantale*
2. *for every arrow $f$ in $G$ there exists two arrows $u$ and $v$ such that $f = uv^*$ with $uu^*$ and $vv^*$ units.*

**Proof :**
If $Q$ is a Grothendieck quantale, then (by $(Q10)$) any element of $Q$ can be written as a supremum of elements of the form $uv^*$ with $uu^* \leqslant 1$ and $vv^* \leqslant 1$, as $Q$ is atomic we can write these $u$ and $v$ as union of atoms and hence any elements of $Q$ can be written as a supremum of elements of the form $uv^*$ where $uu^*$ and $vv^*$ are units. In particular, any $f \in G$ is an atom of $Q$, and hence should be of the form $uv^*$.

Conversely, if $G$ satisfies condition (2) then any element of $Q$ can be written as a union of its atoms, which are all of the form $uv^*$ with $uu^*$ and $vv^*$ units (and hence $\leqslant 1$).
□

**Definition :** *An element $f$ of a hypergroupoid such that $ff^*$ is a unit is called a simple element. An element which can be written in the form $fg^*$ with $f$ and $g$ simple is called a semi-simple element. A hypergroupoid satisfying the condition of the proposition, i.e. such that every element is semi-simple, will be called a semi-simple hypergroupoid.*

Hence an atomic Grothendieck quantale is essentially the same thing as a semi-simple hypergroupoid.
This terminology is borrowed to P.J.Freyd and A.Scedrov (in [26]) and has, as far as we know, nothing to do with the notion of simplicity and semi-simplicity in representation theory.

## 3.3 Hypergroupoid algebra

3.3.1. In this section we consider an atomic topos $\mathcal{T}$, an arbitrary object $X$, the quantale $Q$ of relations on $X$ and the corresponding hypergroupoid $G$. We assume that $G$ (the set of all arrow of $G$) is decidable. This implies that $X$ is a decidable object, indeed:

**Lemma :** *Let $G$ be a decidable hypergroupoid, then its set of units is complemented. And if $G$ is associated to an object $X$ of an atomic topos then $X$ is decidable.*

Of course, if we assume the law of excluded middle in $\mathcal{S}$ this lemma is completely useless.

**Proof :**
Let

$$\Delta^c = \{g \in G | \forall e \in E(G), e \neq x\}.$$

We will prove that $\Delta^c$ is a complement of $E(G)$. They are disjoint and for every $g \in G$ there exists a unique $e$ such that $eg = g$. As $G$ is decidable, either $g = e$ or $g \neq e$. If $g = e$ then $g \in E(G)$. If $g \neq e$ then for all $e'$, one has $e' = g \Rightarrow e' = e$ because of the uniqueness of $e$, and hence $e' = g$ yields a contradiction, so $g \in \Delta^c$.
In particular, as $\mathcal{P}(G)$ is isomorphic to $\mathsf{Sub}(X \times X)$, and as the diagonal subobject of $X$ corresponds to the set of units of $G$, this proves that $X$ is decidable.
$\square$

In this situation, the convolution product defined in section 2.8 gives a convolution on functions on $G$ with value in $\overrightarrow{\mathbb{R}_+^{\infty}}$. We will give necessary and sufficient conditions in order that the convolution product induces an interesting multiplication on some algebra.
As, in this situation, the convolution product depends on $\mathcal{T}$ and not only on $G$, these conditions will be expressed in terms of the logic of $\mathcal{T}$. In the next section we will focus on the case of a semi-simple hypergroupoid, in this situation the topos $\mathcal{T}$ will be canonically determined and it will be possible to reformulate the definition given here more explicitly in terms of $G$.

3.3.2. We will need a few generalities about cardinals of sets in a constructive setting in order to be able to give an internally valid proof of the main results of this section.

**Definition :** *Let $X$ be a decidable set, then the cardinal of $X$ is defined by:*

$$|X| = \left( \sum_{x \in X} 1 \right) \in \overrightarrow{\mathbb{R}_+^{\infty}}$$

We remind the reader that $\overrightarrow{\mathbb{R}_+^{\infty}}$ contains an element $+\infty$. The following lemma gives two properties that completely characterize the cardinal of a set.

**Lemma :**

- *For $n \in \mathbb{N}$ one has $n \leqslant |X|$ if and only if there exists $x_1, \cdots x_n$ pairwise distinct elements of $X$.*
- *For $q \in \mathbb{Q}$, $q < |X|$ if and only if there exists an $n \in \mathbb{N}$ such that $q < n \leqslant |X|$.*

**Proof :**
Let $q \in \mathbb{Q}$, such that $q < |X|$. By definition, there exists $x_1, \ldots, x_n \in X$ pairwise distinct such that $\exists q_1, \ldots, q_n < 1$ pairwise distinct with $q < \sum q_i$. This can be rewritten as $\exists x_1, \ldots, x_n \in X$ such that $q < n$. This proves the second point of the lemma assuming the first.
If there are $n$ distinct elements in $X$, any $q < n$ is also smaller than $|X|$ hence $n \leqslant |X|$. Conversely, if $n \leqslant |X|$ then $(n - \frac{1}{2}) < |X|$, so, by the first half of the proof, there is an integer $m$ with $(n - \frac{1}{2}) < m$ and $x_1, \ldots x_m$ distinct element in $X$. As $n \leqslant m$ one also has $n$ distinct elements, this concludes the proof of the first point of the lemma. $\square$

**Proposition :** *If $X$ is a decidable set, the following conditions are equivalent:*

1. *$X$ is finite.*
2. *$|X|$ is an integer.*
3. *$|X|$ is a (finite) continuous real number (ie an element of $\mathbb{R}_{\mathcal{T}}$).*

For an example of a set $X$ with $|X| < \infty$ but not satisfying these properties, one can take any non-complemented sub-set of a finite (decidable) set.

**Proof :**
1. $\Rightarrow$ 2. is clear because a finite decidable set is isomorphic to $\{1, \ldots, n\}$ (see the notations and conventions section of the thesis) and hence as cardinal $n$.
2. $\Rightarrow$ 3. is also clear.
Assume 3., then there exist $q, q'$ such that $|q - q'| < \frac{1}{2}$ and $q < |X| < q'$. There exists an integer $n$ such that $q < n \leqslant |X| < q'$, and $x_1, \ldots, x_n$ pairwise distinct elements of $X$. Let $x \in X$ then there are two possible cases: either $x = x_i$ for some $i$, or $x$ is distinct from all the other $x_i$ (this is proved recursively on $n$ using the decidability of $X$). But if $x$ is distinct from all the $x_i$ then $(n+1) \leqslant |X|$ and $q \leqslant n < (n+1) \leqslant q$ which yields a contradiction. So $x = x_i$ for some $i$, and $X$ is indeed finite.
$\square$

We also note that the same argument yields the following result: If $X$ is decidable, and we have a function p:$X \to \mathbb{N}^{>0}$ such that $\sum_{x \in X} p(x)$ is an integer, then $X$ is finite.

3.3.3. It might be useful at this point to recall our conventions. In order to have the composition of functions and of relations in the usual order and to have a nice "matrix like" composition of relation in the form $x(R \circ P)y :=$ "$\exists t, xRt$ and $tPy$" compatible with the convolution product of 2.8, we are lead to define a relation $R$ from $X$ to $Y$ when it is a subset of $Y \times X$, and to think of a function $f : X \to Y$ as the relation $\{f(x), x)\}$. In particular, if $g \in G(e, e')$ is an arrow in a hypergroupoid $G$ represented on an object $X$ of a topos, then $g$ will be consider as a sub-object of $e' \times e$ (and $e'$ and $e$ as two sub-object of $X$).

3.3.4. Let $g$ and $g'$ be two arrows in $G$, and $[g]$,$[g']$ the characteristic functions of the singletons $\{g\}$ and $\{g'\}$ the the convolution product $*$ of 2.8 can be computed internally as:

$$([g]*[g'])(x,y) = \sum_{z \in X} [g](x,z)[g'](z,y) = |\{z | xgz \text{ and } zg'y\}|$$

**Definition :** *Let* $\binom{g,g'}{a}$ *denote the evaluation at* $a$ *of the function* $[g]*[g']$. *We also define for* $g \in G(e,e')$,

$$|g|_l = \binom{g^*,g}{e} \quad |g|_r = \binom{g,g^*}{e'}$$

**Proposition :** $\binom{g,g'}{a}$, $|g|_l$ *and* $|g|_r$ *can be computed internally using the formulas:*
*For any* $u \in e$ *the source unit of* $g$:

$$|g|_l = |\{z | zgu\}|.$$

*For any* $v \in e'$ *the target unit of* $g$:

$$|g|_r = |\{z | vgz\}|$$

*For any* $(x,y) \in a$:

$$\binom{g,g'}{a} = |\{t | xgt \text{ and } tg'y\}|$$

**Proof :**
The formula for $\binom{g,g'}{a}$ is essentially its definition. The two other formulas follow easily. The fact that each time the value internally does not depend on any choice of (internal) elements is clear because the various possible choices all belong to a same atom. $\square$

One also mentions the two easy (but important) relations:

$$|g^*|_l = |g|_r, \quad \binom{g,g'}{a} = \binom{g'^*,g^*}{a^*}$$

3.3.5. Here are some of the important combinatorial properties of these coefficients:

**Theorem :** *For all pair $g, g'$ of composable arrows and all $a \in gg'$ one has:*

1. $\binom{g, g'}{a} |a|_l = \binom{g^*, a}{g'} |g'|_l$
2. $\binom{g, g'}{a} |a|_r = \binom{a, g'^*}{g} |g|_r$
3. $|g|_l |g'|_l = \sum_{a \in gg'} \binom{g, g'}{a} |a|_l$

**Proof :**
let $e_1$,$e_2$ and $e_3$ be the units such that $g' \in G(e_1, e_2)$ and $g \in G(e_2, e_3)$.
Let $x \in e_1$ be arbitrary then let:

$$X_a = \{(u, v) | ugv \text{ and } vg'x \text{ and } uax\}$$

$$X = \{(u, v) | ugv \text{ and } vg'x\} = \bigsqcup_{a \in gg'} X_a.$$

The cardinality of $X_a$ can be computed in two different ways, on one side:

$$|X_a| = \sum_{u \text{ s.t. } uax} |\{v | ugv \text{ and } vg'x\} = \sum_{u \text{ s.t. } uax} \binom{g, g'}{a} = |a|_l \binom{g, g'}{a}.$$

On the other side:

$$|X_a| = \sum_{v \text{ s.t. } vg'x} |\{u | vg^*u \text{ and } uax\}| = \sum_{v \text{ s.t. } vg'x} \binom{g^*, a}{g'} = |g'|_l \binom{g^*, a}{g'}.$$

The equality of the two results gives (1.). The result (2.) is the dual (one can use a similar proof or apply $*$ everywhere).
Similarly,

$$|X| = \sum_{v \text{ s.t. } vg'x} |\{u | ugv\}| = \sum_{v, vg'x} |g|_l = |g|_l |g|_r$$

Hence 3. comes from the fact that $X$ is the disjoint union of the $X_a$.
□

3.3.6. **Definition :** *Let $X$ be an object of an atomic topos, and $G$ the hypergroupoid corresponding to the atomic modular quantale $\mathit{Rel}(X)$. We say that $(X, G)$ is locally finite if for all arrows $g : e \to e'$ in $G$ one has internally $\forall x \in e, \{y | ygx\}$ is finite.*

**Lemma :** *If $(X,G)$ is locally finite, then for any two composable arrows $a,b \in G$ the set $ab$ is finite.*

This lemma holds without any decidability assumption on $X$ or $G$, but the hypothesis "composable" is necessary only if one does not assume that the set of units is decidable.

**Proof :**
Let $a : e' \to e$ and $b : e'' \to e'$ two composable arrows. Elements of $ab$ are exactly the arrows $f : e'' \to e$ is the set of atoms of:

$$X_0 = \{(x,y) \in e \times e'' | \exists t \in e', xat \text{ and } tby\}.$$

For each $y \in e''$, the set of $(x,t)$ such that $xat$ and $tby$ is finite because the set of $t$ such that $tby$ is finite and for each $t$ the set of $x$ such that $xat$ is also finite. In particular, $X_0$ can be seen as a finite object in the topos $\mathcal{T}_{/e''}$, and this implies that the set of atoms of $X_0$ (ie $ab$) is finite.
Indeed, the finiteness of $X_0$ in $\mathcal{T}_{/e''}$ implies that the map $\mathcal{T}_{/X_0} \to \mathcal{T}_{/e''}$ is a proper map [14]. As $e''$ is an atom, the topos $\mathcal{T}_{/e''}$ is hyperconnected and hence compact. In particular, by composition of proper maps, the topos $\mathcal{T}_{/X_0}$ is compact. This implies that the localic reflection of $\mathcal{T}_{X_0}$ is also compact and this exactly means that the set of atoms of $X_0$ (i.e. $ab$ ) is finite. $\square$

3.3.7. **Theorem :** *Let $G$ be a decidable hypergroupoid represented in a topos i.e. corresponding to the modular atomic quantale $\mathsf{Rel}(X)$ for some object $X$ of a topos. The following propositions are equivalent:*

- *$(X,G)$ is locally finite.*
- *for all $g \in G, g : e' \to e$ the value of $[g] * [g^*]$ at $e$ is a finite continuous real number.*
- *for all $g,g' \in G$ the set $gg'$ is a finite set and for all $a \in gg'$ $\begin{pmatrix} g,g' \\ a \end{pmatrix}$ is a finite integer.*

*Moreover in this situation, one has a formula of the form:*

$$[g] * [g'] = \sum_{a \in gg'} \begin{pmatrix} g,g' \\ a \end{pmatrix} [a]$$

**Proof :**
If $g$ is an arrow in $G$ from $e'$ to $e$ The value of $[g] * [g^*]$ at $e$ is exactly the internal value of the cardinal of $\{y|xgy\}$ for any $x \in e'$, hence it is a continuous real number if and only if this set is internally finite, and the first two conditions are clearly equivalent.
The third condition clearly implies the second because the value of $[g] * [g^*]$ at $e$ is by definition $\begin{pmatrix} g,g^* \\ e \end{pmatrix}$.

14. The notion of proper map is defined in section one of [55], in addition of this definition, we use the propositions 1.4,2.1,2.4 and 2.5

We assume the first two conditions. In particular the set $\{x|xgy\}$ is a finite subset of a decidable set and hence is complemented in $X$, hence $\{z|xgz$ and $zg'y\}$ is finite as a complemented subset of a finite set.
This proves that the evaluation of $[g] * [g']$ at every point is indeed a positive integer (it is the cardinal of the previous set). The fact that $gg'$ is finite is exactly the lemma of 3.3.6.
□

The situation described in this theorem is basically the best we can hope: one gets a $\mathbb{Z}$-algebra generated (as a group) by the symbols $[g]$ for $g \in G$, the multiplication being given by the third item of the previous theorem. We will call this algebra $A_G$.

Note that, as a set of functions on $G$, $A_G$ is exactly the set of finitely supported functions on $G$ with integral value, and our result is that this subset set is stable by the convolution product of 2.8.

But conversely, if we want to have any interesting convolution structure – coming from the construction done in 2.8 – on a set of functions on $G$ with value in continuous numbers, we need to have the second condition. This proves that the locally finite hypothesis is exactly the good hypothesis for getting an interesting convolution product in the atomic case.

## 3.4 Semi-simple Hypergroupoid algebra

In this section we assume that the object $X$ is now a bound of $\mathcal{T}$, and we call it $B$ following our conventions. The hypergroupoid $G$ is now semi-simple, and $\mathcal{T}$ is fully determined by $G$, so we should be able to express the value of $\begin{pmatrix} g, g' \\ a \end{pmatrix}$ in terms of the structure of $G$. We still assume that $G$ is decidable.

3.4.1. **Proposition :** *In a semi-simple decidable hypergroupoid one has:*

$$\begin{pmatrix} g, g' \\ a \end{pmatrix} = \sup_{\substack{a = xy^* \\ x,y \text{ simples}}} |g^* x \wedge g' y|$$

The sup in the proposition is taken in $\overrightarrow{\mathbb{R}_+^\infty}$, this means in particular that the coefficient is an integer if and only if the supremum is reached.

**Proof :**
Let $q < \begin{pmatrix} g, g' \\ a \end{pmatrix}$. By lemma 3.3.2 and the observation in 3.3.4 that this coefficient is an internal cardinal there exists (at least internally) an $n$ such that $q < n \leqslant \begin{pmatrix} g, g' \\ a \end{pmatrix}$. This means that internally for every $(x, y) \in a$, there exists $(v_1, \dots v_n)$ pairwise distinct in $B$ such that for all $i$, $(xgv_i)$ and $(v_i g' y)$.

This means that there is a surjection $(x,y) : t \twoheadrightarrow a$ and a collection of $n$ maps $v_1, \dots v_n$ from $t$ to $B$ pairwise distinct [15], such that for all $i$, $(x, v_i)$ has value in $g$ and $(v_i, y)$ has value in $g'$.
If we choose any atom on $B$ which maps to $t$, the composite is still a surjection on $a$. Hence we can freely assume that $t$ is a unit of $G$, and that $x, y, v_1, \dots v_n$ are arrows in $G$. The fact that $(x, v_i)$ takes values in $g$ and that $(v_i, y)$ takes values in $g'$ translate into $v_i \in g'y$ and $x \in gv_i$, i.e. for all $i$ from 1 to $n$, $v_i \in g'y \wedge g^*x$. Moreover as $(x,y)$ is a surjection from $t$ to $a$ one has $a = xy^*$, so:

$$q < n \leqslant \sup_{a=xy^*} |g'y \wedge g^*x|.$$

Conversely, if $q < \sup_{a=xy^*} |g'y \wedge g^*x|$, then for some $x, y$ simple such that $a = xy^*$,

$$q < n \leqslant |g'y \wedge g^*x|.$$

So there exist $v_1, \dots, v_n$ pairwise distinct in $g'y \wedge g^*x$. If $n > 0$, this implies that $g, g'$ is composable (if $n = 0$, then $q$ is smaller than any cardinal). Let $e$ be the target of $g$ and the source of $g'$.
In this situation, for any $u \in e$, $v_1(u) \dots v_n(u)$ are $n$ pairwise distinct elements in $\{z | x(u) g z \text{ and } z g' y(u)\}$ and hence $q < n \leqslant \begin{pmatrix} g, g' \\ a \end{pmatrix}$.
And this concludes the proof. $\square$

3.4.2. **Proposition :** *An element $g \in G$ is simple if and only if $|g|_l = 1$.*

**Proof :**
Let $g \in G(e, e')$. Internally one has by proposition 3.3.4:

$$\forall x \in e, |g|_l = |\{z | zgx\}|$$

Hence $|g|_l = 1$ exactly means that for all $x$ there exists a unique $z$ such that $zgx$ which means that $g$ is a partial function, i.e. a simple element. $\square$

3.4.3. **Proposition :** *Let $G$ be a semi-simple decidable hypergroupoid, then $g \in G$ is left finite if and only if there exists a simple arrow $u$ such that $(g, u)$ is composable, $gu$ is finite and contains only simple elements. Moreover in this case $|g|_l$ is the cardinal of the set $(gu)$.*

**Proof :**

15. As we will soon assume that $t$ is an atom the precise meaning of "distinct" is not important.

Translating into an external result the internal formulation of the left finiteness of $g$ would actually give us exactly the statement of this theorem, but this translation requires some work (essentially done in the proof 3.4.1) which can be avoided by the use of the combinatorial identities we already proved.
Assume first that $gu$ is finite and contains only simple elements $\{x_1, \dots, x_n\}$. then by the formula 3. of 3.3.5 and replacing by 1 the left cardinal of simple elements, one gets that:

$$|g|_l = \sum_{i=1}^{n} \binom{g, u}{x_i}$$

But one can see on the formula given in the proposition 3.3.4 that $\binom{g, u}{x_i} \leqslant |u|_l = 1$, hence $\binom{g, u}{x_i} = 1$ and $|g|_l = n$ which implies that $g$ is left finite.
Conversely, assume that $|g|_l = n$ for some $n$. By 3.4.1 one has:

$$|g|_l = \sup_{u \text{ simple}} |gu|$$

In particular as $(n - 1/2) < |g|_l$, there exists $u$ simple such that $|gu| = n$. one has then:

$$|g|_l |u|_l = \sum_{x \in gu} \binom{g, u}{x} |x|_l$$

$$n = \sum_{i=1}^{n} \binom{g, u}{x_i} |x_i|_l$$

This implies first that all $\binom{g, u}{x_i} |x_i|_l$ have an opposite, hence they are all continuous numbers, and hence integers. Moreover as all the $\binom{g, u}{x_i} |x_i|_l$ are $\geqslant 1$ they have to be all equal to 1, hence all the $x_i$ are simple and this concludes the proof.
□

## 3.5 The category of $\mathcal{T}$-groups

In this section, we will show that in the locally finite case, the algebra $A_G$ we obtained from theorem 3.3.7 can be seen as a particular subalgebra of endomorphisms of the free group $\mathbb{Z}X$ generated by $X$ in the logic of $\mathcal{T}$. This gives an abstract interpretation of the algebra $A_G$. We still assume that $X$ is decidable. We will also show that in the semi-simple case (when $X = B$ is a bound) the category of abelian groups of $\mathcal{T}$ embeds as a full subcategory of the category of $A_G$-modules, and that this embedding induces an equivalence between $\mathbb{Q}_{\mathcal{T}}$-vector spaces in $\mathcal{T}$ and full $A_G \otimes \mathbb{Q}$-modules.

3.5.1. Let $E = \mathbb{Z}X$ be the free group generated by $X$ in $\mathcal{T}$ ($E$ is a group object of $\mathcal{T}$). we denote by $(e_x)_{x \in X}$ the generators of $E$. In particular $E \otimes E = \mathbb{Z}(X \times X)$ is the free group generated by $X \times X$, hence as $X$ is assumed to be decidable one can define a bi-linear map $\Delta : E \times E \to \mathbb{Z}_{\mathcal{T}}$ which sends $(e_x, e_{x'})$ to 1 if $x = x'$ and to 0 if $x \neq x'$.
Let $f$ be an endomorphism of $E$. One can associate to $E$ the function of "matrix elements" of $f$, $\rho(f) : (x, x') \mapsto \Delta(e_x, f(e_{x'}))$ from $X \times X$ to $\mathbb{Z}$. The map $\rho : f \mapsto \rho(f)$ is injective, and one has internally of $x \in X$:

$$f(e_x) = \sum_{x' \in X} \rho(f)(x', x) e_{x'}$$

Also, $\rho(f \circ f') = \rho(f) * \rho(f')$ for the convolution of functions on $X \times X$. So we just have to understand the image of $\rho$:

**Proposition :** *A function $f$ from $G$ to $\mathbb{Z}$ belongs to the image of $\rho$ if and only if it verifies the following two properties:*

1. *If $f(g) \neq 0$ then $g$ is left finite.*
2. *For each unit $e$ of $G$, there is at most a finite number of arrows $g \in G$ pointing to $e$ such that $f(g)$ is non zero.*

In particular, property (1.) tells us that if we are not in the locally finite case, then the algebra of group homomorphisms of $E$ is in some sense 'too small'.
Also, in the locally finite case, the algebra $A_G$ is identified with the sub-algebra of endomorphisms of $E$ such that $f(g)$ is non zero only for a finite number of $g \in G$.

**Proof :**
Internally, a function $f : X \times X \to \mathbb{Z}_{\mathcal{T}}$ corresponds to a group homomorphism if and only if (internally) for all $x \in X$ there is only a finite set of $x'$ such that $f(x', x)$ is non zero. Indeed, the corresponding group homomorphism has to send $e_x$ to $\sum_{x'} f(x', x).e_{x'}$.
The cardinality of the set of $x'$ such that $f(x', x)$ is non zero defines a function $\tau$ on $X$, whose value at any atom $e$ of $X$ (ie at any unit of $G$) is given by:

$$\tau(e) = \sum_{g \in G(e', e), f(g) \neq 0} |g|_l$$

Indeed, for any $x \in X$ the set of $x'$ such that $f(x', x)$ is non zero is partitioned by the various $g \in G(e', e)$ such that $(x', x) \in g$ and each of this set has cardinality $|g|_l$ (because the value of $f(x', x)$ only depends on the atom that contains it). So $\tau(e)$ is an integer if and only if each of the $|g|_l$ are integers (this is condition 1.) and if they arise in finite number (this is condition 2.).
$\square$

In the remainder of this chapter we assume that the representation of $G$ in $\mathcal{T}$ is locally finite (as defined in 3.3.6).

3.5.2. Let $F$ be any abelian group object of $\mathcal{T}$. Then $\hom(E,F)$ is a right $\hom(E,E)$ module (where the hom denotes the internal group homomorphisms). The units of $G$ act as family of disjoint projections on $E$, and hence on $\hom(E,F)$. Let:

$$\tilde{F} = \bigoplus_{e \in E(G)} Hom(E,F).e$$

equivalently, $\tilde{F}$ is the subset of $\hom(E,F)$ of elements $x$ such that there exists a finite set $I$ of units such that $x.e_I = x$ where $e_I = \sum_{e \in I} e$.

**Proposition :** *For all abelian group $F$ of $\mathcal{T}$, $\tilde{F}$ is a full*[16] *right $A_G$-module. This gives a functor from group objects of $\mathcal{T}$ to full right $A_G$-modules. Also $\tilde{E}$ is $A_G$ seen as a right $A_G$-module.*

**Proof :**
Clear from the observation that $\tilde{F}$ is the subset of $\hom(E,F)$ of elements $x$ such that there exists a finite set $I$ of units such that $x.e_I = x$ where $e_I = \sum_{e \in I} e \in A_Q$. And $\tilde{E}$ identify with $A_Q$ thanks to 3.5.1 $\square$

Actually, $\tilde{F} = \hom(E,F).A_Q$.

3.5.3. Assume that $G$ is semi-simple and that $X = B$ is a bound of $\mathcal{T}$, hence, the category of atoms of $B$ and morphisms between them (i.e. the category of units of $G$ and simple arrows between them) endowed with the atomic topology is a site of definition of $\mathcal{T}$.
If $F$ is an abelian group object of $\mathcal{T}$ then for each $e$ atom of $B$, $F(e) = \tilde{F}.e$ and the action of a simple arrow $f$ from $e$ to $e'$ is given by the action of $[f]$ on $\tilde{F}$. In particular, the sheaf corresponding to $F$ is fully determined by $\tilde{F}$ and any $A_Q$-linear morphism from $\tilde{F}$ to $\tilde{F'}$ gives rise to a morphism of sheaves and one can conclude that:

**Lemma :** *When $G$ is semi-simple (and locally finite) the functor from abelian $\mathcal{T}$-groups to $A_G$-modules defined in 3.5.2 is fully faithful.*

Unfortunately, if we start from a general $A_G$-module we only get a pre-sheaf over the site of units. In the general case, we have not found a characterization of the $A_G$-modules corresponding to $\mathcal{T}$-group simpler than the definition of a sheaf. But in the case where we assume that all the coefficients $|g|_l$ are invertible, then the action of the $[g]$ for non simple $g$ will automatically turn our pre-sheaf into a sheaf.

**Proposition :** *(Still under the assumption that $G$ is semi-simple and locally finite) Let $M$ be an $A_G$-module such that for every $g \in G$ the integer $|g|_l$ acts (by multiplication) as a bijection on $M$. Then $M$ comes from an abelian $\mathcal{T}$-group. In particular there is an equivalence of categories between $\mathbb{Q}_{\mathcal{T}}$-vector spaces and full right $A_G \otimes \mathbb{Q}$-modules.*

16. A module $M$ over a (possibly non unitary) ring $A$ is said to be full if the map $A \times M \to M$ is surjective.

**Proof :**
We will check that under this assumption the pre-sheaf of $\tilde{M}.e$ is actually a sheaf for the atomic topology.
Let $e$ be a unit of $G$, let $f$ be any simple arrow starting at $e$. And let $m \in M.e$ such that for any two simple arrows $g, h$ targeting $e$ such that $fg = fh$ one has $mg = mh$. We need to prove that there exists a unique $n$ such that $nf = m$.
The uniqueness is easy: if $m = n.f$ then $m.f^* = n.ff^* = |f|_r n$ so as $|f|_r$ is invertible one has $n = \frac{1}{|f|_r} m.f^*$.
Conversely, we will prove that $n = \frac{1}{|f|_r} m.f^*$ provides a solution.

$$n.f = \frac{1}{|f|_r} m.f^* f = \frac{1}{|f|_r} \sum_{a \in f^* f} \binom{f^*, f}{a} m.a$$

let $a \in f^* f$. Then $a$ can be written in the form $a = gh^*$ with $g$ and $h$ simple.

$$gh^* \in f^* f \Rightarrow f \in fgh^* \Rightarrow fg \in fh$$

Hence one has $fg = fh$ and by the assumption on $m$, $m.h = m.g$.
The relation $m.g = m.h$ implies,

$$m.[g][h^*] = m.[h][h^*]$$

$$[h][h^*] = |h|_r.[e]$$

By 3.3.5 one has:

$$[g][h^*] = \frac{|h|_r}{|a|_r} [a]$$

so $m.[g][h^*] = m.[h][h^*]$ becomes:

$$m.a = |a|_r.m$$

and we can conclude that:

$$n.f = \frac{1}{|f|_r} \sum_{a \in f^* f} \binom{f^*, f}{a} |a|_r m = |f^*|_r m = m$$

again by 3.3.5 and the fact that $f$ is simple hence $|f^*|_r = |f|_l = 1$.
□

## 3.6 $A_G$ as a quantum dynamical system

In this section we construct the regular representation of $A_G$. We show that the $C^*$-algebra generated by $A_G$ comes with a canonical action of $\mathbb{R}$. There is also a regular representation of $A_G$, attached to a $KMS_1$ state and defining a $C^*$-algebra $\mathcal{C}^*_{\text{red}}(G)$ by completion.

3.6.1. Let $\mathcal{H}$ be a real Hilbert space in $\mathcal{T}$. Then

$$\tilde{\mathcal{H}} = \bigoplus_{e \in E(G)} \hom(e, \mathcal{H})$$

is an $A_G$ vector space. The internal scalar product on $\mathcal{H}$ gives rise to a scalar product on each of the $\hom(e, \mathcal{H})$ and turns $\tilde{\mathcal{H}}$ into a pre-Hilbert space.

**Proposition :** *In the action of $A_G$ on $\tilde{\mathcal{H}}$ one has:*

$$[g]^* = \frac{|g|_l}{|g|_r}[g^*]$$

*And $[g]$ has norm smaller than $|g|_l$.*

**Proof :**
Let $g \in G$, $v \in \hom(e, \mathcal{H})$ and $v' \in \hom(e', \mathcal{H})$.
If $g$ is not an arrow from $e$ to $e'$, then both $\langle v, v'[g] \rangle$ and $\langle v[g^*], v' \rangle$ are zero (hence equal). If $g$ is an arrow from $e$ to $e'$, then for any $x \in e$ one has:

$$\langle v, v'[g] \rangle (x) = \left\langle v(x), \sum_{y, ygx} v'(y) \right\rangle = \sum_{y, ygx} \langle v(x), v'(y) \rangle$$

But $g$ is an atom of $X \times X$, and $\langle v(x), v'(y) \rangle$ is a function on $X \times X$ so its value does not depend on $x$,$y$ as long as they belong to $g$. Hence, for any $(x, y) \in g$ one has:

$$\langle v, v'[g] \rangle = |g|_l \langle v(x), v'(y) \rangle$$

Similarly:

$$\langle v[g^*], v' \rangle = |g|_r \langle v(x), v'(y) \rangle$$

Finally:

$$\langle v, v'[g] \rangle = \frac{|g|_l}{|g|_r} \langle v[g^*], v' \rangle$$

And the first result follows.
The second result follows from

$$\langle v, v'[g] \rangle = |g|_l . \langle v(x), v'(y) \rangle \leqslant |g|_l \|v\| \|v'\|.$$

As $\|v(x)\| = \|v\|$.
□

3.6.2. **Definition :** *For $g \in G$, We will denote:*

$$\chi(g) = \frac{|g|_l}{|g|_r}$$

One has:

$$\chi(g^*) = \chi(g)^{-1}$$
$$[g]^* = \chi(g)[g^*]$$

and also the more surprising result:

**Proposition :** *For any three arrows $a, g, g'$ of $G$ such that $a \in gg'$:*

$$\chi(a) = \chi(g)\chi(g')$$

**Proof :**
We will just need several applications of the first two points of theorem 3.3.5.

$$\chi(a) = \frac{|a|_l}{|a|_r} = \frac{|a|_l \begin{pmatrix} g, g' \\ a \end{pmatrix}}{|a|_r \begin{pmatrix} g, g' \\ a \end{pmatrix}} = \frac{|g'|_l}{|g|_r} \frac{\begin{pmatrix} g^*, a \\ g' \end{pmatrix}}{\begin{pmatrix} a, g'^* \\ g \end{pmatrix}}$$

but:

$$\frac{|g'|_r}{|g|_l} \frac{\begin{pmatrix} g^*, a \\ g' \end{pmatrix}}{\begin{pmatrix} a, g'^* \\ g \end{pmatrix}} = \frac{|g|_l \begin{pmatrix} g', a^* \\ g^* \end{pmatrix}}{|g'|_r \begin{pmatrix} a^*, g \\ g'^* \end{pmatrix}} = \frac{|g|_l \begin{pmatrix} a, g'^* \\ g \end{pmatrix}}{|g'^*|_l \begin{pmatrix} a^*, g \\ g'^* \end{pmatrix}} = 1$$

So we can conclude that:

$$\chi(a) = \frac{|g'|_l}{|g|_r} \frac{\begin{pmatrix} g^*, a \\ g' \end{pmatrix}}{\begin{pmatrix} a, g'^* \\ g \end{pmatrix}} = \frac{|g'|_l}{|g|_r} \frac{|g|_l}{|g'|_r} = \chi(g)\chi(g')$$

□

3.6.3. **Definition :** *Let $A_{G,\mathbb{Q}}$ be the algebra $A_G \otimes \mathbb{Q}$. It is endowed with the involution $\_^*$ defined by 3.6.1, i.e.:*

$$[g]^* = \chi(g)[g^*]$$

*We also define the elements:*

$$e_g = \frac{1}{|g|_l}[g] \in A_{G,\mathbb{Q}}$$

*which are additive generators such that $(e_g)^* = e_{g^*}$.*

**Proposition :** *If $G$ is semi-simple and locally finite, the functor which sends a $\mathcal{T}$-Hilbert space $\mathcal{H}$ to the completion of $\tilde{H}$ is (one half of) an equivalence of categories between the category of internal Hilbert spaces of $\mathcal{T}$, and the full*[17] *right Hilbert $*$-representations of $A_{G,\mathbb{Q}}$.*

**Proof :**
The proof is really similar to the case of $\mathbb{Q}_{\mathcal{T}}$-vector spaces done in 3.5.2 and 3.5.3. If we start from a $\mathcal{T}$-vector space, we already proved that $A_{G,\mathbb{Q}}$ acts on $\tilde{H}$ by bounded morphisms and in a way compatible with the involution. So it extends to a full Hilbert $*$ representation of $A_{G,\mathbb{Q}}$ on the completion of $H$.

In the other direction, the sheaf of complex numbers on the site of units of $G$ is the constant sheaf. Hence if $\mathcal{H}$ is a Hilbert $*$ representation of $A_{G,\mathbb{Q}}$ then $e \to \mathcal{H}[e]$ defines a pre-sheaf of $\mathbb{C}_{\mathcal{T}}$-modules that we will denote by $H$. The pre-sheaf $H$ is a sheaf by 3.5.3.
For every simple arrow $f : e \to e'$ one has:

$$[f].[f]^* = [f][f^*]\frac{1}{|f|_r} = [e']\frac{|f|_r}{|f|_r} = [e']$$

and the induced map $\mathcal{H}[e'] \to \mathcal{H}[e]$ (i.e. the structural map of $H$) is an isometric injection. This proves that the scalar product $\mathcal{H}[e] \times \mathcal{H}[e] \to \mathbb{C}$ is in fact a morphism of sheaves $H \times H \to \mathbb{C}_{\mathcal{T}}$ and hence this endows $H$ with an internal scalar product.
It remains to show that $H$ is internally complete. Let $\tilde{H}$ be its completion, let $h \in \tilde{H}(e)$.
Then, by (internal) density of $H$ in $\tilde{H}$, for every $n \in \mathbb{N}$, there exists $f : e' \to e$, and $h' \in H(e')$ such that $\|h' - h.f\| < 1/n$. But one can write:

$$\|h'[f]^* - h\| < 1/n$$

and $h'[f]^* \in H(e)$, hence $h$ can be approximated by elements of $H(e)$. As $H(e)$ is complete, this proves that $h \in H(e)$ and hence that $H$ is internally complete.
Finally this is an equivalence, because if we start from a full Hilbert $*$-representation $H$ of $A_{G,\mathbb{Q}}$ then the construction we just made corresponds to that of 3.5.3 applied to $H.A_{G,\mathbb{Q}}$ hence as we applied a completion at the end, we will get $H$ back because $H.A_{G,\mathbb{Q}}$ is dense in $H$ by assumption.
□

17. Here full, mean that $\mathcal{H}.A_{G,\mathbb{Q}}$ is dense in $\mathcal{H}$.

3.6.4. At this point we can either consider the closure of $A_{G,\mathbb{C}} = A_G \otimes \mathbb{C}$ in a specific representation: the one corresponding to the internal Hilbert space $l^2(B)$ of square summable functions on $B$, defining a $C^*$-algebra $\mathcal{C}^*_{\mathrm{red}}(G)$, or we can take the universal $C^*$-algebra generated by $A_{G,\mathbb{Q}}$ that we will denote by $\mathcal{C}^*_{\max}(G)$.
Both these algebras come with a time evolution $(\sigma_t)_{t\in\mathbb{R}}$ given by:

$$\sigma_t([g]) = \chi(g)^{it}[g]$$

This is a morphism of algebras because of 3.6.2.
Let $e$ be an atom of $B$. Then one has a map $e \hookrightarrow B$ which gives rise to a map $e \to l^2(B)$ and hence to a vector of the corresponding representation of $\mathcal{C}^*_{\mathrm{red}}(G)$ that we will simply denote $l^2$. An easy computation shows that the state on $\mathcal{C}^*_{\mathrm{red}}$ induced by this vector is:

$$\eta_e([q]) = \begin{cases} 1 \text{ if } q = e \\ 0 \text{ otherwise.} \end{cases}$$

If the set of units of $G$ is finite then the (renormalized) sum of all the $\eta_e$ is a state, in general we can define it without renormalization as a semi-finite weight (it is finite on the algebra $A_G$), we denote it by $\eta$.

**Proposition :** *The GNS representation induced by $\eta$ is the $l^2$ representation, and $\eta$ verifies the KMS condition.*

The KMS condition is what ensure that the time evolution we constructed on $\mathcal{C}^*_{\mathrm{red}}$ is the modular time evolution of the von Neumann algebra obtained as the weak closure of $\mathcal{C}^*_{\mathrm{red}}$ in the $l^2$ representation with respect to the state/weight $\eta$. More information about this can be found in the end of chapter 2, and the reader can consult [67, Chapter VIII] for a detailed account of this theory.

**Proof :**
The first part is clear: the GNS representation induced by $\eta$ is included in $l^2$ and contains all the vectors corresponding to the $e \in G$ (indeed, $[e]$ gives rise to this vector through the GNS construction). If it were a strict sub-representation then it would correspond internally to a sub Hilbert space of $l^2(B)$ containing all the basis vectors, which is impossible.
For the second part:

$$\eta([q]\sigma_i([q'])) = \chi(q')^{-1}\eta([q][q'])$$

If $q' \neq q*$ then both $\eta([q][q'])$ and $\eta([q'][q])$ are zero (because $e \in qq' \Rightarrow q' = q^*$).
If $q' = q^*$ then

$$\eta([q]\sigma_i([q']) = \chi(q)\eta([q][q^*]) = \chi(q)\begin{pmatrix} q, q^* \\ e \end{pmatrix} = \chi(q)|q|_r = |q|_l = \eta([q'][q])$$

$\square$

## 3.7 The time evolution of an atomic locally separated topos.

In this subsection, we will first show that for a decidable bound $B$ of an atomic topos $\mathcal{T}$, the hypergroupoid of atoms of $B \times B$ is locally finite if and only if the slice topos $\mathcal{T}_{/B}$ is separated (or Hausdorff) in the sense of [55]. Then we will show that an atomic topos admits such a bound if and only if it is locally decidable and "locally separated", that is if there exists an inhabited object $X$ of $\mathcal{T}$ such that $\mathcal{T}_{/X}$ is separated. And finally, that in this case the time evolution constructed in 3.6 is completely canonical when seen as a family of functors on the category of Hilbert space of $\mathcal{T}$ and is described by a canonical principal $\mathbb{Q}_+^*$ bundle $\chi_{\mathcal{T}} : \mathcal{T} \to B\mathbb{Q}_+^*$ attached to every locally separated locally decidable atomic topos $\mathcal{T}$.

3.7.1. A geometric morphism $f : \mathcal{T} \to \mathcal{E}$ is said to be *proper* if $f_*(\Omega_{\mathcal{T}})$ is a compact locale internally in $\mathcal{E}$, and is said to be *separated* if the diagonal map $\mathcal{T} \to \mathcal{T} \times_{\mathcal{E}} \mathcal{T}$ is proper. A topos is said to be compact (resp. separated) if the geometric morphism from $\mathcal{T}$ to the base topos $\mathcal{S}$ is proper (resp. separated). These notions have been introduced and studied by I.Moerdijk and C.C. Vermeulen in [55], they are also discussed in [44, C3.2 and C5.1].

We will say that a topos is *locally separated* if there exists an inhabited object $X$ of $\mathcal{T}$ such that the slice topos $\mathcal{T}_{/X}$ is separated.

3.7.2. We start by the following proposition which relates finiteness conditions to the separation property.

**Proposition :** *An atomic locally decidable topos $\mathcal{T}$ is separated if and only if every atom of $\mathcal{T}$ is internally finite, and the set of atoms of the terminal object $1_{\mathcal{T}}$ is decidable.*

Also, the "only if" part holds without assuming that $\mathcal{T}$ is locally decidable.

**Proof :**
We start by assuming that $\mathcal{T}$ is separated, and that $a \in |\mathcal{T}|$ is an atom. Then the topos $\mathcal{T}_{/a}$ is hyperconnected [18] and hence proper. Proposition II.2.1(iv) of [55] asserts that when one has a commutative diagrame:

$$\begin{tikzcd} \mathcal{T}_{/a} \arrow[r, "g"] \arrow[dr, "h"'] & \mathcal{T} \arrow[d, "f"] \\ & * \end{tikzcd}$$

with $h$ proper and $f$ separated then $g$ is proper. But the map $\mathcal{T}_{/a} \to \mathcal{T}$ is proper if and only if the discrete space $a$ (internally in $\mathcal{T}$) is compact if and only if $a$ is finite.
Moreover, if $\mathcal{T}$ is separated, then its localic reflection is separated (see [55, II.2.5]). As $\mathcal{T}$ is atomic its localic reflection is exactly the set of atoms of its

18. This mean that the locale $p_*(\Omega_{\mathcal{T}_{/a}})$ whose open are subobjects of $a$ is trivial, which is the definition of the fact that $a$ is an atom.

terminal object endowed with the discrete topology, which (by [55, II.1.3(1)]) is separated if and only if this set of atoms is decidable.

Conversely let $\mathcal{T}$ be an atomic topos whose atoms are internally finite and such that the set of atoms of $1_{\mathcal{T}}$ is separated. In order to prove that $\mathcal{T}$ is separated, one has to prove that both its localic reflection, and its hyperconnected map to its localic reflection are separated (see [55, II.2.5]). As we have seen in the previous part, the localic reflection is separated because the set of atoms of the terminal object of $\mathcal{T}$ is decidable. Hence we are brought to prove (internally) that a hyperconnected atomic locally decidable topos with finite atom is separated.
The commutative diagram:

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\Delta"] \arrow[dr, "Id"] & \mathcal{T} \times \mathcal{T} \arrow[d, "\pi_2"] \\ & \mathcal{T} \end{tikzcd}$$

can be seen as a point $\Delta$ of the topos $\mathcal{T} \times \mathcal{T}$ internally in $\mathcal{T}$. As $\mathcal{T} \times \mathcal{T}$ is the pullback of $\mathcal{T}$ by the canonical geometric morphism from $\mathcal{T}$ to the point, it will still be a hyperconnected locally decidable atomic topos internally in $\mathcal{T}$, and it will still have a generating family of finite objects and hence all its atoms will be finite internally in $\mathcal{T}$.
Hence our problem is equivalent to prove (constructively) that if $\mathcal{T}$ is atomic locally decidable and hyperconnected with a point $p$ and that all the atoms of $\mathcal{T}$ are finite then $p$ is proper. But an atomic topos with a point is equivalent to $BG$ for $G$ the localic group of automorphisms of the point, and the fact that the atoms are finite means that all the $G$-transitive sets are finite, and as $G$ has been taken to be set of automorphisms of the point, this will imply that $G$ is compact:
Indeed, the localic monoid of endomorphisms $M(G) = \lim G/U$ constructed in [54] is compact by (localic) Tychonoff's theorem, and separated because thanks to the locale decidability one can restrict to the $U$ such that $G/U$ is decidable, hence separated. In particular the point 1 is closed, and as $G$ can be identified with the subspace of $M(G) \times M(G)$ of $f, g$ such that $fg = 1$ which is a closed subspace in a separated compact space, $G$ is also compact.
Finally, the map $1 \to BG$ is proper because its pullback along itself is the map $G \to *$ which we just showed to be proper, and the map $* \to BG$ is always an open surjection (for exemple by [44, C3.5.6(i)]) hence the fact that proper maps descend along open surjections (see [44, C5.1.7]) allows us to conclude. $\square$

3.7.3. **Theorem :** *Let $\mathcal{T}$ be an atomic topos, and $B$ a bound of $\mathcal{T}$ then the semi-simple hypergroupoid attached to $B$ is decidable and locally finite if and only if $B$ is decidable and $\mathcal{T}_{/B}$ is separated.*

**Proof :**
Let $G$ be the hypergroupoid corresponding to $B$.

Assume first that $G$ is decidable and locally finite. Applying lemma 3.3.1 one obtains that $B$ is decidable, and hence that $\mathcal{T}$ is locally decidable. We will prove that $\mathcal{T}_{/B}$ is separated by applying 3.7.2. The set of atoms of $B$ is finite because it is the subset of units of $G$ which is decidable. Let $u$ be any atom of $\mathcal{T}_{/B}$, as $B$ is a bound of $\mathcal{T}$, $u$ can be covered by an atom $b$ of $B$, hence one has two maps $b \to u \to b'$ where $b'$ denote the image of $u$ in $B$. The map $b \to b'$ corresponds to an arrow in $G$, hence as $G$ is locally finite this map has internally finite fibre, hence $b$ is finite in $\mathcal{T}_{/b'}$ and hence also $u$ as a quotient of $b$. As the set of atoms of $B$ is decidable, this concludes the proof.

Conversely, assume that $B$ is decidable and that $\mathcal{T}_{/B}$ is separated. We will first prove that $G$ is locally finite. Indeed, let $g$ be any arrow in $G$, the set $\{(x,y)|xgy\}$ endowed with its map $(x,y) \mapsto x$ to $B$ is an atom of $\mathcal{T}_{/B}$, hence it is finite in $\mathcal{T}_{/B}$ by proposition 3.7.2. This exactly means that $\forall x \in B, \{y|xgy\}$ is finite, i.e. that $G$ is locally finite. By lemma 3.3.6 this also impllies that for all pairs of composable arrows $f,g$ in $G$ the set $fg$ is finite.

We will now prove that $G$ is decidable. As $\mathcal{T}_{/B}$ is separated, the set of atoms of $B$, i.e. the set of units of $G$ is decidable, and as $B$ is decidable, the set of units of $G$ is complemented in $G$. For any two arrows $f$ and $g$ of $G$, by decidability of the set of units, either they do not have the same source or target, in which case $f \neq g$ or they have the same source and target. In this second case, as the set $f^*g$ is finite its intersection with the set of units is also finite because the set of units is complemented, and hence the intersection of $f^*g$ with the set of units is either empty or inhabited. If it is empty then $f \neq g$, if it is inhabited then $f = g$ by uniqueness of the inverse in an hypergroupoid. In any case, $f = g$ or $f \neq g$, which proves that $G$ is decidable and this concludes the proof. □

3.7.4. Of course an atomic topos $\mathcal{T}$ admits a bound $B$ which is decidable and such that $\mathcal{T}_{/B}$ is separated as in theorem 3.7.3 if and only if $\mathcal{T}$ is locally decidable (each object is a quotient of a decidable object) and locally separated (there exists an inhabited object $X$ such that $\mathcal{T}_{/X}$ is separated). Indeed, assuming that $\mathcal{T}$ is locally decidable and locally separated, if $B_0$ is any bound and $X$ an inhabited object such that $\mathcal{T}_{/X}$ is separated then any decidable cover of $B_0 \times X$ will be a bound with the expected properties.

3.7.5. The time evolution constructed in 2.4 can be seen as a family of functors $\mathcal{H} \to \mathcal{H}^t$ for $t \in \mathbb{R}$ acting on the category of Hilbert spaces of $\mathcal{T}$, corresponding to the functor which sends a representation of $\mathcal{C}^*_{\max}(Q)$ to the representation twisted by $\sigma_t$.
By the previous theorem we know that any atomic locally decidable locally separated topos has such a time evolution. We will show that this time evolution is canonical by giving a construction of it which does not depend on the choice of the bound $B$.

To be more precise, let $\mathcal{T}$ be a locally decidable locally separated topos. We will construct a $\mathbb{Q}_{\mathcal{T}}$ principal bundle $\chi_{\mathcal{T}}$ in $\mathcal{T}$ in the following way. The decidable atoms $a$ of $\mathcal{T}$ such that $\mathcal{T}_{/a}$ is separated form a generating family. Hence to define an object of $\mathcal{T}$ it is enough to define a sheaf for the atomic topology on the full subcategory of these atoms.

If $f : a \to a'$ is map between two decidable atoms with $\mathcal{T}_{/a'}$ separated, $a$ is a finite decidable set internally in $\mathcal{T}_{/a'}$, and its cardinal is (as $a'$ is an atom) an externally defined natural number called the *degree* of $f$, and denoted $deg(f)$.

We define:

$$\hom(a, \chi_{\mathcal{T}}) = \mathbb{Q}_+^*$$

and if $f : a \to a'$ is any map then it acts on $\mathbb{Q}_+^*$ by multiplication by its degree. All these maps are bijective, hence it defines a sheaf. Also $\mathbb{Q}_+^*$ acts on $\chi_{\mathcal{T}}$ by multiplication turning it into a principal $\mathbb{Q}_+^*$ bundle.
We note in particular that if $\mathcal{T}$ is itself separated, then the terminal object of $\mathcal{T}$ is among the atoms of the site we consider and hence $\hom(1, \chi_{\mathcal{T}})$ is inhabited and hence $\chi_{\mathcal{T}}$ is the trivial bundle. But saying that $\chi_{\mathcal{T}}$ is trivial only means that it is possible to construct a global section $d$ of it, which will be a map associating to any decidable atom $a$ such that $\mathcal{T}_{/a}$ is separated a finite rational number $d(a)$ such that if $f : a \to a'$ is a map between two of these atoms then $d(a) = d(a')deg(f)$.

3.7.6. Finally, the time evolution given by any bound can be described in terms of this invariant $\chi_{\mathcal{T}}$. Indeed if $\mathcal{H}$ is an arbitrary Hilbert space on $\mathcal{T}$, and we choose an "admissible" bound $B$, then the effect of the time evolution on $\mathcal{H}$ can be described by the fact that $\mathcal{H}^t$ is the same sheaf as $\mathcal{H}$ on the site of atoms of $B$ but with the action of a map $f : e \to e'$ twisted by $\chi(f)^{it} = (degf)^{-it}$.
Hence if we see $\chi_{\mathcal{T}}$ as a morphism from the topos $\mathcal{T}$ to the classifying space $B\mathbb{Q}_+^*$ of principal $\mathbb{Q}_+^*$ bundles (i.e. the topos of $\mathbb{Q}_+^*$ sets), and if we call $E_t$ the one dimensional Hilbert space in $B\mathbb{Q}_+^*$ defined by $\mathbb{C}$ with its usual Hilbert space structure and endowed with action $q.z = q^{-it}(z)$ then the previous formula for $\mathcal{H}^t$ can be rephrased as:

$$\mathcal{H}^t = \mathcal{H} \otimes \chi_{\mathcal{T}}^*(E_t)$$

where the tensor product is just the internal tensor product of Hilbert spaces in $\mathcal{T}$.

## 3.8 Examples

3.8.1. If $G$ is a discrete group, $\mathcal{T}$ the topos of $G$-sets and $B$ is $G$ endowed with its (left) action on itself, then the corresponding quantale is $\mathcal{P}(G)$, the hypergroupoid is the group $G$, the integral algebra is $\mathbb{Z}[G]$, and the reduced and maximal $C^*$-algebras are the usual reduced and maximal group $C^*$-algebras. In this situation the time evolution is trivial.

3.8.2. The best-known example of this situation is the case of double coset algebras. Let $G$ be a discrete group, and $(K_i)$ a family of subgroups of $G$ (one can generalize to $G$ a localic group and $K_i$ open subgroups). Let $X_i$ be the $G$-set $G/K_i$. The topos of $G$-set is atomic and $\mathsf{Rel}(X_i, X_j)$ can be identified with the subset of $G$ stable by the action of $K_i$ on the left and $K_j$ one the right, hence the atom of $\mathsf{Rel}(X_i, X_j)$ are exactly the $(K_i, K_j)$ double-cosets. Under this identification, the composition of a $(K_i, K_j)$ cosets with a $(K_j, K_t)$ cosets is the set of $(K_i, K_t)$ cosets included in the product of their elements, and the coefficients $\begin{pmatrix} g, g' \\ a \end{pmatrix}$ are exactly the usual coefficients involved in the definition of the double coset modules and double coset algebras (hence we are in the locally finite case if and only if one has the usual almost normality condition).

3.8.3. The previous example in particular gives back the BC-system constructed in [9] with both its time evolution and its integral sub-algebra by considering the topos of continuous actions of the group $G = \mathbb{A}_{\mathbb{Q}}^f \rtimes \mathbb{Q}_+^*$, where $\mathbb{A}_{\mathbb{Q}}^f$ denote the additive group of finite adele of $\mathbb{Q}$, i.e. the restricted product of all $p$-adic completions of $\mathbb{Q}$, with the bound $B = G/\hat{\mathbb{Z}}$.
Unfortunately, trying to replace $\mathbb{Q}$ by another number field in this construction does not seem to give the "good" $BC$-system for number field constructed in [30], and certainly not the good arithmetic subalgebra. Actually the variant of the BC algebra associated to a number field $K$ constructed in [2] corresponds to the topos of continuous $G_K = (\mathbb{A}_K^f) \ltimes K^*$-sets with the bound $B = G_K/\widehat{\mathcal{O}_K}$.

3.8.4. It appears that, when we assume the axiom of choice in the base topos, every locally separated atomic topos is a disjoint sum of classifying toposes of locally profinite groups. In particular the case of double cosets algebra (of a locally pro-finite group) is essentially the only things one can obtain with this theory. Indeed, one can see that a locally finite object of a separated connected atomic topos is a coherent object, and hence that connected separated atomic toposes are coherent toposes. And coherent toposes always have points by a theorem of P.Deligne. In particular, assuming the axiom of choice, any atomic locally separated topos has enough points, and an atomic topos with enough points is a disjoint sum a classifying topos of localic groups. The assumption of local separation finally imply that the groups involved are locally pro-finite.

But as the methods used in this article are constructive (and the argument used in this last paragraph is not), one can find internally to other topos example of toposes where this theory applies and which are not of this form. Moreover the technique we have used here are meant to be generalized to other toposes, and the main goal of this work was to provide a toy model in order to study more general toposes.

# Chapter 2

# Measure theory over toposes

## 1 Introduction

In all this second chapter (except maybe during the introduction, 1 and 1.1), and contrary to the rest of this thesis, the base topos $\mathcal{S}$ will *always be assumed to satisfy the law of excluded middle*, but not necessarily the internal axiom of choice. The main reason for this is that in this chapter we mostly focus on the study of boolean toposes, hence it is natural to do it over a boolean base topos. Also, most of the results presented here seem to require the law of excluded middle in an essential way.

In subsection 1.1 we will briefly review, without proof, measure theory over locales in order to motivate its generalization to toposes. We define the notion of *generalized measure class* on a topos $\mathcal{T}$ as the data of a boolean topos $\mathcal{B}$ endowed with an injective geometric morphism (where injective means a monomorphism in the 2-category of toposes, not an inclusion of toposes) from $\mathcal{B}$ to $\mathcal{T}$. A Measurable field with respect to this generalized measure class is then defined as an object in $\mathcal{B}$. For example, a measurable field of Hilbert spaces over $\mathcal{T}$ is an Hilbert space in the logic of $\mathcal{B}$ (we will call this a $\mathcal{B}$-Hilbert space).

In section 2 we study some general properties of these measurable fields. In general the category of Hilbert spaces over a boolean topos is something that we may want to call a monotone closed $C^*$-category: bounded above directed families of self-adjoint operators have supremum but this category might not have normal representations in the category of Hilbert spaces. If such normal representation exists, then it is a $W^*$-category in the sense of [27]. We show that this is the case for the category of $\mathcal{B}$-Hilbert space, if and only if, for every object $X$ of $\mathcal{B}$, or equivalently for every object $X$ belonging to some fixed generating family of $\mathcal{B}$, the complete boolean algebra $\mathsf{Sub}(X)$ of subobjects of $X$ admit enough locally finite valuations. This equivalence is proved in subsection 2.1, and a boolean topos satisfying this condition is said to be *integrable*. In subsection 2.2 we prove a simple injection/surjection factorisation theorem for geometric morphisms from boolean toposes to arbitrary toposes showing that generalized measure classes can be pushed forward along geometric morphisms and that any generalized measure class on a topos $\mathcal{T}$ can be described internally in $\mathcal{T}$ as a sub-terminal boolean locale.

Von Neumann algebras (and also $W^*$-categories) are well known to have a canonical "modular" time evolution (see for example, [67]). The main result of this chapter is to provide a geometric description of this time evolution in the case of von Neumann algebras arising from a boolean locally separated topos $\mathcal{T}$ in terms of a certain (canonical) $\mathbb{R}^{>0}_{\mathcal{T}}$ bundle over $\mathcal{T}$ analogous to the bundle of locally positive well supported measure over a boolean locale (which is a bundle exactly because of the Radon-Nikodym theorem).

More precisely, in section 3 we define a notion (relatively restrictive) of locally finite well supported measure on a boolean topos, which we called *invariant measure* because on the topos associated to an action of a group $G$ over a boolean locale such an invariant measure indeed corresponds to a $G$-invariant measure. We prove (in 3.2.8) that such an invariant measure can only exists on an integrable and locally separated topos.
For a separated integrable topos, invariant measures correspond exactly to locally finite well supported measure on the localic reflection (see 3.2.6 ). For a more general integrable locally separated topos $\mathcal{T}$, we will show that the existence of invariant measure is controlled by a "modular" $\mathbb{R}^{>0}_{\mathcal{T}}$-principal bundle over $\mathcal{T}$ denoted $\chi$. Invariant measures on $\mathcal{T}$ correspond exactly to global sections of $\chi$, hence, if $\chi$ is trivial there exists invariant measures on $\mathcal{T}$, and two such measures differ by multiplication by a positive function on the localic reflection of $\mathcal{T}$. In particular, (if $\chi$ is trivial) invariant measures on $\mathcal{T}$ also correspond to (locally finite well supported) measures on the localic reflection of $\mathcal{T}$, but this time in a non canonical way. Finally, if $\chi$ is non trivial then there is no such invariant measure, but instead one can use $\chi$ to construct a canonical time evolution of the the category of $\mathcal{T}$-Hilbert spaces, given by tensorisation by a one parameter family of line bundles over $\mathcal{T}$ associated to $\chi$.

This gives a classification in types $I$, $II$ and $III$ of boolean integrable locally separated toposes analogous (but not totally equivalent) to the classification in type of von Neumann algebras: Type $I$ corresponds to separated toposes, type $II$ to toposes which are not separated but which have a trivial modular bundle, and type $III$ to toposes which have a non trivial modular bundle.
Of course in full generality, one obtains that every boolean integrable locally separated topos decomposes in a disjoint sum of a topos of each of the three types by applying this disjunction internally in the localic reflection.

In the last subsection of this chapter, we consider $\mathcal{T}$ a boolean integrable locally separated toposes, $X$ an object of $\mathcal{T}$ such that $\mathcal{T}_{/X}$ is separated and $l^2(X)$ the Hilbert space of square summable sequences. In this situation, we show that the modular time evolution of the von Neumann algebra $A$ of globally bounded endomorphisms of $l^2(X)$ is indeed the time evolution on $A$ described by $\chi$, and that the integration of diagonal matrix elements against an invariant measure on $\mathcal{T}$ gives rise to a trace on $A$.

The following table sum-up the dictionary between topos theory (in the left column) and operator algebra (in the right column) that arise in this chapter.

This is of course just a vague analogy that we have observed while developing this theory, we do not claim that there is any sort of rigorous correspondence here. In particular we think that this dictionary is meant to be made more precise in the future, for example a work in progress (mentioned in the introduction under the name "Non abelian monoidal Gelfand duality") highly suggest that it can be made a lot more precise if we take into account on the right hand side the monoidal structures that arise from the internal tensor product of Hilbert space and the compatibility to these structures.

| | |
|---|---|
| Boolean topos (locally separated) | Monotone complete $C^*$-algebra (up to morita equivalence) |
| Boolean integrable topos (locally separated) | $W^*$-algebra (up to morita equivalence) |
| Localic reflection | Center |
| Measure on an object | Semi-finite normal weight |
| Measure on an object of mass 1 | Normal state |
| Invariant measure | Normal (semi-finite) trace |
| The modular bundle | The $\Delta$ operator of the Tomita-Takesaki construction |
| The family of line bundles $(F_t)_{t\in\mathbb{R}}$ | The modular time evolution |

## 1.1 Review of measure theory on locales

In this subsection, we review the classical measure theory of locales in order to motivate the definition given in the introduction for general toposes. The measure theory of locales is slightly different from usual measure theory of $\sigma$-algebras, but still very similar. The reader can also consult M.Jackson thesis [39] which devoted to a topos theoretic presentation of measure theory and contains a large part of what we will says in this subsection.

A measure on a locale $\mathcal{L}$ will be defined as a (completely continuous) valuation on the frame $\mathcal{O}(\mathcal{L})$ of open sublocales of $\mathcal{L}$. A valuation on a frame $F$ is defined as a function $\mu$ which associates to every element of the frame a positive real number (possibly infinite) and which satisfy:

1. $\mu(\emptyset) = 0$
2. if $a \leqslant b$ then $\mu(a) \leqslant \mu(b)$.
3. $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$.
4. For any directed set $(A_i)_{i\in I}$ of element of $F$ one has $\sup \mu(A_i) = \mu\left(\bigvee A_i\right)$

The fourth condition is sometimes weakened to only assuming this when $A_i$ is a countable family, by analogy with what is done in classical measure theory. Such valuations will be called $\sigma$-continuous valuations. Here are several reasons why we prefer completely continuous valuations to $\sigma$-continuous one:

- In a constructive context – more precisely in the absence of the axiom of countable and dependant choice – the notion of countability is poorly behaved (for example, a countable union of finite sets may fail to be a countable set).
- In the cases where we are able to deal properly with the notion of countability, then a $\sigma$-continuous valuation on a frame $F$ can be seen as a completely continuous valuation on a frame $F'$ which is generated by the elements of $F$ with the relations given by all the intersections and the countable unions which hold in $F$. Hence, when they are well behaved, $\sigma$-countable valuations can be seen as a special case of completely continuous valuations.
- In the context of classical measure theory, where the measures are defined for every measurable subset – and not only on open subset – complete continuity is a too strong condition which excludes all non trivial measures. But when we restrict to open subsets a lot of measures satisfy this complete continuity condition. For example any measure on a Lindelöf topological space or any inner regular measure satisfies this condition.
- A normal( semi-finite) weight on an abelian von Neumann algebra is the same thing as a (locally finite) completely continuous valuation on the frame of its projections.

In the all of this chapter, by *valuation* we will always mean completely continuous valuation, and by a *measure* on a locale $\mathcal{L}$ we mean a completely continuous valuation on the frame $\mathcal{O}(\mathcal{L})$.

A valuation on a frame $F$ (or a measure on a locale $X$ if $F = \mathcal{O}(X)$ ) will be called *locally finite* if every element of $F$ can be written as a union of elements of finite measure, and *finite* if every element of $F$ has a finite valuation.

Even assuming classical logic and the axiom of choice, the question of existence of enough locally finite measures on a boolean locale is difficult. The notion of complete boolean algebra is in general equivalent to the notion of commutative monotone closed $C^*$-algebra (or to the notion of commutative $AW$-algebra): to a commutative monotone closed $C^*$-algebra one can associate the complete boolean algebra of its projections, and to a complete boolean algebra one associates the algebra of complex valued functions on the corresponding boolean locale.
If $A$ is a monotone closed $C^*$-algebra we can denote by $NSp(A)$ the corresponding boolean locale, we call it the *normal spectrum* of $A$ because one can check that $NSp(A)$ classifies the theory of "normal characters of $A$" (which most of the time does not have any models) and a locally finite measure on $NSp(A)$ is the same as a normal semi finite weight on $A$, hence there is enough such measures if and only if $A$ is a von Neumann algebra or equivalently if (in the terminology of [66, Ch III.1]) the Stonean space associated to $NSp(A)$, which is the usual spectrum of $A$, is a hyperstonean space. A Boolean locale satisfying these properties will be called an *integrable* locale (the terminology "measurable" can also be found).

For an example of non-integrable boolean locale, one can take the complete boolean algebra of regular open subset (i.e. satisfying $\neg\neg U = U$) of a compact metric topological space with no isolated points. Any boolean locale over which some uncountable set in the base topos became countable in the topos of sheaves also provide a counterexample.

For a measure $\mu$ on a boolean locale there is a smallest open sublocale $S$, called the *support* of $\mu$, such that $\mu$ is zero on the complement of $S$. A measure is said to be well supported if its support is the maximal open sublocale. Assuming the law of excluded middle, one easily checks that a measure is well supported if and only if each non-empty open sublocale has a non-zero measure. Assuming the axiom of choice, a Boolean locale is integrable if and only if it admits a well supported locally finite measure.

A certain part of measure theory on locales can be developed constructively, and even "geometrically". This shows in particular that integration of a function with respect to a measure behaves well when pulled back along geometric morphisms, or allows one to prove the continuity of certain integrals with parameters in a topos theoretic context. One can consult the work of S.Vickers in [72] and [73] on constructive measure theory. We will use what S.Vickers calls the "lower integral" which allows one to integrate a positive lower semi-continuous function with respect to a (positive) measure which takes value in lower semi-continuous reals.

Unfortunately, if we stick to intuitionist logic it is not possible – or at least we do not know how to do it properly – to define any of the more advanced constructions of measure theory, like the Hilbert space $L^2(X,\mu)$ or the a von Neumann algebra $L^\infty(X,\mu)$ associated to a locale $X$ endowed with a measure $\mu$.
The main reason for this is that (in the case of $L^2$) we want $L^2$ to be a Hilbert space containing the characteristic function of open sublocales (and even generated by them) and such that $\langle \mathbb{I}_U, \mathbb{I}_V \rangle = \mu(U \wedge V)$ but the scalar product has to be a continuous real number, whereas measure of open sublocales are semi-continuous real numbers and asking them to be continuous generally implies the law of excluded middle.

Assuming the law of excluded middle, if $\mu$ is locally finite, one can define $L^2(X,\mu)$ as mentioned earlier: it is the Hilbert space generated by symbols $\mathbb{I}_U$ for $U \in \mathcal{O}(X)$ of finite measure with the scalar product $\langle \mathbb{I}_U, \mathbb{I}_V \rangle = \mu(U \wedge V)$. The formula $M_U(\mathbb{I}_V) = \mathbb{I}_{U \wedge V}$ then defines a globally bounded operator $M_U$ (of norm smaller than one) on $L^2(X,\mu)$, and the family of all the $M_U$ generates a von Neumann algebra $L^\infty(X,\mu)$. This algebra contains the algebra of continuous functions on $X$ and actually all (upper or lower) semi-continuous functions on $X$. It is also endowed with a semi-finite weight (also denoted $\mu$) which extends the integration of lower semi-continuous functions.
Moreover, there is an injective morphism of locales:

$$i : NSp(L^\infty(X,\mu)) \to X.$$

such that the measure on $X$ is the pushforward of the measure on $NSp(L^\infty(X,\mu))$ corresponding to this semi-finite weight. It appears that (assuming the law of excluded middle) for any measure $\mu$ on a locale $X$ there is a unique boolean locale $[X,\mu]$ endowed with a well supported measure $\tilde{\mu}$ and an injective map $i : [X,\mu] \to X$ such that $i_* \tilde{\mu} = \mu$.

This boolean locale $[X,\mu]$ is the space on which are naturally defined objects on $X$ that are only well defined up to something $\mu$-null. For example the elements of the spaces $L^p(X,\mu)$ are defined as particular functions on $[X,\mu]$, and the good notion of $\mu$-measurable fields of Hilbert space over $X$ is an Hilbert space in the logic of $[X,\mu]$: it corresponds exactly to the notion of $L^\infty(X,\mu) - W^*$ module, and hence it corresponds to the traditional notion of measurable field under some countability assumption (as in [22, II.1]).

This is why, by analogy it is natural to define a generalized measure class on a topos $\mathcal{T}$ as an equivalence class of boolean toposes $\mathcal{B}$ endowed with an injective geometric morphism to $\mathcal{T}$, and to say that something is measurable with respect to this class if it exists in $\mathcal{B}$. Proposition 2.2.3 shows that a generalized measure class on a locale in this sense corresponds indeed to a class of boolean locale endowed with an injection into our locale.

# 2 Measurable fields over a topos

## 2.1 Integrable toposes

For a locale $\mathcal{L}$ not every "generalized measure class" on $\mathcal{L}$ as defined earlier indeed comes from some locally finite measure on $\mathcal{L}$. A generalized measure class on $\mathcal{L}$ is obtained as a supremum of generalized measure classes which come from measures on $\mathcal{L}$ if and only if the corresponding boolean locale is integrable.
The goal of this subsection is to define and study the notion of integrable boolean topos generalizing the notion of integrable boolean locale.

2.1.1. Let $\mathcal{T}$ be a topos, and $X$ be an object of $\mathcal{T}$. We call *measure* on $X$ a valuation on $\mathsf{Sub}(X)$. If $f : X \to \mathbb{R}^+_{\mathcal{T}}$ is a morphism from $X$ to the object of positive real numbers of $\mathcal{T}$, then $f$ corresponds externally to a continuous function from the locale associated to the frame $\mathsf{Sub}(X)$ to the locale of real numbers. In particular, if $\mu$ is a measure on $X$ one can define:

$$\int_X f d\mu \qquad \text{Or} \qquad \int_{x\in X} f(x) d\mu$$

as the integral of this corresponding function with respect to $\mu$.
If $f$ is a complex function on $X$ such that $\int_X |f| d\mu$ is finite, then one can also define $\int_X f d\mu$ by decomposing $f$ into the positive and negative part of both its real and imaginary part.

2.1.2. We recall that if $X$ is an object of a topos with a decidable equality, then $l^2(X)$ denote the space a square sumable sequence of complex number indexed by $X$. Equivalently, $l^2(X)$ is the Hilbert space generated by elements $e_x$ for each $x \in X$ such that $\langle e_x, e_{x'}\rangle = 1$ if $x = x'$ and 0 otherwise.

2.1.3. If $\mathcal{T}$ is a boolean topos, and $\mu$ is a measure on an object $X \in \mathcal{T}$ such that $\mu(X) = 1$ then for any operator $h : l^2(X) \to l^2(X)$ which is globally bounded [1] then as:

$$\int_{x \in X} |\langle e_x, he_x \rangle| \leqslant \|h\|_\infty \mu(X)$$

one can define:

$$\eta(h) = \int_{x \in X} \langle e_x, he_x \rangle \, d\mu$$

This defines an external normal state on the algebra of globally bounded endomorphisms of the $\mathcal{T}$-Hilbert space $l^2(X)$.
Conversely, if $\eta$ is any normal state on the algebra of globally bounded endomorphisms of $l^2(X)$ then one has a measure $\mu$ on $X$ defined by the formula $\mu(b) = \eta(1_b)$, where $b$ is a subobject of $X$ and $1_b$ is the endomorphism of $l^2(X)$ corresponding to the multiplication by the characteristic function of $b$. Of course, these two constructions are not inverse from each other but they allow one to prove the following:

**Proposition :** *Let $X$ be an object of a boolean topos $\mathcal{T}$ then the following conditions are equivalent:*

- *$\mathsf{Sub}(X)$ is an integrable boolean algebra.*
- *The $C^*$-algebra of globally bounded endomorphisms of the $\mathcal{T}$-Hilbert space $l^2(X)$ is a von Neumann algebra.*

**Proof :**
Assume first that $\mathsf{Sub}(X)$ is integrable. We will prove that $A = End(l^2(X))$ is a von Neumann algebra by showing that it is monotone closed [2] and admits enough normal states (see [66, III.3.16]).
The fact that $A$ is monotone closed comes from the fact that $\mathcal{T}$ is boolean. Indeed assuming the law of excluded middle, the usual proof that $B(\mathcal{H})$ is (internally) monotone closed for any Hilbert space applies and hence the supremum of a net in $A$ can be computed internally (the fact that an internal supremum yields an external supremum is a general property for supremum in ordered sets).
For the existence of enough normal states, if $h$ is a positive globally bounded self-adjoint operator on $l^2(X)$ such that for all normal state $\eta$ associated to a measure $\mu$ on $X$ one has $\eta(h) = 0$ then the integral of the function $\langle e_x, he_x \rangle$ against any positive measure on $X$ is zero (because of the relation between measure and normal state) hence, if $\mathsf{Sub}(X)$ is integrable this implies that the function $\langle e_x, he_x \rangle$ is zero on $X$ and hence that $h = 0$ because $h$ is positive.
Conversely, if $End(l^2(X))$ is a von Neumann algebra and if $b$ is a non zero subobject of $X$ then there exists a normal state $\eta$ such that $\eta(1_b) > 0$ hence this will give a measure $\mu$ on $X$ such that $\mu(b) > 0$. $\square$

1. that is it is internally bounded and its internal operator norm is bounded externally.
2. This mean that every bounded increasing net of normal operators of $A$ has a supremum

2.1.4. **Definition :** *We will say that a boolean topos $\mathcal{T}$ is integrable if for every object $X$ of $\mathcal{T}$, the complete boolean algebra $\mathsf{Sub}(X)$ is integrable.*

One easily checks that if some boolean locale $\mathcal{L}$ admits a surjection from an integrable boolean locale then $\mathcal{L}$ is integrable. Hence, if a boolean topos admits a generating family of objects $X$ such that $\mathsf{Sub}(X)$ is integrable then it is integrable.
One also has that:

**Proposition :** *Let $\mathcal{T}$ be a boolean topos then the following conditions are equivalent:*

- *$\mathcal{T}$ is integrable.*
- *The $C^*$-category $\mathcal{H}(\mathcal{T})$ of $\mathcal{T}$-Hilbert spaces is a $W^*$-category.*

**Proof :**
The first condition is an immediate consequence of the second condition by proposition 2.1.3. We will assume the first condition and prove the second. As $\mathcal{H}(\mathcal{T})$ has bi-product, it is enough to prove that for every $\mathcal{T}$-Hilbert space $H$, the algebra of globally bounded endomorphisms of $H$ is a von Neumann algebra. Moreover, as in the proof of proposition 2.1.3, it is enough to prove that this algebra admits enough normal states:
For any positive operator $h : H \to H$ of (external) norm $\|h\|_\infty = 1$, the function $\|h\|$ on $1_\mathcal{T}$ has supremum one, and hence it is greater than $1/2$ one some subobject $S \subseteq 1_\mathcal{T}$. Internally, one has, $S$ implies that $\|h\| > 1/2$ which in turn implies that there exists $x \in H$ of norm one such that $\langle x, hx \rangle > 1/2$. Externally, this means that there exists an object $S'$ covering $S$ and a map from $v : S' \to H$ whose image is (internally) formed of norm one vectors such that $\langle x, hx \rangle > 1/2$. Let $\mu$ be any measure of mass one on $S'$ and define for $f : H \to H$:

$$\eta(f) = \int_{x \in S'} \langle v(x), fv(x) \rangle$$

This defines a normal state $\eta$ on the algebra of globally bounded endomorphisms of $H$ (as in 2.1.3) and $\eta(h)$ is greater than $1/2$ because internally $\langle v(x), hv(x) \rangle > 1/2$ for all $x$. $\square$

2.1.5. **Corollary :** *(At least assuming the axiom of choice in the base topos) Let $\mathcal{T}$ be an integrable boolean topos, then there exists a von Neumann algebra $A$ (uniquely determined up to Morita equivalence of von Neumann algebras) such that the category of $\mathcal{T}$-Hilbert spaces is equivalent to the category of Hilbert $A$ modules.*

The axiom of choice is in fact probably unnecessary. The only reason why we need to assume it is that proposition 7.6 of [27] uses the axiom of choice. But it seems possible to give a choice free version of this proposition based on a slightly stronger notion of completeness for $W^*$-categories.

**Proof :**

As the category of $\mathcal{T}$-Hilbert spaces has arbitrary orthogonal sums and splitting of projections (they are computed internally) it is enough, by proposition 7.6 of [27], to check that it has a generator in the sense of [27] 7.3.
We fix $G$ a set of generators of $\mathcal{T}$. Let $E$ be the family of all (equivalences class of) triples $(H, g, f)$ where $H$ is a $\mathcal{T}$-Hilbert space, $g$ is an element of $G$ and $f$ is a map $g \to H$ such that internally the image of this map spans a dense subspace of $H$. To an element $(\mathcal{H}, g, f)$ of $E$ one can associate a continuous function on $g \times g$ defined by $(x, y) \to \langle f(x), f(y) \rangle$, and if $(H, g, f)$ and $(H', g, f')$ define the same function on $g \times g$ then they are isomorphic. In particular the (isomorphism class of) elements of $E$ form a set.
Now, for any $f : A \to B$ an operator between two $\mathcal{T}$-Hilbert spaces, there exists $g \in G$ and a map $\lambda : g \to A$ such that $f \circ \lambda \neq 0$, the adherence of the span of $\lambda$ gives an element $H$ of $E$ and a map $i$ from $H$ to $A$ such that $f \circ i \neq 0$. This proves that elements of $E$ form a family of generators of $\mathcal{H}(\mathcal{T})$, i.e. their orthogonal sum is a generator of this $W^*$-category. This concludes the proof. □

Even if this von Neumann algebra is naturally attached to the topos $\mathcal{T}$ it is in most case "too big" for example, if $\mathcal{T}$ is the classifying topos of a discrete group, then $\mathcal{H}(\mathcal{T})$ is the category of unitary representations of $G$ and the associated von Neumann algebra is the enveloping von Neumann algebra of the maximal $C^*$-algebra of the group, which is an enormous algebra.
It seems that a more reasonable algebra to consider in practice is the algebra of operators on a space $l^2(X)$ for $X$ a separating bound of $\mathcal{T}$. The results of the last section suggest this algebra can be controlled by the geometry of $\mathcal{T}$, whereas in general an algebra of operators on an arbitrary Hilbert space over a topos $\mathcal{T}$ can have nothing in common with the geometry of $\mathcal{T}$. For example, any von Neumann algebra arises as the algebra of globally bounded endomorphisms of some representation of a discrete group.

## 2.2 Booleanization of a topos

The goal of this section is essentially to prove proposition 2.2.3 which allows one to understand internally in $\mathcal{T}$ what is a generalized measure class over $\mathcal{T}$, to push-forward such generalized measure class along geometric morphisms and more generally to provide a way to construct such classes.

2.2.1. The following result is trivial but will be extremely important:

**Lemma :** *A map from a topos to a boolean topos is always an open map*[3].

**Proof :**
Working internally in the target of the arrow it is enough to show that assuming the law of excluded middle, for any topos $\mathcal{T}$ the canonical map to the point is an open map, i.e. (by [44, C3.1.19]) that $\mathcal{T}$ can be defined by a site where each cover is inhabited. Now thanks to the law of excluded middle, it is always the

3. See [44, C3.1] for the definition of open maps of locales and of toposes.

case: it suffices to remove all the objects admitting an empty cover from a site of definition of $\mathcal{T}$. (this argument appears in [44] just before $C3.1.20$). $\square$

In general the notion of surjection between toposes is not well behaved, but open surjections are extremely well behaved, for example they are stable surjections and effective descent morphisms (see [44, C5.1.5 and C5.1.6]). Hence for maps targeting a boolean topos, surjectivity will be enough to have the nice properties of open surjections.

2.2.2. As mentioned in the introduction:

**Definition :** *A generalized class of measure over a topos $\mathcal{T}$ is the data of a boolean topos $\mathcal{B}$ endowed with an injective map $\mathcal{B} \to \mathcal{T}$ modulo isomorphism of $\mathcal{T}$-topos. A generalized measure class is said to be integrable if the corresponding boolean topos is integrable.*

2.2.3. **Proposition :** *Let $f : \mathcal{B} \to \mathcal{T}$ be a geometric morphism from a boolean topos to an arbitrary topos, then it admits a unique factorisation in the form:*

$$\mathcal{B} \twoheadrightarrow \{\mathcal{B}\} \hookrightarrow \mathcal{T}$$

*where $\{\mathcal{B}\}$ is also boolean, the first map is a surjection and the second map an injection.*
*Moreover any injective map from a boolean topos $\mathcal{B}'$ to $\mathcal{T}$, i.e. any generalized measure class on $\mathcal{T}$ corresponds internally in $\mathcal{T}$ to a generalized measure class on the point[4], i.e. to a sub-terminal boolean locale.*

**Proof :**
We will first prove the uniqueness of such a decomposition:
Let

$$\mathcal{B} \twoheadrightarrow \mathcal{B}' \hookrightarrow \mathcal{T}$$

be an eventual decomposition of $f$, then as $\mathcal{B}'$ is boolean, the surjection $\mathcal{B} \twoheadrightarrow \mathcal{B}'$ is an open surjection, and hence an effective descent morphism ([44] C5.1.6), hence $\mathcal{B}'$ can be identified (see [44] B.3.4.12) with the colimit of the following truncated simplicial diagram

$$\mathcal{B} \times_{\mathcal{B}'} \mathcal{B} \times_{\mathcal{B}'} \mathcal{B} \begin{array}{c} \xrightarrow{\pi_{1,2}} \\ \xrightarrow{\pi_{1,3}} \\ \xrightarrow{\pi_{2,3}} \end{array} \mathcal{B} \times_{\mathcal{B}'} \mathcal{B} \begin{array}{c} \xrightarrow{\pi_1} \\ \xleftarrow{\Delta} \\ \xrightarrow{\pi_2} \end{array} \mathcal{B}$$

But as $\mathcal{B}' \to \mathcal{T}$ is injective, fiber product over $\mathcal{B}'$ are the same things as fiber product over $\mathcal{T}$ (they satisfy the same universal property), hence this truncated

4. Which is no longer a boolean locale if the topos $\mathcal{T}$ is not boolean.

simplicial diagram does not depend on $\mathcal{B}'$ but only on $\mathcal{B}, \mathcal{T}$ and the morphism between them. This proves the uniqueness of the decomposition.
We will now prove the existence of such a decomposition, with additionally the injective part which is internally a generalized measure class on the point:
Consider the hyperconnected/localic factorisation [5] of $f : \mathcal{B} \twoheadrightarrow \mathcal{B}_1 \rightarrow \mathcal{T}$. We define $\{\mathcal{B}\}$ as a locale in $\mathcal{T}$ by giving (internally) its frame of open sublocales: it is the smallest complete boolean subalgebra of $\mathcal{O}(\mathcal{B}_1)$, where by "complete boolean subalgebra" we mean that the inclusion is a frame homomorphism, in particular it contains the top and the bottom element.
The map $i : \{\mathcal{B}\} \rightarrow *$ is internally injective (as a map of locale) because if one has two frame homomorphisms from $\mathcal{O}(\{\mathcal{B}\})$ to another frame they have to agree on a complete boolean subalgebra of $\mathcal{O}(\{\mathcal{B}\})$ and as $\mathcal{O}(\{\mathcal{B}\})$ has by construction no non trivial such subalgebras, the two morphisms are in fact equal. This implies that the corresponding (external) map $i : \{\mathcal{B}\} \rightarrow \mathcal{T}$ is also injective (as a map of toposes), indeed, if one has two maps from an arbitrary topos $u, v : \mathcal{E} \rightrightarrows \{\mathcal{B}\}$ such that $i \circ u = i \circ v$ then internally in $\mathcal{T}$, one also has $i \circ u = i \circ v$, the two maps $u, v$ factor into the localic reflection $\mathcal{L}$ of $\mathcal{E}$, and hence coincide (on $\mathcal{L}$, and hence also on $\mathcal{E}$) by the internal injectivity of $i$, and this concludes the proof of the existence of the factorization.
The last part of the proposition follow immediately by applying the factorization to $B'$ and using that the previous construction show that $\{B'\}$ is localic over $\mathcal{T}$. $\square$

2.2.4. The classes of measure on a topos are ordered by "inclusion" of subobjects: one says that $E_1 \leqslant E_2$ if the injection of $E_1$ into $\mathcal{T}$ factors into $E_2$.

**Proposition :** *let $\mathcal{T}$ be a topos and $E \hookrightarrow \mathcal{T}$ be a generalized measure class on $\mathcal{T}$. There is a correspondence between generalized measure classes smaller than $E$ and subobjects of $1$ in $E$.*

**Proof :**
A generalized measure class smaller than $E$ is the same thing as a generalized measure class on $E$. But a generalized measure class on a boolean topos corresponds internally to a generalized measure class on the point, which is internally – because of the law of excluded middle – either empty or non-empty, in which case it is an isomorphism. Externally it means that it is indeed a sub-terminal object. $\square$

2.2.5. In particular the generalized measure classes smaller than a given generalized measure class form a set, and even a complete boolean algebra. In general there might exist a proper class of generalized measure classes on a given topos. In fact one easily obtains the equivalence of the following assertions for a topos $\mathcal{T}$:

5. See [44, A4.6]

- There exists a boolean topos $\mathcal{B} \to \mathcal{T}$ which is universal for map from boolean topos to $\mathcal{T}$.
- The class of generalized measure class on $\mathcal{T}$ is a set.
- There exists a maximal generalized measure class on $\mathcal{T}$.

It is proved by simply observing that a universal boolean topos is the same thing as a maximal generalized measure class. And it is well known that not every topos (in fact not every locale) admits such a universal boolean reflection, see [44, Remark C1.1.21].

2.2.6. We do not know at the present time if –as in the localic case – every topos admits a maximal integrable generalized measure class, that is a generalized measure class $\mathcal{M}$ such that any other generalized measure class on this topos is integrable if and only if it is smaller than $\mathcal{M}$.

# 3 Invariant measure and time evolution of boolean toposes

The main goal of this section is roughly to generalise the results obtained in chapter 1 about the time evolution and the modular principal bundle of locally separated (locally decidable) atomic toposes to the more general framework of boolean locally separated toposes. This will show that, as the theory of von Neumann algebras, measure theory over topos has also its own canonical dynamics. We will also show the relation between the two dynamics through the use of the KMS condition (see 3.5.5).

## 3.1 Introduction and Invariant measures

In this subsection we will introduce the new key concept towards the construction of the dynamics of measure theory over toposes. It is the notion of "invariant measure" over a boolean topos. Our goal here is just to introduce and motivate this definition, hence except the definition of an invariant measure itself, all the material covered in this introduction subsection is voluntarily vague and many things are claimed without proof. Everything will be made more precise and proved properly in the rest of the section.

In chapter 1 we showed that if $\mathcal{T}$ is a locally separated atomic topos [6] then there is a canonical $\mathbb{Q}_+^*$ principal bundle $\delta$ on $\mathcal{T}$ which describes the time evolution predicted by the modular theory of von Neumann algebras of an operator algebra naturally attached to $\mathcal{T}$.

Let us first recall the construction of this bundle $\delta$:
One can prove that, for any pair of atoms $a, b \in |\mathcal{T}|$ such that $\mathcal{T}_{/a}$ and $\mathcal{T}_{/b}$ are separated, and any map $f : a \to b$, the map $f$ has a finite degree, that is,

6. with enough decidable objects, but as we now assume the law of excluded middle in the base topos this is automatic.

there exists (externally) an integer $n$ such that (internally) any element of $b$ has exactly $n$ pre-images by $f$.
If $\mathcal{T}$ is locally separated then the atoms $a$ of $\mathcal{T}$ such that $\mathcal{T}_{/a}$ is separated form a generating family, and hence they form a site of definition $\mathcal{C}$ of $\mathcal{T}$. Objects of $\mathcal{C}$ are the atoms such that $\mathcal{T}_{/a}$ is separated, its morphisms are morphisms in $\mathcal{T}$ and its topology is the restriction of the canonical topology of $\mathcal{T}$ which actually coincides with the atomic topology of $\mathcal{C}$. The bundle $\delta$ is then defined as a sheaf on this site by:

- for any atom $a$ such that $\mathcal{T}_{/a}$ is separated, $\delta(a) = \mathbb{Q}_+^*$.
- for any $f : a \to b$ a map between two such atoms, $f$ acts on $\delta(a) = \delta(b) = \mathbb{Q}_+^*$ by multiplication by the degree of $f$.

And of course, the action of $\mathbb{Q}_+^*$ on this sheaf is the component wise multiplication.

What does it mean for this $\mathbb{Q}_+^*$ bundle to be trivial ?
A principal bundle is trivial if and only if it has a global section, hence one should try to understand what a global section of this bundle is. Working in the site $\mathcal{C}$, the terminal object is the constant sheaf equal to $\{*\}$, hence a global section of a sheaf $\mathcal{F}$, is the data of the choice of an element of $\mathcal{F}(a)$ for each $a \in \mathcal{C}$ such that these choices are compatible with the functoriality.
In the case of $\delta$, a global section is hence the choice of a function which to each atom $a$ of $\mathcal{T}$ such that $\mathcal{T}_{/a}$ is separated associates a positive rational number $\mu(a)$, and if $f : a \to b$ is a map then $\mu(a) = \mu(b) \deg(f)$.
When one has such a function $\mu$ then:

- It is natural to think of $\mu(a)$ as the "mass" of $a$, indeed, if $b \to a$ is an an $n$-fold cover of $b$ then the mass of $b$ is exactly $n$ times the mass of $a$.
- if $w$ is an atom such that $\mathcal{T}_{/w}$ is not separated, then $w$ always admit a cover by an atom $a$ such that $\mathcal{T}_{/a}$ is separated. The map $f : a \to w$ must have an infinite degree, indeed, if it were not the case, $\mathcal{T}_{/w}$ would be separated (it is a consequence of proposition 4.7.2 of chapter 1). Hence in order to preserve a form of relation $\mu(a) = \deg(f)\mu(w)$, it is reasonable to define $\mu(w) = 0$.
- Finally if $X$ is an arbitrary object of $\mathcal{T}$, it is reasonable to define:

$$\mu(X) = \sum_{a \text{ atom of } X} \mu(a)$$

This construction of a function which associates to every object of $\mathcal{T}$ a real number in $[0, \infty]$ satisfying some reasonable conditions with respect to morphisms motivates the following definition:

3.1.1. **Definition :** *Let $\mathcal{T}$ be a boolean topos. An invariant measure*[7] *on $\mathcal{T}$ is a function which to every object $X$ of $\mathcal{T}$ associates a real number $\mu(X) \in [0, \infty]$ such that:*

*(IM1) For each $X \in \mathcal{T}$, the restriction of $\mu$ to sub-objects of $X$ defines a locally finite valuation on $\mathsf{Sub}(X)$.*

7. It should probably be more correct to call this a "well supported locally finite invariant measure".

*(IM2) There exists a generating family of $\mathcal{T}$ of objects $X$ on which the valuation induces on $\mathsf{Sub}(X)$ is well supported.*

*(IM3) For each surjective arrow $f : X \twoheadrightarrow Y$ such that $\mu(X) < \infty$, if we define the function $c_f$ from $X$ to $[0,1]$ by the internal formula[8]:*

$$c_f(x) = \frac{1}{|\{x' \in X | f(x) = f(x')\}|},$$

*then:*

$$\mu(Y) = \int_X c_f d\mu|_{\mathsf{Sub}(X)},$$

*where the $\mu$ in $d\mu$ in fact denote the restriction of $\mu$ to $\mathsf{Sub}(X)$.*

Note that the third condition implies that $\mu(X) = \mu(Y)$ when $X$ and $Y$ are isomorphic.
One can check (but it will be a direct consequence of 3.3.3) that a global section of $\delta$ is exactly the same as an invariant measure which associates a rational mass to every atom. A general invariant measure corresponds to a section of the $\mathbb{R}^{\geqslant 0}_{\mathcal{T}}$ bundle $\delta'$ constructed in the same way as $\delta$ but taking $\delta'(a) = \mathbb{R}^{\geqslant 0}_{\mathcal{T}}$ instead of $\mathbb{Q}^*_+$. It corresponds internally to the "tensorisation" of $\delta$ by $\mathbb{R}^{\geqslant 0}_{\mathcal{T}}$ over $\mathbb{Q}^*_+$.
Also, one notes that if $\mu$ is an invariant measure on a topos $\mathcal{T}$, and $X$ is an object of $\mathcal{T}$ then $\mu$ induces an invariant measure denoted $\mu_X$ on $\mathcal{T}_{/X}$ by defining the mass of an object $Y$ over $X$ to be simply $\mu(Y)$.

3.1.2. Example: We will see in 3.2.6 that an invariant measure on the topos of sheaves on a boolean locale is the same thing as a measure on the locale.
Also, if $G$ is a discrete group acting on a boolean locale $X$, then an invariant measure on the topos $\mathsf{Sh}_G(X)$ of $G$-equivariant sheaves over $X$ is the same thing (in a not completely trivial way) as a $G$ invariant well supported locally finite measure on $X$. A direct proof of this at this point might be a little long, but it is an immediate consequence of results of the section 3.3: as $\mathsf{Sh}(X)$ is separated and etale over $\mathsf{Sh}_G(X)$, the sections of $\chi$ over the corresponding object of $\mathsf{Sh}_G(X)$ are simply the well supported locally finite measures on $X$. One easily deduces from this that the bundle $\chi$ is the sheaf of locally finite well supported measures over $X$ with the natural action of $G$ and hence that a global section of $\chi$ is exactly a locally finite well supported measure on $X$ which is $G$ invariant.

## 3.2 Separated and locally separated boolean toposes

In this subsection we introduce the notion of separated and locally separated toposes due to Moerdjik and Vermeulen in [55], and prove some results about it in relation to the notion of invariant measure.
The main results are that invariant measures can only exist on locally separated toposes (3.2.8) and that on a separated topos $\mathcal{T}$ they naturally correspond to valuations on $\mathsf{Sub}(1_{\mathcal{T}})$ (3.2.6). The key lemma to prove these facts is that (over

8. Where $|\_|$ denote the cardinal of a set.

a boolean base topos) a hyperconnected separated topos is atomic with finite atoms (3.2.4).

3.2.1. The following definitions are due to Moerdijk and Vermeulen (see [55])

**Definition :**

- *A topos is said to be compact if its localic reflection is compact.*
- *A geometric morphism $f : \mathcal{E} \to \mathcal{T}$ is said to be proper if internally in $\mathcal{T}$, the $\mathcal{T}$-topos $\mathcal{E}$ is compact.*
- *A geometric morphism $f : \mathcal{E} \to \mathcal{T}$ is said to be separated if its diagonal map $\Delta : \mathcal{E} \to \mathcal{E} \times_{\mathcal{T}} \mathcal{E}$ is a proper map.*
- *A topos is said to be separated if its canonical geometric morphism to the base topos is separated.*

Being separated is a strong property, for example the topos of $G$-sets for a discrete group $G$ is separated if and only if $G$ is finite.

3.2.2. **Definition :** *An object $X \in |\mathcal{T}|$ is said to be separating if the slice topos $\mathcal{T}_{/X}$ is separated. A topos is said to be locally separated if it admits a separating inhabited object.*

As a slice of a separated topos (by a decidable object) is again separated, in a boolean topos as soon as one has an arrow $X \to Y$ with $Y$ separating, $X$ is again separating. In particular in a locally separated boolean topos every object can be covered by a separating object.

3.2.3. In addition, the class of separating objects also enjoy another stability property:

**Lemma :** *Let $f : X \twoheadrightarrow Y$ be a surjection in $\mathcal{T}$. Assume that $X$ is separating and that internally each fibre of $f$ is finite, then $Y$ is separating.*

**Proof :**
The fact that $f$ is a surjection with finite fibre exactly means that the induced map $\mathcal{T}_{/X} \to \mathcal{T}_{/Y}$ is a proper surjection (indeed, internally in $\mathcal{T}_{/Y}$ this means that the object $X \to Y$ is inhabited and finite). The result then follows immediately from [55, Prop II.2.1.(iii)], asserting that if $f \circ g$ is separated with $g$ a proper surjection then $f$ is separated. $\square$

3.2.4. **Proposition :** *Let $\mathcal{T}$ be a hyperconnected*[9] *separated topos. Then $\mathcal{T}$ is atomic and all its atoms are (internally) finite.*

**Proof :**
Let $\mathcal{T}$ be an hyperconnected topos. Let $\mathcal{B}$ be a non zero boolean locale endowed with a geometric morphism to $\mathcal{T}$. In the logic of $\mathcal{B}$, the pullback of the topos $\mathcal{T}$ is hyperconnected separated and has a point. Applied internally in $\mathcal{B}$, the theorem II.3.1 of [55] shows that the pullback of $\mathcal{T}$ is equivalent to the topos of $G$-sets for $G$ a compact localic groups in $\mathcal{B}$. In particular the pullback of $\mathcal{T}$ is atomic in the logic of $\mathcal{B}$ and as the map from $\mathcal{B}$ to the base topos is an open surjection (because the base topos is boolean) it implies that $\mathcal{T}$ is atomic in the base topos (by [44, C5.1.7]).
The finiteness of the atoms is proved in chapter 1, proposition 3.7.2.
$\square$

Note that the first part of the result (that the topos is atomic) uses the law of excluded middle in an essential way: the problem is that over a non boolean base topos the topos of continuous $G$-sets for $G$ an etale complete localic group is in general not atomic: it is atomic if and only if $G$ is open as a locale (because of [44, C3.5.14]).

3.2.5. The key technical point to deal with boolean separated toposes is to apply 3.2.4 internally into their localic reflection. Indeed, if $\mathcal{T}$ is a boolean separated topos, and if $\mathcal{L}$ denotes its localic reflection (which is a boolean locale) then the canonical map from $\mathcal{T}$ to $\mathcal{L}$ is hyperconnected and separated by [55, II.2.3 or II.2.5], and as $\mathcal{L}$ is boolean one can apply 3.2.4 to see that this map is atomic with (internally in $\mathcal{L}$) finite atoms.

**Definition :** *Let $\mathcal{T}$ be a boolean (separated) topos, we say that an object $U$ of $\mathcal{T}$ is a quasi-atom if it satisfies the following condition:*

- *There is a bijection between subobjects of $U$ and subobjects of its image $[U]$ in $1_{\mathcal{T}}$. This bijection being induced by the pullback and direct image along the canonical morphism from $U$ to $1_{\mathcal{T}}$.*
- *For each $U$ there is (externally) an integer $n$, such that, internally, the map from $U$ to $[U]$ is (internally) a map of degree $n$.*

The first condition alone is exactly the fact that $U$ corresponds to an internal atom of $\mathcal{T}$ defined over $U \subset \mathcal{L}$. The second axiom is related to the fact that atoms of separated toposes are finite.
One then has the following:

**Proposition :** *Let $\mathcal{T}$ be a boolean separated topos, then every object $\mathcal{T}$ is covered by subobjects which are quasi-atoms.*

**Proof :**

9. This means that its localic reflection is the point, or that $\mathsf{Sub}(1_{\mathcal{T}})$ is isomorphic to $\Omega$, see [44, A4.6].

As $\mathcal{T}$ is boolean, it suffices to prove that every non zero object $X$ of $\mathcal{T}$ contains a non-zero quasi-atom. Let $[X]$ be the image of $X$ in $1_{\mathcal{T}}$. Internally in $\mathcal{L}$, $X$ is an object of the $\mathcal{L}$-topos $\mathcal{T}$ such that $[X]$ implies that $X$ is inhabited. As $\mathcal{T}$ is, as an $\mathcal{L}$ topos, hyperconnected and separated 3.2.4 implies that $\mathcal{T}$ is atomic with finite atoms. In particular, still internally in $\mathcal{L}$, $X$ contains a finite atom. This means that $[X]$ can be covered by subobjects $V \subset 1_{\mathcal{T}}$ such that on each subobject $V$ one has a given internal atom of $X$, and, externally, this atom is a subobject $U$ of $X$ whose image in $1_{\mathcal{T}}$ is $V$, whose subobjects identify with subobjects of $V$ and which is internally finite. As $U$ internally has a cardinal (which is a section of $\mathbb{N}^{>0}$ defined over $V$) one can decompose $V$ as a disjoint sum over all positive integer of $V_n$ such that the part of $U$ over $V_n$ is a quasi-atom with degree $n$. As $V_n$ has to be non zero for at least one $n$, the part of $U$ over this $V_n$ gives a non-zero quasi-atom included in $X$. $\square$

Note in particular that quasi-atoms are internally finite objects and that (because subobjects of quasi-atoms identify with subobjects of the terminal object) a boolean separated topos is integrable if and only if its localic reflection is integrable.

3.2.6. **Proposition :** *Let $\mathcal{T}$ be a boolean separated topos, then if $\mu$ is a well supported locally finite measure on $\mathsf{Sub}(1_{\mathcal{T}})$, then:*

$$\tilde{\mu}(X) = \int_{1_{\mathcal{T}}} |X| d\mu,$$

*where $|X|$ denote the internal cardinal of $X$ which is a map in $\mathcal{T}$ from $1_{\mathcal{T}}$ to $\mathbb{N} \cup \{\infty\}$, is an invariant measure on $\mathcal{T}$.*
*Moreover $\mu$ is the restriction of $\tilde{\mu}$ to subobjects of $1_{\mathcal{T}}$ and every invariant measure on $\mathcal{T}$ is of the form $\tilde{\mu}$ for some locally finite well supported valuation on $\mathsf{Sub}(1_{\mathcal{T}})$.*

**Proof :**
We will first show that $\tilde{\mu}$ satisfies the three points $(IM1) - (IM3)$ of the definition 3.1.1.

(IM1) One easily sees that $\tilde{\mu}$ is a valuation on $\mathsf{Sub}(X)$ for any $X \in |\mathcal{T}|$ essentially because the cardinal is internally a valuation on $X$ and that the integral of an internal valuation is again a valuation (of course one needs to know that for our definition of integral the integral is linear and commutes to directed supremum of positive functions).

It remains to prove that this valuation is locally finite. But if one considers a quasi-atom $U$ of $\mathcal{T}$ (as in 3.2.5) such that $\mu([U])$ is finite, then $\tilde{\mu}(U) = n.\mu([U])$ with $n$ the degree of $U \to [U]$. Hence $\tilde{\mu}(U)$ is finite and as every object of $\mathcal{T}$ can be covered by such sub-objects it implies that the restriction of $\tilde{\mu}$ to any object is locally finite.

(IM2) The restriction of $\tilde{\mu}$ to a quasi-atom $U$ is exactly (through the identification of $\mathsf{Sub}(U)$ with $\mathsf{Sub}([U])$) the measure $\mu$ on $[U]$ times $n$. In particular it is well supported as $\mu$ is well supported. As quasi-atoms cover any object, they form a generating family.

(IM3) Let $f : X \twoheadrightarrow Y$ be a surjection between two objects in $\mathcal{T}$ such that $\tilde{\mu}(X) < \infty$. In particular, $|X|$ has to be finite everywhere on $\mathcal{L}$ (because $\mu$ is well supported) and hence, $X$ and $Y$ are internally finite. Let $Y_n$ be internally defined by:

$$Y_n = \{y \in Y || f^{-1}(y)| = n\}.$$

One has $Y = \coprod_n Y_n$. Let also $X_n = f^{-1}Y_n$, one has $X = \coprod X_n$.
One has internally $|X_n| = n.|Y_n|$ hence $\tilde{\mu}(X_n) = n\tilde{\mu}(Y_n)$, and $c_f$ is constant equal to $(1/n)$ over $X_n$. Hence:

$$\int_X c_f d\tilde{\mu} = \sum_n \int_{X_n} \frac{1}{n} d\tilde{\mu} = \sum_n \frac{\tilde{\mu}(X_n)}{n} = \sum_n \tilde{\mu}(Y_n) = \tilde{\mu(Y)}$$

which concludes the proof of this third point.

Now, if $U \in \mathsf{Sub}(1_{\mathcal{T}})$ then $|U|$ is just the characteristic function of $U$ hence $\tilde{\mu}(U) = \mu(U)$.
Finally, if $\nu$ is any invariant measure on $\mathcal{T}$, let $X \in |\mathcal{T}|$ be any object of $\mathcal{T}$ such that $\nu(X) < \infty$, then let $f$ be the function on $\mathcal{L}$ defined internally by $f = |X| \in \mathbb{N} \cup \{\infty\}$. For each $n \in \mathbb{N} \cup \{\infty\}$, let $U_n$ the subobject of $1_{\mathcal{T}}$ on which $f = |X| = n$. One has:

$$\frac{\nu(X \times U_n)}{n} = \nu(U_n)$$

And $\nu(U_\infty) = 0$ because $\nu(X) < \infty$. Hence:

$$\nu(S) = \sum_{n \in \mathbb{N} \cup \{\infty\}} \nu(S \times U_n) = \sum_{n \in \mathbb{N} \cup \{\infty\}} \frac{\nu(U_n)}{n} = \int f d\nu$$

As (from $(IM1)$) any object of $\mathcal{T}$ can be written as a union of objects $X$ such that $\nu(X) < \infty$ this formula holds for any object of $\mathcal{T}$. Hence $\nu$ is of the form $\tilde{\mu}$ for $\mu$ the restriction of $\nu$ on $\mathsf{Sub}(1_{\mathcal{T}})$. The valuation $\mu$ is well supported because if $\mu(U) = 0$ for some $U \subset 1_{\mathcal{T}}$ then for any object $S$ over $U$ one has $\nu(S) = 0$. If $U$ is non zero it will be in contradiction with $(IM2)$. Hence $\mu$ is a locally finite well supported valuation on $\mathsf{Sub}(1_{\mathcal{T}})$ and $\nu = \tilde{\mu}$. $\square$

3.2.7. **Lemma :** *Let $\mathcal{T}$ be a topos which is boolean, hyperconnected and has a generating family of finite objects. Then $\mathcal{T}$ is atomic and separated.*

**Proof :**
Let $X$ be a finite object of $\mathcal{T}$. As $\mathcal{T}$ is boolean every subobject of $X$ is finite and $\mathcal{P}(X)$ is also finite.
A finite object of $\mathcal{T}$ can only have a finite number of global sections, indeed as $1_{\mathcal{T}}$ has no non trivial subobject, two global sections of an object are either equal or internally distinct, hence an infinite number of distinct global sections would give internally an infinite number of distinct elements of $X$ which is impossible if $X$ is finite.

In particular, the set $\mathsf{Sub}(X)$ of global sections of $\mathcal{P}(X)$ is a finite boolean algebra, hence it is atomic. This proves that $\mathcal{T}$ is atomic. As $\mathcal{T}$ is generated by finite objects, every atom of $\mathcal{T}$ will be finite and hence by proposition 4.7.2 of chapter 1 it is separated. □

3.2.8. **Theorem :** *Let $\mathcal{T}$ be a boolean topos equipped with an invariant measure $\mu$ in the sense of 3.1.1, then:*

- *$\mathcal{T}$ is a integrable topos.*
- *The measure induced by $\mu$ on a object $X$ of topos is well supported if and only if $X$ is separating.*
- *The topos $\mathcal{T}$ is locally separated.*

**Proof :**
The first point is immediate: by definition $\mathcal{T}$ as a generating family of objects admitting well supported locally finite measures. Hence it has a generating family of objects $X$ such that $\mathsf{Sub}(X)$ is integrable, and as we noted in 2.1.4, this proves that $\mathcal{T}$ is integrable. The third point is an immediate corollary of the second: as $\mathcal{T}$ has a generating family of objects $X$ on which the measure $\mu$ is well supported it has a generating family of separating objects.
We now prove the point 2.. Let $X \in |\mathcal{T}|$ be a separating object of $\mathcal{T}$. The induced measure $\mu_X$ is a measure on the separated topos $\mathcal{T}_{/X}$ hence by 3.2.6 the restriction of $\mu$ to $\mathsf{Sub}(X)$ is well supported.
Conversely, let $X \in |\mathcal{T}|$ such that $\mu$ is well supported on $X$. Let $p : Y \to X$ be any object over $X$ such that $\mu(Y) < \infty$. Let $V = \{x \in X | p^{-1}(x) \text{ is infinite } \}$, and let $Y' = p^{-1}(V)$. As $\mu(Y')$ is finite one can apply $(IM3)$ to $f : Y' \to V$, it is a surjection, hence from the definition of $V$ one has $c_f = 0$, and hence $\mu(V) = 0$. As $\mu$ is well supported on $X$ this implies that $V = 0$, and hence each fibre of $p$ is finite: $(Y, p)$ is a finite object of $\mathcal{T}_{/X}$, which is hence generated by finite objects.
Let $\mathcal{L}$ be the localic reflection of $\mathcal{T}_{/X}$. One can apply 3.2.7 to $\mathcal{T}_{/X}$ internally in $\mathcal{L}$: over $\mathcal{L}$, $\mathcal{T}_{/X}$ is hyperconnected boolean and still generated by finite objects, hence $\mathcal{T}_{/X} \to \mathcal{L}$ is separated, as $\mathcal{L}$ is a boolean locale it is regular hence separated, and hence $\mathcal{T}_{/X}$ is separated, which concludes the proof.
□

## 3.3 The modular $\mathbb{R}^{\geq 0}_{\mathcal{T}}$ principal bundle.

In the previous sub-section we proved that in order to admit an invariant measure a topos must at least be integrable and locally separated, and that when it is separated its invariant measures are naturally parametrized by valuations on its localic reflection. The next question is naturally "What are the invariant measures on a Boolean integrable locally separated topos ?".

3.3.1. For this subsection, $\mathcal{T}$ will denote a boolean integrable locally separated topos. If $X$ is a separating object of $\mathcal{T}$, we will denote by $\chi(X)$ the set of locally finite well supported measures on $X$. Let $f : X \to Y$ be a map between two separating objects of $\mathcal{T}$ and $\mu \in \chi(Y)$, by 3.2.6 this defines an invariant measure $\tilde{\mu}$ on $\mathcal{T}_{/Y}$ and the restriction of $\tilde{\mu}$ to $\mathsf{Sub}(X)$ is a locally finite well supported measure, that we will denote by $f^*\mu$. This turns $\chi$ into a functor on the full category of separating objects.

**Proposition :** *The functor $\chi$ defined this way on the full category of separating objects is a sheaf for the restriction of the canonical topology of $\mathcal{T}$.*

**Proof :**
Let $f : X \twoheadrightarrow Y$ be a surjection between two separating objects. Let $P = X \times_Y X$ and $\pi_1, \pi_2$ the two maps from $P$ to $X$. As $P$ is defined over $X$ it is also a separating object. Let $\mu \in \chi(X)$ such that $\pi_1^*\mu = \pi_2^*\mu$, in order to conclude one needs to prove that $\mu = f^*\nu$ for some uniquely defined $\nu \in \chi(Y)$, i.e. that there exists a unique measure $\nu$ on $Y$ such that the restriction to $X$ of the corresponding invariant measure on $\mathcal{T}_{/Y}$ is $\mu$.
Applying 3.2.5 to the object $X$ of the separated topos $\mathcal{T}_{/Y}$, the object $X$ can be covered by quasi-atoms of $\mathcal{T}_Y$. For any $U \subset X$ a quasi-atom of $\mathcal{T}_{/Y}$ of degree $n$, any possible measure $\nu$ restricted to the image $f_!(U)$ of $U$ is necessarily given by $(1/n)\mu$ through the identification of $\mathsf{Sub}(U)$ with $\mathsf{Sub}(f_!(U))$. This proves that the measure $\nu$ is unique, all we have to check is that it exists by proving that if one chooses two quasi-atoms $U$ and $U'$ (of degree $n$ and $n'$) of $\mathcal{T}_{/Y}$ included in $X$ then the two possible values for $\nu$ they impose coincide on the intersection of their image in $Y$.
Let $V \subset f_!(U) \wedge f_!(U')$. As $V$ is a subobject of both $f_!(U)$ and $f_!(U')$ there exist two (uniquely defined) subobjects $U_1 \subset U$ and $U_1' \subset U'$ such that $f_!(U_1) = f_!(U_1') = V$.
Let $T = U_1 \times_Y U_1' \subset P$. By the hypothesis on $\mu$ one has $\pi_1^*\mu(T) = \pi_2^*\mu(T)$. As $f_!(U_1) = f_!(U_1')$, the map $\pi_1 : T \to X$ factors into $U_1'$ and has degree $n$ as a map from $T$ to $U_1'$ hence:

$$\pi_1^*\mu(T) = n\mu(U_1')$$

Similarly:

$$\pi_2^*\mu(T) = n'\mu(U_1)$$

where $n'$ denotes the degree of the map $U' \to f_!U'$.
Finally:

$$\frac{\mu(U_1)}{n} = \frac{\mu(U_1')}{n'}$$

i.e. the two possible definitions of $\nu(V)$ agree and this concludes the proof. (of course one still has to check that the measure $\nu$ constructed indeed satisfies $f^*\nu = \mu$ but it is immediate from its construction). $\square$

3.3.2. The previous proposition shows that, if $\mathcal{T}$ is integrable boolean and locally separated, there exists an object $\chi$ of $\mathcal{T}$ such that for each separating object $X$, morphisms from $X$ to $\chi$ identify with measures on $X$ and the "pullback" operation $f^*\mu$ defined for measures on separating objects corresponds simply to the composition $\mu \circ f$ (where $\mu$ denote the map with value in $\chi$).

3.3.3. If we are given an invariant measure $\mu$ on $\mathcal{T}$, then for each separating object $X$ of $\mathcal{T}$ there is a canonical measure $\mu_X$ on $X$ and for any map between two separating objects $f : X \to Y$ one has $f^*\mu_Y = \mu_X$ hence $\mu$ defines a global section on $\chi$.

**Theorem :** *For any boolean integrable locally separated topos $\mathcal{T}$ this construction defines a bijection between invariant measures on $\mathcal{T}$ and global sections of $\chi$.*

**Proof :**
Let $m : 1 \to \chi$ be a global section of $\chi$. If $X$ is a separating object of $\mathcal{T}$, then the map from $X$ to $\chi$ constant equal to $m$ defines a valuation $m_X$ on $\mathsf{Sub}(X)$. All these $m_X$ are compatible in the sense that if $f : X \to Y$ is a map between separating objects then $f^*m_Y = m_X$. If $Z$ is a general object of $\mathcal{T}$ one defines $Z_s$ to be the biggest separating object included in $Z$, and we define $\mu(Z) = m_{Z_s}(Z_s)$. We will now check that $\mu$ is indeed an invariant measure on $\mathcal{T}$:

(IM1) For any $Z \in |\mathcal{T}|$ and for any $Z' \subset Z$, one has $Z'_s = Z' \wedge Z_s$ hence $\mu(Z') = m_{Z_s}(Z' \wedge Z_s)$. Hence $\mu$ restricted to $\mathsf{Sub}(Z)$ corresponds to the valuation $m_{Z_s}$ on $Z_s$ and 0 on its complementary. Hence it is a locally finite valuation.

(IM2) In particular, on any separating object $X$, the restriction of $\mu$ to $\mathsf{Sub}(X)$ is $m_X$. Hence it is a well supported measure.

(IM3) The third axiom holds almost by definition when all the objects involved are separating, because of the relation $f^*m_Y = m_x$. The general idea is that the non separating objects do not change anything on any side of the equality because the measure will be zero on them, and because a separating object cannot be sent on a non-separating with a finite degree because of 3.2.3.

More precisely, let $Z$ and $Z'$ be two objects of $\mathcal{T}$ with $Z$ of finite measure, and let $f : Z \twoheadrightarrow Z'$ be a surjection. One decomposes $Z = Z_s \coprod Z_{ns}$ and $Z' = Z'_s \coprod Z'_{ns}$ where $Z_{ns}$ denotes the complementary of $Z_s$ in $Z$. One then decomposes $Z_s$ and $Z_{ns}$ in $Z_s = Z_s^s \coprod Z_s^{ns}$ and $Z_{ns} = Z_{ns}^s \coprod Z_{ns}^{ns}$ such that $Z_s^s$ and $Z_{ns}^s$ are mapped by $f$ into $Z'_s$ and that $Z_s^{ns}$ and $Z_{ns}^{ns}$ are mapped into $Z'_{ns}$.

$Z_{ns}^s$ is mapped into $Z'_s$ which is separating, hence $Z_{ns}^s$ is separating, and as it is a subobject of $Z_{ns}$ it has to be empty.

If $c_f$ (as in $(IM3)$) is non zero on some subset $Y$ of $Z_s^{ns}$ then the image of $Y$ by $f$ is a separating object by lemma 3.2.3, and hence, as it is a subobject of $Z'_{ns}$ it has to be empty. Hence $c_f$ is zero on $Z_s^{ns}$.

Finally:

$$\int_Z c_f d\mu = \int_{Z_s} c_f d\mu = \int_{Z_s^{ns}} c_f d\mu + \int_{Z_s^s} c_f d\mu$$

But as $c_f$ is zero on $Z_s^{ns}$, the integral $\int_{Z_s^{ns}} c_f d\mu$ is null. Hence:

$$\int_Z c_f d\mu = \int_{Z_s^s} c_f d\mu$$

But if we denote $f'$ the map $Z_s^s \to Z'_s$ then $f'$ is a surjection because $Z'_s$ is covered by the union of $Z_s^s$ and $Z_{ns}^s$, this second term being empty. Also, $c_{f'}$ is just the restriction of $c_f$ to $Z_s^s$, and $f'$ is a map between separating objects hence $(IM3)$ holds for it. Hence:

$$\int_Z c_f d\mu = \int_{Z_s^s} c_{f'} d\mu = \mu(Z'_s) = \mu(Z')$$

which concludes the proof that this associates an invariant measure to any global section of $\chi$.

Now the measure $\mu$ restricted to a separating object $X$ is by definition the measure $m_X$, hence the global section of $\chi$ associated to $\mu$ is $m$. And conversely, as any invariant measure $\mu$ is non zero only on the separating part of objects, it is reconstructed from our previous construction applied to the corresponding global section of $\chi$. This proves that these two constructions are inverse from each other. $\square$

3.3.4. Let $X$ be a separating object of $\mathcal{T}$. We denote by $\mathcal{L}_X$ the localic reflection of $\mathcal{T}_{/X}$, i.e. $\mathcal{O}(\mathcal{L}_X) = \mathsf{Sub}(X)$. Each morphism $\mu$ from $X$ to $\chi$ and each morphism $f$ from $X$ to $\mathbb{R}_{\mathcal{T}}^{\geq 0}$ corresponds to a well supported locally finite measure $\mu$ on $\mathcal{L}_X$ and a positive real valued function $f$ on $\mathcal{L}_X$. The product $f\mu$ is then again a well supported locally finite measure on $\mathcal{L}$ which in turns defines a morphism from $X$ to $\chi$, also denoted $f\mu$.
One can check that this product is functorial and hence it defines a product $\mathbb{R}_{\mathcal{T}}^{\geq 0} \times \chi \to \chi$.

**Proposition :** *Let $\mathcal{T}$ be a boolean integrable locally separated topos. Then for this action of $\mathbb{R}_{\mathcal{T}}^{\geq 0}$, the object $\chi$ is a $\mathbb{R}_{\mathcal{T}}^{\geq 0}$ principal bundle on $\mathcal{T}$.*

**Proof :**
$\chi$ is inhabited because if $X$ is a separating inhabited object of $\mathcal{T}$ any locally finite well supported valuation on $\mathsf{Sub}(X)$ gives a morphism from $X$ to $\chi$. Such a valuation exists assuming the axiom of choice because $\mathcal{T}$ is integrable, but they also exist locally on $X$ without assuming the axiom of choice, and this local existence is enough.
Let $a, b$ be two sections of $\chi$ on some object $Y \in \mathcal{T}$, and let $X$ be a separating object over $Y$, then over $X$, there exists a real valued function $f$ such that

$fa = b$ by the Radon-Nikodym theorem. Hence internally, $\forall a, b \in \chi \exists f \in \mathbb{R}^{>0}$ such that $a = fb$. Which proves that $\chi$ is a principal bundle. $\square$

From 3.3.3 and 3.3.4 one deduces that if there exists an invariant measure $\mu$ on a topos $\mathcal{T}$ then $\chi$ admits a global section and hence is isomorphic to the trivial bundle, and in this case, once we fix a global section of $\chi$, invariant measures on $\mathcal{T}$ are parametrized by positive real valued functions on $1_{\mathcal{T}}$ (i.e. on the localic reflection of $\mathcal{T}$). If on the contrary there is no invariant measure on $\mathcal{T}$, then $\chi$ is a non trivial $\mathbb{R}^{>0}_{\mathcal{T}}$ principal bundle and we will see in the next subsection that it can be used to describe a modular "time evolution".

3.3.5. $\chi$ will be called the *modular* principal bundle. For example, If $G$ is a locally profinite group, then the topos of $G - set$ is boolean (even atomic) locally separated (because if $U$ is a compact open subgroup of $G$ then $G/U$ is separating object). In this case $\chi$ is exactly $\mathbb{R}^{>0}$ endowed with action of $G$ by multiplication by the modular function $\delta : G \rightarrow \mathbb{R}^{>0}$. In particular, $\chi$ is trivial if and only if the group $G$ is unimodular, and in this case the invariant measure is given for any $U$ subgroups of $G$ by $\mu(G/U) = 1/\nu(U)$ where $\nu$ denotes the Haar measure of $G$.

3.3.6. We conclude this section by a lemma about the pullback of measures between separating objects which will be useful later.

**Lemma :** *Let $f : X \rightarrow Y$ be a map between two separating objects of a boolean topos. Let $\mu$ be a well supported*[10] *measure on $Y$ and let $f^*\mu$ be the pull back of $\mu$ as in 3.3.1. Then, for any positive function $h$ on $X$ one has:*

$$\int_{x \in X} h(x) df^*\mu = \int_{y \in Y} \left( \sum_{x \in f^{-1}(y)} h(x) \right) d\mu$$

*If $h$ is a function with value in $\mathbb{C}$ such that:*

$$\int_{x \in X} |h(x)| df^*\mu < \infty$$

*then the previous formula still holds.*

**Proof :**
In the case of positive functions, the result holds by definition when $h$ is the characteristic function of some subset of $X$ and the two sides are linear and compatible with supremums in $h$. Hence it holds for arbitrary functions.
For a complex function the point is to apply the result separately to each of the positive and negative part of the real and imaginary part of $h$. In order to do so, one needs to check that each of this four terms is finite, but this is implied by the condition of finiteness on the integral of $|h|$. $\square$

10. If $\mu$ is not well supported one can always replace $Y$ be the support of $\mu$.

## 3.4 Time evolution of Hilbert bundles.

3.4.1. From the modular principal bundle $\chi$ one can define the time evolution of Hilbert spaces of $\mathcal{T}$ in the same way as we did in chapter 1.
Let $F_t$, for $t \in \mathbb{R}$, be the sheaf defined as:

$$F_t = \{f : \chi \to \mathbb{C} | \forall \alpha \in \chi, \forall r \in \mathbb{R}^{>0}, f(r.\alpha) = r^{-it} f(\alpha)\}$$

$F_t$ is a $\mathbb{C}$-vector space for the natural action of $\mathbb{C}$ and if $a$,$b$ are two elements of $F_t$ then $\overline{a(\alpha)}b(\alpha)$ does not depend on $\alpha \in \chi$, indeed if $\alpha'$ is another element of $\chi$, then $\alpha' = r.\alpha$ for $r \in \mathbb{R}^{>0}$ and

$$\overline{a(\alpha')}b(\alpha') = r^{it}r^{-it}\overline{a(\alpha)}b(\alpha) = \overline{a(\alpha)}b(\alpha$$

Defining $\langle a, b\rangle$ as the value of $\overline{a(\alpha)}b(\alpha)(\alpha)$ gives a scalar product on $F_t$. Finally, the internal choice of any $\alpha \in \chi$ defines (by evaluation at $\alpha$ an isometric isomorphism $F_t \to \mathbb{C}$. Hence:

**Proposition :** *$F_t$ is internally a one dimensional Hilbert space for each $t \in \mathbb{R}$.*

This means that externally the $F_t$ are line bundles over $\mathcal{T}$.

3.4.2. If $f \in F_t$ and $g \in F_{t'}$ then their pointwise multiplication satisfies the relation:

$$f(r.\alpha)g(r.\alpha) = r^{-i(t+t')}f(\alpha)g(\alpha)$$

Hence this defines a map $F_t \otimes F_{t'} \to F_{t+t'}$. One easily checks that this map is isometric and hence one has:

**Proposition :** *The map $f \otimes g \to f.g$ induces a natural isomorphism $F_t \otimes F_{t'} \simeq F_{t+t'}$.*

3.4.3. The time evolution is then defined by: $\sigma_t(\mathcal{H}) = F_t \otimes \mathcal{H}$ for $\mathcal{H}$ an Hilbert space of $\mathcal{T}$. Equivalently, one can define:

$$\sigma_t\mathcal{H} = \{f : \chi \to \mathcal{H} | \forall \alpha \in \chi, \forall r \in \mathbb{R}^{>0}, f(r.\alpha) = r^{-it} f(\alpha)\}$$

The $\sigma_t$ are functors on the category of Hilbert spaces of $\mathcal{T}$.

## 3.5 The algebra $\mathcal{B}(l^2(X))$.

3.5.1. In this subsection, $X$ denotes a *separating* object of a boolean integrable locally separated topos $\mathcal{T}$. We also choose $\mu$ a locally finite well supported measure on $X$, which corresponds to a morphism $\lambda$ from $X$ to $\chi$.

We denote by $l^2(X)$ the Hilbert space internally defined as the set of square summable $X$-indexed sequences, with (internally) its generator $(e_x) \in l^2(X)$ for each $x \in X$. We also denote by $A = \mathcal{B}(l^2(X))$ the (external) algebra of globally bounded operators on $l^2(X)$. It is a von Neumann algebra (see 2.1.3).

3.5.2. $\mu$ can be used to construct a normal locally finite weight on $A$, also denoted $\mu$ defined by:

$$\forall f \in A^+, \mu(f) = \int_{x \in X} \langle x, fx \rangle \, d\mu$$

$\mu$ is locally finite because if $V$ is a subset of finite measure of $X$, and if $P_V \in A$ denotes the orthogonal projection on $l^2(V) \subset l^2(X)$ then $P_V f P_V$ has measure smaller than the measure of $V$ times the norm of $f$. And letting $V$ vary among all finite measure subsets of $X$, $P_V f P_V$ weakly converges to $f$.

3.5.3. $\lambda$ can be used to construct an isomorphism $\phi_t : l^2(X) \simeq \sigma_t l^2(X)$. Indeed, one can define (internally):

$$\phi_t(e_x) := \alpha \mapsto \left( \frac{\alpha}{\lambda(x)} \right)^{-it} e_x$$

where $\alpha/\lambda(x)$ denotes the unique element $r$ of $\mathbb{R}^{>0}$ such that $\alpha = r.\lambda(x)$. Defined this way $\phi_t(e_x)$ indeed satisfies the relation $\phi_t(e_x)(r.\alpha) = r^{-it}\phi_t(e_x)$, showing that it is an element of $\sigma_t(l^2(X))$. Moreover the $\phi_t(e_x)$ are of norm one and pair-wise orthogonal, hence $\phi_t$ indeed defines an isometric map $l^2(X) \to \sigma_t(l^2(X))$. It satisfies in particular, $\phi_0 = Id$ and $\sigma_t(\phi_{t'}) \circ \phi_t = \phi_{t+t'}$ hence $\sigma_t \phi_{-t}$ constitutes an inverse for $\phi_t$ showing that it is an isomorphism.

3.5.4. Finally, as $l^2(X)$ is “fixed” by the time evolution (as attested by the isomorphism $\phi_t$) one obtains an action of $\mathbb{R}$ directly on $A$, via:

$$\forall a \in A, \theta_t(a) = \phi_t^{-1} \sigma_t(a) \phi_t$$

One easily checks that it is an action of $\mathbb{R}$ (either directly or from the following proposition). This time evolution on $A$ can be more explicitly described on the matrix elements by:

**Proposition :** *For $a$ an element of $A$ and $x, y$ internal elements of $X$:*

$$\langle e_y, \theta_t(a) e_x \rangle = \left( \frac{\lambda(y)}{\lambda(x)} \right)^{-it} \langle e_y, a e_x \rangle$$

With $\frac{\lambda(y)}{\lambda(x)}$ denoting the unique element $r(x,y)$ of $\mathbb{R}^{>0}$ such that $r(x,y)\lambda(x) = \lambda(y)$.

**Proof :**
One has by definition of $\theta$ and as the $\phi_t$ are isometric:

$$\langle e_y, \theta_t(a) e_x \rangle = \langle \phi_t(e_y), \sigma_t(a) \phi_t(e_x) \rangle$$

But:

$$\phi_t(e_x) = \alpha \mapsto \left( \frac{\alpha}{\lambda(x)} \right)^{-it} e_x$$

And similarly for $y$, hence:

$$\begin{aligned}\langle \phi_t(e_y), \sigma_t(a)\phi_t(e_x)\rangle &= \left\langle \left(\frac{\alpha}{\lambda(y)}\right)^{-it} e_y, \left(\frac{\alpha}{\lambda(x)}\right)^{-it} ae_x \right\rangle \\ &= \left(\frac{\lambda(y)}{\lambda(x)}\right)^{-it} \langle e_y, ae_x\rangle\end{aligned}$$

□

3.5.5. Finally all these structures satisfy the modular, or Kubo-Martin-Schwinger, condition:

**Proposition :** *For each $u \in A_+$ such that $\mu(u)$ is finite one has $\mu(\theta_t(u)) = \mu(u)$.*
*Let $u, v \in A$ such that $\mu(u^*u)$,$\mu(uu^*)$,$\mu(v^*v)$ and $\mu(vv^*)$ are all finite. Then there exists a complex function $F_{u,v}$ defined on $\{z \in \mathbb{C} | Im(z) \in [-1, 0]\}$ and holomorphic on its interior such that for all real numbers $t$:*

$$F_{u,v}(t) = \mu(\theta_t(u)v) \qquad F_{u,v}(t - i) = \mu(v\theta_t(u))$$

This proves that $\theta_t$ is indeed the modular group of automorphisms of the algebra $A$, associated to the semi finite normal weight $\mu$. See [67, Ch. VII]. Also, if $\chi$ is trivial, and $\lambda$ is chosen to be a constant function from $X$ to $\chi$, then the formula 3.5.4 shows that $\theta_t$ is the identity for all $t$ and hence this result shows that $\mu$ is a normal semi-finite trace on $A$.

**Proof :**
From the formula given in 3.5.4 one can see that $\theta_t$ left unchanged the diagonal coefficients of $u$, and as $\mu$ is defined as the integral of the diagonal coefficients one immediately has that $\mu(\theta_t(u)) = \mu(u)$.
Let, for any $a \in A$, the function $a_x^y$ of matrix coefficients be defined internally by $a_x^y = \langle e_y, ae_x\rangle$ (it is a function on $X \times X$). A general formal computation gives that for $a, b \in A$:

$$\begin{aligned}\mu(ab) &= \int_{x\in X} \langle e_x, abe_x\rangle \, d\mu \\ &= \int_{x\in X} \left\langle e_x, \sum_{y\in X} ab_x^y e_y \right\rangle d\mu \\ &= \int_{x\in X} \sum_{y\in X} b_x^y a_y^x d\mu \\ &= \int_{(x,y)\in X^2} b_x^y a_y^x d\pi_1^*\mu\end{aligned}$$

The last equality, corresponds to lemma 3.3.6, and holds only if $b_x^y a_y^x$ is a positive function, or if the integral is finite when we replace $b_x^y a_y^x$ by $|b_x^y a_y^x|$.
In particular, it holds when $a, b$ are $(u^*, u)$,$(u, u^*)$,$(v^*, v)$ or $(v, v^*)$, and this together with the finiteness hypothesis on $u$ and $v$ shows that all the four integrals of $|a_x^y|^2$ and $|b_x^y|^2$ with respect to both $d\pi_1^*\mu$ and $d\pi_2^*\mu$ on $X \times X$ are finite. Also

note that under the correspondence between measures on $X \times X$ and functions from $X \times X$ to $\chi$, the measure $\pi_1^* \mu$ corresponds to the function $(x,y) \mapsto \lambda(x)$ and $\pi_2^* \mu$ to the function $(x,y) \mapsto \lambda(y)$. Hence one has:

$$\pi_2^* \mu = \left( \frac{\lambda(y)}{\lambda(x)} \right) \pi_1^* \mu \tag{2.1}$$

For any complex number $z$ such that $Im(z) \in [-1, 0]$ one has:

$$\begin{aligned} \left| \left( \frac{\lambda(y)}{\lambda(x)} \right)^{iz} u_y^x v_x^y \right| &= \left( \frac{\lambda(y)}{\lambda(x)} \right)^{-Im(z)} |u_y^x| |v_x^y| \\ &\leqslant |u_y^x|^2 + |v_x^y|^2 + \left( \frac{\lambda(y)}{\lambda(x)} \right) |u_y^x|^2 + \left( \frac{\lambda(y)}{\lambda(x)} \right) |v_x^y|^2 \end{aligned} \tag{2.2}$$

And the four terms on the right have a finite integral on $X \times X$ with respect to $\pi_1^* \mu$ (because of (2.1) for the last two), hence one can define the following function $F_{u,v}$ which is finite and continuous for $Im(z) \in [-1, 0]$ and the previous formal computation can be applied to it.

$$F_{u,v}(z) = \int_{(x,y) \in X \times X} \left( \frac{\lambda(y)}{\lambda(x)} \right)^{iz} u_y^x v_x^y d\pi_1^* \mu$$

Putting together the formal computation done for $\mu(ab)$ and the expression given in 3.5.4 for the matrix coefficients of $\theta_t(a)$, one has for $t$ real $F_{u,v}(t) = \mu(\theta_t(u)v)$, and using equation (2.1) one gets that:

$$\begin{array}{rcl} F_{u,v}(t-i) & = & \int_{(x,y) \in X \times X} \left( \frac{\lambda(y)}{\lambda(x)} \right)^{it} u_y^x v_x^y \frac{\lambda(y)}{\lambda(x)} d\pi_1^* \mu \\ & = & \int_{(x,y) \in X \times X} \left( \frac{\lambda(y)}{\lambda(x)} \right)^{it} u_y^x v_x^y d\pi_2^* \mu \\ & = & \int_{(y,x) \in X \times X} \left( \frac{\lambda(y)}{\lambda(x)} \right)^{it} u_y^x v_x^y d\pi_1^* \mu \\ & = & \mu(v \theta_t(u)) \end{array}$$

It remains to be proven that $F_{u,v}$ is holomorphic. Let $V_n$ be the subobject of $X \times X$ on which the function $\frac{\lambda(y)}{\lambda(x)}$ is between $(1/n)$ and $n$. one has $\bigcup V_n = X \times X$ and consider:

$$F_{u,v}^n = \int_{(x,y) \in V_n} \left( \frac{\lambda(y)}{\lambda(x)} \right)^{iz} u_y^x v_x^y d\pi_1^* \mu$$

The functions $F_{u,v}^n$ are holomorphic, and the inequality (2.2) shows that $F_{u,v}^n$ converges to $F_{u,v}$ uniformly in $z$ on all its domain of definition, showing that $F_{u,v}$ is holomorphic on the interior of its domain.

□

# Chapter 3

# Localic Banach space

## 1 Introduction

In [4], C.J.Mulvey and B.Banaschewski showed [1] that the usual Gelfand duality between abelian $C^*$-algebras and compact (Hausdorff) topological spaces can be extended into a "constructive" Gelfand duality between $C^*$-algebras and compact completely regular locales. A locale (see 2.1) is almost the same as a topological space, but may fail to have points. A locale which has enough points is called a spatial locale and is the same thing as a (sober) topological space. Assuming the axiom of choice, any locally compact locale has enough points; hence the result of Mulvey and Banaschewski gives back the usual Gelfand duality when assuming the axiom of choice. But the constructive version can be applied to a broader context: internal application to topos of sheaves over a topological space relates continuous fields of abelian $C^*$-algebra and proper maps to the base space. This can also be applied to to more general toposes.

At the end of their proof of the constructive Gelfand duality, Mulvey and Banaschewski suggested that "compact completely regular" is not the most natural condition one would have expected. It would be nicer to weaken this condition into "compact regular" (which is the same as compact separated, see [44] C3.2.10). Unfortunately, when a locale is not completely regular it might fail to have continuous $\mathbb{C}$-valued functions, and hence the associated $C^*$-algebra has no reason to keep track of enough informations about $X$. They suggest that their result should be extended into a duality between compact regular locales and a notion of localic $C^*$-algebras yet to be defined. This is a natural idea because when $X$ is a compact regular locale, one can still define a locale $[X, \mathbb{C}]$ of functions from $X$ to $\mathbb{C}$ and complete regularity only concerns the existence of points for this locale. The main goal of this chapter is to define this notion of localic $C^*$-algebras (which we will call $C^*$-locales) and to prove this conjectured duality.

Two other reasons for developing a theory of localic $C^*$-algebras and more gen-

1. To be more accurate, they only showed this result internally in Grothendieck toposes, using at some points an external argument relying on the axiom of choice (the Barr covering theorem). A completely internal and constructive proof has been given later by T.Coquand and B.Spitters in [20].

erally of localic Banach spaces (called Banach locales) are the following. In [53] I.Moerdijk showed (using the result of A.Joyal and M.Tierney in [45]) that Grothendieck toposes can be identified with a full subcategory of the 2-category of localic groupoids (that is groupoids in the category of locale, morphisms between them being the localic principal bi-bundles, see [11] for more details). A Banach space in the logic of the topos which corresponds to a localic groupoid $G_1 \rightrightarrows G_0$ is essentially a continuous field of Banach spaces $\mathcal{B}$ over $G_0$ endowed with a continuous action of $G_1$ such that there are enough local sections of $\mathcal{B}$ which have an open stabilizer. This hypothesis of open stabilizers is, from the point of view of analysis and geometry, a little too restrictive and is related to the requirement of existence of points. Hence one could expect that a good notion of Banach locale could remove it. Also for the purpose of non-commutative geometry one would like to be able to study equivariant bundle on general localic (topological) groupoids and not just those which correspond to toposes. For example the groupoid defined by $G_0$ being a point and $G_1$ being a connected locally compact topological group does not correspond to a topos but is an important groupoid for non-commutative geometry. In order to define the notion of Banach space over an arbitrary localic groupoid an important point is that this notion should descend along open surjections (see 2.5). Unfortunately, there is no such descent property for Banach spaces and $C^*$-algebras. However, as locales descend along open (or proper) surjections and as the pullback of Banach spaces is a pullback of the localic completion, we will be able to prove that Banach locales and $C^*$-locales have this descent property, and form in fact the "stackification" of the notion of Banach spaces and $C^*$-algebras, i.e. the smallest generalization of the notion which have this descent property.

Section 2 reviews some well known facts and definitions, mostly about the theory of locales, in order to fix the notation and prove some basic but important results for the rest of the paper. In section 3 we will develop the theory of metric locales in a constructive context (the classical theory is already known and can be found for example in [62]). We also show how to construct a classifying locale $[X, Y]_1$ for metric maps between two complete metric locales, which was apparently not known even in the classical case. In section 4 we apply the theory of section 3 in order to define Banach locales and $C^*$-locales and prove the announced result, although most of the technical difficulties lie in section 3. Finally in section 5 we prove a theorem asserting that for a large class of Grothendieck toposes (satisfying some technical conditions), which includes all paracompact topological spaces, there is no difference between Banach spaces and Banach locales. As mentioned in the general introduction, this last result is a topos theoretical adaptation of a theorem of Douady and Dal Soglio-Hérault asserting that over a paracompact topological spaces every Banach-bundle has enough continuous sections.

# 2 Notations and Preliminaries

## 2.1 The category of locales

We will start by briefly introducing the notion of locale, essentially in order to fix the notation and the vocabulary. A short introduction to this subject can be found in the first two sections of [8], a more complete one in part $C$ (especially in $C1$) of [44] and an extremely complete (but non constructive) one in [62].

2.1.1. A *frame* $\mathcal{F}$ is an ordered set which admits arbitrary supremums, also called union, (hence it also admits arbitrary infimums, called intersections) and such that binary intersections distribute over arbitrary unions, that is:

$$v \wedge \bigvee_{u \in I} u_i = \bigvee_{i \in I} (v \wedge U_i)$$

for each $v \in \mathcal{F}$ and each family $(U_i)$ of elements of $\mathcal{F}$. A frame homomorphism is a function which commutes to arbitrary unions (in particular to the bottom element, which is the supremum of the empty set) and to finite intersection (i.e. to both binary intersection and the top element which is the intersection of the empty family).

2.1.2. The category of *locales* is defined as the opposite category of the category of frames. But we will adopt "topological" notations for them:

- If $X$ is a locale, the corresponding frame is denoted by $\mathcal{O}(X)$.
- If $f : X \to Y$ is a morphism of locales, we denote by $f^*$ the corresponding frame homomorphism from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$.
- An element $U \in \mathcal{O}(X)$ is called an open sublocale of $X$, the top element of $\mathcal{O}(X)$ is denoted $X$.
- As $f^*$ commutes to arbitrary supremums, it has a right adjoint denoted $f_*$.

2.1.3. A *sublocale* of a locale $X$ is (an equivalence class of) a locale $Y$ endowed with a morphism $f : Y \to X$ such that $f^*$ is a surjective frame homomorphism (such a morphism is called an *inclusion*). A morphism of locale $f$ is said to be *surjective* if the corresponding frame homomorphism is injective. In particular, the injection/surjection factorisation of frame homomorphisms induces a unique (up to unique isomorphism) factorisation of every morphism of locale $f : X \to Y$ in a surjection followed by an inclusion:

$$X \twoheadrightarrow f_!(X) \hookrightarrow Y.$$

The sublocale $f_!(X)$ is called the image [2] of $f$. More generally if $S$ is any sublocale of $X$ we denote by $f_!(S)$ the image of the restriction of $f$ to $S$ and this is called the image of $S$ by $f$.

2.1.4. If $f : X \to Y$ is a morphism of locales and $S$ is a sublocale of $Y$ then the categorical pullback $f^{-1}(S)$ is a sublocale of $X$ and one has an adjunction formula:

$$A \subset f^{-1}(B) \Leftrightarrow f_!(A) \subset B$$

for any sublocale $A$ of $X$ and $B$ of $Y$.

2. From a purely categorical point of view, we should call it the regular image of $X$.

2.1.5. If $U$ is an element of the frame $\mathcal{O}(X)$ then it corresponds to a sublocale (also denoted $U$) of $X$ which is defined by the frame $\mathcal{O}(U) = \{v \in \mathcal{O}(X) | v \leqslant U\}$ and which is sent into $X$ by the morphism corresponding to $i^*(V) = V \wedge U$ for any $V \in \mathcal{O}(X)$. Hence, the elements of $\mathcal{O}(X)$ correspond to particular sublocales of $X$, which justifies the term "open sublocales" for elements of $\mathcal{O}(X)$. Also, through this identification, one has $f^*(U) = f^{-1}(U)$.

2.1.6. To any locale $X$ one can associate the topos of sheaves on $X$, denoted $\mathsf{Sh}(X)$. If $X$ and $Y$ are two locales, the category of geometric morphisms from $\mathsf{Sh}(X)$ to $\mathsf{Sh}(Y)$ is (equivalent to) the ordered set of locale morphisms from $X$ to $Y$ ordered by the pointwise ordering of the corresponding frame homomorphism (this is called the specialisation order). Hence locales can be seen as a specific kind of toposes.

2.1.7. An extremely important result of the theory of locales is that there is an equivalence of category between $X$-locales, that is locales in the logic of $\mathsf{Sh}(X)$ and locales $Y$ endowed with a morphism to $X$. This allows one to turn any reasonable property of locales into a property of geometric morphisms, corresponding to the relative notion, for example one says that a map $Y \to X$ is proper if the $X$-locale corresponding to $Y$ is compact in the logic of $\mathsf{Sh}(X)$. The same holds for toposes: a $\mathcal{T}$-topos is the same thing as a topos $\mathcal{E}$ endowed with a geometric morphism to $\mathcal{T}$.

2.1.8. At several points of this chapter we will deal (in simple situations) with locales as if they had points in order to define a map between two locales or to give constraints on some map. This kind of expression should of course not be interpreted in terms of points of a locale $X$ but in terms of "generalized points", that is morphisms from $T$ to $X$ for an arbitrary locale $T$, and all the constructions done on these points should be interpreted in the logic of $\mathsf{Sh}(T)$. If all the constructions on these generalized elements are "geometric" (that is compatible with the pullback from $\mathsf{Sh}(T)$ to $\mathsf{Sh}(T')$ for any locale $T'$ over $T$) then these constructions yield a morphism of functor, or relation between such morphisms and hence by the Yoneda lemma this indeed gives a morphism of locales or conditions between such morphisms.

## 2.2 Presentations, classifying spaces and pullbacks by geometric morphisms.

2.2.1. One says that a (Grothendieck) topos $\mathcal{T}$ classifies some structure $T$, or that $\mathcal{T}$ is the classifying topos of the theory $T$ if for any Grothendieck topos $\mathcal{E}$ there is an equivalence of categories (natural in $\mathcal{E}$) between the geometric morphisms from $\mathcal{E}$ to $\mathcal{T}$ and the model of the structure $T$ in the logic of $\mathcal{E}$. In particular, it requires that a model of $T$ can be pulled back along geometric morphisms. One can consult part $D$ of [44] for the general theory of classifying topos. Roughly this notion induces an equivalence between Grothendieck toposes and something called "geometric theory" up to a notion of equivalence.

2.2.2. One says that a locale $\mathcal{L}$ classifies some theory $T$ if the topos $\mathsf{Sh}(\mathcal{L})$ classifies the theory $T$. Locales are the classifying spaces of what is called propositional geometric theory. That is theory over a signature (see [44]D1.1.1) which contains no sorts. In particular it contains no function symbol and all the relations symbol it contains have no free variable and are called propositions.

2.2.3. A frame can be described by generators and relations. Basically, if one chooses a family of generators $(U_i)_{i \in I}$ and a family of relations $R$ of the form $T \leqslant T'$ where $T$ and $T'$ are expressions formed inductively using the $U_i$, the symbols $\top$ and $\perp$ (denoting the top element and the bottom element), arbitrary unions and finite intersections, then there exists a (unique up to unique isomorphism) frame $\mathcal{F} = \{I|R\}$ satisfying the following universal property: a morphism from $\mathcal{F}$ to an arbitrary frame $\mathcal{G}$ is the data of a family $V_i \in \mathcal{G}$ of elements indexed by $I$, such that all the relations in $R$ are satisfied by the $V_i$.

2.2.4. Another way to state the universal property of the frame $\mathcal{F} = \{I|R\}$ is to say that the corresponding locale is the classifying space of the propositional geometric theory on the basic propositions $(U_i)$ with axioms $T \Rightarrow T'$. As the $T$ are formed with the $U_i$, $\perp$ and $\top$ using finite intersection and arbitrary union they are interpreted as geometric propositions in the sense of [44, D1.1.3(f)].

2.2.5. There are other kinds of presentations of locales, for example the notion of Grothendieck site: if $\mathcal{C}$ is a category which is a pre-order (that is every of its morphisms set has at most one element) and $J$ is any Grothendieck topology on $\mathcal{C}$, then the category of sheaves for this topology is the category of sheaves over some locale $\mathcal{L}$, and we say that $(\mathcal{C}, J)$ is a site for $\mathcal{L}$. There are lots of other notions of presentation like "sup-lattice presentation" (see [45]) and "preframe presentation" (see [42]). See also [70] for a comparison of these last two approaches.

2.2.6. If $\mathcal{L}$ is a locale in the logic of some topos $\mathcal{T}$ and if $f : \mathcal{E} \to \mathcal{T}$ is a geometric morphism then, $f^*\mathcal{O}(\mathcal{L})$ is in general not a frame in $\mathcal{E}$, but it can be completed in a frame, giving rise to locale called $f^{\#}(\mathcal{L})$ in $\mathcal{E}$. More precisely, if one takes any sort of presentation of $\mathcal{L}$, then one can pullback the presentation through $f$ and construct a locale $\mathcal{L}'$ in $\mathcal{E}$. One can then check from the universal property that (for any reasonable kind of presentations) one has the following pullback diagram:

$$\begin{tikzcd}
\mathsf{Sh}_{\mathcal{E}}(\mathcal{L}') \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
\mathsf{Sh}_{\mathcal{T}}(\mathcal{L}) \arrow[r] & \mathcal{T}
\end{tikzcd}$$

which shows that $\mathcal{L}'$ does not depend on any choice of the presentation, and hence can be denoted $f^{\#}(\mathcal{L})$. This construction is very practical as it allows one to easily obtain presentations of $f^{\#}(X)$, or to easily know what will be the pullback when $\mathcal{L}$ is given by a presentation or as the classifying locale of some theory.

## 2.3 Positivity

2.3.1. **Definition :**

- *A locale $\mathcal{L}$ is said to be positive, if whenever we can write $\mathcal{L}$ as a union of open sublocales:*

$$\mathcal{L} = \bigvee_{i \in I} u_i$$

*the set of indices $I$ has to be inhabited. In this case, we write $\mathcal{L} > \emptyset$.*
- *A locale $\mathcal{L}$ is said to be locally positive if every open sublocale can be written as a union of positive open sublocales.*

If one assumes the law of excluded middle, then an open sublocale is positive if and and only if it is non-zero and every locale is locally positive (any non-zero element is the union of just itself, and the zero element is the empty union). But without the law of excluded middle this becomes a non trivial property.

2.3.2. If $X$ is a locale (preferably locally positive) we will denote by $\mathcal{O}(X)^+$ the subset of positive open sublocales of $X$.

2.3.3. Local positivity is closely related to the notion of open map:

**Proposition :** *Let $f : \mathcal{L} \to \mathcal{M}$ be a morphism of locale, then the following conditions are equivalent:*

- *For any $U$ open sublocale of $\mathcal{L}$, its image $f_!(U)$ is an open sublocale of $\mathcal{M}$;*
- *The frame morphism $f^* : \mathcal{O}(\mathcal{M}) \to \mathcal{O}(\mathcal{L})$ has a left adjoint $f_\circ$ (i.e. $f_\circ(U) \leqslant V$ if and only if $U \leqslant f^*(V)$) which satisfies the additional identity:*

$$f_\circ(U \wedge f^*(V)) = (f_\circ U) \wedge V;$$

- *$\mathcal{L}$ is locally positive as a $\mathcal{M}$-locale.*

*Moreover in this situation, $f_\circ$ is the same as $f_!$ (restricted to open sublocales) and it corresponds to the internal map which associates to every $U \in \mathcal{O}(\mathcal{L})$ the $\mathcal{M}$-proposition " $U$ is positive ".*

For a proof, see [8]1.6.1 and 1.6.2 for the equivalence of the first two points, and see [44] C3.1.17 for the last point.

2.3.4. The following lemma will often be useful to prove that some locales are locally positive:

**Lemma :** *Let $X$ be a locale, and $p$ the morphism from $X$ to the point $\{*\}$. Assume that there is a basis $(b_i)_{i \in I}$ of $X$ and a collection of propositions $(w_i)_{i \in I}$ such that:*

$$w_i \Rightarrow (b_i) > \emptyset$$
$$b_i \leqslant p^* w_i$$

*Then $X$ is positive, $w_i$ is equivalent to $b_i > \emptyset$ and an arbitrary open sublocale of $X$ is positive if and only if it contains one of the $b_i$ such that $b_i > \emptyset$.*

**Proof :**
As the $b_i$ form a basis, any $U \in \mathcal{O}(X)$ can be written as:

$$U = \bigvee_{\substack{i \in I \\ b_i \leqslant U}} b_i$$

but as $b_i \leqslant p^*(w_i) = \bigvee_{w_i} \top$ one has:

$$U = \bigvee_{\substack{i \in I \\ b_i \leqslant U}} p^*(w_i) \wedge b_i = \bigvee_{\substack{i \in I \\ b_i \leqslant U \text{ and } w_i}} b_i$$

as $w_i$ implies that $b_i$ is positive, this is an expression of $U$ as a supremum of positive open sublocales, proving that $X$ is locally positive. Now $w_i \Rightarrow b_i > \emptyset$ and as $b_i = \bigvee_{w_i} b_i$ one also has $b_i > \emptyset \Rightarrow w_i$, which proves the equivalence between $w_i$ and "$b_i$ is positive". Finally if $U$ is positive, then from the previous expression of $U$ as a union, there exists an $i$ such that $b_i \leqslant U$ and $w_i$ hence $b_i$ is positive, and conversely if $U$ contains a positive $b_i$ then $U$ is itself positive.
□

2.3.5. **Proposition :** *A locale $\mathcal{L}$ is locally positive if and only if it can be defined by a Grothendieck site where each covering is inhabited. In this situation, an open $U$ of $\mathcal{L}$ is positive if and only if it contains one of the representable.*

This is essentially the localic version of [44, C3.1.19].

2.3.6. **Proposition :** *Let $X$ be a locally positive locale in a topos $\mathcal{T}$ and $f : \mathcal{E} \to \mathcal{T}$ a geometric morphism. Then $f^{\#}(X)$ is also locally positive, and (internally in $\mathcal{E}$) an open $f^*(U) \in f^*(\mathcal{O}(X))$ is positive if and only if $f^*($"$U > \emptyset$"$)$.*

**Proof :**
If one has a site of definition $(C, J)$ for $\mathcal{L}$ in which each covering relation is inhabited then $f^*(C, J)$ also has this property and it is a site of definition for $f^{\#}(\mathcal{L})$. Hence this is an immediate corollary of the previous proposition. □

2.3.7. Once we replace the idea of "having points" by "being positive and locally positive" to state that a locale is inhabited one can obtain a constructive version of "the axiom of choice" in the form of:

**Proposition :** *Let $I$ be a set with decidable equality and let $(X_i)_{i\in I}$ be a family of positive and locally positive locales. Then $\prod_{i\in I} X_i$ is positive and locally positive.*

**Proof :**
Open surjections are stable by composition and pullback ([44, C3.1.11]), hence if $X_1, \dots X_n$ are locally positive locales, then $\prod_{i=1}^{n} X_i$ also is. In general, a base of open sublocales of $\prod_{i\in I} X_i$ is given by the finite intersections of open sublocales of the form $\pi_i^*(U)$ for $U$ as open of $X_i$. If $I$ is decidable, each of these open sublocales can be rewritten as an intersection $\pi_{i_1}^*(U_1) \wedge \dots \wedge pi_{i_k}^*(U_k)$ with the $i_j$ pairwise distinct. Moreover, as each $X_i$ is locally positive such an open sublocale can be written as a union of sublocales of the same form but with the $U_i$ positive. As locales these open sublocales can be identified with $\prod_{i=1}^{n} U_i$ which is positive if each $U_i$ is positive, hence one has given a basis of positive open sublocales of the product $\prod_{i\in I} X_i$. This concludes the proof. $\square$

2.3.8. Note that the hypothesis that $I$ has a decidable equality cannot be removed, and in fact cannot be weakened too much as the following proposition shows:

**Proposition :** *Let $I$ be a set which is a quotient of a set $J$ with a decidable equality. Then $\prod_{i\in I} X_i$ is positive and locally positive for any family of positive and locally positive locale $X_i$ if and only if $I$ has a decidable equality.*

Together with lemma 2.4.7 this shows that we essentially only have a (localic) axiom of choice indexed by decidable sets. This should probably be related to Diaconescu theorem asserting that the full axiom of choice imply the law of excluded middle (the law of excluded middle is equivalent to every set having a decidable equality).
This proposition is not used anywhere in this thesis but answer a question asked to us by the reviewer S.Vickers.

**Proof :**
Let $p : J \twoheadrightarrow I$ be the surjection. Let $(X_i)$ be the discrete (hence locally positive) locale whose points are $p^{-1}(i)$. As $p$ is a surjection, each $p^{-1}(i)$ is inhabited and hence the $X_i$ are also positive.
Let $Y = \prod_i X_i$, i.e. $Y$ is the space of sections of $p$. Let $f$ be the canonical map from $Y$ to the base topos. In the logic of $Y$ the map $f^*(p) : f^*(J) \rightarrow f^*(I)$ has a section (the universal section of $p$) hence $f^*(I)$ can be identified with a subobject of $f^*(J)$, in particular $f^*(I)$ is decidable. Now if we assume that $Y$ is positive and locally positive then $f$ is an open surjection and hence $f^*$ preserves (and reflects) all first order logic (and not just geometric logic) and as " $I$ is decidable" is a statement of first order logic the decidability of $f^*(I)$ imply the decidability of $I$. $\square$

2.3.9. We also have a constructive version of the axiom of dependent choice:

**Proposition :** *Let $X$ be an inhabited set equipped with a relation $R$ such that for each $x \in X$ there exists $y \in X$ with $xRy$. Then the sublocale of $X^{\mathbb{N}}$ which classifies the sequences $(x_n)$ such that for each $n$ one has $x_n R x_{n+1}$ is positive and locally positive.*

This is proved in [56] as lemma $C$.

## 2.4 Positivity and fiberwise density.

2.4.1. **Definition :** *A geometric morphism $f : \mathcal{M} \to \mathcal{L}$ is said to be fiberwise dense (or to have a fiberwise dense image) if for any proposition $U$, one has the relation:*

$$p^*(U) = f_* f^* p^*(U)$$

*where $p$ denotes the canonical map $\mathcal{L} \to \{*\}$ and $U$ is identified with an open sublocale of $\{*\}$.*
*A sublocale $S \subset \mathcal{L}$ is said to be fiberwise closed if it is fiberwise dense in no other sublocale of $\mathcal{L}$.*

We will soon see that in the presence of the law of excluded middle these are equivalent to the more classical notions of density and closeness, but in general fiberwise density only implies density, and closeness only implies fiberwise closeness. For this reason they have also been called "strongly dense" and "weakly closed", but we prefer the terminology "fiberwise" which is more uniform, more specific and allows less confusions. This name "fiberwise" comes from the fact that, when interpreted internally in $\mathsf{Sh}(X)$ for a (nice enough) topological space $X$, it indeed corresponds to a notion of fiberwise density (and fiberwise closeness) of morphisms of locales over $X$ whereas the usual notion of density would correspond to simple density, without taking the basis into account.
Of course every sublocale $S$ admits a fiberwise closure $\overline{S}$ which is the smallest fiberwise closed sublocale containing $S$, or equivalently, the unique fiberwise closed sublocale in which $S$ is fiberwise dense.

2.4.2. In the case of locally positive locales, the fiberwise density takes the following simpler form.

**Proposition :** *Let $f : X \to Y$ be a map with $X$ locally positive. Then the following conditions are equivalent:*
*(a) $f$ is fiberwise dense.*
*(b) $Y$ is locally positive, and for any positive open sublocale $U$ of $Y$, $f^*(U)$ is positive.*

In presence of the law of excluded middle, every locale is locally positive and a positive open sublocale is just a non-zero open sublocale. Hence the previous proposition asserts (in presence of the law of excluded middle) that $f$ is fiberwise dense if for every non zero open sublocale $f^*(U)$ is also non zero, which is a classical characterisation of a dense map.

**Proof :**

Let $p$ denotes the canonical map from $Y$ to $\{*\}$.

Assume $(a)$. Let $U$ be an open sublocale of $Y$. As $X$ is locally positive one has

$$f^*(U) \leqslant f^*p^*(\text{``}f^*(U) > \emptyset\text{''})$$

By adjunction one gets

$$U \leqslant f_*f^*p^*(\text{``}f^*(U) > \emptyset\text{''})$$

but by strong density of $f$, $f_*f^*p^* = p^*$, also, in full generality $f^*(U) > \emptyset \Rightarrow U > \emptyset$ hence, by 2.3.4 this proves both that $Y$ is locally positive, and that the positivity of $U$ implies the positivity of $f^*(U)$.

Now assume $(b)$. Let $V$ be some proposition, in general, one has $p^*(V) \leqslant f_*f^*p^*(V)$. We need to prove the reverse inequality. First assume that for some positive element $u$ one has $u \leqslant f_*f^*p^*(V)$ then $f^*(u) \leqslant f^*p^*(V)$ but, by hypothesis, $f^*(u)$ is also positive, and hence this inequality implies that $V$ holds. In particular, $u \leqslant p^*(V)$.

We have proved that, for all positive $u$ such that $u \leqslant f_*f^*p^*(V)$ one has also $u \leqslant p^*(V)$. By locale positivity of $Y$ this concludes the proof. □

2.4.3. **Corollary :** *Let $f : X \to Y$ be a surjection with $X$ locally positive, then $Y$ is locally positive.*

**Proof :**

A surjection is in particular a fiberwise dense map. □

2.4.4. **Proposition :** *A fiberwise dense sublocale of a locally positive sublocale is also locally positive.*

**Proof :**

Let $X$ be a locally positive locale and $S$ be a fiberwise dense sublocale of $X$. Let $i$ be the inclusion of $S$ into $X$, $f$ the canonical map from $X$ to the point and $g$ the canonical map from $S$ to the point ($g = f \circ i$). We define $g_! = f_! i_*$ and we check that $g_!$ is indeed a left adjoint for $g^*$. As $S$ is fiberwise dense, one has: $i_*g^* = i_*i^*f^* = f^*$ hence:

$$\begin{aligned} g_!(u) \leqslant v \quad &\Leftrightarrow \quad i_*(u) \leqslant f^*(v) = i_*g^*(v) \\ &\Leftrightarrow \quad u \leqslant g^*(v) \text{ because } i_* \text{ is injective.} \end{aligned}$$

This concludes the proof (there is no need to check the identity $g_!(a \wedge g^*b) = g_!(a) \wedge b$ because it is automatic when $g$ is a map to $\{*\}$). □

2.4.5. **Proposition :** *If $g : X \to Y$ is a fiberwise dense map between two locally positive locales, then any pullback of $g$ by a geometric morphism is also fiberwise dense.*

**Proof :**
Let $f : \mathcal{E} \to \mathcal{T}$ be a geometric morphism between two toposes, and assume that $g : X \to Y$ is a fiberwise dense geometric morphism between two locally positive locales in the logic of $\mathcal{T}$.
The function $f^{\#}(g) : f^{\#}(X) \to f^{\#}(Y)$ can be constructed the following way: $f^*(\mathcal{O}(X)^+)$ and $f^*(\mathcal{O}(Y)^+)$ are basis of positive open sublocales of $f^{\#}(X)$ and $f^{\#}(Y)$, and $\tilde{g} = f^{\#}(g)$ is defined by the fact that $\tilde{g}^*$ send $f^*(\mathcal{O}(Y)^+)$ to $f^*(\mathcal{O}(X)^+)$ by $f^*(g^*)$ (because as $g$ is fiberwise dense it preserve positivity by 2.4.2). But as an open sublocale $U$ of $f^{\#}(Y)$ is positive if and only if it contains an element of $f^*(\mathcal{O}(Y)^+)$ this proves that if $U$ is positive then $g^*(U)$ is positive, and hence concludes the proof by 2.4.2.
□

A counterexample to this proposition without the local positivity assumption can be found in [44] right after corollary $C.1.2.16$.

2.4.6. **Definition :** *A locale $\mathcal{L}$ is said to be weakly spatial if there exists a fiberwise dense map $P \to \mathcal{L}$ with $P$ a spatial locale (or simply, with $P$ a set).*

By 2.4.2, a weakly spatial locale is automatically locally positive, and a locally positive locale is weakly spatial if and only if every positive open sublocale has a point.

2.4.7. **Lemma :** *Let $X$ be any object of the base topos, then there exists a positive locally positive locale $\mathcal{L}$, with $p$ the canonical geometric morphism from $\mathsf{Sh}(\mathcal{L})$ to the base topos, such that $p^*X$ is the quotient of an object $I$ of $\mathsf{Sh}(\mathcal{L})$ which has decidable equality.*

**Proof :**
If the base topos is a Grothendieck topos over a boolean topos, then any covering of the base topos by a locale will qualify because over a locale any sheaf is a quotient of a decidable sheaf.
In the more general situation of an elementary topos, one can take $\mathcal{L}$ to be the classifying space for partial surjective maps from $\mathbb{N}$ to $X$. It is always a positive locally positive locale (see [45]$V$.3 just after proposition 2), and in $\mathsf{Sh}(\mathcal{L})$ the object $p^*X$ is naturally a quotient of a subobject of $\mathbb{N}$, which is decidable. □

2.4.8. **Proposition :** *Let $X$ be a locally positive locale (of the base topos), then there exists a topos $\mathcal{T}$ (even a locale) such that the canonical geometric morphism $p : \mathcal{T} \to *$ is an open surjection and such that $p^{\#}(X)$ is weakly spatial in $\mathcal{T}$.*

This result will be extremely important in the rest of the chapter: indeed weak spatiality will play the same role as spatiality for complete metric spaces (see 3.6), and as locales descend along open surjections this result will roughly allow us to assume whenever needed that all the metric locales involved come from metric sets.

**Proof :**
Thanks to the previous lemma, one can construct a locale $\mathcal{L}$ in which one has a basis $(U_i)_{i \in I}$ of positive open sublocales of $p^{\#}(X)$ indexed by a set with decidable equality. By 2.3.7:

$$Y = \prod_{i \in I} U_i$$

is a positive locally positive locale, and corresponds to an open surjection (also denoted $p$) $p : \mathsf{Sh}_{\mathcal{L}}(Y) \to \mathcal{L} \to *$. We will now prove that $p^{\#}(X)$ is weakly spatial.
Internally in $\mathcal{L}$, there is a canonical map $s_i : Y \to X \times Y$ defined as the composition of the i-th projection and the inclusion of $U_i$ into $X$ on the first component and the identity of $Y$ on the second component. This defines a map of locale over $Y$:

$$s : \coprod_{i \in I} Y \to X \times Y = p^{\#}(X)$$

which internally in $\mathsf{Sh}_{\mathcal{L}}(Y)$ gives a map $s$ from $f^*(I)$ to $p^{\#}(X)$ such that for each $i$, $s(i)$ is a point of $U_i$. As any positive open sublocale of $p^{\#}(X)$ contains one of the $U_i$, it shows that $p^{\#}(X)$ is weakly spatial. $\square$

## 2.5 Descent theory

In this section we will consider a kind of structure interpretable in a topos that can be pulled back along geometric morphisms. The term "structure" has to be interpreted in an extremely loose sense, as the main example we have in mind is the category of locales internal to a topos with the pullback as constructed in 2.2.6.
More precisely we just want to have a contravariant functor $\mathcal{C}$ from the 2-category of toposes to the 2-category of categories, like for example the functor which sends every topos $\mathcal{T}$ to the category of internal locales of $\mathcal{T}$. We will denote by $f^*$ the action of a geometric morphism $f$ on $\mathcal{C}$.

Let $f : \mathcal{E} \to \mathcal{T}$ be a geometric morphism, and let $c \in |\mathcal{C}(\mathcal{E})|$. A descent data on $c$ is the data of an isomorphism $\epsilon : \pi_1^*(c) \to \pi_2^*(c) \in \mathcal{C}(\mathcal{E} \times_{\mathcal{T}} \mathcal{E})$, such that if $\Delta$ denotes the diagonal map $\Delta : \mathcal{E} \to \mathcal{E} \times_{\mathcal{T}} \mathcal{E}$ then $\Delta^*(\epsilon)$ identifies with the identity map of $c$, and if $\pi_{1,2}, \pi_{1,3}$ and $\pi_{2,3}$ denote the three projections

$\mathcal{E} \times_{\mathcal{T}} \mathcal{E} \times_{\mathcal{T}} \mathcal{E} \to \mathcal{E} \times_{\mathcal{T}} \mathcal{E}$ and $\pi_1$, $\pi_2$ and $\pi_3$ the three projections from $\mathcal{E} \times_{\mathcal{T}} \mathcal{E} \times_{\mathcal{T}} \mathcal{E}$ to $\mathcal{E}$ then one has a commutative diagram:

$$\begin{tikzcd} \pi_1^*(c) \arrow[r, "\pi_{12}^*\epsilon"] \arrow[dr, "\pi_{13}^*\epsilon"] & \pi_2^*(c) \arrow[d, "\pi_{23}^*\epsilon"] \\ & \pi_3^*(c) \end{tikzcd}$$

We define $Des(f, \mathcal{C})$ to be the category of objects of $\mathcal{C}(\mathcal{E})$ endowed with a descent data (and morphisms being the morphisms in $\mathcal{C}(\mathcal{E})$ whose pullback along $\pi_1$ and $\pi_2$ commute to the $\epsilon$). If $c_0 \in \mathcal{C}(\mathcal{T})$ then $f^*c$ is naturally endowed with a descent data and this defines a functor from $\mathcal{C}(\mathcal{T})$ to $Des(f, \mathcal{C})$. One says that objects of $\mathcal{C}$ descend along $f$, or that $f$ is a descent morphism [3] for $\mathcal{C}$ if this functor induces an equivalence between $\mathcal{C}(\mathcal{T})$ and $Des(f, \mathcal{C})$.
It is for example proved in [45] that both objects and locales descend along open surjections. That is, for $\mathcal{C}(\mathcal{T}) = \mathcal{T}$ and $\mathcal{C}(\mathcal{T})$ being the category of internal locales of $\mathcal{T}$ the geometric morphisms which are open and surjective are descent morphisms.

In another language, the fact that objects of $\mathcal{C}$ descend along all open surjections, or more generally along all geometric morphisms belonging to some Grothendieck topology one the 2-category of topos exactly means that $\mathcal{C}$ is a *stack* for this topology.

## 2.6 Spaces of numbers

2.6.1. As mentioned in the introduction we are assuming that the base topos has a natural number object denoted by $\mathbb{N}$. This essentially means that $\mathbb{N}$ is a model of Peano arithmetic, see [44, A2.5 and D5.1] for more details on this notion. From a natural number object one easily defines the set $\mathbb{Z}$ of relative integers (as two copies of $\mathbb{N}$) with all its operations. And the set $\mathbb{Q}$ of rational numbers is then defined as the quotient of $\mathbb{Z} \times \mathbb{N} - \{0\}$ by the proportionality relation.

2.6.2. When working in a Grothendieck topos, or more generally in a topos $\mathcal{T}$ equiped with a geometric morphism to the topos of sets, the three objects $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are simply the constant sheaves with value $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$. In particular any first order property [4] provable in classical set theory also holds in $\mathcal{T}$. When we are working in a general elementary topos (with a natural number object) as

3. We follow the terminology of [44], it is in fact more common to say that $f$ is an effective descent morphism.

4. Obviously this is true for geometric properties, but as the topos of sets is boolean any first order property can be written in the form of several geometric properties by explicitly adding a proposition for each required negation and a property stating that this is indeed the negation of something. This process is essentially what is called Morleyization of a theory, see the statement and the proof of [44, D.1.5.13]

we are doing in this chapter, one can still prove that $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ have decidable equality and that anything which is provable in Peano arithmetic (with the law of excluded middle) also holds internally.

2.6.3. The definition of real numbers becomes a little more complex. Of course as we do not assume the axiom of dependent choice the definition of $\mathbb{R}$ by Cauchy sequences should be avoided. A definition by one-sided Dedekind cuts gives a set of real numbers with good properties as an ordered set, but poor algebraic properties (it is in general not even possible to define the opposite of an element). On the contrary a definition by two-sided Dedekind cuts gives good algebraic properties (the reals no longer form a field, but they still form a locale ring) but fails to yield good order properties: the fact that every bounded set admits a supremum no longer holds. The two-sided Dedekind cuts will be used when we need to perform algebraic operations on the numbers, for example when they arise as scalars of a vector space. And the one-sided Dedekind cuts will be used when we are only interested in the order relation: for example the distance function will take value in one sided Dedekind cuts as suggested by earlier works of C.J Mulvey. Moreover, we will not consider the "set of real numbers" as set but always as a locale. It appears that without the law of excluded middle, the topological space $\mathbb{R}$ can loose some of its good properties (like local compactness) whereas the locale of (two-sided) real numbers always has good topological properties (See [44, D4.7]).

2.6.4. More precisely, a Dedekind real number or a continuous real number is a pair $(L, U)$ of subsets of $\mathbb{Q}$ such that:

- Both $L$ and $U$ are inhabited.
- If $q \in U$ and $q' > q$ then $q' \in U$, and symmetrically if $q \in L$ and $q' < q$ then $q' \in L$
- If $q \in U$ then there exists $q'$ such that $q' < q$ and $q' \in U$ and symmetrically if $q \in L$ then there exists $q'$ such that $q' > q$ and $q' \in L$
- for all $q < q'$ one has either $q \in L$ or $q' \in U$

Of course once we have defined the embeddings of $\mathbb{Q}$ and proved the basic properties of this set, when $x$ is a Dedekind real number the corresponding $U$ and $L$ are simply $U = \{q | x < q\}$ and and $L = \{q | q < x\}$. See [44, D4.7] for the structure on two-sided Dedekind cuts.
Finally $\mathbb{R}$ will denote the classifying locale of the geometric propositional theory of Dedekind real numbers. When it is spatial (for example in presence of the law of excluded middle) it is the set of real numbers endowed with its classical topology. In any case, it agrees with the localic completion (as we define in 3.3.12) of $\mathbb{Q}$ for the Archimedean distance.

2.6.5. Similarly we will define a locale $\overleftarrow{\mathbb{R}_+^\infty}$ in which the distance function will take value. As earlier work of C.J.Mulvey showed we only care about knowing when a distance is smaller than some rational number, hence $\overleftarrow{\mathbb{R}_+^\infty}$ will be defined as the classifying locale of the theory of $P \subset \mathbb{Q}_+^*$ such that if $q \in P$ and $q < q'$ then $q' \in P$ and if $q \in P$ then there exists $q' < q$ such that $q' \in P$.

As $P$ is defined as a subset of positive rational numbers, $\overleftarrow{\mathbb{R}_+^\infty}$ corresponds only to non-negative numbers, and as we do not ask $P$ to be inhabited, $\overleftarrow{\mathbb{R}_+^\infty}$ contains a point $+\infty$ (corresponding to $P = \emptyset$). The topology on $\overleftarrow{\mathbb{R}_+^\infty}$ is the topology of upper semi-continuity i.e. the basic open sublocales are the $[0, q[$ for $q$ a rational (or real) number.

2.6.6. On a topological space (or more generally in a Grothendieck topos) Dedekind real numbers correspond to continuous functions to $\mathbb{R}$, whereas points of $\overleftarrow{\mathbb{R}_+^\infty}$ correspond to non negative upper semi-continuous (possibly infinite) functions. This explains why Dedekind reals are called "continuous" real numbers, and why points of $\overleftarrow{\mathbb{R}_+^\infty}$ can be called upper semi-continuous real numbers.

## 2.7 $[X, \mathbb{R}]$ is locally positive

The goal of this subsection is to show that, when $X$ is a compact regular locale, the locale $[X, \mathbb{R}]$ is locally positive (and hence also $[X, \mathbb{C}] \simeq [X, \mathbb{R}]^2$).

If $U$ and $V$ are two open sublocales of $X$ we write $U \ll V$ if $U$ is way below $V$, i.e. if when $V \leqslant \bigvee_{i \in I} U_i$ then there exists a finite subset $J \subset I$ such that $U \leqslant \bigvee_{j \in J} U_j$. We write $U \prec V$ when $U$ is rather below $V$, i.e. when $V \vee \neg U = X$, where $\neg U$ is the biggest open sublocale disjoint from $U$. A locale $X$ is regular when $\forall V \in \mathcal{O}(X)$, $V = \bigvee_{U \prec V} U$. In a compact regular locale the two relations $\prec$ and $\ll$ are equivalent.

In [37] one can find a description of the geometric theory classified by $[X, \mathbb{R}]$. This description shows that the open sublocales of the form $(U, q, q') = \{f | U \ll f^*(]q, q'[)\}$ [5] for $U$ an open sublocale of $X$ and $q$, $q'$ two rational numbers form a pre-basis of the topology of $[X, \mathbb{R}]$.

As:

$$U \ll f^*(]q, q'[) \Leftrightarrow (U \ll f^*(]q, +\infty[)) \wedge (U \ll f^*(]-\infty, q'[)),$$

$[X, \mathbb{R}]$ has a basis of open sublocales of the form

$$B = \left( \bigwedge_{i=1}^{n} (U_i, u_i, -) \right) \wedge \left( \bigwedge_{j=1}^{m} (V_j, v_j, +) \right), \tag{3.1}$$

where $U_i$ and $V_i$ are open sublocales of $X$, $u_i$ and $v_i$ are rational numbers, $(U_i, u_i, -)$ denotes $\{f | U \ll f^*(]-\infty, u_i[)\}$ and $(V_j, v_j, +)$ denotes $\{f | V_j \ll f^*(]v_j, +\infty[)\}$.

5. Of course, we do not mean the set of points $f$ of $[X, \mathbb{R}]$ satisfying this properties, but the open sublocale classifying such functions $f$.

2.7.1. **Definition :** *An open sublocale of the form given in* (3.1) *will be called a basic sublocale. A basic sublocale will be said to be admissible if it satisfies the following condition:*

$$\forall i \in 1, \dots, n, j \in 1, \dots, m, (u_i \leqslant v_j) \Rightarrow (\neg U_i) \vee (\neg V_j) = X.$$

We will show in 2.7.5 that a basic open is admissible if and only if it is positive, hence the property of being admissible is indeed a property of the open sublocale $B$, and not of its representation. But, while we have not proven this, we will assume that each time we consider a basic open $B$, it is given with a representation in the form of (3.1) and say that it is admissible if and only if its representation is.

2.7.2. The following lemma is in some sense a constructive form of Urysohn's lemma, asserting that compact regular locales are in fact completely regular.

**Lemma :** *Let $X$ be a compact regular locale, and let $U$,$V$ be two open sublocales of $X$ such that $U \ll V$. Then there exists a positive locally positive locale $\mathcal{L}$, such that in the logic of $\mathcal{L}$ there exists a continuous function from $X$ to* $[0,1]$ [6] *such that $f$ restricted to $U$ is zero and $f$ is constant equal to one on $\neg V$.*

**Proof :**
The classical proof of the Urysohn lemma for locale (see for example [62, Chap. XIV]) goes as follows: In a compact regular locale the relation $U \prec V$ defined as $\neg U \vee V = \top$ (ie, the closure of $U$ is included in $V$) is equivalent to the relation $U \ll V$ defined by: if $(V_i)$ is a directed set of open subspaces of $X$ such that $V \leqslant \bigvee V_i$ then $U \leqslant V_j$ for some $j$ (or equivalently the closure of $U$ is compact and included in $V$). The relation $\prec$ in general does not interpolate, but in a locally compact locale the relation $\ll$ always does, ie if $a \ll b$ then there exists $c$ such that $a \ll c \ll b$. In particular in a compact regular space the relation $\prec$ interpolates and (using the axiom of choice) one can construct a $\mathbb{Q}$-indexed family of open subspaces $U_q$ such that $U_0 = U$ , $U_1 = V$ and if $q < q'$ then $U_q \prec V_{q'}$, and we define $U_q = \emptyset$ when $q < 0$ and $U_q = X$ when $q > 1$. This defines a "scale" (see [62] XIV.5.2 ) which in turns defines a function from $X$ to $[0,1]$ with the required property (see [62]XIV.5.2.2).

The only part of the previous proof which is not constructive is the application of the axiom of dependent choice to construct the sequence $U_q$. By applying 2.3.9 one can construct a locale $\mathcal{L}$ in which there exists such a sequence and then finish the proof in the logic of $\mathcal{L}$ by constructing the function we are looking for. The only thing we need to check is that if $x \prec y$ then their pullback to $\mathcal{L}$ also satisfy this identity, but as it can equivalently be defined by " $\exists c$ such that $x \wedge c = \emptyset$ and $c \vee y = \top$ " this is immediate.
□

6. That is externally a function from $\mathcal{L} \times X$ to $[0,1]$.

2.7.3. **Proposition :** *If $X$ is compact completely regular and $B$ is an admissible basic sublocale of $[X,\mathbb{R}]$, then $B$ has a point. If $X$ is just compact regular and $B$ is admissible then $B$ is positive.*

**Proof :**
Assume that $X$ is completely regular, and let us first remark that when $X$ is a compact completely regular locale, if $U$ and $V$ are two open sublocales of $X$ such that $(\neg U) \vee (\neg V) = X$, then, as $U \ll (\neg V)$, it is possible to construct a continuous function $f : X \to [0,1]$ such that $f$ restricted to $U$ is constant equal to 0 and $f$ restricted to $V \subseteq \neg\neg V$ is constant equal to 1.

Now let
$$B = \left( \bigwedge_{i=1}^{n} (U_i, u_i, -) \right) \wedge \left( \bigwedge_{j=1}^{m} (V_j, v_j, +) \right)$$
be an admissible basic sublocale of $[X,\mathbb{R}]$.
Let $\epsilon$ be a positive rational number smaller than all the positive differences between two numbers of the form $u_i$ or $v_i$. For each couple $(i,j)$ we choose a continuous function $f_{i,j} : X \to \mathbb{R}$ such that:

- If $v_j < u_i$ then $f_{i,j}$ is the constant function equal to $\frac{v_j+u_i}{2}$
- If $u_i \leqslant v_j$ then $(\neg U_i) \vee (\neg V_j) = X$ and $f_{i,j}$ is a continuous function such that $f$ is constant equal to $u_i - \epsilon$ on $U_i$, $f$ is constant equal to $v_j + \epsilon$ on $V_j$ and $f$ takes value in $[u_i - \epsilon, v_j + \epsilon]$. (such a function exists by the previous remark).

Then,
$$f = \max_{1 \leqslant j \leqslant m} \min_{1 \leqslant i \leqslant n} f_{i,j},$$
is a point of $B$. Indeed:

- Let $i \in \{1, \dots, n\}$, then (on $U_i$), since for each $j$, $f_{i,j}$ is smaller than $u_i - \frac{\epsilon}{2}$, the infimum $\inf_{i'=1}^{n} f_{i',j}$ is smaller than $u_i - \frac{\epsilon}{2}$ and $f$ smaller than $u_i - \frac{\epsilon}{2}$ on $U_i$ as a (finite) supremum of a quantities smaller than $u_i - \frac{\epsilon}{2}$.
- Let $j \in \{1, \dots, m\}$, then (on $V_j$), as for each $i$, $f_{i,j}$ is greater than $v_j + \frac{\epsilon}{2}$, the infimum $\inf_{i=1}^{n} f_{i,j}$ is greater than $v_j + \frac{\epsilon}{2}$. And finally $f$ is greater than $v_j + \frac{\epsilon}{2}$ on $V_j$.

This concludes the proof when $X$ is completely regular. We now assume that $X$ is only regular. Then all the functions $f_{i,j}$ we used in the first part can be instead constructed in the logic of positive locally positive locales $\mathcal{L}_{i,j}$ using 2.7.2. The product $\mathcal{L}$ of all these $\mathcal{L}_{i,j}$ is also positive and locally positive by 2.3.7, and in the logic of $\mathcal{L}$, all the functions $f_{i,j}$ we used in the first part exist and hence one can construct the function $f$ which is going to be a point of $B$ in the logic of $\mathcal{L}$ exactly as we did above. This defines a map $\mathcal{L} \to B$ and, as $\mathcal{L}$ is positive, this proves that $B$ is positive and concludes the proof. $\square$

2.7.4. **Lemma :** *Let $p$ denote the canonical map from $[X,\mathbb{R}]$ to the point. Let $B$ be a basic sublocale then one has:*

$$B \leqslant p^*(\text{“ } B \text{ is admissible ”})$$

*where we identify the proposition "$B$ is admissible" with a subset of $\{*\}$ and hence with an open sublocale of the point.*

**Proof :**
We will prove that in the theory classified by $[X, \mathbb{R}]$ (describe in [37]) the proposition asserting that $B$ is admissible can be deduced from the proposition corresponding to $B$.
Maybe we should first recall the axioms of the theory describe in [37]:

1. $a \ll f^*(b) \vdash a' \ll f^*(b')$ when $a' \leqslant a$ and $b \leqslant b'$.
2. $\vdash \emptyset \ll f^*(b)$ for all $b$, and $\vdash a \ll f^*(\mathbb{R})$ for all $a \ll X$.
3. $a' \ll f^*(b)$ , $a \ll f^*(b) \vdash a \cup a' \ll f^*(b)$
4. $a \ll f^*(b)$, $a \ll f^*(b') \vdash a' \ll f^*(b \cap b')$ for any $a' \ll a$.
5. $a \ll f^*(b) \vdash \bigvee_{a \ll a'} a' \ll f^*(b)$
6. If $b = \bigcup_{i \in I} b_i$ then one has the axiom:

$$a \ll f^*(b) \vdash \bigvee_{\substack{J \subset I \text{ finite} \\ a = \bigcup_{j \in J} a_j}} \left( \bigwedge_{j \in J} (a_j \ll f^*(b_j)) \right)$$

Also note that axiom 6. applied with $I = \emptyset$ gives the axiom $a \ll f^*(\emptyset) \vdash a = \emptyset$.

Let $B$ be as in (3.1) and let $i$ and $j$ such that $u_i \leqslant v_j$.
By (3.1) and axiom 1. one has:

$$B \vdash (U_i \ll f^*(]-\infty, u_i[)) \wedge (V_j \ll f^*(]v_j, +\infty[)),$$

By axiom 5. one also has:

$$(U_i \ll f^*(]-\infty, u_i[)) \vdash \bigvee_{U_i \ll U} (U \ll f^*(]-\infty, u_i[))$$

$$(V_j \ll f^*(]v_j, +\infty[)) \vdash \bigvee_{V_j \ll V} (V \ll f^*(]v_j, +\infty[))$$

Putting together axioms 1. and 4. one gets that for any $C \ll U \wedge V$ one has:

$$(U \ll f^*(]-\infty, u_i[)) \wedge (V \ll f^*(]v_j, +\infty[)) \vdash C \ll f^*(]-\infty, u_i[\wedge]v_j, +\infty[).$$

Moreover, $]-\infty, u_i[\wedge]v_j, +\infty[= \emptyset$ as $u_i \leqslant v_j$, hence by the case $I = \emptyset$ of 6. one deduce that for all $C \ll U \wedge V$ one has:

$$B \vdash C = \emptyset$$

Now as $X$ is locally compact there exists $U'$ and $V'$ such that $U_i \ll U' \ll U$, $V_i \ll V' \ll V$ and as $X$ is regular $U' \wedge V' \ll U \wedge V$. Hence taking $C = U' \wedge V'$ (and removing the $'$) one obtain that:

$$B \vdash \bigvee_{\substack{U_i \ll U \\ V_j \ll V}} (V \wedge U = \emptyset)$$

but for any $U_i \ll U$ and $V_j \ll V$ if $(V \wedge U = \emptyset)$ then $\neg U \vee \neg V = X$ because

$$\begin{aligned} X &= (\neg U_i \vee U) \wedge (\neg V_j \vee V) \\ &= (\neg U_i \wedge \neg V_j) \vee (\neg U_i \wedge V) \vee (U \wedge \neg V_j) \vee (U \wedge V) \end{aligned}$$

The last term of the union can be removed by assumption, and we can duplicate the first, obtaining

$$\begin{aligned} X &= [(\neg U_i \wedge \neg V_j) \vee (\neg U_i \wedge V)] \vee [(U \wedge \neg V_j) \vee (\neg U_i \wedge \neg V_j)] \\ &= [(\neg U_i) \wedge (\neg V_j \vee V)] \vee [(\neg V_j) \wedge (\neg U_i \vee U)] \\ &= \neg U_i \vee \neg V_j \end{aligned}$$

Hence $B \vdash \neg U_i \vee \neg V_j = X$. As this is true for any $(i,j)$ such that $u_i \leqslant v_j$ we get the desired result.
□

2.7.5. Combining all these results we obtain:

**Theorem :** *If $X$ is a compact regular locale, then a basic sublocale $B$ of $[X,\mathbb{R}]$, is admissible if and only it is positive. In particular, $[X,\mathbb{R}]$ is locally positive and the admissible basic sublocales form a basis of positive open sublocales.*

**Proof :**
It suffices to apply Lemma 2.3.4 with $b_i$ the basic open sublocales and $w_i$ the propositions "$b_i$ is admissible". Proposition 2.7.3 shows that $w_i$ implies $b_i > \emptyset$ and 2.7.4 is exactly the second condition. □

2.7.6. We also obtain the following

**Proposition :** *Let $X$ be a compact regular locale, $X$ is completely regular if and only if $[X,\mathbb{R}]$ is weakly spatial.*

**Proof :**
If $X$ is completely regular, then 2.7.3 shows that each admissible has a point. But by 2.7.5 they form a basis of positive open, hence this proves that points of $[X,\mathbb{R}]$ are dense. Conversely, if $[X,\mathbb{R}]$ is weakly spatial and $U$,$V$ are two open sublocales of $X$ such that $U \prec V$, then there exists $W$ such that $U \prec W \prec V$ and the basic open:

$$B = (U, 0, -) \wedge (\neg W, 1, +)$$

is admissible because $\neg U \vee \neg\neg W \geqslant \neg U \vee W = X$. Hence it is positive and hence it has a point. But a point of $B$ is a function from $X$ to $\mathbb{R}$ such that $f$ is negative on $U$ and greater than one on $\neg W$. As $\neg W \vee V = X$ the function $f$ shows that $U$ is "completely below $V$", and this proves that $X$ is completely regular. □

# 3 Constructive theory of metric locales

## 3.1 Pre-metric locale

As our major concern is the study of localic Banach spaces, we will only consider metrics on a locale which are defined by a distance function. However, it should be noted that the point 9 of the series of propositions given in 3.1.4 shows that one can specify a distance by giving the diameter $\delta(U)$ of each open sublocale $U$, and the classical theory [7] which can be found for example in the chapter XI of [62] suggests that a definition by diameters should also be possible.

3.1.1. **Definition :** *A pre-distance $d$ on a locale $X$ is a function*

$$d : X \times X \to \overleftarrow{\mathbb{R}_+^\infty}$$

*which is symmetric ($d(x,y) = d(y,x)$), satisfies the triangular inequality $d(x,y) \leqslant d(x,z) + d(z,y)$ and such that $d(x,x) = 0$*
*A pre-metric locale is a locally positive locale $X$ endowed with a pre-distance.*

We insist on the fact that our pre-metric locale are always assumed to be locally positive. We do not know exactly which parts of the theory of metric locales it is possible to develop without this hypothesis (without it, one should at least avoid everything which uses the construction $B_q\mathcal{L}$ of 3.1.2 but it seems that what is left is relatively well behaved without it). In any case, the theory is at least easier, and probably nicer with this local positivity assumption. Theorem 2.7.5 shows that this case is enough for the Gelfand duality, and as locale positivity descend along open surjections and is automatic for metric sets it is also enough to obtain good descent properties.

Of course, the formulas $d(x,y) = d(y,x)$ and $d(x,y) \leqslant d(x,z) + d(z,y)$ have to be interpreted in a diagrammatic way or in terms of generalized points. In particular, if we define

$$\Delta_q := \{(x,y) | d(x,y) < q\} = d^*\left(\overleftarrow{[0,q[}\right)$$

then the symmetry means that $\Delta_q$ is invariant by exchange of the two factors, $d(x,x) = 0$ means that for all $q$, $\Delta_q$ contains the diagonal embeddings of $X$, and finally the triangular inequality means that:

$$\pi_{1,2}^*(\Delta_q) \wedge \pi_{2,3}^*(\Delta_{q'}) \leqslant \pi_{1,3}^*(\Delta_{q+q'})$$

Where $\pi_{i,j}$ denote the various projections from $X^3$ to $X^2$.

7. Which has not been done constructively yet as far the author knows.

3.1.2. **Definition :** *Let $X$ be a pre-metric locale, and $\mathcal{L}$ and $\mathcal{M}$ be two sublocales of $X$. then*

- *We say that $\delta(\mathcal{L}) < q$ if $\mathcal{L} \times \mathcal{L} \subseteq \Delta_{q'}$ for some positive rational number $q' < q$. One easily sees that $\delta(\mathcal{L})$ is indeed an element of $\overleftarrow{\mathbb{R}_+^\infty}$;*
- *We say that $\mathcal{L} \triangleleft_q \mathcal{M}$ if $\pi_1^*(\mathcal{L}) \wedge \Delta_q \leqslant \pi_2^*(\mathcal{M})$. We say that $\mathcal{L} \triangleleft \mathcal{M}$ if $\mathcal{L} \triangleleft_q \mathcal{M}$ for some positive rational $q$;*
- *if $q$ is a positive rational number then $B_q\mathcal{L} = (\pi_2)_!(\pi_1^*(\mathcal{L}) \wedge \Delta_q)$.*

These should be interpreted as: $\delta$ is the diameter of a sublocale, $B_q$ is the $q$ neighborhood of a sublocale and $\mathcal{L} \triangleleft_q \mathcal{M}$ means that the $q$ neighborhood of $\mathcal{L}$ is included in $\mathcal{M}$.

3.1.3. We will denote by $\mathcal{O}(X)^{<q}$ the set of open sublocales $U$ of $X$ such that $\delta(U) < q$, and $\mathcal{O}(X)^{+,<q}$ will be simply the subset $\mathcal{O}(X)^+ \cap \mathcal{O}(X)^{<q}$ of positive elements of $\mathcal{O}(X)^{<q}$.

3.1.4. **Proposition :**

1. *$B_q\mathcal{L} \subseteq \mathcal{M}$ if and only if $\mathcal{L} \triangleleft_q \mathcal{M}$.*
2. *If $\mathcal{L} \subseteq \mathcal{M}$ then $\delta(\mathcal{L}) \leqslant \delta(\mathcal{M})$.*
3. *If $\mathcal{L} \triangleleft \mathcal{M}$ then $\mathcal{L} \subseteq \mathcal{M}$. In particular for all positive rational numbers $q$ one has $\mathcal{L} \subseteq B_q\mathcal{L}$.*
4. *If $\mathcal{L} \triangleleft_q \mathcal{M}$ and $\mathcal{L}' \triangleleft_q \mathcal{M}'$ then $\mathcal{L} \wedge \mathcal{L}' \triangleleft_q \mathcal{M} \wedge \mathcal{M}'$ and $\mathcal{L} \vee \mathcal{L}' \triangleleft_q \mathcal{M} \vee \mathcal{M}'$.*
5. $\delta\left(\bigvee_{i \in I} \mathcal{L}_i\right) = \sup_{i,j \in I} \delta(\mathcal{L}_i \vee \mathcal{L}_j)$
6. *If $\mathcal{L} \wedge \mathcal{M}$ contains a positive and locally positive sublocale then $\delta(\mathcal{L} \vee \mathcal{M}) \leqslant \delta(\mathcal{L}) + \delta(\mathcal{M})$.*
7. *Let $(\mathcal{L}_i)_{i=0\dots n}$ be a finite sequence of sublocales such that for all $i$, $\mathcal{L}_{i-1} \wedge \mathcal{L}_i$ contains a positive and locally positive sublocale then:*
$$\delta\left(\bigvee_{i=0}^{n} \mathcal{L}_i\right) \leqslant \sum_{i=0}^{n} \delta(\mathcal{L}_i)$$
8. *For any $q > 0$, $\mathcal{O}(X)^{<q}$ is a basis of the topology of $X$.*
9. $\Delta_q = \bigvee_{U \in \mathcal{O}(X)^{<q}} U \times U$
10. *If $\mathcal{L}$ is locally positive, then*
$$B_q\mathcal{L} = \bigvee_{\substack{U \in \mathcal{O}(X)^{<q} \\ U \wedge \mathcal{L} > \emptyset}} U.$$
*In particular, if $\mathcal{L}$ is locally positive, $B_q\mathcal{L}$ is open.*
11. *If $\mathcal{L}$ is locally positive then*
$$B_{q'}(B_q(\mathcal{L})) \subseteq B_{q+q'}(\mathcal{L}).$$

*12. If $\mathcal{L}$ is locally positive then $\delta(B_q\mathcal{L}) \leqslant 2q + \delta(\mathcal{L})$.*

**Proof :**

1. This is simply the adjunction between $(\pi_2)_!$ and $(\pi_2)^*$.
2. If $\mathcal{L} \subseteq \mathcal{M}$ and if $\delta(\mathcal{M}) < q$ then there exists a positive rational $q' < q$ such that $\mathcal{L} \times \mathcal{L} \subseteq \mathcal{M} \times \mathcal{M} \subseteq \Delta_{q'}$ hence $\delta(\mathcal{L}) < q$.
3. Assume that $\pi_1^*(\mathcal{L}) \wedge \Delta_q \subseteq \pi_2^*(\mathcal{M})$ for some positive rational number $q$, and let $i : X \to X \times X$ be the diagonal embedding, then:

$$i^*(\pi_1^*(\mathcal{L}) \wedge \Delta_q) \subseteq i^*\pi_2^*(\mathcal{M}) = \mathcal{M}$$

And:

$$i^*(\pi_1^*(\mathcal{L}) \wedge \Delta_q) = i^*\pi_1^*(\mathcal{L}) \wedge i^*\Delta_q = \mathcal{L} \wedge X = \mathcal{L}$$

hence $\mathcal{L} \subseteq \mathcal{M}$. The second part of the result then follows from the fact that as $B_q\mathcal{L} \subseteq B_q\mathcal{L}$, one has $\mathcal{L} \lhd_q B_q\mathcal{L}$.
4. Assume that $\pi_1^*\mathcal{L} \wedge \Delta_q \subseteq \pi_2^*\mathcal{M}$ and that $\pi_1^*\mathcal{L}' \wedge \Delta_q \subseteq \pi_2^*\mathcal{M}'$, then:

$$\pi_1^*(\mathcal{L} \wedge \mathcal{L}') \wedge \Delta_q = \pi_1^*(\mathcal{L}) \wedge \Delta_q \wedge \pi_1^*(\mathcal{L}') \wedge \Delta_q \subseteq \pi_2^*(\mathcal{M}) \wedge \pi_2^*(\mathcal{M}')$$

hence $\mathcal{L} \wedge \mathcal{L} \lhd_q \mathcal{M} \wedge \mathcal{M}$.
And for the union:

$$\begin{array}{rcl} \pi_1^*(\mathcal{L} \vee \mathcal{L}') \wedge \Delta_q & = & (\pi_1^*(\mathcal{L}) \vee \pi_1^*(\mathcal{L}')) \wedge \Delta_q \\ & = & (\pi_1^*\mathcal{L} \wedge \Delta_q) \vee (\pi_1^*\mathcal{L}' \wedge \Delta_q) \\ & \subseteq & \pi_2^*(\mathcal{M}) \vee \pi_2^*(\mathcal{M}'), \end{array}$$

which gives the result.
The fact that intersections distribute over finite unions of sublocales and that pullbacks preserve finite unions of sublocales can be found in [44] C1.1.15 and C.1.19, but formulated in terms of frames instead of locales (i.e. union of sublocales correspond to intersection of nuclei, and pullback of a sublocale to a pushout).
5. Clearly, $\sup_{i,j\in I} \delta(\mathcal{L}_i \vee \mathcal{L}_j) \leqslant \delta\left(\bigvee_i \mathcal{L}_i\right)$ because $\mathcal{L}_i \vee \mathcal{L}_j \subseteq \bigvee \mathcal{L}_i$. Let $q$ such that $\sup_{i,j\in I} \delta(\mathcal{L}_i \vee \mathcal{L}_j) < q$ i.e. there exists $q' < q$ such that for all $i, j$, $\delta(\mathcal{L}_i \vee \mathcal{L}_j) < q'$. But as

$$\left(\bigvee_{i\in I} \mathcal{L}_i\right) \times \left(\bigvee_{j\in I} \mathcal{L}_j\right) = \bigvee_{i,j} \mathcal{L}_i \times \mathcal{L}_j$$

and for all $i, j$, $\mathcal{L}_i \times \mathcal{L}_j \subseteq \Delta_{q'}$, one obtains

$$\left(\bigvee_{i\in I} \mathcal{L}_i\right) \times \left(\bigvee_{j\in J} \mathcal{L}_j\right) \subseteq \Delta_{q'},$$

which concludes the proof.

6. Assume that $\mathcal{L} \times \mathcal{L} \subseteq \Delta_q$ and $\mathcal{M} \times \mathcal{M} \subseteq \Delta_{q'}$, we will prove that, under the assumption of the proposition, $(\mathcal{L} \vee \mathcal{M}) \times (\mathcal{L} \vee \mathcal{M}) \subseteq \Delta_{q+q'}$.
   As $(\mathcal{L} \vee \mathcal{M}) \times (\mathcal{L} \vee \mathcal{M}) = (\mathcal{L} \times \mathcal{L}) \vee (\mathcal{L} \times \mathcal{M}) \vee (\mathcal{L} \times \mathcal{M}) \vee (\mathcal{M} \times \mathcal{M})$ and $(\mathcal{L} \times \mathcal{L})$ and $(\mathcal{M} \times \mathcal{M})$ are already known to be subsets of $\Delta_{q+q'}$, we only have to prove it for $(\mathcal{L} \times \mathcal{M})$ and $(\mathcal{M} \times \mathcal{L})$. In $X^3$ one has:

$$\begin{aligned} \mathcal{M} \times (\mathcal{L} \wedge \mathcal{M}) \times \mathcal{L} \subseteq \pi_{1,2}^*(\mathcal{M} \times \mathcal{M}) \wedge \pi_{2,3}^*(\mathcal{L} \times \mathcal{L}) \quad &\subseteq \quad \pi_{1,2}^*(\Delta_q') \wedge \pi_{2,3}^*(\Delta_q) \\ &\subseteq \quad \pi_{1,3}^*(\Delta_{q'+q}) \end{aligned}$$

   Applying $(\pi_{1,3})_!$ yields the result because as $(\mathcal{L} \times \mathcal{M})$ contains some positive and locally positive sublocale, the projection $\pi_{1,3}$ from $\mathcal{L} \times (\mathcal{L} \wedge \mathcal{M}) \times \mathcal{M}$ to $\mathcal{L} \times \mathcal{M}$ is a surjection.
7. It is immediate by induction on $n$ using the previous point.
8. Thanks to the point 2. it is enough to check that $\mathcal{O}(X)^{<q}$ covers $X$. Take a covering of $\Delta_{q/2}$ by open sublocales of the form $U_i \times V_i$, then pulling back along the diagonal embeddings of $X$ into $\Delta_{q/2}$ one has:

$$X = \bigvee_i U_i \wedge V_i$$

   but $(U_i \wedge V_i)^2 \leqslant U_i \times V_i \leqslant \Delta_{q/2}$ hence $\delta(U_i \wedge V_i) < q$ which concludes the proof.
9. Thanks to the previous point, for any $q' < q$, $\Delta_{q'}$ can be written as a union of $U_i \times V_i$ with $\delta(U_i) < q'$ and $\delta(V_i) < q'$. If $U_i \times V_i \subseteq \Delta_{q'}$. then so does $V_i \times U_i$, and hence, in our situation:

$$(U_i \cup V_i)^2 = (U_i \times U_i) \cup (V_i \times U_i) \cup (U_i \times V_i) \cup (V_i \times V_i) \subseteq \Delta_{q'}$$

   Hence $\delta(U_i \cup V_i) < q$ and the $(U_i \cup V_i)^2$ cover $\Delta_{q'}$. This being done for an arbitrary $q' < q$, these open sublocales also cover $\Delta_q$, because as the $\Delta_q$ are defined by a function from $X \times X$ to $\overleftarrow{\mathbb{R}_+^\infty}$ one has

$$\Delta_q = \bigvee_{q'<q} \Delta_{q'}$$

10. Applying the definition of $B_q V$ using that $\pi_1^*(\mathcal{L}) = \mathcal{L} \times X$ and the previous point gives directly

$$B_q\mathcal{L} = (\pi_2)_! \left( \bigvee_{\delta(U)<q} (\mathcal{L} \wedge U) \times U \right) = \bigvee_{\substack{\delta(U)<q \\ \mathcal{L} \wedge U > \emptyset}} U.$$

11. From the previous point

$$B_q(B_{q'}\mathcal{L}) = \bigvee_{\substack{v \in \mathcal{O}(X)^{<q} \\ v \wedge B_{q'}\mathcal{L} > \emptyset}} v$$

   But, still by the previous point, an open sublocale $v$ of $X$ satisfies $v \wedge B_{q'}\mathcal{L} > \emptyset$ if and only if there exists $v' \in \mathcal{O}(X)^{<q'}$ such that $v' \wedge \mathcal{L} > \emptyset$ and

$v \wedge v' > \emptyset$. For any open sublocale of this sort, one has $\delta(v \vee v') < q+q'$ by point 6. Hence $v \vee v'$ is a positive open sublocale such that $\delta(v \vee v') < q+q'$ and $(v \vee v') \wedge \mathcal{L} > \emptyset$. In particular $v \leqslant v \vee v' \leqslant B_{q+q'}\mathcal{L}$.
This proves that $B_q(B_{q'}\mathcal{L}) \leqslant B_{q+q'}\mathcal{L}$.

12. From point 10 one has

$$B_q\mathcal{L} = \bigvee_{\substack{v \in \mathcal{O}(X)^{<q} \\ v \wedge \mathcal{L} > \emptyset}} v.$$

Hence from point 5 one has

$$\delta(B_q\mathcal{L}) = \sup_{\substack{v,v' \in \mathcal{O}(X)^{<q} \\ v \wedge \mathcal{L}, v' \wedge \mathcal{L} > \emptyset}} \delta(v \vee v').$$

But for any two such $v, v'$ one has by point 7: $\delta(v \vee v') \leqslant \delta(v \vee v' \vee \mathcal{L}) \leqslant \delta(\mathcal{L}) + \delta(v) + \delta(v') \leqslant \delta(\mathcal{L}) + 2q$. One obtains the result by taking the supremum.

□

3.1.5. Usually, the distance function $d : X \times X \to \overleftarrow{\mathbb{R}_+^\infty}$ is expected to be in fact a continuous map from $X \times X$ to $\mathbb{R}$, and not only a semi-continuous map as our definition of distance suggest it. The reason for our choice is that we know (see for example [12]) that the norm on a Banach space has to take value in $\overleftarrow{\mathbb{R}_+^\infty}$, even if we want to think of it as a function which is continuous [8]. Classically, the continuity is a consequence of the triangular inequality, and the following proposition gives a constructive interpretation of this result, restoring a form of "fiberwise continuity" of $d$.

**Proposition :** *Let $\overline{\Delta_q}$ be the fiberwise closure of $\Delta_q$ in $X \times X$. Then for all $q < q'$ one has $\overline{\Delta_q} \subseteq \Delta_{q'}$.*

**Proof :**
Let $q'$ be a rational such that $q < q'$ and let $\epsilon = \frac{q'-q}{2}$. As $\Delta_q$ is by definition fiberwise dense in $\overline{\Delta_q}$, Proposition 2.4.2 implies that $\overline{\Delta_q}$ is locally positive, and in particular one can write that

$$\overline{\Delta_q} \leqslant \bigvee_{\substack{v,v' \in \mathcal{O}(X)^{<\epsilon} \\ v \times v' \wedge \overline{\Delta_q} > \emptyset}} v \times v'.$$

But, still by 2.4.2 and by fiberwise density of $\Delta_q$ in $\overline{\Delta_q}$, for any two such $v, v'$ one has $v \times v' \wedge \Delta_q > \emptyset$ and hence there exists $U$ such that $\delta(U) < q$ and $(v \times v') \wedge (U \times U)$ is positive. This implies that $v \wedge U$ and $v' \wedge U$ are positive and hence, by point 7 of 3.1.4, that $\delta(v \vee v') \leqslant \delta(v) + \delta(v') + \Delta(U) < q + 2\epsilon = q'$. Therefore,

$$v \times v' \subseteq (v \vee v') \times (v \vee v') \subseteq \Delta_{q'},$$

and this concludes the proof.
□

8. as opposed to semi-continuous.

3.1.6. **Definition :** *Let $X$ be a pre-metric locale, we will say that $X$ has a continuous distance if the pre-distance function $d : X \times X \rightarrow \overleftarrow{\mathbb{R}_+^\infty}$ internally corresponds to a continuous real number, i.e. if the pre-distance function factors into $X \times X \rightarrow \overline{\mathbb{R}+} \rightarrow \overleftarrow{\mathbb{R}_+^\infty}$. In this situation we define $\Theta_q$ to be the open sublocale of $X \times X$ corresponding to $\{(x,y) | d(x,y) > q\}$.*

3.1.7. Assuming the law of excluded middle, we indeed obtain continuity:

**Proposition :** *Assuming the law of excluded middle in the base topos, any pre-metric locale has a continuous distance.*

**Proof :**
If one assumes the law of excluded middle in the base topos then any fiberwise closed sublocale is in fact a closed sublocale. In particular, there exists open sublocales $\Theta'_q$ of $X \times X$, which are the complementary open sublocales of the (closed) sublocales $\overline{\Delta_q}$. From the fact, proved in 3.1.5 that for any $q < q'$ one has the relation

$$\Delta_q \leqslant \overline{\Delta_q} \leqslant \Delta_{q'}$$

and we deduce

$$\Delta_q \wedge \Theta'_q = \emptyset$$

$$\Delta_{q'} \vee \Theta'_q = X \times X$$

and $\overline{\Delta_q} \leqslant \overline{\Delta_{q'}}$ gives $\Theta'_q \geqslant \Theta'_{q'}$.
If we define, $\Theta_q = \bigvee_{q<q'} \Theta'_{q'}$, then $\Delta_q$ and $\Theta_q$ define a map from $X \times X$ to $\overline{\mathbb{R}+}$ which yields the desired factorisation. $\square$

3.1.8. **Proposition :** *Let $f : X \rightarrow Y$ be a map between two pre-metric locales. Then the following conditions are equivalent:*

*(a) For any positive rational $q$, $\Delta_q \subseteq (f \times f)^*(\Delta_q)$*

*(b) For any locally positive sublocale $\mathcal{L}$ of $X$, $\delta(f_!\mathcal{L}) \leqslant \delta(\mathcal{L})$.*

*(c) For any $U \in \mathcal{O}(X)^{<q_1}$, $v_1 \in \mathcal{O}(Y)^{<q_2}$, $v_2 \in \mathcal{O}(Y)^{<q_3}$ such that $f^*(v_1) \wedge U$ and $f^*(v_2) \wedge U$ are positive, one has $\delta(v_1 \vee v_2) < q_1 + q_2 + q_3$.*

*(d) For any $U \in \mathcal{O}(X)$ and any positive rational $q$:*

$$\delta(B_q f_! U) \leqslant \delta(U) + 2q.$$

*(e) For any open sublocale $U$ of $X$ such that $\delta(U) < q$ there exists an open sublocale $V$ of $Y$ such that $\delta(V) < q$ and $U \subseteq f^*(V)$.*

*A map satisfying these conditions is called a metric map.*

Of course, condition $(a)$ is the point free formulation of the usual $d(f(x), f(y)) \leqslant d(x,y)$.

**Proof :**

$(a) \Rightarrow (b)$ Let $q$ such that $\delta(\mathcal{L}) < q$, i.e. there exists $q' < q$ such that $\mathcal{L} \times \mathcal{L} \subseteq \Delta_{q'}$. Hence,

$$\mathcal{L} \times \mathcal{L} \subseteq (f \times f)^*(\Delta_{q'})$$

This proves that the image $(f \times f)_!(\mathcal{L} \times \mathcal{L})$ in $X \times X$ is included in $\Delta_{q'}$. Unfortunately, as a product of surjections may fail to be a surjection, it is not enough to conclude directly that $f_!(\mathcal{L}) \times f_!(\mathcal{L}) \subseteq \Delta_{q'}$. But we can still conclude using the fact that as $\mathcal{L}$ and $f_!(\mathcal{L})$ are both locally positive, then by 2.4.5 (applied twice) the map $f : \mathcal{L} \times \mathcal{L} \to f_!(\mathcal{L}) \times f_!(\mathcal{L})$ is always fiberwise dense. This implies that $\Delta_{q'}$ is fiberwise dense in $f_!(\mathcal{L}) \times f_!(\mathcal{L})$, and by 3.1.5 that:

$$f_!(\mathcal{L}) \times f_!(\mathcal{L}) \subseteq \overline{\Delta_{q'}} \subseteq \Delta_q$$

which concludes the proof.

$(b) \Rightarrow (c)$ by 2.4.3 $\mathcal{L} = f_!(U)$ is locally positive because $U$ is and $f : U \to f_!(U)$ is a surjection. Also, $\delta(f_!(U)) < q_1$ by $(b)$. Hence one obtains $(c)$ by applying point 7 of 3.1.4 (with n=2), together with the fact that $f^*v \wedge U > \emptyset$ is equivalent to $v \wedge f_!U > \emptyset$ because $f : U \to f_!U$ is a surjection and hence in particular a fiberwise dense map.

$(c) \Rightarrow (d)$ One has

$$B_q f_! U = \bigvee_{\substack{v \in \mathcal{O}(Y)^{<q} \\ f^*(v) \wedge U > \emptyset}} v$$

The same argument as given for point 12 of 3.1.4 allow one to conclude.

$(d) \Rightarrow (e)$ If $\delta(U) < q$ then there exists a positive $\epsilon$ such that $\delta(U) < q - 2\epsilon$. Take $V = B_\epsilon f_! U$ yields the result as $U \leqslant f^* f_! U \leqslant f^* B_\epsilon f_! U = f^* V$.

$(e) \Rightarrow (a)$ Using $(e)$ one gets immediately the inclusion

$$\Delta_q = \bigvee_{U \in \mathcal{O}(X)^{<q}} U \times U \subseteq \bigvee_{V \in \mathcal{O}(Y)^{<q}} f^*(V) \times f^*(V) = (f \times f)^*(\Delta_q)$$

□

3.1.9. **Proposition :** *Let $f : X \to Y$ be a map between two pre-metric locales, let $\epsilon$ and $\eta$ be two positive rational numbers, then the following conditions are equivalent:*

*(a)* $\Delta_\eta \leqslant (f \times f)^* \Delta_\epsilon$

*(b) If $U \in \mathcal{O}(X)$ and $\delta(U) < \eta$ then $\delta(f_!(U)) < \epsilon$*

*(c) If $U \in \mathcal{O}(X)$ and $\delta(U) < \eta$ then there exists $V \in \mathcal{O}(Y)$ such that $\delta(V) < \epsilon$ and $U \leqslant f^*(V)$.*

The point of this proposition is to define a uniform map:

**Definition :** *One says that a map $f$ is a uniform map if for all $\epsilon$ there exists $\eta$ satisfying the conditions of the previous proposition.*

**Proof :**
The proof essentially follows the same lines as the proof of 3.1.8:

$(a) \Rightarrow (b)$ The argument for $(a) \Rightarrow (b)$ in 3.1.8 applies in exactly the same way here.

$(b) \Rightarrow (c)$ If $\delta(f_!(U) < \epsilon$, then there exists $q$ such that $\delta(B_q f_!(U)) < \epsilon$ hence one can take $V = B_q f_!(U)$.

$(c) \Rightarrow (a)$ One has

$$\Delta_\eta = \bigvee_{\delta(U)<\eta} U \times U$$

but for each $U$ such that $\delta(U) < \eta$, there exists $V$ such that $\delta(V) < \epsilon$ and $U \leqslant f^*(V)$, hence

$$\Delta_\eta \leqslant \bigvee_{\delta(V)<\epsilon} f^*V \times f^*V = (f \times f)^*(V \times V)$$

□

3.1.10. **Definition :** *A map between two pre-metric locales is said to be "compatible with $\lhd$" if $U \lhd V$ implies $f^*U \lhd f^*V$.*

Metric maps and uniform maps are in particular compatible with $\lhd$ because if $f$ is uniform and if $\pi_1^*U \wedge \Delta_\epsilon \leqslant \pi_2^*(V)$ then, letting $\eta$ such that

$$\Delta_\eta \leqslant (f \times f)^* \Delta_\epsilon$$

as we have

$$(f \times f)^*(\pi_1^*(U) \wedge \Delta_\epsilon) \leqslant (f \times f)^* \pi_2^* V$$

we obtain

$$\pi_1^*(f^*U) \wedge \Delta_\eta \leqslant \pi_1^*(f^*U)) \wedge (f \times f)^* \Delta_\epsilon \leqslant \pi_2^* f^* V$$

i.e. $f^*U \lhd_\eta f^*V$

3.1.11. **Definition :** *A map $f : X \to Y$ between two pre-metric locales is called an isometric map if $d(f(x), f(y)) = d(x, y)$, i.e. if $\Delta_q = (f \times f)^*(\Delta_q)$.*

We can easily see (by the same kind of argument that 3.1.8) that this is equivalent to the fact that $\delta(\mathcal{L}) = \delta(f_! \mathcal{L})$ for all sublocales of $X$.

**Lemma :** *If $f$ is an isometric map $X \to Y$ then for any locally positive sublocale $\mathcal{L}$ of $X$*

$$\mathcal{L} \leqslant f^*(B_q f_! \mathcal{L}) \leqslant B_q \mathcal{L}$$

**Proof :**
The first inequality immediately follows from the fact that $f_! \mathcal{L} \leqslant B_q f_! \mathcal{L}$. For the second, as $f_!(\mathcal{L})$ is locally positive (because of 2.4.3) one can write that

$$B_q f_! \mathcal{L} = \bigvee_{\substack{v \in \mathcal{O}(Y)^{<q} \\ v \wedge f_!(\mathcal{L}) > \emptyset}} v.$$

By 2.4.2, $v \wedge f_!(\mathcal{L})$ is positive if and only if $f^*(v) \wedge \mathcal{L}$ is. Also, as $f$ is isometric, for any $v \in \mathcal{O}(Y)^{<q}$ , one has $f^*(v) \in \mathcal{O}(X)^{<q}$. Finally

$$f^*(B_q f_! \mathcal{L}) = \bigvee_{\substack{v \in \mathcal{O}(Y)^{<q} \\ f^*(v) \wedge \mathcal{L} > \emptyset}} f^*(v) \leqslant \bigvee_{\substack{w \in \mathcal{O}(X)^{<q} \\ w \wedge \mathcal{L} > \emptyset}} w = B_q \mathcal{L}.$$

□

3.1.12. We now consider two toposes $\mathcal{E}$ and $\mathcal{T}$, a geometric morphism $f : \mathcal{E} \to \mathcal{T}$ and $X$ a pre-metric locale in $\mathcal{T}$. As $f^\#$ is a functor from locale in $\mathcal{T}$ to locale in $\mathcal{E}$ commuting to projective limit and $f^\#(\overleftarrow{\mathbb{R}_+^\infty}_{\mathcal{T}}) \simeq \overleftarrow{\mathbb{R}_+^\infty}_{\mathcal{E}}$, we obtain a map $f^\#(d) : f^\#(X) \times f^\#(X) \to \overleftarrow{\mathbb{R}_+^\infty}$. Moreover all the axioms asserting that $d$ is a pre-distance can be pulled back turning $f^\#(X)$ into a pre-metric locale.

**Proposition :** *Let $\mathcal{L}, \mathcal{M}$ be a sublocales of $X$, then (as sublocales of the pre-metric locale $f^\#(X)$) one has:*
- *If $\delta(\mathcal{L}) < q$ then $\delta(f^\#(\mathcal{L})) < q$.*
- *If $\mathcal{L} \lhd_q \mathcal{M}$ then $f^\#(\mathcal{L}) \lhd_q f^\#(\mathcal{M})$.*
- *If $\mathcal{L}$ is locally positive then $B_q f^\#(\mathcal{L}) = f^\#(B_q \mathcal{L})$.*

**Proof :**
$f^\#$ is a functor commuting to all projective limits, in particular pullbacks, products and intersections, and by definition of the metric $f^\#(\Delta_q) = \Delta_q$ hence

$$\mathcal{L} \times \mathcal{L} \subseteq \Delta_{q'}$$

implies

$$f^\#(\mathcal{L}) \times f^\#(\mathcal{L}) \subseteq \Delta_{q'}$$

and

$$\pi_1^*(\mathcal{L}) \wedge \Delta_q \subseteq \pi_2^*(\mathcal{M})$$

implies

$$\pi_1^*(f^\#(\mathcal{L})) \wedge \Delta_q \subseteq \pi_2^*(f^\#(\mathcal{M}))$$

which proves the first two points.
The third point is harder because in general the pullback $f^\#$ does not commute with the direct image functor $(\pi_2)_!$. But if we assume that $\mathcal{L}$ is locally positive, then the map

$$\pi_1^*(\mathcal{L}) \wedge \Delta_q \to B_q\mathcal{L}$$

is the restriction of the projection from $\mathcal{L} \times X$ to $X$ and hence is an open map. In particular (as we know that it is a surjection by definition) it is an open surjection and hence its pullback by $f^\#$ is again an open surjection. In particular, the maps

$$\pi_1^*(f^\#(\mathcal{L})) \wedge \Delta_q \to f^\#(B_q\mathcal{L}) \to f^\#(X)$$

form a factorisation surjection/inclusion and, by uniqueness of such a factorisation, we obtain the third point. $\square$

3.1.13. We also note that if we define $\mathcal{C}(\mathcal{T})$ to be the category of pre-metric locales and metric maps internal to $\mathcal{T}$, then open surjections are descent morphisms for $\mathcal{C}$ (see 2.5) : If $f : \mathcal{E} \to \mathcal{T}$ is an open surjection and $(X, d)$ is a pre-metric locale in $\mathcal{E}$ endowed with a descent data then it is in particular a descent data on $X$ as a locale, so as locale descend along open surjections, $X$ comes from a locale $X'$ in $\mathcal{T}$. As the $\epsilon : \pi_1^*X \to \pi_2^*X$ is an isomorphism in the category of metric maps it is an isometric map and hence the distance is a morphism in $Des(f, \mathcal{C})$ and hence also descends into a function $d' : X' \times X' \to \overleftarrow{\mathbb{R}_+^\infty}$. All the axioms defining a pre-distance are equality relations (and inequality for the specialisation order), hence as they are satisfied by the pullback of $(X', d')$ along an open surjection they are also satisfied by $(X', d')$. Hence $(X, d)$ is the pullback of the pre-metric locale $(X', d')$. This proves that the functor $\mathcal{T} \to Des(f, \mathcal{C})$ is essentially surjective, but it is also fully faithful for similar reasons: a metric map commuting to descent data is in particular a map of locales commuting to descent data, and as $f$ is an open surjection a map $h$ is metric if and only if $f^*(h)$ is metric.

## 3.2 Metric locales

3.2.1. If $(X, d)$ is a pre-metric locale, then the various properties given in 3.1.4 show that, essentially, the "topology defined by $d$" (whatever the precise meaning of this is) is coarser than the topology of $X$, but nothing forces them to agree. For example, a metric set in the usual sense (with a distance function taking value in $\overleftarrow{\mathbb{R}_+^\infty}$), gives a pre-distance on a discrete locale, and the topology defined by $d$ can disagree with the discrete topology. That is why we require the following additional property:

**Definition :** *A Metric locale is a pre-metric locale $X$ such that for all $U \in \mathcal{O}(X)$,*

$$U = \bigvee_{\substack{V \in \mathcal{O}(X) \\ V \triangleleft U}} V.$$

This definition is equivalent to the fact that the family $(B_q V)_{V \in \mathcal{O}(X), q \in \mathbb{Q}_+^*}$ forms a basis of the topology. Indeed $V \triangleleft_q U$ is equivalent to $B_q V \leqslant U$ and $B_q V = \bigvee B_{q'} V$ for $q' < q$, hence this asserts that the open balls form a basis of the topology.
Also if $X$ is metric and $f$ is a geometric morphism then $f^{\#}(X)$ is also metric because the $B_q V$ for $V \in f^*(\mathcal{O}(X))$ form a basis of $f^{\#}(X)$.

**Proposition :** *A Metric locale satisfies the following separation axiom: the diagonal embedding*

$$X \rightarrow \bigwedge_q \Delta_q$$

*is an isomorphism (where the intersection is an intersection of sublocale).*

The intuitive reason for this is that if we consider two points $(x, y)$ in $\bigwedge_q \Delta_q$ then by definition $d(x, y) = 0$. If the open balls form a basis of the topology then for any open $U$, $x \in U$ if and only if $y \in U$, but for points of a locale this implies that $x = y$. The following proof is just the translation of this argument in terms of generalized points.
**Proof :**
Consider $f : Y \rightarrow \bigwedge_q \Delta_q$ a map, and let $f_1$ and $f_2$ be the two components $Y \rightarrow X$ of $f$. Let $U$,$V$ be two open sublocales of $X$ such that $U \triangleleft_q V$. Then

$$\pi_1^*(U) \wedge \Delta_q \leqslant \pi_2^*(V).$$

Applying $f^*$ to each side gives

$$f_1^*(U) \wedge f^*(\Delta_q) \leqslant f_2^*(V),$$

and as $f^*(\Delta_q) = Y$ by hypothesis, one has $f_1^*(U) \leqslant f_2^*(V)$.
Finally, writing $V = \bigvee_{U \triangleleft V} U$ one has:

$$f_1^*(V) = \bigvee_{U \triangleleft V} f_1^*(U) \leqslant f_2^*(V).$$

The converse inequality follows by symmetry and hence $f_1 = f_2$ i.e. $f$ factors into the diagonal embedding, and this concludes the proof. $\square$

In particular, as by 3.1.5,

$$\bigwedge \Delta_q = \bigwedge \overline{\Delta_q}$$

The diagonal embedding of a metric locale is fiberwise closed, one says that metric locales are *fiberwise separated*.

3.2.2. **Proposition :** *Let $X$ be a metric locale, and $Y$ a pre-metric locale. Let $f : X \to Y$ be an isometric map. Then $X$ is a sublocale of $Y$ i.e. $f^*$ is onto. More generally, if we only assume that $X$ is pre-metric then we obtain the inequalities*

$$\forall U \in \mathcal{O}(X), \bigvee_{V \lhd U} V \leqslant f^* f_*(U) \leqslant U$$

The proposition follows from Lemma 3.1.11:

**Proof :**
Let $U$ be any open sublocale of $X$, such that

$$U = \bigvee_{V \lhd U} V$$

For any $V \lhd_q U$ one has by Lemma 3.1.11

$$V \leqslant f^*(B_q f_! V) \leqslant U$$

hence

$$U = \bigvee_{q, V \lhd_q U} f^*(B_q f_! V) = f^* \left( \bigvee_{q, V \lhd_q U} B_q f_! V \right)$$

In particular, if $X$ is metric, then this works for an arbitrary $U$ and $f^*$ is surjective.
If $X$ is no longer metric, then let $U' = \bigvee_{V \lhd U} V$, then $U'$ satisfy $U' = \bigvee_{V \lhd U'} V$ and hence the first part can be applied to $U'$ and there exists $V$ such that $U' = f^*(V)$. In particular, as $f^*(V) \leqslant U$ we obtain that $V \leqslant f_*(U)$ and hence

$$U' = f^*(V) \leqslant f^*(f_*(U)).$$

The inequality $f^*(f_*(U)) \leqslant U$ being always true this concludes the proof.
□

3.2.3. The following proposition allows one to extend by density relations between continuous functions with values in metric locale.

**Proposition :** *Let $f, g : X \rightrightarrows Y$ be two maps of locales with $Y$ a metric locale (or more generally a fiberwise separated locale). Assume that $f$ and $g$ coincide on some fiberwise dense sublocale $T \subset X$. Then $f = g$.*

**Proof :**
Let $V$ be the pullback of the diagonal of $Y$ by the map $(f, g) : X \to Y \times Y$. As fiberwise closeness is stable under pullback (see [44] C1.2.14(v) ), $V$ is a fiberwise closed sublocale of $X$, containing the fiberwise dense sublocale $T$, hence $V = X$, and this concludes the proof. □

3.2.4. We will also sometimes need to extend by continuity "metric relations" between functions, which will generally be about comparing functions with value in $\overleftarrow{\mathbb{R}_+^\infty}$. As $\overleftarrow{\mathbb{R}_+^\infty}$ is not fiberwise separated, it is not possible to apply directly the previous result. However, one has the following statement:

We will say that a function from $m : X \to \overleftarrow{\mathbb{R}_+^\infty}$ is admissible if there exist two families of functions $f_1, \dots f_n$ and $g_1, \dots, g_n$ from $X$ to pre-metric locales $X_1, \dots X_n$ and a commutative diagram:

$$\begin{tikzcd}
(\overline{\mathbb{R}_+})^n \arrow[r] \arrow[d] & \overline{\mathbb{R}_+} \arrow[d] \\
\left(\overleftarrow{\mathbb{R}_+^\infty}\right)^n \arrow[r, "\lambda"] & \overleftarrow{\mathbb{R}_+^\infty}
\end{tikzcd}$$

(where the vertical arrows are the canonical maps) such that:

$$m(x) = \lambda(d(f_1(x), g_1(x)), \dots, d(f_n(x), g_n(x))).$$

It is probably possible to use a more general definition of "admissible" map, but this one will be enough for all the applications appearing here.

**Proposition :** *Assume that one has two admissible maps $m_1, m_2 : X \rightrightarrows \overleftarrow{\mathbb{R}_+^\infty}$ such that one has an inequality $m_1 \leqslant m_2$ on some fiberwise dense sublocale $S$ of $X$ a locally positive locale, then the inequality holds one the whole $X$.*

**Proof :**
The idea is to pullback everything to some boolean locale $\mathcal{B}$. In the logic of $\mathcal{B}$, thanks to 3.1.7 the admissible functions $m_1$ and $m_2$ will factor as functions $X \rightrightarrows \overline{\mathbb{R}}$ still satisfying an inequality over $S$. The pullback of $S$ is still fiberwise dense in the pullback of $X$ because of 2.4.5, but, contrary to $\overleftarrow{\mathbb{R}_+^\infty}$, $\mathbb{R}$ is (fiberwise) separated and hence one can conclude that in the category of sheaves over $\mathcal{B}$ the pullbacks of $m_1$ and $m_2$ agree on the pullback of $X$ by 3.2.3. This implies that (in the base topos) one has a diagram:

$$\begin{tikzcd}
\mathcal{B} \times X \arrow[r, shift left, "m_1 \leqslant m_2"] \arrow[r, shift right] \arrow[d, "\pi_2"] & \mathcal{B} \times \overleftarrow{\mathbb{R}_+^\infty} \arrow[d, "\pi_2"] \\
X \arrow[r, shift left, "m_1, m_1"] \arrow[r, shift right] & \overleftarrow{\mathbb{R}_+^\infty}
\end{tikzcd}$$

In order to conclude that $m_1 \leqslant m_2$ it is enough to choose $\mathcal{B}$ such that $\pi_2 : \mathcal{B} \times X \to X$ is surjective. It is possible, indeed, if one chooses a boolean locale $\mathcal{B}$ which covers $X$, i.e. with a surjective map $s : \mathcal{B} \twoheadrightarrow X$ then:

$$\begin{tikzcd}
\mathcal{B} \times \mathcal{B} \arrow[r] \arrow[d, "\pi_2"] & \mathcal{B} \times X \arrow[d, "\pi_2"] \\
\mathcal{B} \arrow[r, "s"] & X
\end{tikzcd}$$

The projection $\pi_2 : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ is a surjection because it has a section, the map $s : \mathcal{B} \to X$ is surjective by hypothesis, hence the diagonal map is surjective. This implies that the map $\pi_2 : \mathcal{B} \times X \to X$ is surjective and hence it concludes the proof. $\square$

Of course the same result where the inequality is replaced by an equality also holds by two applications of this result.

## 3.3 Completion of a metric locale

In this subsection we will define the completion of pre-metric locale as the space of minimal Cauchy filters. The same idea has been previously used by S.Vickers in [71].

3.3.1. **Definition :** *Let $X$ be a pre-metric locale. A basis $B$ of $X$ is said to be a metric basis if and only if $B$ contains only positive elements, and if $V \in B$ implies $B_q V \in B$.*

This definition can easily be changed without altering the main result of this chapter, we have chosen it only because it is the simplest notion we have found which is strong enough to assert that the basis will be well behaved and weak enough so that the natural examples we will encounter in practice satisfy this definition, like for example the basis of all open balls on a normed space.
Of course if $B$ is an arbitrary basis of $X$ (composed of positive elements) one can consider the metric basis generated by $B$ by adding to $B$ all the elements of the form $B_{q_1} \dots B_{q_n} V$ for $V \in B$ and $(q_i)$ a finite sequence of positive rational numbers. Also, if $B$ is a metric basis on $X$ in a topos, then the pullback of $B$ by any geometric morphism $f : \mathcal{E} \to \mathcal{T}$ is a metric basis of the pullback of $X$.

3.3.2. **Definition :** *Let $X$ be a pre-metric locale endowed with a metric basis $B$, a $B$-Cauchy filter on $X$ is a subset $\mathcal{F} \subseteq B$ such that:*

*(CF1) For all $V \in \mathcal{F}$ and $U \in B$ such that $V \leqslant U$ one has $U \in \mathcal{F}$. .*

*(CF2) If $U, V \in \mathcal{F}$ then there exists $W \in B$ such that $W \leqslant U$ and $W \leqslant V$ and $W \in \mathcal{F}$.*

*(CF3) For all positive rational numbers $q$, there exists $U \in \mathcal{F}$ such that $\delta(U) < q$.*

*A $B$-Cauchy filter is said to be regular if it satisfies additionally:*

*(CF4) For all $U \in \mathcal{F}$ there exists $V \in \mathcal{F}$ such that $V \lhd U$.*
*A Cauchy filter on $X$ (without specifying the basis) is a $B$-Cauchy filter on $X$, for $B = \mathcal{O}(X)^+$.*

We insist on the fact that $B$ (as a metric basis) is always assumed to be a subset of $\mathcal{O}(X)^+$. This is why there is no axiom asserting that $\emptyset$ is not an element of $\mathcal{F}$, or that all the elements of $\mathcal{F}$ are positive.

3.3.3. **Proposition :** *Any $B$-Cauchy filter $\mathcal{F}$ contains a unique regular Cauchy filter which is $\mathcal{F}^r = \{V \in B | \exists u \in \mathcal{F}, u \lhd V\}$.*

**Proof :**
One easily checks that $\mathcal{F}^r$ is a regular $B$-Cauchy filter. Conversely, let $\mathcal{F}'$ be a regular $B$-Cauchy filter included in $\mathcal{F}$, then for any $U \in \mathcal{F}'$ there exists by $(CF4)$ an element $V \in \mathcal{F}$ such that $V \leqslant B_q V \leqslant U$, hence $U \in \mathcal{F}^r$, which proves that $\mathcal{F}' \subset \mathcal{F}^r$. Let now $U \in \mathcal{F}^r$, by definition there exists $V \in \mathcal{F}$ such that $V \leqslant B_q V \leqslant U$, by $(CF3)$ there exists $W \in \mathcal{F}'$ such that $\delta(W) < q$ and by $(CF2)$ there must be an element $\tau$ of $\mathcal{F}$ such that $\tau \leqslant W$ and $\tau \leqslant V$. In particular, $W \wedge V > \emptyset$ and hence (by the point 10 of 3.1.4) $W \leqslant B_q V \leqslant U$ and $U \in \mathcal{F}'$ which concludes the proof. $\square$

Hence regular Cauchy filters correspond to the notion of minimal Cauchy filter, this explains why we will later construct the completion of a locale as the classifying space of regular Cauchy filters, by analogy with the classical construction of the completion of a uniform space as a uniform structure on the set of minimal Cauchy filters (see [10, Chap. II.7]).

3.3.4. **Lemma :** *Let $X$ be a pre-metric locale endowed with a metric basis $B$, and let $\mathcal{F}$ be a regular Cauchy filter on $X$. Then for any $U \in \mathcal{F}$, there exists $V \in B \wedge \mathcal{F}$ such that $V \leqslant U$.*

**Proof :**
Let $U \in \mathcal{F}$, by $(CF4)$ there exists $U' \lhd_q U$ such that $U' \in \mathcal{F}$. Also by $(CF3)$ there exists an element $W \in \mathcal{F}$ such that $\delta(W) < (q/3)$ and as $B$ is a basis and $W$ is positive there exists $b \leqslant W$ with $b \in B$. Let $V = B_{q/3} b$, then, by the point 12 of 3.1.4, one has $\delta(V) < q$ , also $V \in B$ because $B$ is metric, $W \leqslant V$ because $b \wedge W = b$ is positive and $\delta(W) < q/3$ and hence $V \in \mathcal{F}$. Also by $(CF2)$ there exists $V' \in \mathcal{F}$ such that $V' \leqslant V \wedge U'$, as $V'$ is positive this implies that $V \leqslant B_q U' \leqslant U$. As $V \in B \wedge \mathcal{F}$, this concludes the proof. $\square$

3.3.5. **Corollary :** *The map $\mathcal{F} \to B \wedge \mathcal{F}$ induces a bijection between the set of regular Cauchy filters on $X$ and the set of regular $B$-Cauchy filters on $X$.*

We also mention that, as the following proof will show, this proposition holds for any family $B$ satisfying the conclusion of the previous lemma (3.3.4) even if it is not a metric basis or even if it is not a basis at all.

**Proof :**
Let $\mathcal{F}$ be a regular Cauchy filter on $X$. We will first prove that $\mathcal{F}' = \mathcal{F} \wedge B$ is a regular $B$-Cauchy filter, this is essentially immediate by Lemma 3.3.4:

- If $U \leqslant V$ with $V \in \mathcal{F}'$ and $U \in B$ then $U \in \mathcal{F}$ and hence $U \in \mathcal{F}'$ because $\mathcal{F}$ satisfy $(CF1)$.
- If $U, V \in \mathcal{F}'$ then there exists $W \in \mathcal{F}$ such that $W \leqslant U \wedge V$ and by the lemma there exists $W' \in \mathcal{F}'$ such that $W' \leqslant W \leqslant U, V$.
- There exists $U \in \mathcal{F}$ such that $\delta(U) < q$ and (by the lemma) a $U' \leqslant U$ such that $U' \in \mathcal{F}'$, hence $\delta(U') < q$.
- Let $U \in \mathcal{F}'$, there exists $V \in \mathcal{F}$ such that $V \triangleleft U$, then any $V' \leqslant V$ with $V' \in \mathcal{F}'$ (again given by the lemma) works.

Now $\mathcal{F}$ can be reconstructed from $\mathcal{F}'$ by the lemma together with $(CF1)$:

$$\mathcal{F} = \{U | \exists U' \in \mathcal{F}', U' \leqslant U\}.$$

And if you take $\mathcal{F}'$ to be any regular $B$-Cauchy filter, then the previous formula defines a $\mathcal{F} \subseteq \mathcal{O}(X)^+$ which is easily checked to be a regular Cauchy filter as well, and by $(CF1)$ $\mathcal{F}' = \mathcal{F} \wedge B$. This concludes the proof. $\square$

3.3.6. Let $X$ be a pre-metric locale, and $B$ be a metric basis on $X$, the theory of regular $B$-Cauchy filters as defined in 3.3.2 is clearly a propositional geometric theory with basic propositions indexed by $B$. Hence it has a classifying space $\widetilde{X}_B$.
If $X$ is a pre-metric locale in a topos $\mathcal{T}$ and if $f : \mathcal{E} \to \mathcal{T}$ is a geometric morphism, then $f^{\#}(\tilde{X}_B) \simeq \widetilde{f^{\#}(X)}_{f^*(B)}$ because the pullback of a classifying locale classifies the pullback of the theory and the pullback of the theory of regular $B$-Cauchy filter is exactly the theory of regular $f^*(B)$-Cauchy filter on $f^{\#}(X)$. But by 3.3.5 the points of $\widetilde{X}_B$ do not depend on $B$, and hence by the observations we just made, their points on any topos over the base topos do not depend on $B$, and all the $\widetilde{X}_B$ are isomorphic.

**Definition :** *The completion $\widetilde{X}$ of $X$ is defined as the classifying locale $\widetilde{X}_B$ of the theory of regular $B$-Cauchy filters on $X$ for any metric basis $B$ of $X$.*

Also if $U$ is any positive open sublocale of $X$ we denote by $U^\sim$ the open sublocale of $\widetilde{X}$ corresponding to the proposition "$U \in \mathcal{F}$". It is a general fact about classifying spaces that the $U^\sim$ form a pre-basis of the topology of $X$, but the axiom $(CF2)$ show that for any metric basis $B$ of $X$, the $U^\sim$ with $U \in B$ form a basis of $\widetilde{X}$. If $U$ is not necessarily positive, one can still defined $U^\sim$ by

$$U^\sim = \bigvee_{\substack{V \leqslant U \\ V > \emptyset}} V^\sim.$$

When $U > \emptyset$, the two possible definitions of $U^\sim$ are compatible because

$$\bigvee_{\substack{V \leqslant U \\ V > \emptyset}} V^\sim = U^\sim$$

3.3.7. **Proposition :** *Let $Y$ be a locale, a morphism $f$ from $Y$ to $\widetilde{X}$ corresponds to a map $\tau : B \to \mathcal{O}(Y)$ such that:*

1. *$\tau$ is non-decreasing.*
2. $\tau(U) \wedge \tau(V) \leqslant \bigvee_{\substack{W \in B \\ W \leqslant U \wedge V}} \tau(W)$
3. $\bigvee_{\substack{U \in B \\ \delta(U) < q}} (\tau(U)) = Y$
4. $\tau(U) \leqslant \bigvee_{\substack{V \in B \\ V \triangleleft U}} \tau(V)$

*Moreover this correspondence is characterized by the relation $\tau(U) = f^*(U^\sim)$. Also if $\tau$ only satisfies the first three properties, then there exists a unique $\tau^r$ such that $\tau^r$ satisfy the four properties and $\tau^r \leqslant \tau$ for the pointwise ordering and one has*

$$\tau^r(U) = \bigvee_{\substack{V \in B \\ V \triangleleft U}} \tau(V)$$

**Proof :**
A morphism from $Y$ to $\widetilde{X}$ is the data of a regular Cauchy filter on $X$ in the internal logic of $Y$. i.e. for each $U \in B$ one should have a proposition $\tau(U) :=$ "$U \in \mathcal{F}$" satisfying (internally) the axiom $(CF1-5)$. The four properties given for $\tau$ corresponds exactly to the externalisation of the four axioms $(CF1-4)$ (in the right order).

If $\tau$ only satisfies the first three properties then it is just a $B$-Cauchy filter on $X$ and in this case one can apply 3.3.3 and there is a unique regular $B$-Cauchy filter $\tau^r \leqslant \tau$ and it is indeed given by

$$\tau^r(U) = \bigvee_{\substack{V \in B \\ V \triangleleft U}} \tau(V)$$

which is the direct translation of $U \in \tau^r$ if there exists $V \triangleleft U$ with $V \in \tau$.
□

Of course, the inequalities in the axioms 2. and 4. are in fact equalities because the axiom 1. implies the reverse inequalities.

3.3.8. **Proposition :** *There is a map $i$ from $X$ to $\widetilde{X}$ defined by*

$$i^*(U^\sim) = \bigvee_{V \lhd U} V.$$

*Moreover, for any $U \in \mathcal{O}(X)$,*

$$i_*(U) = U^\sim$$

**Proof :**
The inclusion map $e : \mathcal{O}(X)^+ \to \mathcal{O}(X)$ clearly satisfies the first three points of 3.3.7. Hence the map

$$e^r(U) = \bigvee_{V \lhd U} V$$

satisfies the four points of 3.3.7 and hence there is a map $i : X \to \widetilde{X}$ such that for any $U \in \mathcal{O}(X)^+$ one has $i^*(U^\sim) = e^r(U)$. But as $U^\sim$ is defined as $\bigvee_{\substack{V \leqslant U \\ V > \emptyset}} V^\sim$ this formula immediately extends to an arbitrary $U$.
We still have to prove that $i_*(U) = U^\sim$. As $i^*(U^\sim) \leqslant U$, one has $U^\sim \leqslant i_*(U)$. Let $V$ an arbitrary open sublocale of $X$ such that $V^\sim \leqslant i_* U$ hence,

$$\bigvee_{V' \lhd V} V' \leqslant U$$

Consider an arbitrary Cauchy filter $F$ on $X$ such that $V \in F$. Then there exists $V' \lhd V$ such that $V' \in F$ and hence $U \in F$. This proves that $V^\sim \leqslant U^\sim$ and hence, as $V^\sim \leqslant U^\sim$ imply $V^\sim \leqslant i_*(U)$ one has $V^\sim \leqslant i_* U$ if and only if $V^\sim \leqslant U^\sim$ hence as the $V^\sim$ form a basis of $\widetilde{X}$ this proves that $i_*(U) = U^\sim$.
$\square$

3.3.9. **Proposition :** *The canonical map $i : X \to \widetilde{X}$ is fiberwise dense and $\widetilde{X}$ is locally positive.*

**Proof :**
The $(B_q V)^\sim$ for $q$ a positive rational number and $V$ a positive element of $\mathcal{O}(X)$ form a basis of $\widetilde{X}$. Indeed, the $U^\sim$ for $U \in \mathcal{O}(X)^+$ form a basis, and for any $U \in \mathcal{O}(X)$ by $(CF4)$,

$$U^\sim = \bigvee_{\substack{V \lhd U \\ V > \emptyset}} V^\sim = \bigvee_{B_q V \leqslant U} (B_q V)^\sim.$$

Moreover,

$$i^*((B_q V)^\sim) = \bigvee_{U \lhd B_q V} U \geqslant \bigvee_{q' < q} B_{q'} V = B_q V.$$

Hence one has a basis of elements of $\widetilde{X}$ whose pre-image by $i$ are positive. This implies that $\widetilde{X}$ has a basis of positive elements and that for each positive element of $\widetilde{X}$ its pre-image along $i$ is positive, which concludes the proof. $\square$

3.3.10. **Proposition :** *There is a distance function $d$ on $\widetilde{X}$ such that*

$$\Delta_q = \bigvee_{U \in \mathcal{O}(X)^{<q}} U^{\sim} \times U^{\sim}.$$

One might note that this definition of the distance on $\widetilde{X}$ is the point-free formulation of the more usual definition:

$$d(\mathcal{F},\mathcal{F}') < q \text{ if and only if } \exists u \in \mathcal{F} \wedge \mathcal{F}' \text{ with } \delta(u) < q$$

which is equivalent if interpreted in terms of generalized points.

**Proof :**

Let $U \in \mathcal{O}(X)$ such that $\delta(U) < q$. Then there exists $q'$ such that $\delta(U) < q'$ and $U^{\sim} \times U^{\sim} \leqslant \Delta_{q'}$. Hence

$$\Delta_q = \bigvee_{q'<q} \Delta_{q'},$$

which proves that this formula defines a function $d : \widetilde{X} \times \widetilde{X} \rightarrow \overleftarrow{\mathbb{R}_+^{\infty}}$. This function is clearly symmetric, and the diagonal embeddings factor into $\Delta_q$ because the $U^{\sim}$ with $\delta(U) < q$ cover $\widetilde{X}$ by axiom $(CF3)$. The last point to check is the triangular inequality, but:

$$\pi_{1,2}^*(\Delta_q) \wedge \pi_{2,3}^*(\Delta_{q'}) = \bigvee_{\substack{\delta(U)<q \\ \delta(U')<q'}} U^{\sim} \times (U^{\sim} \wedge U'^{\sim}) \times U'^{\sim}$$

$$(\pi_{1,3})_! \left(\pi_{1,2}^*(\Delta_q) \wedge \pi_{2,3}^*(\Delta_{q'})\right) = \bigvee_{\substack{\delta(U)<q \\ \delta(U')<q' \\ U \wedge U' > \emptyset}} U^{\sim} \times U'^{\sim}.$$

Since $U^{\sim} \times U'^{\sim} \leqslant (U \vee U')^{\sim} \times (U \vee U')^{\sim}$ and as we are restricted to the case $U \wedge U' > \emptyset$, one has $\delta(U \vee U') < q + q'$ by point 6 of 3.1.4, hence $U^{\sim} \times U'^{\sim} \subset \Delta_{q+q'}$ and

$$(\pi_{1,3})_! \left(\pi_{1,2}^*(\Delta_q) \wedge \pi_{2,3}^*(\Delta_{q'})\right) \leqslant \Delta_{q+q'},$$

which is the triangular inequality. The last point to prove is that this pre-distance is a distance. This a consequence of the following lemma. □

**Lemma :** *For any $U \in \mathcal{O}(X)$ one has $B_q(U^{\sim}) \leqslant (B_q U)^{\sim}$. In particular, if $U \triangleleft_q V$ then $U^{\sim} \triangleleft_q V^{\sim}$.*

**Proof :**

Indeed, for any $W \in \mathcal{O}(X)$ such that $\delta(W) < q$ and $U^{\sim} \wedge W^{\sim}$ is positive, $(CF2)$ proves that $U \wedge W$ is positive, hence, from the definition of $\Delta_q$:

$$B_q(U^{\sim}) = (\pi_2)_!(\pi_1^*(U^{\sim})\Delta_q) = \left( \bigvee_{\substack{\delta(W)<q \\ U^{\sim} \wedge W^{\sim} > 0}} W^{\sim} \right) \leqslant (B_q U)^{\sim}$$

which concludes the proof of the lemma. $\square$

This lemma allows to finish the proof of the proposition, indeed, by $(CF4)$, $V^\sim = \bigvee_{U \triangleleft V} U^\sim$, hence any $V \in \mathcal{O}(\widetilde{X})$ can be written as

$$V = \bigvee_{U^\sim \leqslant V} U^\sim = \bigvee_{A^\sim \triangleleft U^\sim \leqslant V} A^\sim.$$

3.3.11. **Proposition :** *Let $S \to Y$ be a fiberwise dense isometric map between two pre-metric locales, let $X$ be any pre-metric locale and $f : S \to \widetilde{X}$ be a uniform map. Then there exists a unique extension $\widetilde{f} : Y \to \widetilde{X}$.*

**Proof :**
The uniqueness of the extension follows from the fact that $\widetilde{X}$ is metric (3.3.10) and the result of 3.2.3, so we only have to prove the existence. We will use 3.3.7 for this. Let $\tau : \mathcal{O}(X)^+ \to \mathcal{O}(Y)$ defined by:

$$\tau(U) = i_* f^*(U^\sim)$$

where $i$ denote the embeddings of $S$ into $Y$.
We will first check that $\tau$ satisfies the first three properties of 3.3.7:

1. $i_*, f^*$ and $U \mapsto U^\sim$ are all order preserving. Hence $\tau$ is order preserving.
2. One has $U^\sim \wedge V^\sim = (U \wedge V)^\sim$ (essentially by $(CF2)$) hence as $i_*$ and $f^*$ also commute to binary intersection one has: $\tau(U) \wedge \tau(V) = \tau(U \wedge V)$. This is not enough to conclude immediately the proof of this point because $U \wedge V$ might fail to be positive. Fortunately, if one assumes that $\tau(W) = i_* f^*(W^\sim)$ is positive, then $i^* i_* f^*(W^\sim)$ is also positive because $i$ is fiberwise dense, which implies that $f^*(W^\sim)$ is positive (because it is bigger than $i^* i_* f^*(W^\sim)$) and hence that $W^\sim$ is positive, which finally implies that $W$ is positive (by 3.3.9 and 3.3.8). Hence one can write that

   $$\tau(U) \wedge \tau(V) = \tau(U \wedge V) = \bigvee_{\tau(U \wedge V) > \emptyset} \tau(U \wedge V) \leqslant \bigvee_{U \wedge V > \emptyset} \tau(U \wedge V),$$

   which proves points 2.
3. We fix $q$ a positive rational number, and (as $f$ is uniform) $\eta$ such that $\Delta_\eta \leqslant (f \times f)^* \Delta_{q/3}$ (see 3.1.9).

   Let $U \in \mathcal{O}(S)^{+,<\eta}$ then (by 3.1.9) there exists $W \in \mathcal{O}(\widetilde{X})^{<q/3}$ such that $U \leqslant f^*(W)$.

   In particular $W$ is also positive and hence, by $(CF3)$ and the fact that the $V^\sim$ form a basis of $\widetilde{X}$, there exists $V_0 \in \mathcal{O}(X)^{+,<q/3}$ such that $V_0^\sim \leqslant W$. We define $V = B_{q/3} V_0$. One has $\delta(V) < q$ (by 3.1.4.12) and $W \leqslant V^\sim$ (by the lemma proved in 3.3.10), in particular $U \leqslant f^*(V^\sim)$. This proves that

$$\bigvee_{U \in \mathcal{O}(S)^{+,<\eta}} i_* U \leqslant \bigvee_{V \in \mathcal{O}(X)^{+,<q}} i_* f^*(V^\sim) = \bigvee_{V \in \mathcal{O}(X)^{+,<q}} \tau(V), \qquad (3.2)$$

Finally

$$Y = \bigvee_{V \in \mathcal{O}(Y)^{+,<\eta}} V \leqslant \bigvee_{V \in \mathcal{O}(Y)^{+,<\eta}} i_* i^* V = Y.$$

As $i$ is an isometric map, for any $V \in \mathcal{O}(Y)^{<\eta}$ one has $i^* V \in \mathcal{O}(S)^{<\eta}$. Hence

$$Y = \bigvee_{V \in \mathcal{O}(Y)^{+,<\eta}} i_* i^* V \leqslant \bigvee_{U \in \mathcal{O}(S)^{+,<\eta}} i_* U. \qquad (3.3)$$

The inequalities (3.2) and (3.3) together conclude the proof of the third point.

Hence from 3.3.7 there is a map $\tilde{f} : Y \to \widetilde{X}$ such that $\tilde{f}^*(U^\sim) = \tau^r(U) = \bigvee_{V \lhd U} i_* f^* V^\sim$. It remains to be proved that $\tilde{f}$ is indeed an extension of $f$, i.e. that $\tilde{f} \circ i = f$.

$$i^* \tilde{f}^*(U^\sim) = \bigvee_{V \lhd U} i^* i_* f^*(V^\sim) \leqslant \bigvee_{V \lhd U} f^*(V^\sim) = f^*(U^\sim)$$

Because $\bigvee_{V \lhd U} V^\sim = U^\sim$ by $(CF4)$. One the other hand, from the non-metric part of 3.2.2

$$i^* \tilde{f}^*(U^\sim) = \bigvee_{V \lhd U} i^* i_* f^*(V^\sim) \geqslant \bigvee_{\substack{V \lhd U \\ V' \lhd f^*(V^\sim)}} V'.$$

As $f^*$ is uniform it is compatible with $\lhd$, hence the set of $V'$ appearing in the last union contains all the $f^*(W^\sim)$ for $W \lhd V$ hence

$$i^* \tilde{f}^*(U^\sim) \geqslant \bigvee_{\substack{V \lhd U \\ W \lhd V}} f^*(W^\sim) = f^*(U^\sim),$$

which proves $i^* \tilde{f}^*(U^\sim) = f^*(U^\sim)$ and concludes the proof.
□

We also note that if the map $f$ is metric (resp. isometric), the extension $\widetilde{f}$ will also be metric (resp. isometric) by an application of 3.2.4.

3.3.12. **Theorem :** *Let $X$ be a pre-metric locale, then the following conditions are equivalent:*

1. *The map $X \to \widetilde{X}$ is an isomorphism;*
2. *$X \simeq \widetilde{Y}$ for some $Y$;*
3. *For any $S \to Y$ a strongly dense isometric map between pre-metric locales, and any map from $S$ to $X$ there exists a map from $Y$ to $X$ making the triangle commute;*

4. *Any strongly dense isometric map from $X$ to a metric locale $Y$ is an isomorphism.*

*A locale satisfying these conditions is called a complete metric locale.*

**Proof :**
1. $\Rightarrow$ 2. is clear.
2. $\Rightarrow$ 3. is a direct consequence of 3.3.11.
4. $\Rightarrow$ 1. is also clear because the map from $X$ to $\widetilde{X}$ is a dense isometric map.
3. $\Rightarrow$ 4. remains to be proved. Let $f : X \to Y$ be a strongly dense isometric map. The identity map from $X$ to $X$ can be extended into a map $g$ from $Y$ to $X$ by 3., such that $g \circ f = Id_X$. As, $f \circ g$ restricted to $X$ is the inclusion from $X$ to $Y$, $f \circ g$ is the identity of $Y$ by fiberwise density of $X$ into $Y$ and fiberwise separation of $Y$ ( 3.2.3) hence $g$ is an inverse for $f$, and they are isomorphisms. $\square$

It is immediate from point 3. that a locally positive fiberwise closed sublocale of a complete locale is also complete.

3.3.13. **Proposition :** *If $X$ is a pre-metric locale in a topos $\mathcal{T}$ and $f : \mathcal{E} \to \mathcal{T}$ is an open (or proper) surjection such that $f^{\#}(X)$ is complete then $X$ is complete.*

**Proof :**
The pullback along $f$ of the canonical map $X \to \widetilde{X}$ is the canonical map $f^{\#}(X) \to \widetilde{f^{\#}(X)}$. Hence as $f^{\#}$ is a descent functor for the categories of locales, it is in particular conservative and if the pullback map is an isomorphism, the map $X \to \widetilde{X}$ is also an isomorphism. $\square$

An immediate corollary of this result is that if $\mathcal{C}(\mathcal{T})$ is the category of complete metric locales and metric maps between them then objects of $\mathcal{C}$ descend along open surjections. Indeed, it is a full subcategory of the category of pre-metric locales, for which open surjections are descent morphisms as observed in 3.1.13, and this just states that $(X', d')$ is complete if it descends from a complete locale $(X, d)$.

3.3.14. **Proposition :** *Let $X$ be a pre-metric locale and let $X_d$ be the regular image of $X$ into $\widetilde{X}$ then $\mathcal{O}(X_d)$ identifies with the set of $U \in \mathcal{O}(X)$ such that*

$$U = \bigvee_{V \triangleleft U} V$$

*and any map compatible with $\triangleleft$ from $X$ to a metric locale $Y$ factors into $X_d$.*

**Proof :**
The regular image of $i : X \to \widetilde{X}$ is identified as a frame with the image of $i^* : \mathcal{O}(\widetilde{X}) \to \mathcal{O}(X)$ which is clearly (by 3.3.8) the set of open sublocales defined

in the proposition. If one has any map $f$ from $X$ to a metric locale $Y$ compatible with $\triangleleft$ then for any $U \in \mathcal{O}(Y)$,

$$U = \bigvee_{V \triangleleft U} V$$

Hence,

$$f^*(U) = \bigvee_{V \triangleleft U} f(V)^*$$

as $f^*(V) \leqslant f^*(U)$ this proves that $f^*(U) \in \mathcal{O}(X_d)$. Hence $f$ factors into $X_d$.
□

## 3.4 Product of metric locales

3.4.1. Let $\mathcal{L}$ and $\mathcal{M}$ be two pre-metric locales, one defines a pre-distance on $\mathcal{L} \times \mathcal{M}$ in the following way: $\Delta_q^{\mathcal{L}\times\mathcal{M}} \subset (\mathcal{L} \times \mathcal{M}) \times (\mathcal{L} \times \mathcal{M})$ is the intersection of the pullback $\pi_{1,3}^*(\Delta_q^{\mathcal{L}})$ and $\pi_{2,4}^*(\Delta_q^{\mathcal{M}})$ (where the exponent on $\Delta$ indicate to which locale it is related). This corresponds to taking $d((l,m),(l',m')) = \max(d(l,l'), d(m,m'))$, and the classical argument can be adapted (in terms of generalised points) to prove that this is indeed a pre-distance on $\mathcal{L} \times \mathcal{M}$.

**Proposition :** *$\mathcal{M} \times \mathcal{L}$ endowed with the previously constructed distance function is the categorical product of $\mathcal{M}$ and $\mathcal{L}$ in the category of pre-metric locales and metric maps.*

**Proof :**
The projection $\pi_1 : \mathcal{L} \times \mathcal{M} \to \mathcal{L}$ satisfies $\Delta_q \subset \pi_1^*(\Delta_q)$ by construction of the distance function on $\mathcal{L} \times \mathcal{M}$, hence it is a metric map. In particular if $f : X \to \mathcal{M} \times \mathcal{L}$ is a metric map then the two component $f_1$ and $f_2$ are metric maps. Conversely, assume that $f_1$ and $f_2$ are metric maps. Then

$$(f \times f)^*(\Delta_q^{\mathcal{L}\times\mathcal{M}}) = (f \times f)^*(\pi_{1,3}^*(\Delta_q^{\mathcal{L}}) \wedge \pi_{2,4}^*(\Delta_q^{\mathcal{M}})).$$

But $\pi_{1,3}(f \times f) = f_1 \times f_1$ and $\pi_{2,4}(f \times f) = f_2 \times f_2$, hence,

$$(f \times f)^*(\Delta_q^{\mathcal{L}\times\mathcal{M}}) = (f_1 \times f_1)^*(\Delta_q^{\mathcal{L}}) \wedge (f_2 \times f_2)^*(\Delta_q^{\mathcal{M}})$$

As we assume that both $f_1$ and $f_2$ are metric,

$$\Delta_q^X \subset (f_1 \times f_1)^*(\Delta_q^{\mathcal{L}}) \wedge (f_2 \times f_2)^*(\Delta_q^{\mathcal{M}}),$$

This proves that $f$ is also metric and concludes the proof of the proposition.
□

3.4.2. **Proposition :** *The product of two complete metric locales is a complete metric locale. More generally the completion of* $\mathcal{L}\times\mathcal{M}$ *is canonically isomorphic to* $\widetilde{\mathcal{L}}\times\widetilde{\mathcal{M}}$*.*

**Proof :**
Assume that $\mathcal{L}$ and $\mathcal{M}$ are complete. Let $S \to Y$ be a strongly dense map, and let $f : S \to \mathcal{L}\times\mathcal{M}$ be an isometric map. Then by the previous result and Proposition 3.3.11 there is a map $\widetilde{f} : Y \to \mathcal{L}\times\mathcal{M}$ extending $f$. Hence $\mathcal{L}\times\mathcal{M}$ is complete.
For the second part, $\mathcal{L}\times\mathcal{M} \to \widetilde{\mathcal{L}}\times\widetilde{\mathcal{M}}$ is a fiberwise dense isometric map with $\widetilde{\mathcal{L}}\times\widetilde{\mathcal{M}}$ complete, hence $\widetilde{\mathcal{L}}\times\widetilde{\mathcal{M}}$ is the completion of $\mathcal{L}\times\mathcal{M}$. $\square$

## 3.5 The locale $[X,Y]_1$ of metric maps

In this subsection we show that it is possible to construct a classifying space $[X,Y]_1$ of metric maps between two metric locales $X$ and $Y$, at least when $Y$ is complete. The key observation underlying this construction is that (in a classical settings) on the set of metric functions the topology of point-wise convergence on any dense subsets is equivalent to the compact-open topology, and that when we endow this set of metric functions with this topology the composition law is bi-continuous. This suggests that this topology classifies metric functions. The general idea of this section is to give a point-free formulation of this topology, by replacing the basic open "$f(x) \in V$" by "$U \wedge f^{-1}(V) > \emptyset$" for $U$ a small neighborhood of $x$.

3.5.1. **Definition :** *Let* $X$ *and* $Y$ *be two pre-metric locales. Let* $A$ *be a basis*[9] *of positive open of* $X$ *and* $B$ *be a metric basis of* $Y$*. We define* $[X_A,Y_B]_1$ *as the classifying space of the propositional geometric theory on propositions* $(U,V)$ *for* $U\in A$ *and* $V \in B$ *with the axioms:*
*(MM1) For all* $U' \leqslant U$ *and* $V' \leqslant V$

$$(U',V') \vdash (U,V)$$

*(MM2) For all* $V\in B$*,*$U \in A$ *and any positive rational number* $q$ *one has*

$$(U,V) \vdash \bigvee_{\substack{u\leqslant U\\ \delta(u)<q}} (u,V);$$

*(MM3) For all* $U\in A$ *and all* $q$ *positive:*

$$\vdash \bigvee_{\substack{V\in B\\ \delta(V)<q}} (U,V);$$

9. One can actually see that we do not even need $A$ to be a basis. All we need is that for all positive rational $q$ the set of $a\in A$ such that $\delta(a)<q$ cover $X$.

*(MM4)* *For all* $U \in A$, $V \in B$

$$(U,V) \vdash \bigvee_{\substack{V' \in B \\ V' \lhd V}} (U,V');$$

*(MM5)* *Let* $W_1, W_2, \tau \in A$, $q_1, q_2 \in \mathbb{Q}$, $V_1, V_2, V_1', V_2' \in B$ *such that*

$$\begin{array}{cc} \delta(W_1) < q_1 & \delta(W_2) < q_2 \\ V_1' \lhd_{q_1} V_1 & V_2' \lhd_{q_2} V_2 \\ \tau \leqslant W_1 & \tau \leqslant W_2 \end{array}$$

*then*

$$(W_1, V_1') \wedge (W_2, V_2') \vdash \bigvee_{\substack{V \in B \\ V \leqslant V_1 \wedge V_2}} (\tau, V)$$

*(MM6)*

$$(U,V) \wedge (U,V') \vdash \delta(V \vee V') \leqslant \delta(U) + \delta(V) + \delta(V').$$

3.5.2. The main result of this section is

**Theorem :** *The locale* $[X_A, Y_B]_1$ *we just constructed does not depend on* $A$ *and* $B$ *and classifies metric maps between* $X$ *and* $\widetilde{Y}$. *With the propositions* $(U,V)$ *corresponding to* $U \wedge f^*(V^\sim) > \emptyset$. *This locale will be denoted* $[X,Y]_1$

Its proof will occupy us for the rest of this subsection.

3.5.3. If $f$ is a geometric morphism from $\mathcal{E}$ to $\mathcal{T}$, then, by the same argument as in 3.3.6:

$$f^{\#}([X_A, Y_B]_1) \simeq [f^{\#}(X)_{f^*(A)}, f^{\#}(Y)_{f^*(B')}]_1$$

So it suffices to show that the points of $[X_A, Y_B]_1$ correspond to metric functions from $X$ to $\widetilde{Y}$ to obtain the announced result.

3.5.4. **Proposition :** *Let* $f : X \rightarrow \widetilde{Y}$ *be a metric map and let:*

$$(U,V)_f := \text{“}U \wedge f^*(V^\sim) > \emptyset\text{”}$$

*For* $U \in A$ *and* $V \in B$. *Then this defines a point of* $[X_A, Y_B]_1$.

**Proof :**
Axiom $(MM1)$ is immediate. $(MM2)$ holds because for any $V \in B, U \in A$, if $f^*(V^\sim) \wedge U$ is positive then one can write $U$ as a union of $u \in A$ such that $u \leqslant U$ and $\delta(u) < q$ and the locale positivity of $X$ allows one to conclude. Axiom $(MM3)$ and $(MM4)$ hold because the corresponding unions holds in $\widetilde{Y}$.

We now prove axiom $(MM5)$: Let $W_1, W_2, \tau, q_1, q_2, V_1, V'_1, 2_2, V'_2$ satisfying the hypothesis of $(MM5)$. We also assume that $(W_1, V'_1)_f$ and $(W_2, V'_2)_f$ holds. Then as $f$ is metric and $V'_i \lhd_{q_i} V_i$ then $V'^{\sim}_i \lhd_{q_i} V^{\sim}_i$ one has

$$f^*(V'^{\sim}_i) \lhd_{q_i} f^*(V^{\sim}_i).$$

As $\delta(W_i) < q_i$ and $W_i \wedge f^*(V_i) > \emptyset$ this implies that

$$W_i \subseteq f^*(V^{\sim}_i),$$

and hence, as $\tau \leqslant W_1 \wedge W_2$, that

$$\tau \subseteq f^*(V^{\sim}_1 \wedge V^{\sim}_2).$$

As $\tau$ is positive (the presentation of $X$ is assumed to be locally positive) and $V^{\sim}_1 \wedge V^{\sim}_2$ is covered by the $V^{\sim}$ for $V \subseteq V_1 \wedge V_2$ this concludes the proof of $(MM5)$.

We now prove $(MM6)$. Let $U, V$ and $V'$ such that $U \wedge f^*(V^{\sim}) > \emptyset$ and $U \wedge f^*(V'^{\sim}) > \emptyset$. Let $q$ and $q'$ such that $\delta(V) < q$ and $\delta(V') < q'$. Let also $\epsilon$ be a positive rational number such that $\delta(V) < q - 2\epsilon$ and $\delta(V') < q' - 2\epsilon$. Let $W = B_\epsilon V$ and $W' = B_\epsilon V'$, in particular $\delta(W) < q$ and $\delta(W') < q'$.
One has, by the assumption on $V$ and $V'$ and the fact that $f$ is metric (see 3.1.8 proposition $(c)$):

$$\delta(W^{\sim} \vee W'^{\sim}) \subseteq \delta(W^{\sim}) + \delta(W'^{\sim}) + \delta(U)$$

Let $i$ be the isometric map $Y \to \widetilde{Y}$ of 3.3.8, i.e.

$$i^*(V^{\sim}) = \bigvee_{U \lhd V} U.$$

In particular, as $W$ and $W'$ are open balls, one has $i^*(W^{\sim}) = W$ and $i^*(W'^{\sim}) = W'$, and $i^*(W^{\sim} \vee W'^{\sim}) = W \vee W'$, and as $i$ is isometric, this implies that $\delta(W \vee W') \leqslant \delta(W^{\sim} \vee W'^{\sim})$.
Moreover since $\delta(W) < q$ then by definition of the distance on $\widetilde{Y}$, $W^{\sim} \times W^{\sim} \subseteq \Delta_q$, and hence $\delta(W^{\sim}) \leqslant q$. One deduces from this that

$$\delta(V \vee V') \leqslant \delta(W \vee W') \leqslant \delta(W^{\sim} \vee W'^{\sim}) \leqslant \delta(W^{\sim}) + \delta(W'^{\sim}) + \delta(U) \leqslant q + q' + \delta(U),$$

which concludes the proof as it has been done for arbitrary $q$ and $q'$ bigger than $\delta(V)$ and $\delta(V')$.
$\square$

3.5.5. **Definition :** *To any point $p$ of $[X_A, Y_B]_1$ we associate the function $\tau_p : B \to \mathcal{O}(X)$ defined by:*

$$\tau_p(V) := \bigvee_{\substack{\delta(W)<q \\ V' \lhd_q V \\ p \in (W,V')}} W$$

*where $V'$ runs through elements of $B$, $W$ through elements of $A$, and $q$ through positive rational numbers.*

**Proposition :** *If $f$ is a metric map from $X$ to $\widetilde{Y}$ and $p$ is the point of $[X_A, Y_B]$ associated to $f$ in 3.5.4 then*

$$\tau_p(V) = f^*(V^\sim).$$

**Proof :**
One has by definition:

$$\tau_p(V) = \bigvee_{\substack{\delta(W)<q \\ V' \lhd_q V \\ f^*(V'^\sim) \wedge W > \emptyset}} W.$$

Hence, as for any $W$ appearing in the supremum one has $W \leqslant f^*(V^\sim)$, we obtain that $\tau_p(V) \leqslant f^*(V^\sim)$.
Conversely,

$$f^*(V^\sim) = \bigvee_{V' \lhd_q V} f^*(V'^\sim) = \left( \bigvee_{\substack{V' \lhd_q V \\ \emptyset < W \leqslant f^*(V'^\sim) \\ \delta(W)<q}} W \right) \leqslant \tau_p(V'^\sim).$$

□

3.5.6. **Lemma :** *Let $p$ be any point of $[X_A, Y_B]_1$, then:*

$$p \in (U,V) \Leftrightarrow U \wedge \tau_p(V) > \emptyset$$

**Proof :**
Assume first that $\tau_p(V) \wedge U > \emptyset$. Then there exists $W$ and $V'$ such that $\delta(W) < q$, $V' \lhd_q V$ , $(W, V')$ and $W \wedge U > \emptyset$. Applying $(MM5)$, one obtains that there exists $V'' \leqslant V$ such that $p \in (W \wedge U, V'')$ and hence $p \in (U,V)$.
Conversely assume that $p \in (U,V)$, then (by $(MM4)$) there exists $V' \in B$ and a positive $q$ such that $V' \lhd_q V$ and $p \in (U,V')$. Also by $(MM2)$ there exists $W \in A$ such that $\delta(W) < q$ and $p \in (W,V')$. But this implies that $W \leqslant \tau_p(V)$ and as $W \leqslant U$ and $W > \emptyset$ one concludes that $U \wedge \tau_p(V) > \emptyset$. □

3.5.7. At this point, all that remains to be checked in order to prove 3.5.2 is that for any point $p$, the map $\tau_p$ defines a metric map $X \to \widetilde{Y}$.

**Proposition :** *The map $\tau_p : B \to \mathcal{O}(X)$ satisfies the four conditions of 3.3.7 and in particular there is a (unique) map $f : X \to \widetilde{Y}$ such that $f^*(V^\sim) = \tau_p(V)$.*

**Proof :**
We recall that

$$\tau_p(V) := \bigvee_{\substack{\delta(W)<q \\ V' \triangleleft_q V \\ p \in (W,V')}} W$$

Also the point $p$ being fixed, we will write $\tau$ instead of $\tau_p$ and $(U,V)$ instead of $p \in (U,V)$.

1. if $U \leqslant V$ then any $W$ appearing in the supremum defining $\tau(U)$ also appears in the one defining $\tau(V)$ with the same $V'$ and $q$. Hence $\tau$ is order preserving.
2. $$\tau(V_1) \wedge \tau(V_2) = \bigvee W_1 \wedge W_2$$

   where the union runs over all $W_1, W_2 \in A$ such that there exist $q'_1, q'_2$ positive rational numbers, and $V'_1, V'_2 \in B$ such that

   $$\delta(W_i) < q'_i;$$

   $$V'_i \triangleleft_{q'_i} V_i;$$

   $$(W_i, V'_i).$$

   For any such $W_1$ and $W_2$ there exists a positive rational number $\epsilon$ such that $\delta(W_i) < q'_i - \epsilon$. Let $q_i = q'_i - \epsilon$. One has in particular $\delta(W_i) < q_i$ and

   $$V'_i \triangleleft_{q_i} B_{q_i} V'_i \triangleleft_\epsilon V_i.$$

   Moreover $W_1 \wedge W_2$ can be written as the union of $\tau \in A$ such that $\tau \leqslant W_1 \wedge W_2$ and $\delta(\tau) < \epsilon$. Finally, one can apply $(MM5)$ (taking $B_{q_i} V'_i$ instead of $V_i$) to obtain that there exists $V$ such that

   $$V \leqslant (B_{q_1} V'_1 \wedge B_{q_2} V'_2) \triangleleft_\epsilon V_1 \wedge V_2$$

   and

   $$(\tau, V).$$

   This proves that $\tau \leqslant \tau(B_\epsilon V)$ with $B_\epsilon B \leqslant V_1 \wedge V_2$ and $B_\epsilon V \in B$ because $B$ is metric, and hence concludes the proof that.

   $$\tau(V_1) \wedge \tau(V_2) \leqslant \bigvee_{\substack{V \in B \\ V \leqslant V_1 \wedge V_2}} \tau(V).$$

3. Let $q$ be any positive rational number. Let $W \in A$ such that $\delta(W) < q/3$. Then by $(MM3)$ there exists $V' \in B$ such that $\delta(V') < q/3$ and $(W, V')$. Let $V = B_{q/3}V' \in B$, one has: $\delta(W) < q/3$, $V' \lhd_{q/3} V$, $(W, V')$, hence $W \leqslant \tau(V)$ with $\delta(V) < q$ this proves that

$$W \leqslant \bigvee_{\substack{V \in B \\ \delta(V) < q}} \tau(V)$$

As we have done this for an arbitrary $W$ with $\delta(W) < q/3$ this concludes the proof.

4. Let $V \in B$, let $W$ appearing in the union defining $\tau(V)$, i.e. there exists a positive rational $q$, and a $V' \in B$ such that $\delta(W) < q$ and $V' \lhd_q V$. But, there exists a positive rational number $\epsilon$ such that $\delta(W) < q - \epsilon$, and $V' \lhd_{q-\epsilon} B_{q-\epsilon}V' \lhd_\epsilon V$. Hence

$$W \leqslant \tau(B_{q-\epsilon}V' \leqslant \bigvee_{\substack{U \in B \\ U \lhd V}} \tau(U).$$

Finally, we obtain

$$\tau(V) \leqslant \bigvee_{\substack{U \in B \\ U \lhd V}} \tau(U).$$

□

The fact that the map $f$ induced by $\tau_p$ is metric follow from axiom $(MM6)$ using the characterization $(c)$ of metric maps given in 3.1.8, hence this concludes the proof of theorem 3.5.2.

## 3.6 Case of metric sets

3.6.1. We define a (pre)metric set as set $X$ endowed with a distance function $d : X \times X \to \overleftarrow{\mathbb{R}_+^\infty}$ satisfying the usual axioms for a (pre)distance:

- $d(x, x) = 0$
- $d(x, y) = d(y, x)$
- $d(x, z) \leqslant d(x, y) + d(y, z)$

With additionally, $d(x, y) = 0 \Rightarrow x = y$ for a metric set.
A (pre)metric set can be seen as a pre-metric locale by seeing its underlying set as a discrete locale. It is in general not a metric locale even if we start with a metric set.

3.6.2. We will say that a metric set $(X, d)$ is complete if the natural map $i : X \to \widetilde{X}$ identifies $X$ with the points of $\widetilde{X}$. As points of $\widetilde{X}$ identify with regular Cauchy filters one easily checks that this is equivalent to the usual (Cauchy filter based) definition of completeness.

3.6.3. **Theorem :** *There is an equivalence of categories between the category of weakly spatial complete metric locales (with metric maps) and complete metric sets (with metric maps).*

**Proof :**
The functors are given by the following construction: to a complete metric set $X$ one associates its localic completion $\widetilde{X}$, which is weakly spatial, because $X$ is fiberwise dense in it, and to a weakly spatial complete metric locale one associates its set of points endowed with the induced distance. These two constructions are functorial on metric maps.
By definition of a complete metric set it identifies with the set of points of its localic completion, and conversely, if $\mathcal{L}$ is a weakly spatial complete metric locale and $X$ is its set of points endowed with the induced distance, then $X \to \mathcal{L}$ is a fiberwise dense isometric map from $X$ to a complete locale, hence $\mathcal{L}$ is isomorphic to the completion of $X$. This proves that the two functors are inverse from each other on objects. They are also inverse of each other on morphisms, tautologically on one side and by 3.2.3 on the other side. $\square$

3.6.4. The internal application of the fact that the set of points of a complete metric locale is complete in the classical sense can prove directly a result of completeness of the space of functions with values in a complete locale for the uniform distance. This cannot be stated directly in terms of completeness of some metric locale because in general (if the initial space is not locally compact) the space of functions is not a locale, but one has:

**Proposition :** *Let $(f_i)_{i \in I}$ be a Cauchy net of functions between two locales $X$ and $Y$, with $Y$ a complete metric locale. This means that $I$ is a directed (filtering) ordered set and that for all positive rational number $\epsilon$ there exists $i_0 \in I$ such that $\forall i, j \geqslant i_0$, the map $(f_i, f_j)$ factors into $\Delta_\epsilon \subset Y \times Y$.*
*Then the net $f_i$ converges to some (uniquely defined) function $f : X \to Y$. This mean that there is a unique function $f : X \to Y$ such that for all positive rational number $\epsilon$ there exists $i_0 \in I$ such that $\forall i \geqslant i_0$, the map $(f, f_i)$ factors into $\Delta_\epsilon$.*

**Proof :**
The net of functions $f_i : X \to Y$ can be interpreted as a net of points of $p^{\#}Y$ in the logic of $X$ (where $p$ is the map $X \to *$). And the fact that it is externally a Cauchy net immediately gives that it is internally a Cauchy net. The usual proof that completeness by filter imply completeness by net is completely constructive [10] and hence the fact that $p^{\#}Y$ is complete implies the convergence of the net $f_i$. Uniqueness of the limit implies that the limit is a global point of $p^{\#}Y$ in $X$, and hence a map from $X$ to $Y$. One then easily check that the internal convergence together with the external Cauchy condition imply the external convergence. $\square$

10. On the contrary, the converse relies on the axiom of choice.

3.6.5. In particular the category of complete metric sets identifies with the full subcategory of the category of complete metric locales composed of weakly spatial locales, and by 2.4.8 any complete metric locale becomes weakly spatial (hence identifies with a complete metric set) after a pullback to some open locale. We already mentioned that if one defines $\mathcal{C}(\mathcal{T})$ as the category of complete metric locales over $\mathcal{T}$, then, it is a stack for the topology whose covering are open surjections.
From these observations one can deduce that the stack of internal complete metric locales is the stackification (the analogue of sheafication for stack and pre-stack) of the pre-stack of complete metric sets, that is the universal extension of the notion of complete metrics sets for the descent properties along open surjection.

At this point one could obtain the localic Gelfand duality of 4.2.5 directly by observing that the notion of compact regular locale is obtained as the stackification of the notion of compact completely regular locale, and apply the constructive Gelfand duality between compact regular locale and $C^*$-algebra to show that the two pre-stacks are equivalent. This will also avoid the use any of the material of section 3.5, but it will give an extremely uncomfortable definition of the spectrum of a localic $C^*$-algebra. This is why we prefer explicitly constructing the spectrum (in 4.2.3, using the construction of 3.5) before applying the descent argument to show the Gelfand duality.

# 4 Banach locales and $C^*$-locales

## 4.1 Banach locales and completeness

4.1.1. **Definition :** *A pre-Banach locale is a locally positive locale $\mathcal{H}$ endowed with:*

- *A commutative group law: $+ : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, with neutral element $0 : * \to \mathcal{H}$ and an inversion: $x \mapsto -x : \mathcal{H} \to \mathcal{H}$.*
- *An action of $\mathbb{Q}[i]$ (endowed with the discrete topology), $\mathbb{Q}[i] \times \mathcal{H} \to \mathcal{H}$, satisfying the usual axioms of a (unital) module.*
- *A norm function $\|.\| : \mathcal{H} \to \overleftarrow{\mathbb{R}_+^\infty}$*

*where the norm function is expected to satisfy the following conditions:*

- $\forall x, y \in \mathcal{H} \|x + y\| \leqslant \|x\| + \|y\|$
- $\forall \lambda \in \mathbb{Q}[i], \forall x \in \mathcal{H}, \|\lambda x\| = |\lambda| \|x\|$
- $\|0\| = 0$
- $\mathcal{H} = \bigvee_{n \in \mathbb{N}} \{x | \|x\| < n\}$

Of course, all the conditions stated in this definition have to be interpreted either in diagrammatic terms or in terms of generalized elements.

4.1.2. **Proposition :** *Let $(\mathcal{H}, \|.\|)$ be a pre-Banach locale. Let $s$ and $p$ denote the maps $\mathcal{H} \times \mathcal{H} \to \mathcal{H}$ defined by:*

$$s(x,y) = x - y$$

$$p(x,y) = x + y$$

*Let $m$ denote the map $x \mapsto -x$ and $n$ be the norm map, $n : \mathcal{H} \to \overleftarrow{\mathbb{R}_+^\infty}$.*
*Finally we will denote $B_q 0 = n^*([0,q[)$ (point 5 ensures that there is no possible confusion).*
*Then, one has the following facts:*

1. *The map $n \circ s$ is a pre-distance on $\mathcal{H}$.*
2. *The maps $s$ and $p$ are open maps.*
3. *The open sublocales $\Delta_q$ coincide with $s^*(B_q 0)$.*
4. *If $\mathcal{L}$ is any sublocale of $\mathcal{H}$ then $B_q \mathcal{L}$ coincide with both $p_!(\mathcal{L} \times B_q 0)$ and $s_!(\mathcal{L} \times B_q 0)$.*
5. *$B_q 0$ is the same things as $B_q\{0\}$.*

**Proof :**

1. A proof by generalized points will be exactly the same as the usual proof that $d(x,y) = \|x - y\|$ is a distance on a normed space.
2. We will consider two maps $\mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ given by

$$\tau_p = (p, m \circ \pi_1);$$
$$\tau_s = (\pi_1, s).$$

   These maps correspond in term of generalized points to the maps $\tau_p(x,y) = -x + y, -y)$ and $\tau_s(x,y) = (x, x - y)$, and they are both involutive and hence bijective. The maps $s$ and $p$ are then obtained as $\pi_2 \circ \tau_s$ and $\pi_1 \circ \tau_p$, but as $\mathcal{H}$ is locally positive, both $\pi_1$ and $\pi_2$ are open maps. Hence by composition $s$ and $p$ are open maps.
3. $\Delta_q$ is by definition $d^*([0,q[)$, but as $d = n \circ s$, one has $\Delta_q = s^* n^*([0,q[) = s^*(B_q 0)$.
4. The involutive map $\tau_s$ introduced in the proof of point 2 exchange $\pi^*(\mathcal{L}) \wedge \Delta_q$ with $\mathcal{L} \times B_q 0$, indeed:

$$\tau_s^*(\mathcal{L} \times B_q 0) = pi_1^*(\mathcal{L}) \wedge s^*(B_q 0) = \pi_1^*(\mathcal{L}) \wedge \Delta_q.$$

   Hence $\pi_{2!}(\pi_1^*(\mathcal{L}) \wedge \Delta_q) = (\pi_2 \circ \tau_s)_!(\mathcal{L} \times B_q 0)$ and $\pi \circ \tau_s = s$, which shows that $B_q \mathcal{L} = s_!(\mathcal{L} \times B_q 0)$.
   It also coincides with $p_!(\mathcal{L} \times B_q 0)$ because as $n \circ m = n$ one has $m^*(B_q 0) = B_q 0$, and as $s = p \circ (Id, m)$ this concludes the proof.
5. From the previous result, $B_q\{0\}$ identifies with $p_!(\{0\} \times B_q 0)$ but $p$ acts on $\{0\} \times B_q 0$ as the inclusion of $B_q 0$ in $\mathcal{H}$ (this is the definition of 0 being the neutral element), hence $p_!(\{0\} \times B_q 0) = B_q 0$ and this concludes the proof.

□

4.1.3. **Proposition :** *Let $\mathcal{H}$ be a pre-Banach locale, the following conditions are equivalent:*

*(LB1) The open sublocales $B_q 0$ form a basis of neighborhoods of* $0$.

*(LB2) $\mathcal{H}$ is metric for the distance induced by $\|.\|$.*

A pre-Banach locale satisfying either $(LB1)$ or $(LB2)$ is called a Banach locale, we will soon see that there is no need for a completeness assumption: it will be automatic.

**Proof :**
We will use the same notation $s, p$ as in proposition 4.1.2. Assume $(LB1)$, and let $U$ be any open of $\mathcal{H}$. Consider the open sublocale $p^*U \subset \mathcal{H} \times \mathcal{H}$, and decompose it as a union of basic open sublocales

$$p^*U = \bigvee_{i \in I} A_i \times B_i$$

where $A_i$ and $B_i$ are open sublocales of $\mathcal{H}$. Let $i$ such that $(A_i \times B_i) \wedge U \times \{0\}$ is positive. Then $B_i \wedge \{0\}$ is also positive, hence $0 \in B_i$, and from the hypothesis, there exists $q$ such that $B_q 0 \leqslant B_i$. This implies that for each $i$ such that $0 \in B_i$, as $A_i \times B_q 0 \leqslant p^*U$ one has $B_q A_i = p_!(A_i \times B_q 0) \leqslant U$ hence $A_i \lhd_q U$.
Now as $U \times \{0\}$ is locally positive and a subset of $p^*(U)$:

$$U \times \{0\} \leqslant \bigvee_{\substack{i \in I \\ (A_i \times B_i) \wedge (U \times \{0\}) > \emptyset}} \leqslant \bigvee_{\substack{i \in I \\ 0 \in B_i}} A_i \times B_i$$

Applying $\pi_{1!}$ one gets (as any $B_i$ having a point is positive) that

$$U \leqslant \bigvee_{\substack{i \in I \\ 0 \in B_i}} A_i \leqslant \bigvee_{\substack{i \in I \\ A_i \lhd U}} A_i,$$

which concludes the proof of the first implication.
Assume now $(LB2)$, let $U$ be an arbitrary neighborhood of $0$, then as $\mathcal{H}$ is metric, there exists an open sublocale $V$ such that $0 \in V$ and $V \lhd U$. In particular, there exists $q$ such that $B_q V \leqslant U$, and as $0 \in V$ one has:

$$B_q 0 \subset B_q V \subset U$$

which proves $(LB1)$ and concludes the proof of the proposition.
□

4.1.4. **Proposition :** *Let $\mathcal{H}$ be a pre-Banach locale, then its completion $\widetilde{\mathcal{H}}$ is naturally endowed with a structure of Banach locale such that the map $\mathcal{H} \to \widetilde{\mathcal{H}}$ is a linear isometric map.*

**Proof :**
Everything comes more or less immediately from 3.3.11 for the construction of operations and from 3.2.3 and 3.2.4 for the verification of the axioms:

Indeed, as $\mathcal{H}\times\mathcal{H}$ has a fiberwise dense image in $\widetilde{\mathcal{H}}\times\widetilde{\mathcal{H}}$, the canonical (uniform) map $p:\mathcal{H}\times\mathcal{H}\to\mathcal{H}\to\widetilde{\mathcal{H}}$ extends into a map $\widetilde{\mathcal{H}}\times\widetilde{\mathcal{H}}\to\widetilde{\mathcal{H}}$. Similarly, the opposite map $m:\mathcal{H}\to\mathcal{H}$ is isometric and hence extends into a map $m:\widetilde{\mathcal{H}}\to\widetilde{\mathcal{H}}$ and one checks all the group axioms on $\widetilde{\mathcal{H}}$ because they hold in $\mathcal{H}$, that $\widetilde{\mathcal{H}}$ is metric and that $\mathcal{H}^n$ has a fiberwise dense image in $\widetilde{\mathcal{H}}^n$.
The action of the locale of complex numbers on $\widetilde{\mathcal{H}}$ is obtained in the same way: for each $\lambda\in Q[i]$ the multiplication by $\lambda$ is a uniform map $\mathcal{H}\to\mathcal{H}$ and hence extends into a map $\widetilde{\mathcal{H}}\to\mathcal{H}$, giving a map $\mathbb{Q}[i]\times\widetilde{\mathcal{H}}\to\widetilde{\mathcal{H}}$ and all the axioms of compatibility with the group law are also satisfied by a density argument.
Finally, we already know that there is a distance function on $\widetilde{\mathcal{H}}$ we only have to check that $\|x\|=d(0,x)$ is a norm and that $d(x,y)=\|x-y\|$. But this also immediately comes from a density argument by 3.2.4. $\square$

4.1.5. **Corollary :** *Banach locale are complete metric locales.*

**Proof :**
Let $\mathcal{H}$ be a Banach locale, in particular $\mathcal{H}$ is a metric locale and hence by 3.2.2 it identifies with a sublocale of $\widetilde{\mathcal{H}}$. More precisely, as the inclusion is a linear map, $\mathcal{H}$ identifies with a localic subgroup of a locally positive localic group $\widetilde{\mathcal{H}}$, hence thanks to the constructive version of the closed subgroups theorem proved by P.T.Johnstone in [43], one concludes that $\mathcal{H}$ is fiberwise closed (weakly closed in the terminology of [43]) in $\widetilde{\mathcal{H}}$ and hence is also complete (see the remark at the end of 3.3.12). $\square$

4.1.6. In particular, the action of $\mathbb{Q}[i]$ on a Banach locale extends to an action of its completion $\mathbb{C}$. Indeed (assuming that $\mathcal{H}$ is complete), the map $B_n0\times\mathbb{Q}[i]\to\mathcal{H}$ is uniform (it is $n$-Lipschitz) and hence it extends into $\overline{B_n0}\times\mathbb{C}\to\mathcal{H}$. One has a family of compatible maps $B_n0\times\mathbb{C}\to\mathcal{H}$ which gives rise to a map $\mathcal{H}\times\mathbb{C}\to\mathcal{H}$.

4.1.7. Similarly to what is done in section 3.6, a pre-Banach space in the usual (constructive) sense is exactly the same as a pre-Banach locale whose underlying locale is a discrete topological space. To such a Banach space one can associate its completion which is going to be a Banach locale. Conversely to any Banach locale one can associate its space of points which is a Banach space, and these two constructions induce an equivalence between the category of weakly spatial Banach locales (and linear map) and the category of Banach spaces (with bounded linear map).

## 4.2 The Localic Gelfand duality

4.2.1. **Definition :** *A $C^*$-locale (or localic $C^*$-algebra) is a Banach locale $\mathcal{C}$, endowed with an involution $*: \mathcal{C} \to \mathcal{C}$ and a product $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which satisfy the usual axioms for a $C^*$-algebra:*

- *$\mathcal{C}$ is a $\mathbb{C}$ algebra (i.e. the product is associative, distributes over the addition and is compatible with the action of $\mathbb{C}$).*
- *The $*$ involution is $\mathbb{C}$ anti-linear and satisfies $(ab)^* = b^*a^*$.*
- *One has: $\|ab\| \leqslant \|a\|\|b\|$.*
- *One has: $\|a^*a\| = \|a\|^2$.*

All the axioms are equalities (or inequalities with respect to the specialization order), hence are clearly preserved by pullback and therefore if $\mathcal{C}$ is a $C^*$-algebra and $f$ is a geometric morphism to the base topos then $f^{\#}(\mathcal{C})$ is also a $C^*$-locale. And if $\mathcal{C}$ is a (pre)-Banach locale endowed with an $*$ map and a map $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that for some open surjection $f$, $f^{\#}(\mathcal{C})$ is a $C^*$-algebra for those structure then $\mathcal{C}$ is a $C^*$-algebra.

The main result of this section will be an anti-equivalence of categories between the categories of abelian unital $C^*$-locales and compact regular locales. The "difficult" part lies in the construction of the two functors, and the proof that they are compatible with pullback along geometric morphisms. Indeed once it is done, one can apply 2.4.8 to reduce the proof of the equivalence to the case of spatial $C^*$-algebras and completely regular compact locales which is already known ([4] [20]). Actually, even the construction of the two functors could be avoided since we know that the notion of $C^*$-locale is the "stackification" of the notion of $C^*$-algebra (it is a direct consequence of the observations made in 3.6.5), and one can prove (applying 2.7.6) a similar result for compact regular locales and compact completely regular locales. Hence the already known equivalence between unital abelian $C^*$-algebras and compact completely regular locales immediately yields the equivalence between the "stackified" notions, but we think that it is important to have an explicit construction of these functors without having to use descent theory.

4.2.2. **Proposition :** *Let $X$ be a compact regular locale, then $[X, \mathbb{C}]$ is a $C^*$-algebra, for the addition, product and involution given by the addition, the product and the complex conjugation of $\mathbb{C}$, and the norm given by:*

$$B_q 0 = [X \ll f^* D_q]$$

*where $D_q$ denotes the open disc of radius $q$ in $\mathbb{C}$, and $[X \ll f^* D_q]$ denotes the basic open which classifies the $f$ such that $X \ll f^* D_q$.*

**Proof :**
$[X, \mathbb{C}]$ is indeed locally positive by 2.7.5. For the rest, we recall that Hyland gave in [37] a description of the theory classified by $[X, Y]$ in terms of the basic propositions $[U \ll f^* V]$ for $U \in \mathcal{O}(X)$ and $V \in \mathcal{O}(Y)$. From this description, we immediately obtain that:

$$\bigvee_{q'<q} B_{q'}0 = B_q 0;$$

$$\bigvee_n B_n 0 = [X,\mathbb{C}].$$

Also, as 0 is the point corresponding to the function constant equal to 0, one has indeed $0 \in B_q 0$.

Hence the $B_q 0$ indeed define a function $\|.\| : [X,\mathbb{C}] \to \overleftarrow{\mathbb{R}_+^\infty}$ such that $\|0\| = 0$, and such that $\bigvee_n B_n 0 = [X,\mathbb{C}]$.

All the algebraic axioms (including the triangular inequality) are checked on generalized point exactly as one does for classical points in the usual (constructive) case.

A basic open $[U \ll f^* V]$ (for $U$ positive) contains 0 if $U \ll \bigvee_{0\in V} X$, but this implies that there exists a finite set $F$ included in $\{0 \in V\}$ such that $U \leqslant \bigvee_{f\in F} X$. A finite set is inhabited or empty, hence either $F$ is empty and $U = \emptyset$ or $F$ is inhabited and $0 \in V$. In the first case $[U \ll f^*V] = [X,\mathbb{C}]$ contains all the $B_q 0$. In the second case one has a $q$ such that $D_q \ll V$ and hence $0 \in B_q 0 = [X \ll f^*(D_q)] \leqslant [U \ll f^*(V)]$ which proves that the $B_q 0$ form a basis of neighborhood of 0, and hence $[X,\mathbb{C}]$ is a Banach locale.

□

4.2.3. We now want to construct the spectrum of a $C^*$-locale. We will start by defining the locale Fn $\mathcal{H}$ of linear forms of norm smaller than 1 on a Banach locale $\mathcal{H}$ (the spectrum being the space of characters, it will be a sublocale of this locale). It generalizes the locale Fn $E$ constructed in [60] and [19].

**Proposition :** *Let $\mathcal{H}$ be a Banach locale. There exists a sublocale Fn $\mathcal{H} \subset [\mathcal{H},\mathbb{C}]_1$ which classifies the linear forms of norm smaller or equal to one on $\mathcal{H}$. If $\mathcal{C}$ is a unital commutative $C^*$-locale, then there exists a sublocale Spec $\mathcal{C} \subset [\mathcal{C},\mathbb{C}]_1$ which classifies characters of $\mathcal{C}$.*

**Proof :**

One can for example define the locale Fn $\mathcal{H}$ as the intersection of the equalizer of the following two diagrams:

$$[\mathcal{H},\mathbb{C}]_1 \rightrightarrows [D_1 \times \mathcal{H},\mathbb{C}]_1$$

where $D_1$ denotes the open unit ball in $\mathbb{C}$ and the two maps are the maps defined on generalized elements by: $f \mapsto ((\lambda,x) \mapsto \lambda f(x))$ and $f \mapsto ((\lambda,x) \mapsto f(\lambda x))$, and where the distance on $D_1 \times \mathcal{H}$ is the max distance.

And,

$$[\mathcal{H},\mathbb{C}]_1 \rightrightarrows [\mathcal{H} \times \mathcal{H},\mathbb{C}]_1$$

where $\mathcal{H}\times\mathcal{H}$ is endowed with the norm $\|x_1\|+\|x_2\|$ and the two maps are given by: $f \mapsto ((x,y) \mapsto f(x+y))$ and $f \mapsto ((x,y) \mapsto f(x)+f(y))$.

A map $X \to$ Fn $\mathcal{H}$ is then exactly the data (internally to $X$) of a metric map from $\mathcal{H} \to \mathbb{C}$ which is additive and linear with respect to complex numbers smaller than 1. As it is also linear with respect to integers, it is linear on $nD_1$ for all $n$ and this forms an open cover of $\mathbb{C}$ so it concludes the proof.

If now $\mathcal{C}$ is a unital $C^*$-locale, then one defines Spec $\mathcal{C}$ as the intersection of the two previous equalizers with the pullback of $\{1\} \subset \mathbb{C}$ by the map of evaluation on the unit on $[\mathcal{C}, \mathbb{C}]_1$ and with the equalizer of the following diagram:

$$[\mathcal{C}, \mathbb{C}]_1 \rightrightarrows [B_1 0 \times B_1 0, \mathbb{C}]$$

where $B_1 0$ is the open unit ball of $\mathcal{C}$, and the distance $B_1 0 \times B_1 0$ is given by the max distance. The two maps are given by $f \mapsto ((x,y) \mapsto f(x)f(y))$ and $f \mapsto ((x,y) \mapsto f(xy))$.
A map factoring into Spec $\mathcal{C}$ exactly corresponds to an internal character of $\mathcal{C}$.
□

4.2.4. The following result is a localic version of the Banach-Alaoglu theorem.

**Proposition :** *Let $\mathcal{H}$ be a Banach locale, $\mathcal{C}$ a unital commutative $C^*$-locale, then the locales Fn $\mathcal{H}$ and Spec $\mathcal{C}$ are compact regular locales.*

**Proof :**
Compact regular locales descend along open surjections: for example because for a locale being compact and regular is the same thing as having a map to the point which is both proper and separated (see [44] C.3.2.10) and because both proper maps and separated maps descend along open morphisms, (see [44]C5.1.7). Hence it is enough to prove that some pullback of Fn $\mathcal{H}$ and Spec $\mathcal{C}$ by an open surjection is compact and regular to conclude. In particular, by 2.4.8 one can freely assume that $\mathcal{H}$ and $\mathcal{C}$ are weakly spatial and hence that it is the completion of some Banach space $H$ or $C^*$-algebra $C$. But in this situation, a linear form or a character on the Banach locale is exactly the same as a linear form or a character on the set of points (by extension to the completion) and hence (the pullback of) Fn $\mathcal{H}$ and Spec $\mathcal{C}$ classify the same theory as the locale Fn $H$ and Spec $C$ (also called MFn $C$) studied in [60] and [4] for the case of Grothendieck toposes, and in [19] and [20] for general elementary toposes. These references prove that these locales are indeed compact (completely) regular. □

4.2.5. **Theorem :** *The previous two constructions $X \to [X, \mathbb{C}]$ and $\mathcal{C} \to$ Spec $\mathcal{C}$ induce an anti-equivalence of categories between unital abelian $C^*$-locales and compact regular locales.*

**Proof :**
These two constructions are defined in terms of the theory they classified and hence we can easily check that they are preserved by pullback along geometric morphisms. They correspond to the well known notion of (completion of the) space of continuous functions on $X$ and spectrum of a $C^*$-algebra when $X$ is

completely regular and when $\mathcal{C}$ is weakly spatial. Moreover the two canonical maps "evaluation at $x \in X$" from $X$ to Spec $[X, \mathbb{C}]$ and "evalution at $c \in \mathcal{C}$" from $\mathcal{C}$ to [Spec $\mathcal{C}, \mathbb{C}$] are preserved by pullback (a proof by generalized points shows it immediately).
Hence, applying 2.4.8 one can pullback (along an open surjection) those two maps to a similar situation but with $\mathcal{C}$ and $[X, \mathbb{C}]$ weakly spatial (hence with $X$ completely regular by 2.7.6). We can then conclude that the pullback (along an open surjection) of the two canonical maps are isomorphisms from the usual constructive Gelfand duality (proved in [4] for Grothendieck toposes, and generalized in [20] to arbitrary elementary toposes). And hence, as pullback by an open surjection is conservative, these two canonical maps are isomorphisms. This proves that the two constructions are inverse from each other, the fact that they form an equivalence of categories follows immediately from the same argument. $\square$

# 5 A spatiality theorem

## 5.1 Definition and statement of the theorem

Assuming the axiom of dependent choice, there is no difference between complete metric locales and complete metric sets: indeed one can construct a point in a non empty open sublocale of a complete metric locale by choosing decreasing sequences of non-empty open sublocales whose diameters tend to zero, which will generate a Cauchy filter. Moreover, in the appendix of [25] it is proved by Douady and Dal Soglio-Hérault that over a paracompact topological space, a notion which looks a lot like our notion of Banach locale is equivalent to the usual notion of continuous field of Banach spaces, which is known to be equivalent to the notion of weakly spatial Banach locales. This suggests that the fact that every Banach locale is weakly spatial can be true far beyond the case of a topos where one can apply the axiom of dependent choice.
The goal of this section is to generalize the arguments of Douady and Dal Soglio-Hérault to prove that, for a class of toposes satisfying a certain technical hypothesis generalizing paracompactness, there is indeed no difference between Banach locales and Banach spaces.
The argument given in [25] relies on constructing $\epsilon$-continuous sections of a Banach Bundle, where $\epsilon$-continuous roughly means that the eventual discontinuities are $\epsilon$-small. But in our context this technique cannot be adapted directly because it relies on discontinuous functions, which does not make sense between locales. Fortunately, one can apply their arguments by replacing the construction of $\epsilon$-continuous functions by constructing $\epsilon$-small positive open sublocales of the bundle which, informally, corresponds to tubular neighborhoods of these $\epsilon$-continuous sections.

5.1.1. **Definition :** *Let $\mathcal{T}$ be a topos, we say that the terminal object of $\mathcal{T}$ is numerically projective if for each inhabited object $X \twoheadrightarrow 1$ there exists a decidable* *object $I$ of $\mathcal{T}$, a map $I \to X$ and a function $\lambda : I \to [0,1]$ where $[0,1]$ denotes the sheaf of continuous real numbers between $0$ and $1$, such that internally:*

$$\sum_{i \in I} \lambda(i) = 1$$

*(where the sum converges in the sense of continuous real numbers.)*
*We say that a general object $X \in \mathcal{T}$ is numerically projective if the terminal object of $\mathcal{T}_{/X}$ is numerically projective.*

We see this definition as a technical tool. These "numerically projective" objects are exactly what we need in order to make the proof of the spatiality theorem work. At the present moment nothing indicates that this definition has some deeper meaning, but we hope that the spatiality theorem can be formulated under more natural hypothesis.

5.1.2. Here are the three main examples of numerically projective objects we have in mind:

**Proposition :**

1. *If $X$ is a locale (or a topological space) which admits enough partitions of unity in the sense that*[11] *for every open covering $(U_v)_{v \in V}$ of $X$ there is a family of functions $\lambda_v$ from $X$ to $[0,1]$ such that for each $v$, $\lambda_v$ has support included in $U_v$ and $\sum_v \lambda_v = 1$ (with a locally uniform convergence) then $1$ is numerically projective in $\mathsf{Sh}(X)$.*
2. *If $P$ is a (externally) projective object of a topos $\mathcal{T}$ then $P$ is numerically projective in $\mathcal{T}$.*
3. *If $\mathcal{T}$ has a generating family of inhabited finite decidable objects, then every object of $\mathcal{T}$ is numerically projective.*

**Proof :**

1. Let $\mathcal{F}$ be any inhabited object of $\mathsf{Sh}(X)$. Then $\mathcal{F}$ can be covered by a sheaf of the form $I = \coprod_{v \in V} U_v$ where the $U_v$ are open sublocales of $X$. Then $I$ is a decidable object (it is a subobject of $p^*(V)$) and, as $I$ cover $\mathcal{F}$ and $\mathcal{F}$ is inhabited, $I$ is also inhabited and hence, the $U_v$ form a cover of $X$.

   By hypothesis one can construct a partition of the unity $\lambda_v$ subordinated to $U_v$. This can be internally interpreted as a function from $p^*(V)$ to $[0,1]$ supported by $I$ and such that $\sum_{v \in V} \lambda_v = 1$. As the function is supported by $I$, this also means that $\sum_{i \in I} \lambda_i = 1$ and concludes the proof.
2. If $P$ is projective, then for any object $\mathcal{F} \twoheadrightarrow P$, there exists a section $P \to \mathcal{F}$. As $P$ is always a decidable object in $\mathcal{T}_{/P}$ one can take $I = P$ and $\lambda_i$ to be the function constant equal to one on $P$.

11. Assuming the axiom of choice, this is the case when $X$ is paracompact and regular/Hausdorff.

3. Let $\mathcal{T}$ be a topos with a generating family of finite decidable objects. We will prove that the terminal object is numerically projective. Let $\mathcal{F}$ be an inhabited object of $\mathcal{T}$. Then there exists an inhabited finite decidable object $T$ with a map into $\mathcal{F}$. One can take $I = T$ and $\lambda_i$ the function on $T$ internally defined as the constant equal of $1/|T|$. This proves that 1 is numerically projective. Now as the hypothesis on $\mathcal{T}$ are stable by slice, this proves that the results holds for any objects of the topos.

□

5.1.3. **Definition :** *Let $\mathcal{T}$ be a topos, $X$ an object of $\mathcal{T}$ and $U, V$ be two subobjects of $X$. One says that $U$ is* rather below *$V$ and denote $U \prec V$ if there exists (externally) a subobject $W$ of $X$ such that $U \wedge W = \emptyset$ and $U \vee W = X$*
*We say that $V$ is regular in $X$ if*

$$V = \bigvee_{\substack{U \in Sub(X) \\ U \prec V}} U.$$

5.1.4. We are now ready to state our spatiality theorem:

**Theorem :** *Let $\mathcal{T}$ be a topos, assume that there exists a generating family of objects $A_i$ of $\mathcal{T}$ such that for each $i$, $A_i$ is a regular sub-object of a numerically projective object $P_i$. Also assume that the axiom of dependent choice holds in the base topos. Then any Banach locale of $\mathcal{T}$ is internally weakly spatial.*

This theorem applies to any topos of sheaves over a regular space $X$ which have partitions of unity (that is over any paracompact Hausdorff space if one assume the axiom of choice) but also to all toposes having enough projective objects, or to the locally separated atomic toposes studied in chapter 1 or more generally to the boolean locally separated toposes studied in chapter 2 (both because they admit enough slice generated by finite decidable objects), to classifying toposes of a compact or locally compact (pro-discrete) groups, and essentially to any topos with good enough analytical properties
It implies that in each of these situations there is essentially no difference between Banach locales and usual Banach spaces. The proof of this theorem will end in 5.3.2

## 5.2 Convexity and barycentre

Before proving Theorem 5.1.4, we will need some generalities about the theory of convex open sublocales of Banach locales and barycentres in a constructive context. The results of this subsection are seen as useful lemmas for the proof of 5.3.1, in particular we have not tried to make them optimal and most of them can probably be improved (for example it is unlikely that it is indeed necessary to take the closure in 5.2.6 or 5.2.8).

5.2.1. **Definition :** *Let $U$ be an open sublocale of a Banach locale $\mathcal{H}$, we say that $U$ is convex if the map*

$$\begin{array}{rccc} p_2 : & \mathcal{B} \times \mathcal{B} & \rightarrow & \mathcal{B} \\ & (x,y) & \rightarrow & \dfrac{x+y}{2} \end{array}$$

*sends $U \times U$ on $U$.*

5.2.2. **Proposition :** *Let $U$ be an arbitrary open sublocale of $\mathcal{H}$. We define a sequence of open sublocales:*

$$U_0 = 0$$

$$U_{n+1} = U_n \cup p_{2!}(U_n \times U_n)$$

*(where $p_2$ is the map $p_2(x,y) = (x+y)/2$). We also define:*

$$\mathsf{Conv}\ U := \bigvee_{n \in \mathbb{N}} U_n.$$

*Then $\mathsf{Conv}\ U$ is the smallest convex open sublocale of $\mathcal{H}$ containing $U$.*

**Proof :**
The $U_n$ are all open since $p_2$ is an open map (see 4.1.2 points 2), hence $\mathsf{Conv}\ U$ is open. If $W$ is a convex open sublocale containing $U$ then by an immediate induction $W$ contain $U_n$ for each $n$ and hence $\mathsf{Conv}\ U \subset W$.
It remains to show that $\mathsf{Conv}\ U$ is convex, but as $p_2$ is an open map one can write

$$p_{2!}(\mathsf{Conv}\ U \times \mathsf{Conv}\ U) = \bigvee_{n,m \in \mathbb{N}} p_{2!}(U_n \times U_m).$$

For each $n, m$, if $t$ denotes the maximum of $n$ and $m$ one has

$$p_{2!}(U_n \times U_m) \leqslant p_{2!}(U_t \times U_t) \leqslant U_{t+1} \leqslant \mathsf{Conv}\ U,$$

hence $p_{2!}(\mathsf{Conv}\ U \times \mathsf{Conv}\ U) \leqslant \mathsf{Conv}\ U$, which concludes the proof. $\square$

5.2.3. **Proposition :** *Let $U$ be an open sublocale of a Banach locale $\mathcal{H}$ such that $\delta(U) < q$. Then $\delta(\mathsf{Conv}\ U) < q$.*

**Proof :**
We will work in $\mathcal{H} \times \mathcal{H}$ endowed with the product distance (as defined in 3.4). It is a norm which turns it into a Banach locale.
For any $q$, $\Delta_q$ is convex by triangular inequality. And for any open sublocale of the form $U \times V$ of $\mathcal{H} \times \mathcal{H}$ one has:

$$\mathsf{Conv}\ (U \times V) = \mathsf{Conv}\ U \times \mathsf{Conv}\ V.$$

Indeed one easily checks by induction that $(U \times V)_n = U_n \times V_n$
In particular, if $U \times U \subset \Delta_{q'}$ for some $q' < q$ then $\mathsf{Conv}\ U \times \mathsf{Conv}\ U \subset \Delta_{q'}$ which concludes the proof. $\square$

5.2.4. **Proposition :** *Let $\mathcal{H}$ be a Banach locale then any open sublocale of $\mathcal{H}$ can be written as a union of convex open sublocale.*

**Proof :**
Let $U \subset \mathcal{H}$, let $q$ and $V$ such that $V \triangleleft_q U$. We write $V$ as a union of positive open sublocales $v_i \subset V$ such that $\delta(v_i) < q$ and we set $w_i = \mathsf{Conv}\ v_i$.
Then from the previous result $\delta(w_i) < q$ , and hence as $\emptyset < v_1 \leqslant w_i \wedge V$ and $V \triangleleft_q U$, one has $w_i \subset U$ and as $v_i \subset w_i$ one has:

$$V \leqslant \bigvee_{i \in I} w_i$$

As the $V$ such that $V \triangleleft U$ form a covering of $U$, the various $w_i$ we will obtain will give a covering of $U$ by convex open sublocales. $\square$

5.2.5. **Proposition :** *If $U$ is a convex open sublocale of $\mathcal{H}$ then $B_q U$ is convex for any $q$.*

**Proof :**
$B_q U$ is the direct image of the convex open sublocale $B_q 0 \times U$ by the map $p : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and the map $p$ and $p_2$ commute. This immediately implies that $B_q U$ is also stable by $p_2$. $\square$

5.2.6. In order to use the existence of partitions of unity, we want to be able to take a barycentre of the form $\sum_{i \in I} \lambda_i x_i$ for $I$ a decidable set, $(x_i)$ a family of elements of $\mathcal{H}$ and $(\lambda_i)$ a family of real numbers such that $\sum_{i \in I} \lambda_i = 1$. Even if the family $(\lambda_i)$ is fixed there is no hope of defining this as map from $\mathcal{H}^I$ to $\mathcal{H}$. But if we assume that the element $(x_i)$ are uniformly bounded then it is possible to give a constructive definition of this map, which will give us a map $\Sigma_I : (B_q 0)^I \to \overline{B_q 0}$.
More precisely, for any $S \subset I$ finite, one can define a map:

$$\Sigma_S : (B_q 0)^I \to (B_q 0)^S \to B_q 0 \subset \overline{B_q 0}$$

where the first map is the projection, the second map is $(x_s)_{s \in S} \mapsto \sum_{s \in S} \lambda_s x_S$, and $\overline{b_q 0}$ denotes the fiberwise closure of $B_q 0$. As $I$ is decidable, any finite subset $S$ of $I$ is complemented and we define

$$c(S) = \sum_{s \notin S} \lambda_s = 1 - \sum_{s \in S} \lambda_s.$$

For any $S \subset S'$ two finite sets, one has $d(\Sigma_S, \Sigma_{S'}) < qc(S)$ i.e. the map $(\Sigma_S, \Sigma_{S'})$ factors into $\bigvee_{q' < qc(S)} \Delta_{q'}$.
This relation (and the fact that $c(S)$ can be made arbitrarily small) asserts that if we think of the $\Sigma_S$ as a family of points of $B_q 0 \subset \overline{B_q 0}$ in the logic of $(B_q 0)^I$ it is a Cauchy net, as $\overline{B_q 0}$ is complete it is sill complete in the logic of $(B_q 0)^I$ and hence this net has a limit, which is a function

$$\Sigma_I = \lim_{S \subset I} \Sigma_S : (B_q 0)^I \to \overline{B_q 0}.$$

It should be possible to replace $\overline{B_q 0}$ by $B_q 0$ but this would require a little more work and is not necessary here.

5.2.7. **Lemma :** *Let $\lambda_i$ be a family of (continuous) real numbers between* 0 *and* 1 *such that $\sum_{i \in I} \lambda_i = 1$. Let $\epsilon$ be a positive rational number. There exists a family $(q_i)_{i \in I}$ of rational numbers such that:*

- *Only a finite number of $q_i$ are non-zero.*
- *Each $q_i$ is between* 0 *and* 1.
- *Each $q_i$ is of the form $n_i/2_i^k$ with $n_i$ and $k_i$ integers.*
- $\sum_i q_i = 1$.
- $\sum_i |q_i - \lambda_i| < \epsilon$.

**Proof :**
If $\epsilon > 2$ then the last condition is always true, and one can choose any family $q_i$ such that $\sum_i q_i = 1$. We will assume that $\epsilon \leqslant 2$. Let $S \subset I$ be a finite family such that:

$$\sum_{s \in S} \lambda_s > 1 - \frac{\epsilon}{4}$$

$S$ is a finite subset of a decidable subset, hence $S$ is complemented in $I$, and is isomorphic to $\{1, \dots, n\}$ for some integer $n$. As $\sum_{s \in S} \lambda_S > 1/2$, $n$ cannot be 0, and hence $S$ is inhabited. For each, $s \in S$ let $q_s$ be a rational number between 0 and 1 and of the form $a/2^b$ such that $0 \leqslant \lambda_s - q_s < \frac{\epsilon}{4n}$.
Then $e = \sum_{s \in S} q_s$ is again a rational number of the form $a/2^b$, and is greater than $1 - \epsilon/2$ and smaller than 1. We choose $s_0 \in S$ and we replace $q_{s_0}$ by $q_{s_0} + (1 - e)$. Finally, (as $S$ is complemented) we define $q_i = 0$ when $i$ is not in $s$. One has:

- $q_i$ is non zero only when $i$ is in $S$, hence there is only a finite number of non zero $q_i$.
- Each $q_i$ is nonnegative.
- The denominator of each $q_i$ is a power of two.
- $\sum_{i \in I} q_i = \sum_{s \in S} q_s = 1$ because of the modification of $q_{s_0}$, in particular, each $q_i$ is smaller than or equal to 1.

- $$\sum_i |q_i - \lambda_i| = \left( \sum_{i \notin S} \lambda_i \right) + \left( \sum_{\substack{s \in S \\ s \neq s_0}} \lambda_s - q_s \right) + |\lambda_{s_0} - q_{s_0}|$$

  The first term is smaller than $\epsilon/4$, the second term is smaller than $\frac{(n-1)\epsilon}{4n}$ and the last one is smaller than $\epsilon/2 + \epsilon/(4n)$, hence the sum is smaller than $\epsilon$ which concludes the proof.

□

5.2.8. **Proposition :** *Let $U$ be a convex open sublocale of $B_q0$, then the image of the sublocale $U^I \subset (B_q0)^I$ by $\Sigma_I$ is included in the fiberwise closure $\overline{U}$ of $U$.*

**Proof :**
Let $\epsilon$ be any positive rational number, and let $q_i$ be a family of rational numbers as in Lemma 5.2.7. We denote by $\Sigma_q$ the map $(B_q0)^I \to B_q0$ corresponding to the barycentre with coefficient $q_i$.
As only a finite number of $q_i$ are non zero, and all their denominators are powers of two, the map $\Sigma_q$ can be expressed as a composition of various maps $p_2$. Hence as $U$ is convex, it is stable by $p_2$ and hence $\Sigma_q$ send $U^I$ into $U$.
Let $S$ be a finite subset of $I$ containing all the $i \in I$ such that $q_i$ is non zero, then $d(\Sigma_S, \Sigma_q) < 2q\epsilon$. And as $\Sigma_I$ is the limit of the $\Sigma_S$, this proves that $d(\Sigma_I, \Sigma_q) < (2q+1)\epsilon$. In particular, $\Sigma_I$ can be approximated (uniformly) as close as we want by functions $\Sigma_q$ such that when restricted to $U^I$, $\Sigma_Q$ takes value in $U$. Hence $\Sigma_I$, when restricted to $U^I$ takes value in $\overline{U}$ and this concludes the proof. □

5.2.9. **Lemma :** *Let $\mathcal{H}$ be a Banach locale, we endow $\mathcal{H} \times \mathcal{H}$ with the max norm. Then the function $\Sigma_I^{\mathcal{H}\times\mathcal{H}} : (B_q0 \times B_q0)^I \to (B_q0 \times B_q0)$ of $\mathcal{H} \times \mathcal{H}$ is just the product of two maps $\Sigma_I : (B_q0)^I \to B_q0$ of $\mathcal{H}$.*

**Proof :**
The result holds for immediate diagrammatic reason for all the maps $\Sigma_S$ when $S \subset I$ is finite, and passes to the limit $\Sigma_I = \lim_{S \subset I} \Sigma_S$. □

5.2.10. **Proposition :** *Let $(U_i)_{i\in I}$ be a family of open sublocales of $B_q 0$ such that for every $i$, $\delta(U_i) < \epsilon$. Then the image of $\prod_{i\in I} U_i$ by $\Sigma_I$ is of diameter smaller than $\epsilon'$ for any $\epsilon < \epsilon'$.*

**Proof :**
In $\mathcal{H} \times \mathcal{H}$, the sublocale $X = \Delta_\epsilon \wedge (B_q 0)^2$ is open and convex. Each of the $U_i \times U_i$ is included in $X$. Hence by the previous result, if we apply $\Sigma_I$ in $\mathcal{H}\times\mathcal{H}$, the image of $\prod_{i\in I}(U_i \times U_i)$ will be included in $\overline{X}$.
Now let $Y$ be the image of $\prod_{i\in I} U_i$ by $\Sigma_I$ then the image of $\prod_{i\in I}(U_i \times U_i)$ by $\Sigma_I$ is (by 5.2.9) the same thing as the image of $\left(\prod_{i\in I} U_i\right)^2$ by $(\Sigma_I)^2$ which is fiberwise dense in $Y \times Y$, hence $Y \times Y$ is also included in $\overline{X}$, and as $\overline{X}$ is included in $\Delta_{\epsilon'}$ for any $\epsilon' > \epsilon$, this concludes the proof.
□

## 5.3 Proof of the spatiality theorem

5.3.1. Here is the key lemma:

**Lemma :** *Consider:*

- *A topos $\mathcal{T}$ in which 1 is numerically projective.*
- *A Banach locale $\mathcal{H}$ in $\mathcal{T}$.*
- *Two positive rational numbers $a$ and $e$ and a positive integer $n$.*
- *A global section $U$ of the sheaf of positive convex open sublocales of $\mathcal{H}$ such that $U \subset B_n 0$*

*Then there exists a global section $V$ of the sheaf of convex open sublocales of $\mathcal{H}$ such that $V$ is included in $B_e U$ and $\delta(V) < a$.*

**Proof :**
Internally in $\mathcal{T}$, one can prove that there exists a positive open sublocale $V$ such that $\delta(V) < a/2$ and $V \subset U$. Indeed such a $V$ covers $U$ and $U$ is positive. In particular, externally, there exists an object $P \twoheadrightarrow 1$ which parametrizes a family of such possible choices. Applying the fact that 1 is numerically projective, one can construct an object $I \to P$ and a family of real numbers $\lambda : I \to [0, 1]$ such that internally $\sum_{i\in I} \lambda_i = 1$. Also, internally in $\mathcal{T}$ one has a family $(V_i)_{i\in I}$ of open sublocales of $\mathcal{H}$ such that $V_i \subset U$ and $\delta(V_i) < a/2$.
We define $Y$ to be the image of $\prod_{i\in I} V_i$ by the map

$$\Sigma_I : (B_n 0)^I \to \overline{B_n 0}$$

induced (following 5.2.6)by the family $\lambda_i$.
Then $Y$ is included in $\overline{U}$ and of diameter smaller $v$ for any $v$ greater than $a/2$, in particular $\delta(Y) < (a/2)+(a/5)$. Moreover as the image of something positive and locally positive, $Y$ is positive and locally positive. Let $e' = \min(a/5, e/2)$ then one can define:

$$V = \mathsf{Conv}\ B_{e'} Y$$

one has the following properties:

- $V$ is given by a global section because its construction depends on no internal choices.
- $$\delta(V) = \delta(B_{e'}Y) < 2e' + \frac{a}{2} + \frac{a}{5} \leqslant \frac{2a}{5} + \frac{a}{2} + \frac{a}{5} = a$$
- $Y \subset \overline{U} \subset B_{e/2}U$ hence,
$$B_{e'}Y \subset B_{e/2}B_{e/2}U \subset B_eU.$$

As $B_eU$ is convex, one concludes that $V \subset B_eU$.
This concludes the proof.
□

5.3.2. We can now prove theorem 5.1.4

**Proof :**
Let $\mathcal{T}$ be a topos satisfying all the hypothesis of theorem 5.1.4, and $\mathcal{H}$ be a Banach locale in $\mathcal{T}$.
It is enough to prove internally that for any (positive) integer $n$, for any positive rational number $\epsilon$ and for any positive convex open sublocale $U \subset B_nO$ the open sublocale $B_\epsilon U$ has a point. Indeed, as $\mathcal{H}$ is metric these $B_\epsilon U$ form a basis of the topology. Also, as $\mathbb{N}_\mathcal{T}$ and $\mathbb{Q}_\mathcal{T}$ are just $p^*(\mathbb{N})$ and $p^*(\mathbb{Q}_\mathcal{T})$ for $p$ the canonical geometric morphism to the base topos, it is enough to prove this for any $n$ and any $\epsilon$ fixed in the base topos. We fix $n$ and $\epsilon$ (externally) for the rest of the proof.

More precisely, we will prove the following external statement:

For each $i$, for each section $U$ of $\mathcal{O}^{conv,+}(\mathcal{H})$ over $A_i$ such that $U \subset B_n0$, for each $V_i \prec A_i$, there exists a point $p : P_i \rightarrow \mathcal{H}$ such that over $V_i$, the point $p$ belongs to $B_\epsilon U$.
As the $V_i$ such that $V_i \prec A_i$ cover $A_i$, this indeed proves that for any positive convex open sublocale defined over $A_i$ we indeed have internal existence of a point. As the $A_i$ form a generating family it is indeed enough in order to conclude the proof of the theorem

Let $A_i, V_i$ and $U$ be as previously stated. Let $W_i$ such that $V_i \wedge W_i = \emptyset$ and $A_i \vee W_i = \top$. We extend $U$ to $U'$ a positive convex open sublocale included in $B_n0$ defined over $P_i$ by defining: (for all $x \in P_i$)

$$U'_x = \mathsf{Conv}\left(\left(\bigvee_{x \in A_i} U_x\right) \vee (B_n0 \wedge \text{“}x \in W_i\text{”})\right)$$

$U'_x$ is positive for all $x$ because either $x \in A_i$ and the left side is positive, either $x \in W_i$ and the right side is positive. $U'_x \subset B_n0$ because each side does and $B_n0$ is convex. And for all $x \in V_i$ one has $U'_x = U_x$.
Hence one obtains a section $U'$ over $P_i$ of the sheaf of positive convex open sublocales of $B_n0$ which coincide with $U$ over $V_i$.

As $P_i$ is numerically projective, one can then apply Lemma 5.3.1 (and the axiom of dependent choice) to construct a sequence $U_n$ of sections over $P_i$ of the sheaf of positive convex open sublocales such that:

$$U_0 \subset B_{\epsilon/3} U'$$

$$\delta(U_n) < \frac{1}{n+1}$$

$$U_n \subset B_{l+n+1}$$

$$U_n \subset B_{\epsilon 2^{-n-1}} U_{n-1}$$

And then we define, $V_n = B_{\epsilon 2^{-n-1}} U_n$.
One has

$$V_0 \subset B_{5\epsilon/6} U';$$

$$V_n \subset B_{\epsilon 2^{-n-1}} B_{\epsilon 2^{-n-1}} U_{n-1} \subset B_{\epsilon 2^{-n}} U_{n-1} = V_{n-1};$$

$$\delta(V_n) < \frac{1}{n+1} + \epsilon 2^{-n}.$$

If we consider (internally) the set of convex open $U$ of $\mathcal{H}$ such that $V_n \triangleleft U$ for some $n$, then this defines a $P_i$ indexed family regular Cauchy filter on $\mathcal{H}$ and by completeness of $\mathcal{H}$ this defines a $P_i$-indexed point of $\mathcal{H}$, which will belong to $\overline{B_{5\epsilon/6} U'} \subset B_\epsilon U'$. When one restricts it to $V_i$ it belongs to $B_\epsilon U$, and this concludes the proof.
□

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