Subcritical bootstrap percolation

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Pecolation Today seminar, online

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Definition

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- Graph – $\mathbb{Z}^2$

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- Graph
- Initial condition – $A_0 \sim \bigotimes_{x \in \mathbb{Z}^2} \mathrm{Bernoulli}(q)$

Definition

- Graph
- Initial condition
- Update family – finite collection $\mathcal{U}$ of finite subsets of $\mathbb{Z}^2 \setminus \{0\}$

Definition

- Graph
- Initial condition
- Update family
- Bootstrap dynamics – for $t \in \mathbb{N}$

$$A_{t+1} = A_t \cup \{x \in \mathbb{Z}^2 : \exists U \in \mathcal{U}, x + U \subset A_t\}$$

Definition

- Graph
- Initial condition
- Update family
- Bootstrap dynamics
- Closure – $[A] = \bigcup_{t \in \mathbb{N}} A_t$

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- Closure
- Percolation event – $[A] = \mathbb{Z}^2$

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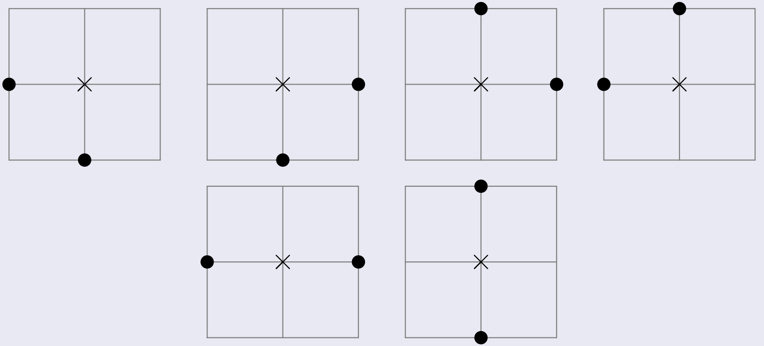
- Percolation event
- Critical probability –

$$q_c = \sup\{q \in [0,1] : \mathbb{P}_q(0 \notin [A]) > 0\}$$

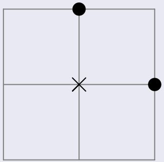
2-neighbour [Chalupa-Leath-Reich'79]

$\mathcal{U} = \binom{N(0)}{2}$:



Oriented percolation

$\mathcal{U} = \{\{(1,0),(0,1)\}\}$:



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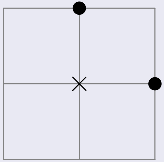

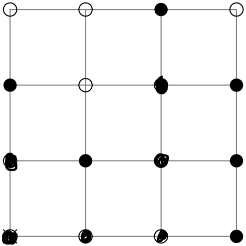

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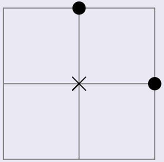
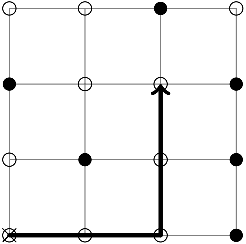
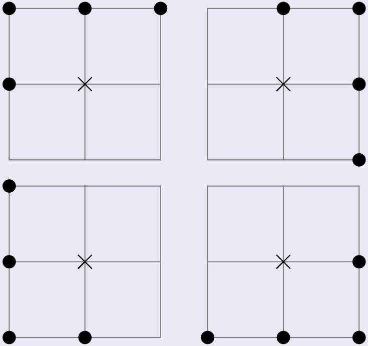




Infection time $\tau_0 = 5$.

Spiral [Toninelli-Biroli-Fisher'07]



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A direction $u \in S^1$ is *unstable* if $[\mathbb{H}_u] = \mathbb{Z}^2$, where $\mathbb{H}_u = \{x \in \mathbb{Z}^2 : \langle x, u \rangle < 0\}$ is the half-plane directed by u.

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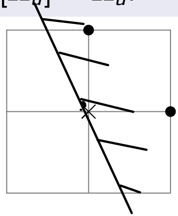

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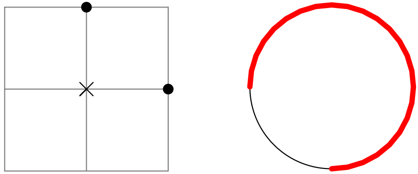

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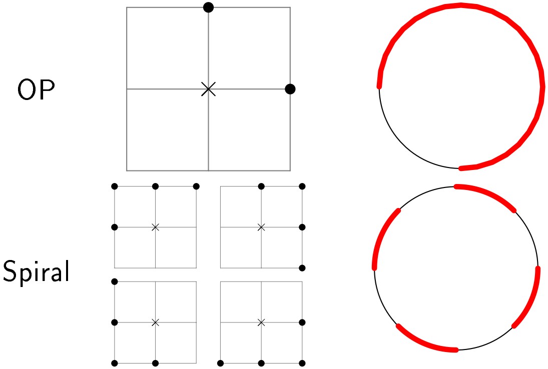

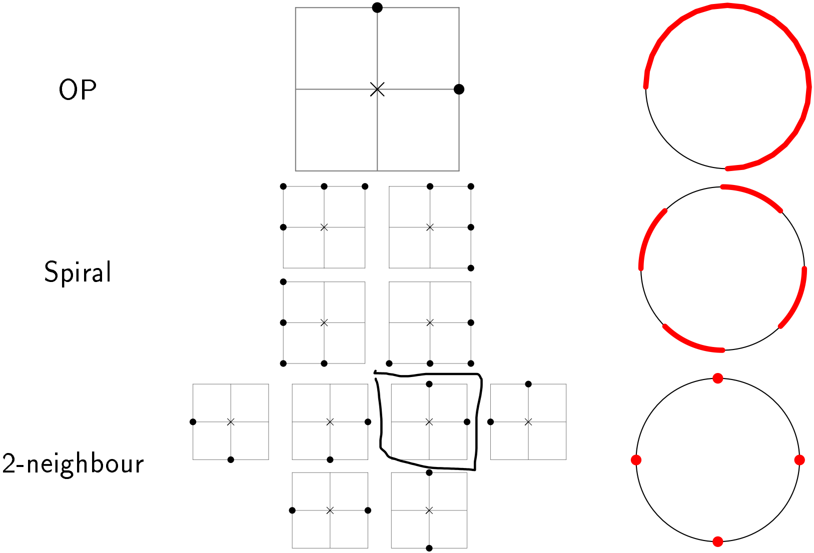
OP
Spiral
2-neighbour

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An update family $\mathcal{U}$ is *subcritical* if every semicircle contains infinitely many stable directions.

OP

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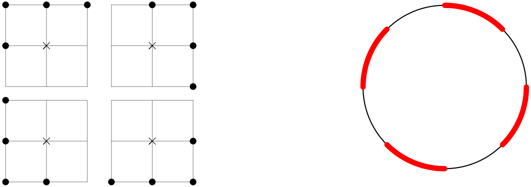

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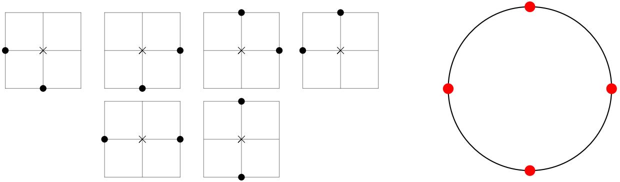

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Theorem (Balister–Bollobás–Przykucki-Smith'16)

An update family $\mathcal{U}$ *is subcritical iff* $q_c > 0$.

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Moreover, $q_c = 1$ *iff all directions are stable.*

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In this talk we study non-trivial subcritical models, that is

$$q_c \in (0, 1).$$

1 Critical densities

$$\theta_n(q) = \mathbb{P}_q(0 \notin [A \cap B_n])$$

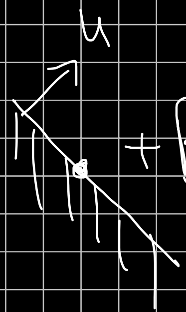
$B_n \to [-n,n]^2$

$$\theta(q) = \lim_{n\to\infty} \theta_n(q)$$

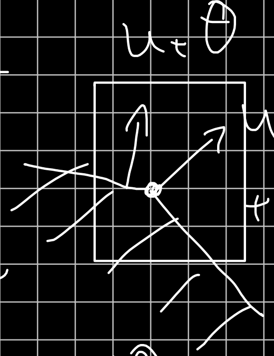
$$q_c = \sup\{q : \theta(q) > 0\}$$

$$\tilde{q}_c = \inf\{q : \limsup_{n\to\infty} \frac{\log \theta_n(q)}{n} < 0\}$$

Morally,

$\sup\{q : \mathbb{P}_q(0 \notin [$ 

$+ \mathrm{Ber}(q)]) > 0\}$

crit prob. of $\mathbb{P}(0 \notin [$ 

$+ \mathrm{Ber}(q)])$ has exp decay

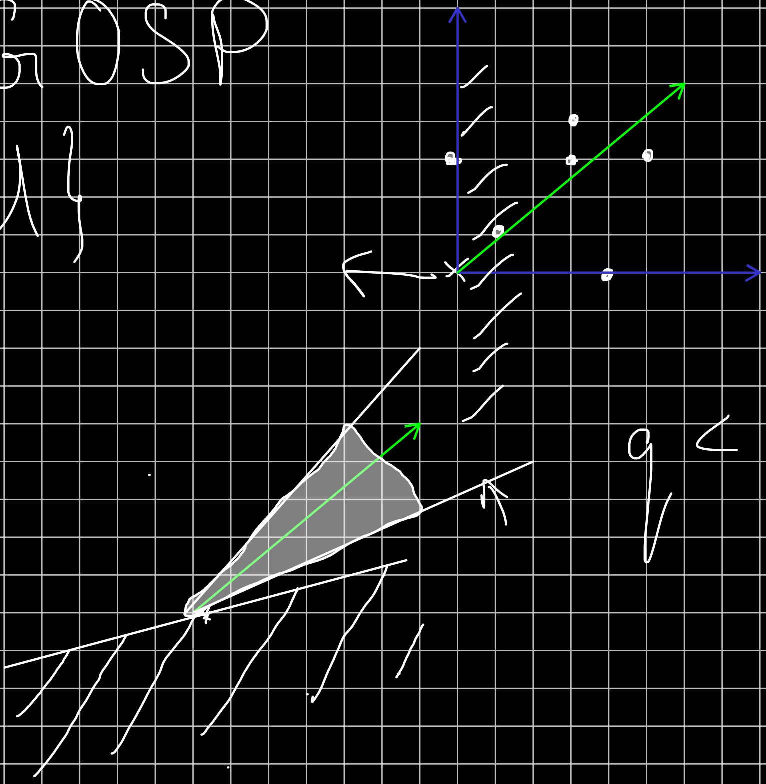
is denoted d_u^θ

The critical density of u

is $d_u = \limsup_{\theta\to 0} d_u^\theta$

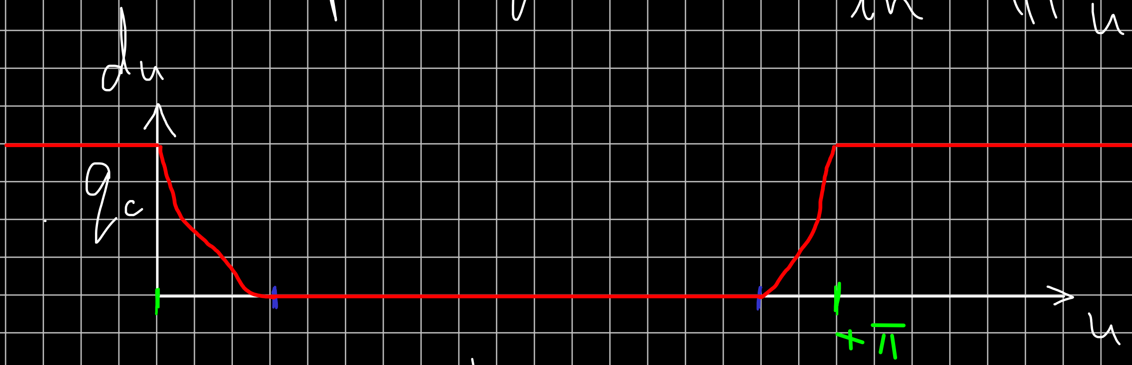
Example GOSP

$\mathcal{U} = \{U\}$



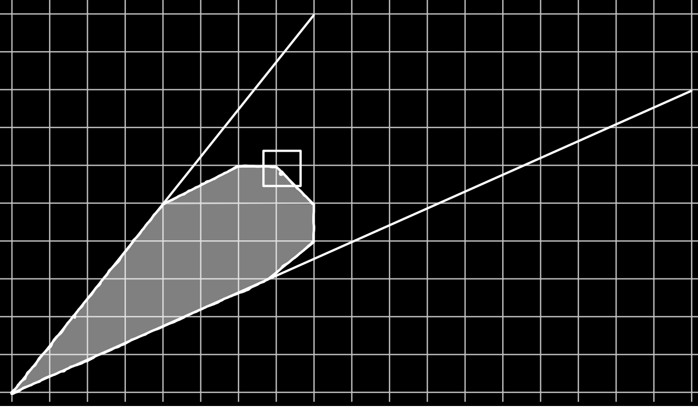
$q < q_c$

$d_u = \sup\{ q : \text{cone is not contained in } H_u \}$



2. Result

Th. (H'21) $\tilde{q}_c = \inf_{C\text{-semi-dir}} \sup_{u \in C} d_u$



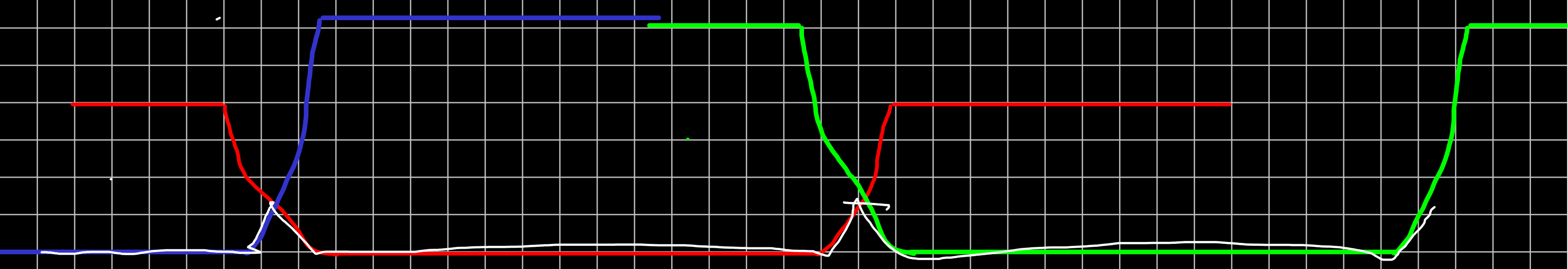
3. Applications

a. Using GOSP to control general $\mathcal{U}$

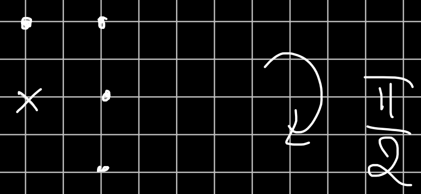
if $\mathcal{U}' \subset \mathcal{U}$, then $d_u(\mathcal{U}') \geq d_u(\mathcal{U})$

Corollary: $\tilde{q}_c(\mathcal{U}) \leq \inf \sup \min_i d_u(\mathcal{U}_i)$

for $\mathcal{U}_i \subset \mathcal{U}$



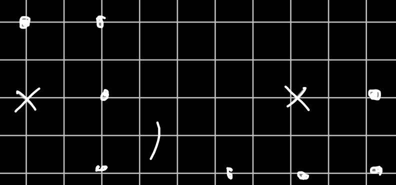
6. Spiral



$$\tilde q_c(\mathcal{U}) \le \inf_{C\text{-semi-circ.}} \sup_{u \in C} \min_i d_u(\mathcal{U}_i)$$

$\hookrightarrow$ 2-rule

$\mathcal{U}_1$ {



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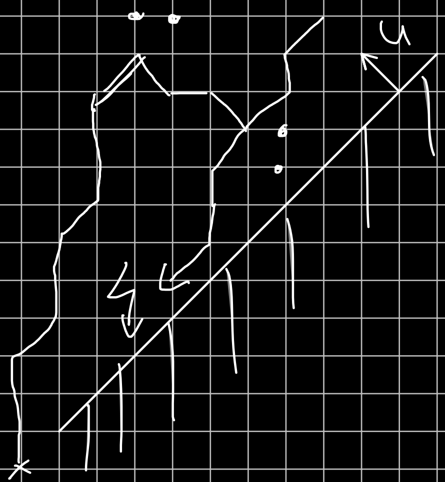
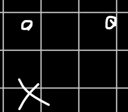


$d_u(\mathcal{U}) = d_u($



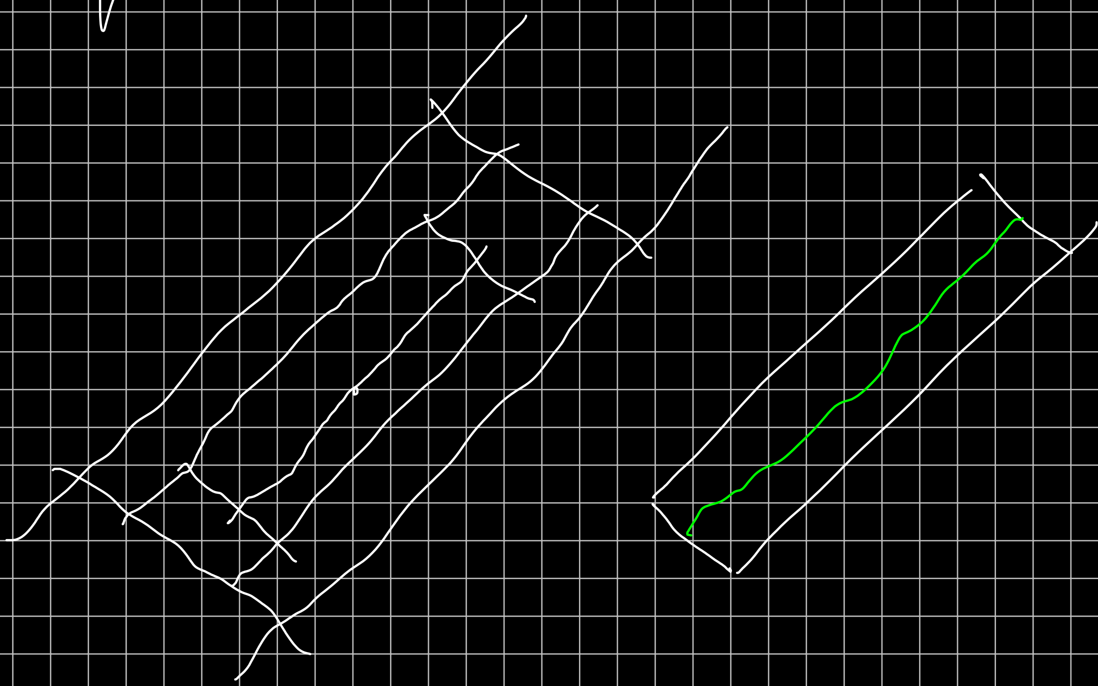
$)$

Th (Toninelli, Biroli '08) Spiral

$$q_c = \tilde q_c = 1 - p_c^{OP}$$

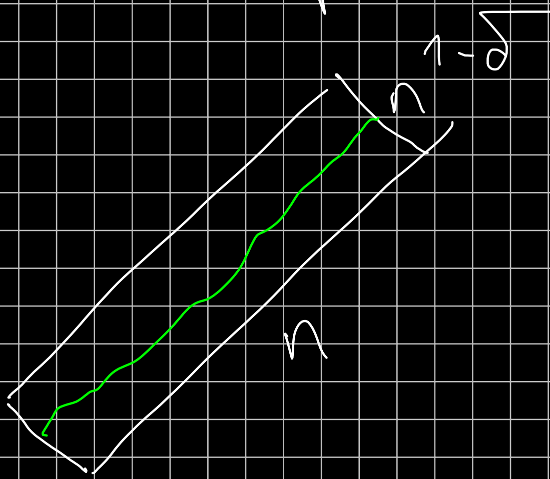
Th (Toninelli, Biroli '08) Spiral

$$\theta(q_c) > 0$$



Th. (Duminil-Copin, Tassion, Teixeira '17)

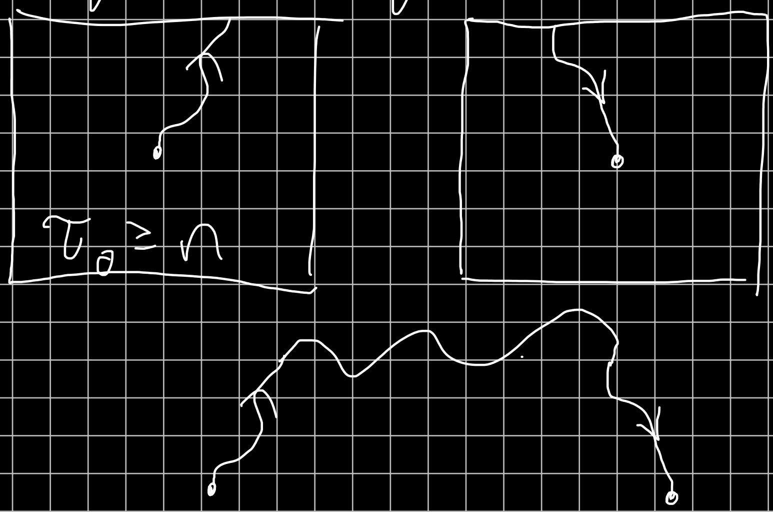
$$\mathbb{P}^{op}_{p_c}\Big(\quad \Big) > 1-\varepsilon$$

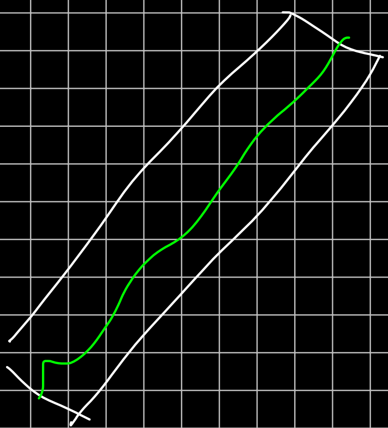


4. Conclusion

a. Open problems

- $q_c = \tilde{q}_c$

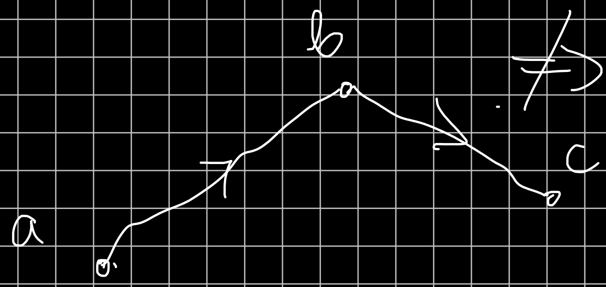


- 

for GOSP

6. Warnings about subcrit. bootstrap

- "no FKG"



$\nexists\, a \to c$

- "no BK / domain Markov"