Universality in bootstrap percolation and kinetically constrained models[1]

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Bootstrap percolation

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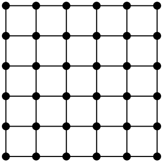

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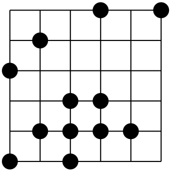

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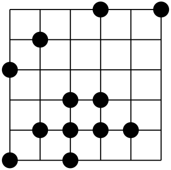

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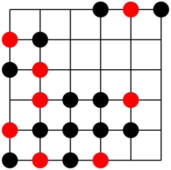

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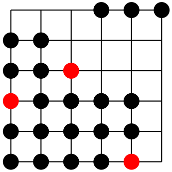

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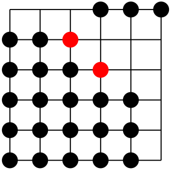

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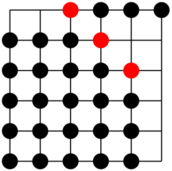

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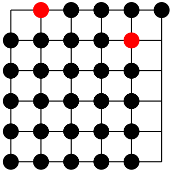

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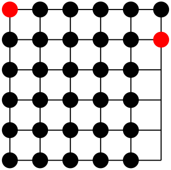

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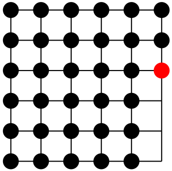

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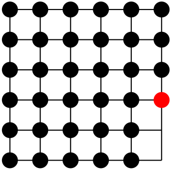

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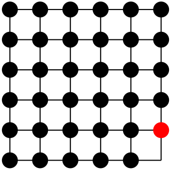

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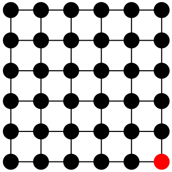

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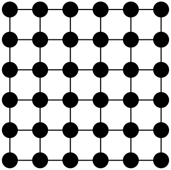

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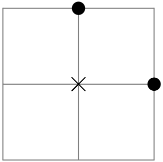

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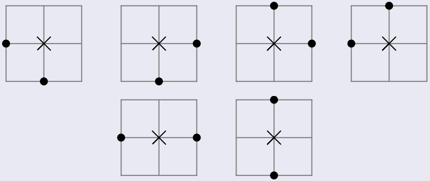

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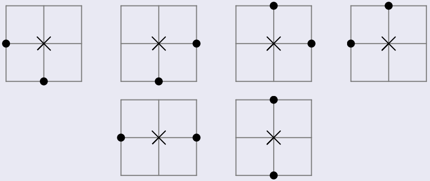

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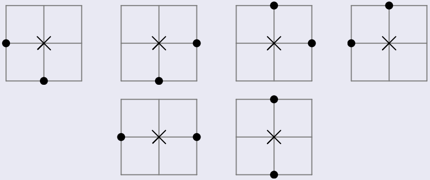
2-neighbour bootstrap percolation



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Examples

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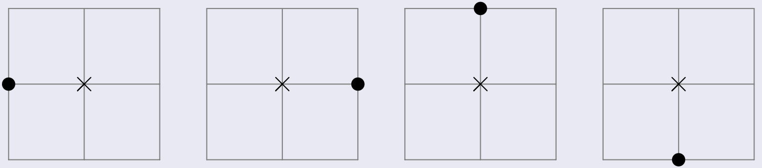

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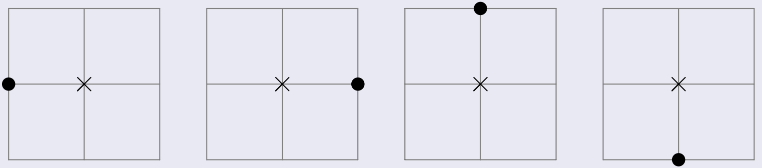

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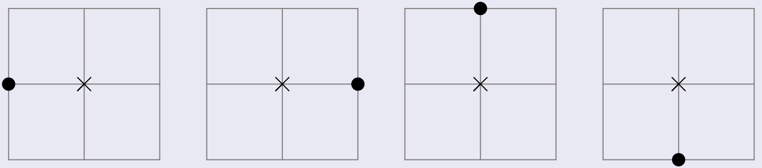

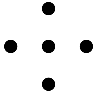

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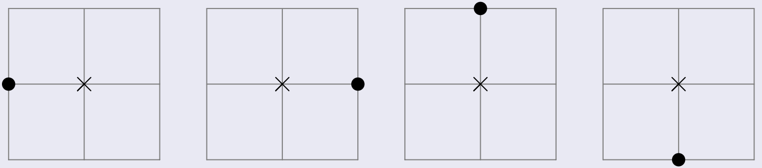

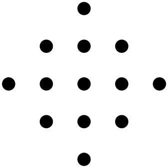

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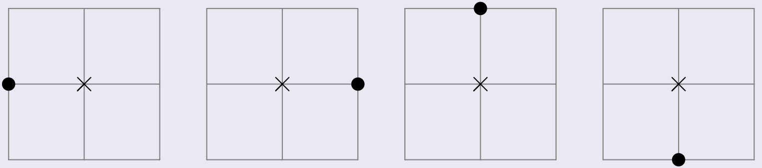

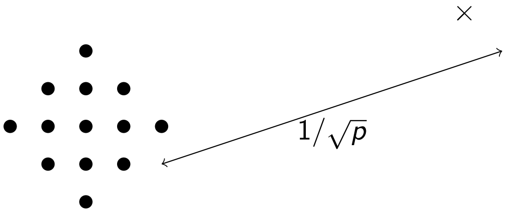

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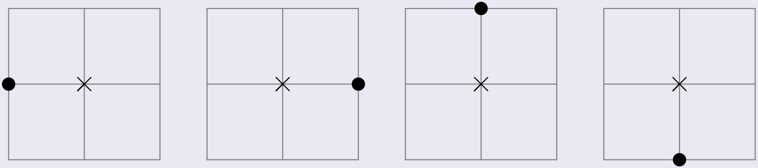

$p_{\mathrm{c}} = 0$ $\tau_0 \approx 1/\sqrt{p}$

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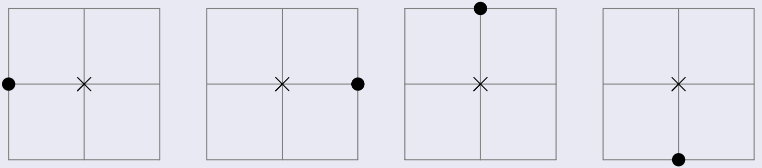

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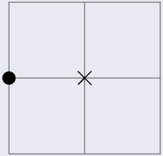

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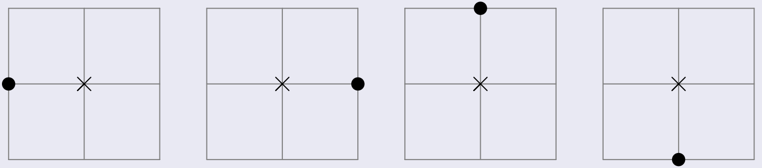

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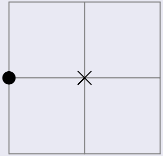

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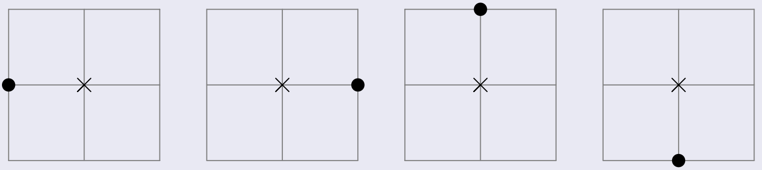

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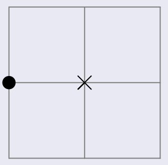

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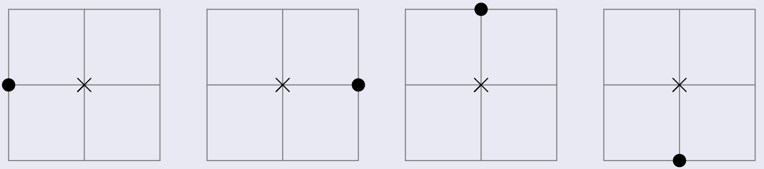

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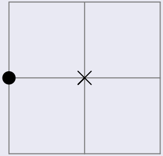

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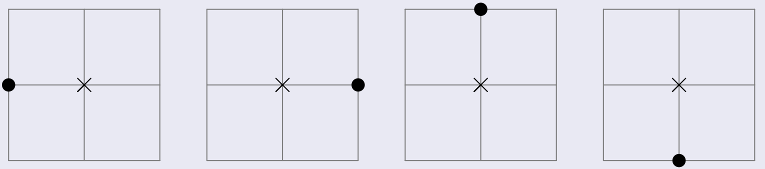

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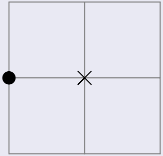

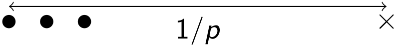

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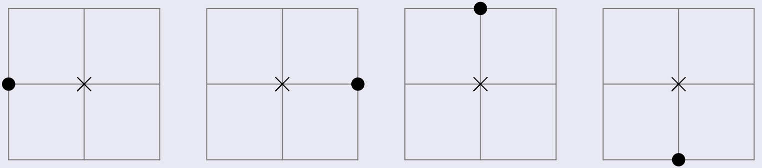

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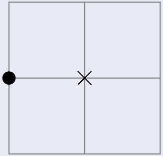

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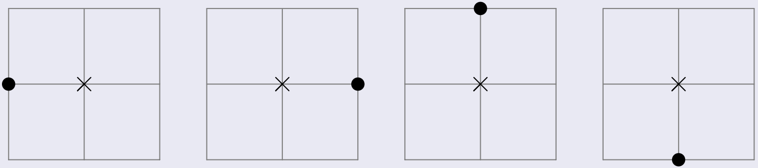

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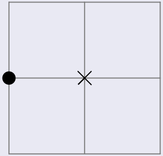

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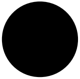

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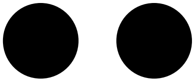

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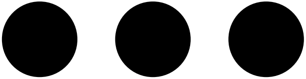

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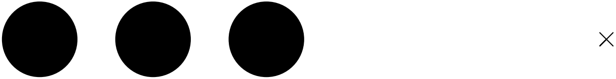

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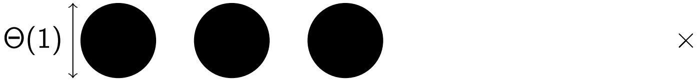

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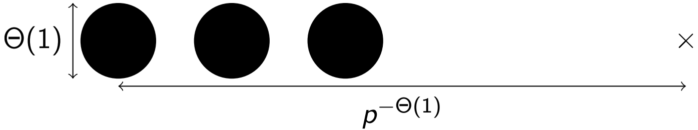

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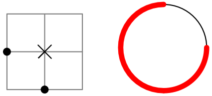

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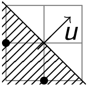

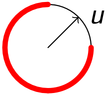

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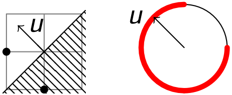

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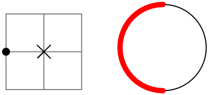

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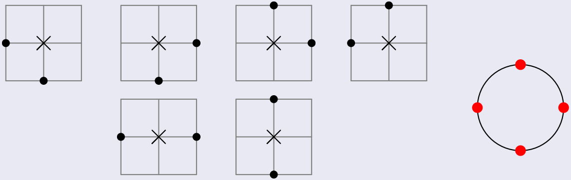
2-neighbour bootstrap percolation



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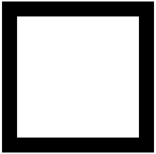

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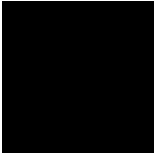

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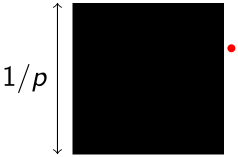

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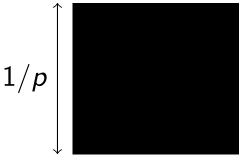

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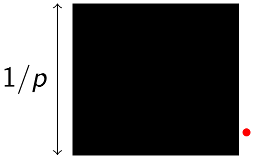

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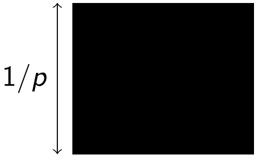

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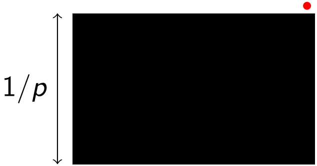

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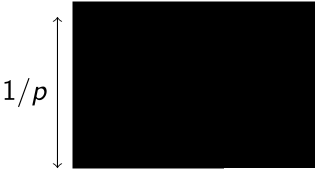

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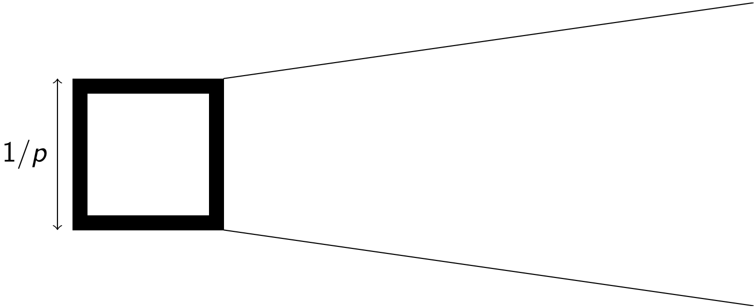

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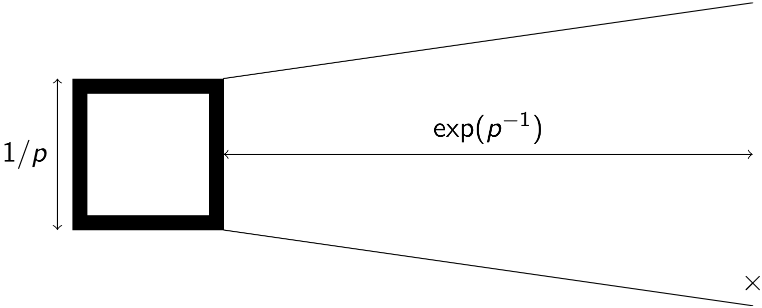

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Theorem (BSU15, Bollobás–Duminil-Copin–Morris–Smith'14+)

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Theorem (H–Mezei'20)

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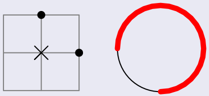
North-East/Oriented percolation



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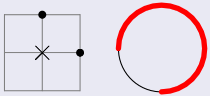

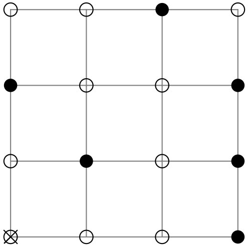

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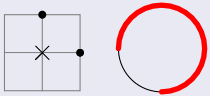

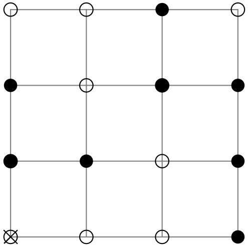

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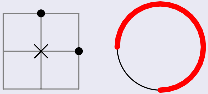

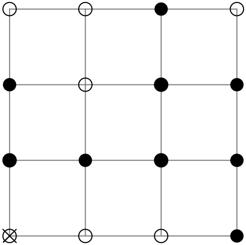

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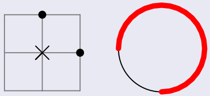

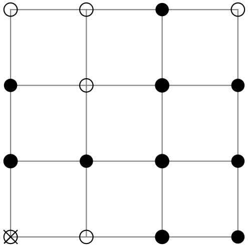

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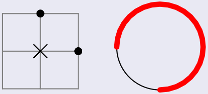

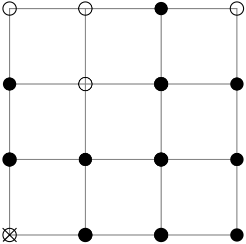

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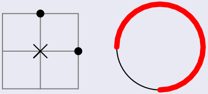

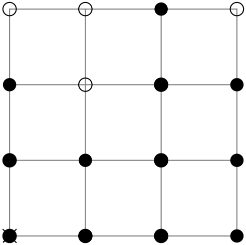

$$\tau_0 = 5$$

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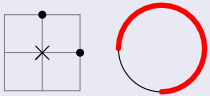

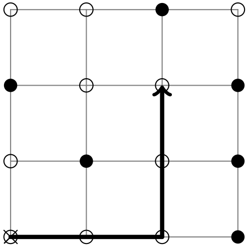

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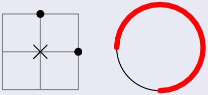

$$p_c \in (0, 1)$$

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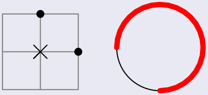

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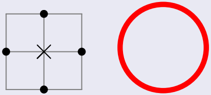
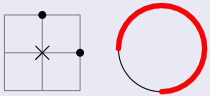
4-neighbour bootstrap percolation



North-East/Oriented percolation



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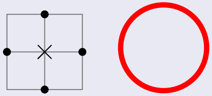

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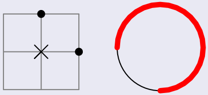

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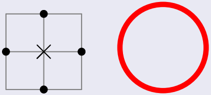

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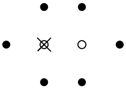

Definition (Subcritical family)

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Theorem (H'22)

For all $\mathcal{U}$ supported in a half-space the conjecture holds.

Bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Update rule: $U \subset \mathbb{Z}^2 \setminus \{0\}$, $U \neq \varnothing$, $|U| < \infty$.
- Update family $\mathcal{U} \neq \varnothing$: finite set of update rules.
- In $\mathcal{U}$-bootstrap percolation infections never heal and at each step we infect all $x \in \mathbb{Z}^2$ such that

$$\exists U \in \mathcal{U}, \forall u \in U : x + u \text{ is } \bullet .$$

- Infection time: $\tau_0 = \inf\{t \in \mathbb{N} : 0 \text{ is } \bullet\} \in \mathbb{N} \cup \{\infty\}$.
- Density of $\bullet$: $p \in [0, 1]$.
- Initial distribution: $\pi = Ber(p)^{\otimes \mathbb{Z}^2}$.
- Critical probability: $p_{\mathrm{c}} = \inf\{p \in [0, 1] : \pi(\tau_0 = \infty) = 0\}$.

Kinetically constrained models

- Geometry: $\mathbb{Z}^2$.
- State space: $\Omega = \{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- Update rule: $U \subset \mathbb{Z}^2 \setminus \{0\}$, $U \neq \varnothing$, $|U| < \infty$.
- Update family $\mathcal{U} \neq \varnothing$: finite set of update rules.
- In $\mathcal{U}$-KCM infections can heal and at rate 1 we update to $Ber(p)$ all $x \in \mathbb{Z}^2$ such that

$$\exists U \in \mathcal{U}, \forall u \in U : x + u \text{ is } \bullet.$$

- Infection time: $\tau_0 = \inf\{t \in \mathbb{R}_+ : 0 \text{ is } \bullet\} \in \mathbb{R}_+ \cup \{\infty\}$.
- Density of $\bullet$: $p \in [0, 1]$.
- Initial and stationary distribution: $\pi = Ber(p)^{\otimes \mathbb{Z}^2}$.
- Critical probability: $p_c = \inf\{p \in [0, 1] : \mathbb{P}_\pi(\tau_0 = \infty) = 0\}$.

Theorem (Cancrini–Martinelli–Roberto–C. Toninelli'08)

For any $\mathcal{U}$ the following are equivalent:

- $\pi(\tau_0 = \infty) = 0$ *in $\mathcal{U}$-bootstrap percolation;*
- $\mathbb{P}_\pi(\tau_0 = \infty) = 0$ *in the $\mathcal{U}$-KCM;*
- 0 *is a simple eigenvalue of the generator of the $\mathcal{U}$-KCM;*
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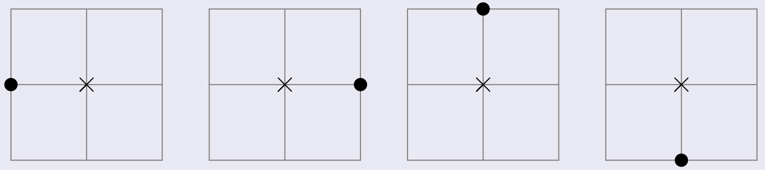
Theorem (CMRT08,H21)

For any $\mathcal{U}$ the following are equivalent:

- *in $\mathcal{U}$-bootstrap percolation τ_0 has an exponential moment;*
- *in $\mathcal{U}$-KCM τ_0 has an exponential moment;*
- $T_{\mathrm{rel}} < \infty$ *for the $\mathcal{U}$-KCM.*

1-neighbour KCM



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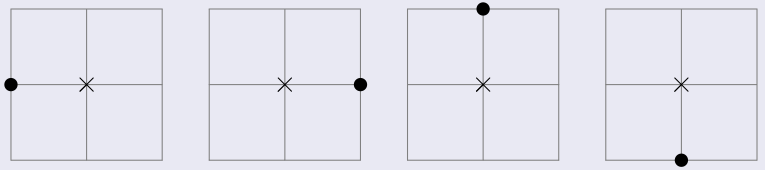

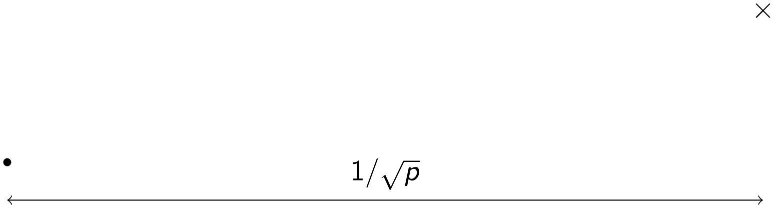

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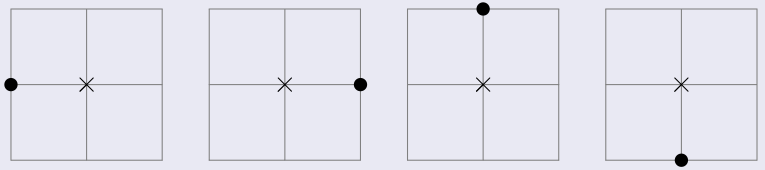

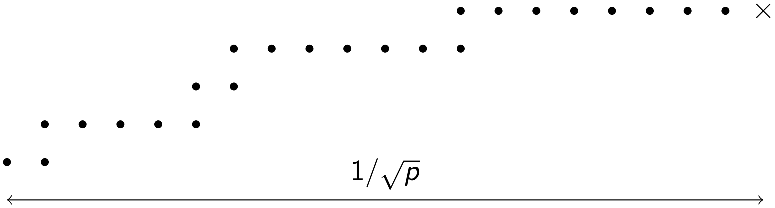

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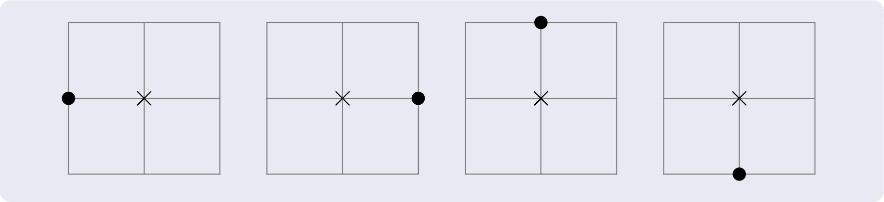

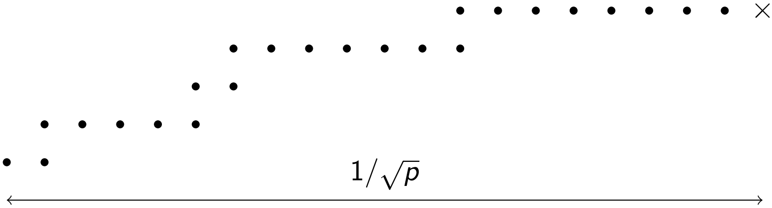

$$\tau_0 \leqslant \exp(1/\sqrt{p})$$

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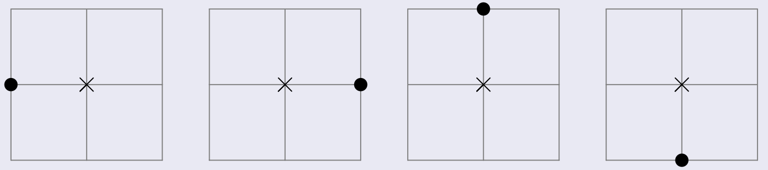

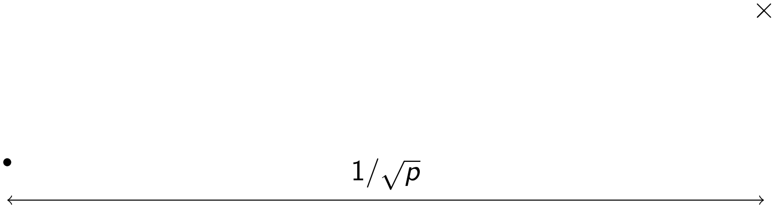

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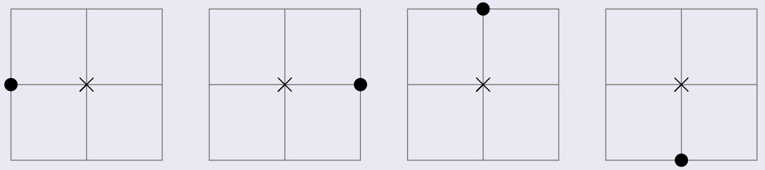

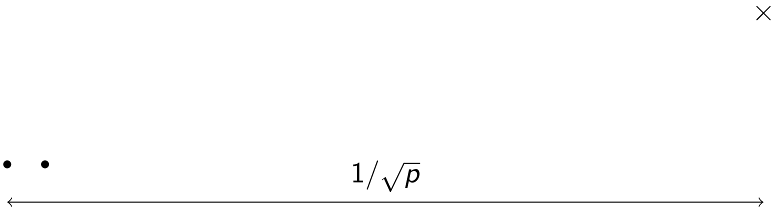

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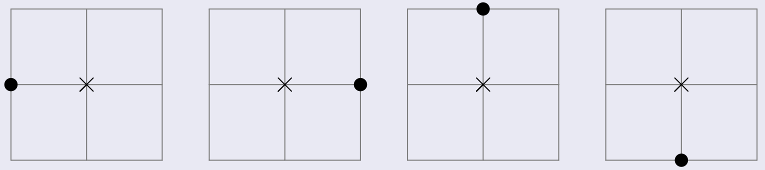

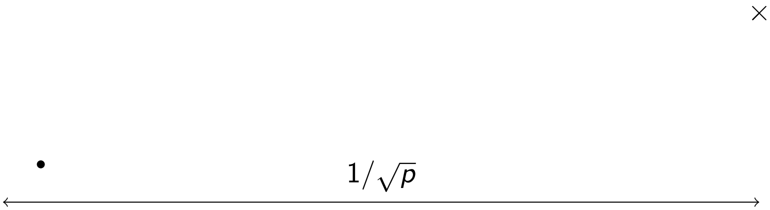

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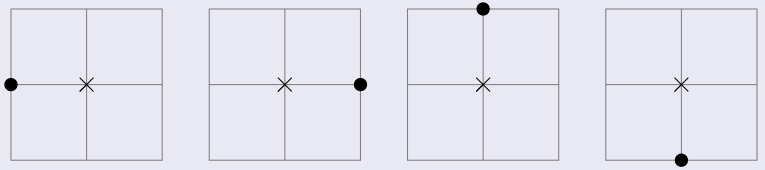

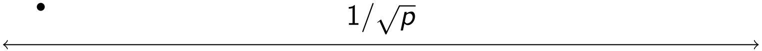

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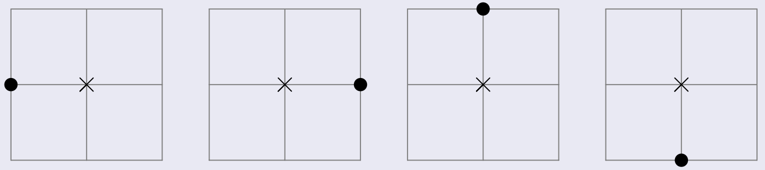

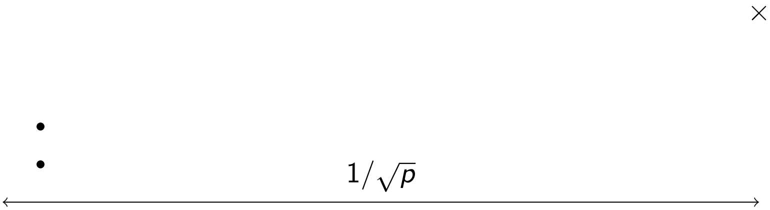

$$\tau_0 \leqslant (1/\sqrt{p})^2/p = 1/p^2$$

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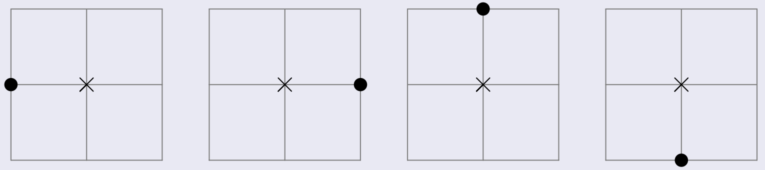
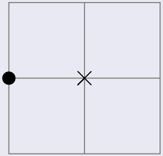


Theorem (CMRT08,Shapira'20)

For the 1-neighbour KCM we have

$$T_{\mathrm{rel}} = \begin{cases} \Theta(p^{-3}) & d = 1, \\ p^{-2+o(1)} & d = 2, \\ \Theta(p^{-2}) & d \geqslant 3. \end{cases}$$

East



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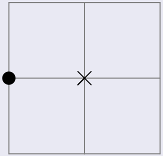

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Starting from an infection at 0 and using at most n infections simultaneously, we can bring an infection only as far as $2^{n-1} - 1$.

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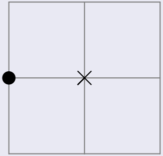

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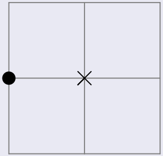

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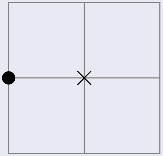

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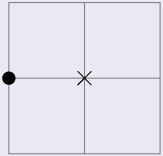

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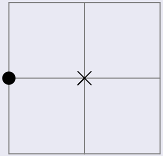

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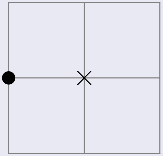

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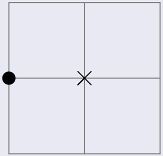

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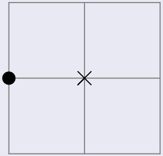

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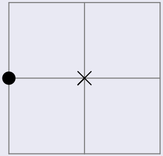

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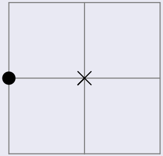

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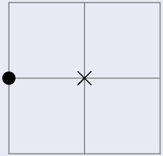

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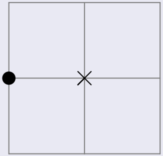

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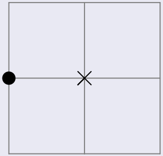

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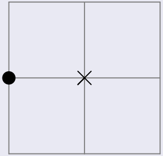

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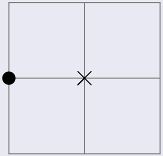

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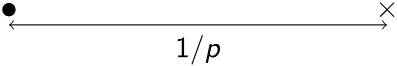

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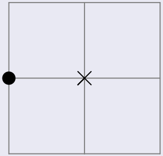

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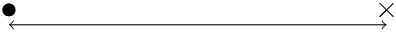
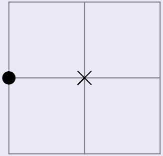


$$1/p$$

$$\tau_0 \approx p^{-\log(1/p)} = \exp\left(\log^2(1/p)\right)$$

East



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Theorem (Aldous–Diaconis'02, CMRT08)

$$T_{\mathrm{rel}} = \exp\left(\frac{\log^2(1/p)}{2\log 2 + o(1)}\right)$$

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Definition (Rooted)

A supercritical family $\mathcal{U}$ is *rooted* if there exist two non-opposite stable directions and *unrooted* otherwise.

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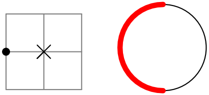

Supercritical KCM

Definition (Rooted)

A supercritical family $\mathcal{U}$ is *rooted* if there exist two non-opposite stable directions and *unrooted* otherwise.

Theorem (Martinelli–C. Toninelli'19, Martinelli–Morris–C. Toninelli'19, Marêché'20, Marêché–Martinelli–C. Toninelli'20)

For a supercritical KCM we have

- $T_{\mathrm{rel}} = p^{-\Theta(1)}$ *if* $\mathcal{U}$ *is unrooted;*
- $T_{\mathrm{rel}} = \exp(\Theta(\log^2(1/p)))$ *if* $\mathcal{U}$ *is rooted.*

Critical KCM

Theorem (MT19, MMT19)

For critical $\mathcal{U}$-KCM we have $\tau_0 = \exp(p^{-\Theta(1)})$.

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Theorem (MMT19, H–Marêché–C. Toninelli'20)

For a critical $\mathcal{U}$-KCM with infinite number of stable directions and difficulty α we have $\tau_0 = \exp(p^{-2\alpha+o(1)})$.

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- East-like movement of 'droplets'.
- East-like deterministic logarithmic bottleneck.

Conjecture (MMT19)

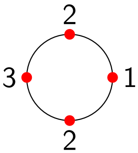
The correct exponent is between α and 2α. It corresponds to either moving 1-neighbour-like or East-like (whichever is more efficient).

Theorem (MMT19, HMT20)

For a critical $\mathcal{U}$-KCM with infinite number of stable directions and difficulty α we have $\tau_0 = \exp(p^{-2\alpha+o(1)})$.

Theorem (H–Martinelli–C. Toninelli'21)

For a critical $\mathcal{U}$-KCM with finite number of stable directions and difficulty α we have $\tau_0 = \exp(p^{-\alpha+o(1)})$.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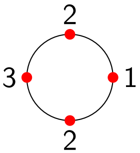

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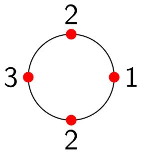

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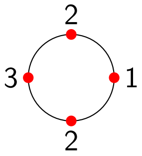

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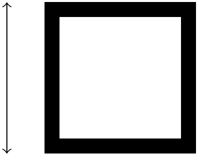
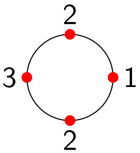
$\frac{\log(1/p)}{p}$





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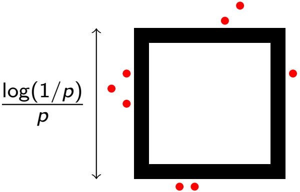

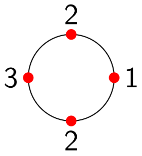

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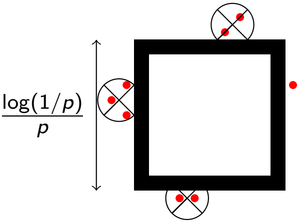

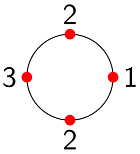

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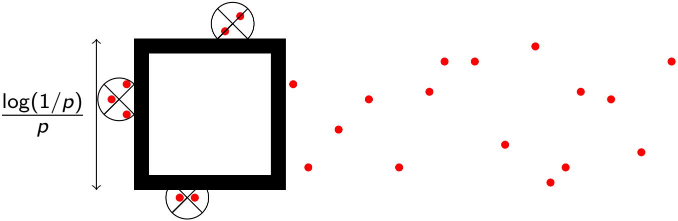

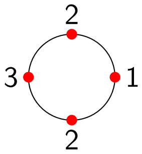

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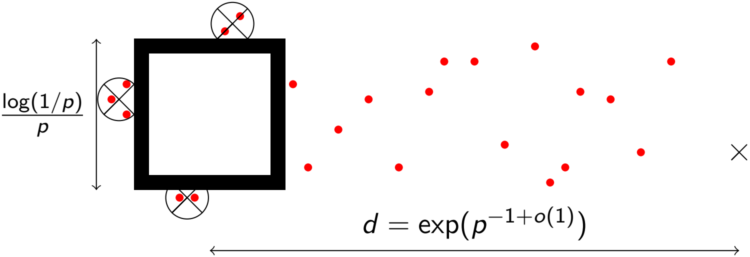
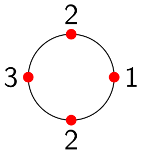


$$\tau_0 \leqslant \left(\exp\left(p^{-1+o(1)}\right)\right)^{\log d} = \exp\left(p^{-2+o(1)}\right)$$



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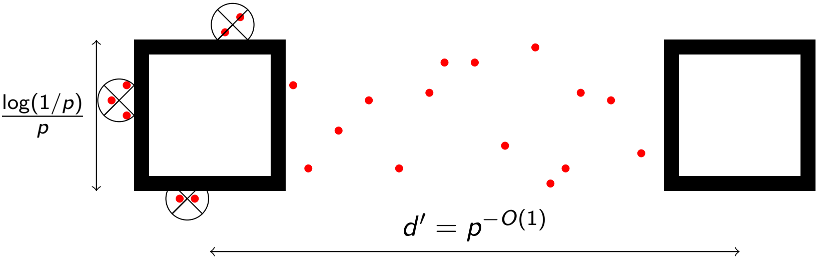

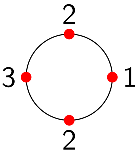

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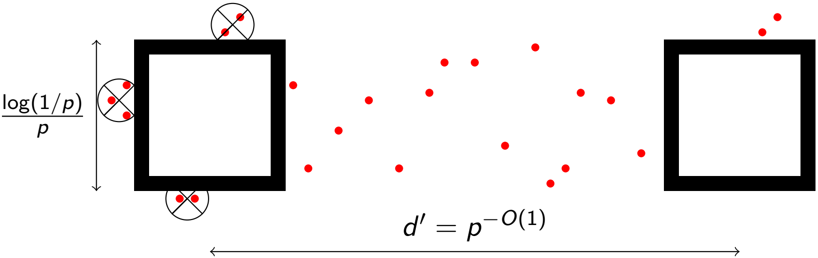

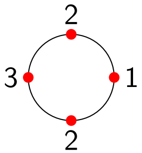

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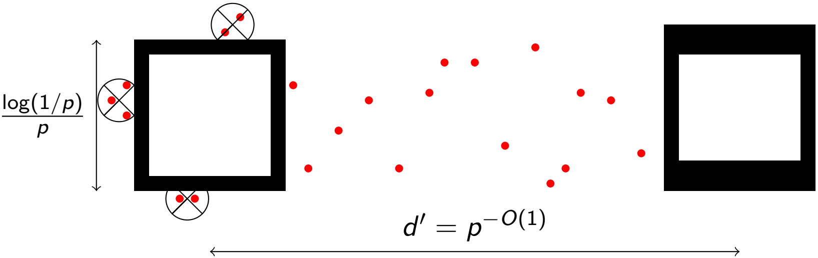

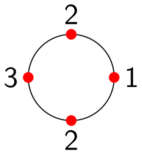

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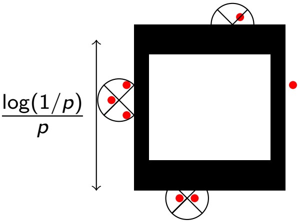

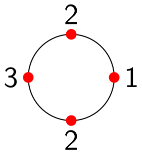

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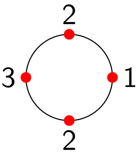

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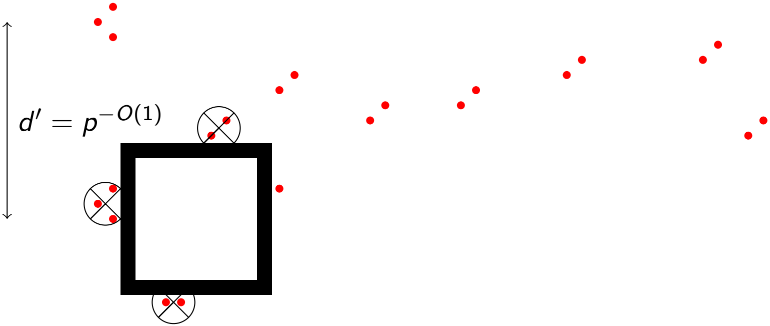

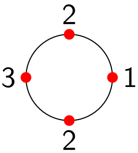

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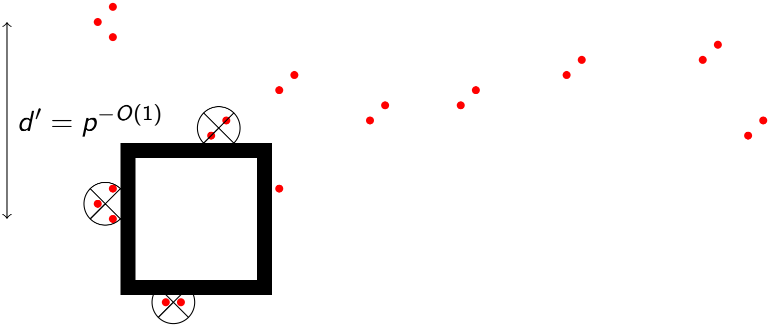

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Refined universality

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Fix a critical update family. A direction u is *hard* if $\alpha(u) > \alpha$. The family is *unbalanced* if there are two opposite hard directions and *balanced* otherwise.

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Theorem (BDCMS14+)

For critical $\mathcal{U}$-bootstrap percolation with difficulty α we have

$$\tau_0 = \exp\left(\Theta(1)\frac{1}{p^\alpha}\left(\log\frac{1}{p}\right)^{\gamma'}\right),$$

where $\gamma' = 0$ if $\mathcal{U}$ is balanced and $\gamma' = 2$ if it is unbalanced.

Definition

A critical update family is *rooted* if it has two non-opposite hard directions and *unrooted* otherwise. Families with one hard direction are *semi-directed*, while those with no hard directions are *isotropic*.

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Theorem (MMT19, HMT'21,H–Marêché'22, H'21+)

For critical $\mathcal{U}$-KCM with difficulty α we have

$$\tau_0 = \exp\left(\Theta(1)\left(\frac{1}{p^\alpha}\right)^\beta\left(\log\frac{1}{p}\right)^\gamma\left(\log\log\frac{1}{p}\right)^\delta\right),$$

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where the exponents β, γ, δ are given in the table below.

	Infinite stable dir.	Finite stable dir.	
		Rooted	Unrooted
Unbalanced	$2,4,0$	$1,3,0$	$1,2,0$
Balanced	$2,0,0$	$1,1,0$	$1,0,1$ S.-dir. Iso. $1,0,0$

Higher dimensions

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Theorem (3 × Balister–Bollobás–Morris–Smith'22+)

Bootstrap percolation universality statements for supercritical and subcritical families extend to higher dimensions modulo adapting the definition as needed. For every critical family there exists an integer $1 \leqslant r \leqslant d-1$ *such that*

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For KCM the analogous universality result (with a rooted/unrooted distinction for supercritical families) is not known. More precisely, the upper bounds for supercritical and critical families are still missing.

Out of equilibrium

Out of equilibrium

Theorem (H–F. Toninelli'22+)

For any update family $\mathcal{U}$ which is not trivial subcritical there exist $p_0 < 1$ and $C > 0$ such that for any $p > p_0$ the following holds for the $\mathcal{U}$-KCM on $[-L, L]^d$ with infected boundary condition. For any $\delta \in (0,1)$ and L large enough depending on δ

$$t_{\mathrm{mix}}(\delta)/L \in [1/C, C].$$

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- Limit shapes, cutoff.
- Sharp phase transition of (2d) subcritical models.
- Determine or even conjecture which subcritical models exhibit a continuous phase transition.

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Thank you.

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