Bootstrap percolation is local

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joint with Augusto Teixiera

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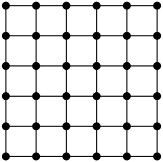

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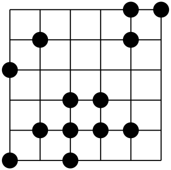

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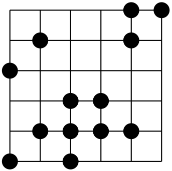

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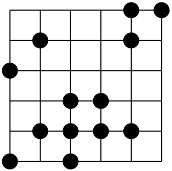

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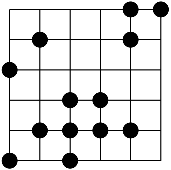

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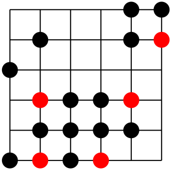

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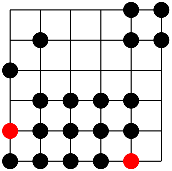

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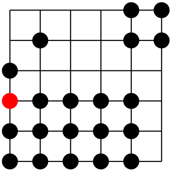

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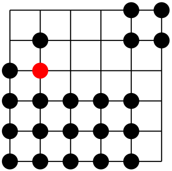

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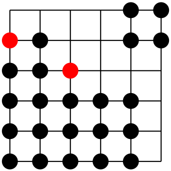

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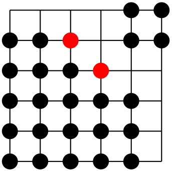

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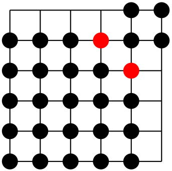

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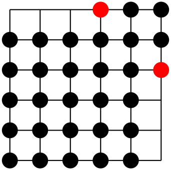

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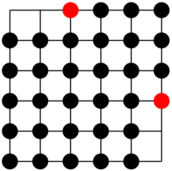

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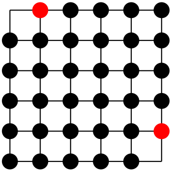

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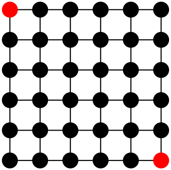

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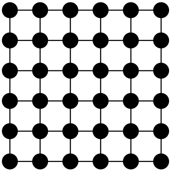

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Exercise 3.

Prove that there are $(n/(e+o(1)))^{2n}$ configurations as in Exercise 1.

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- Low temperature regime: all bounds will hold a.a.s. as $p \to 0$.

Previous results

- [Van Enter'87] For all $p > 0$ we have $\tau < \infty$ a.s.

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- [Van Enter'87] Trivial transition
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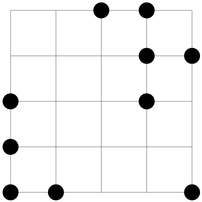

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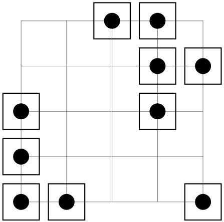

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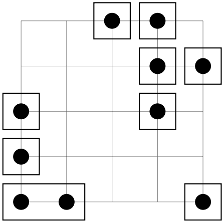

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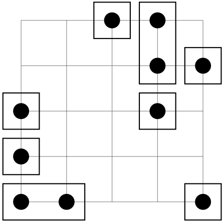

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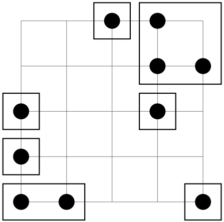

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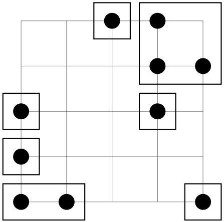

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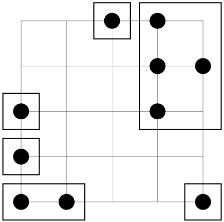

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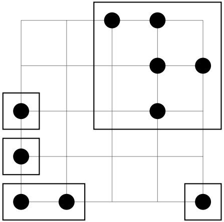

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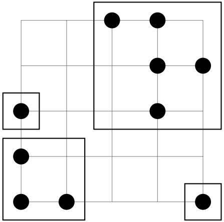

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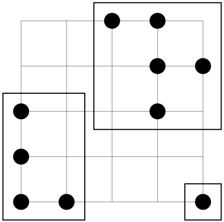

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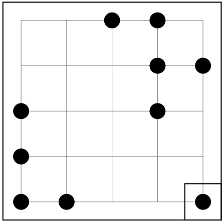

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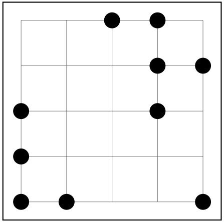

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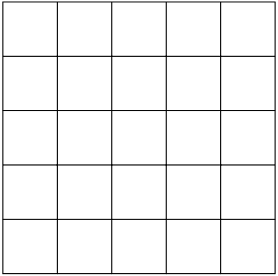

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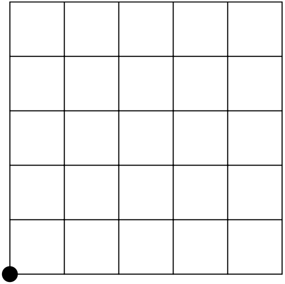

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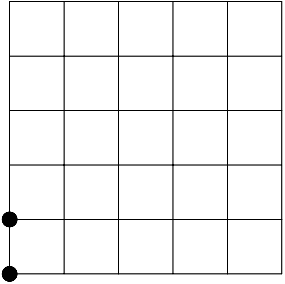

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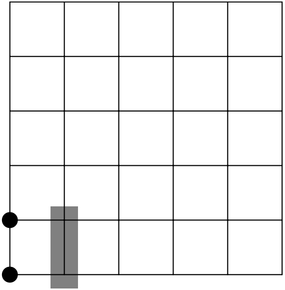

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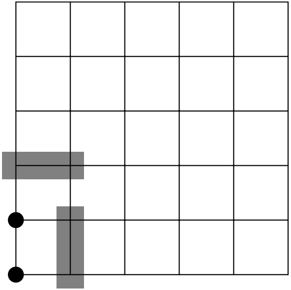

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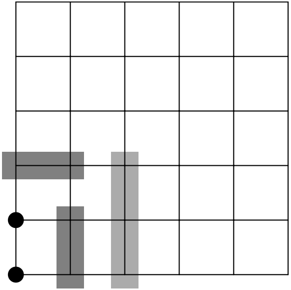

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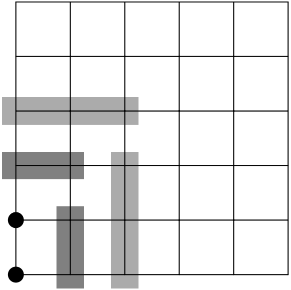

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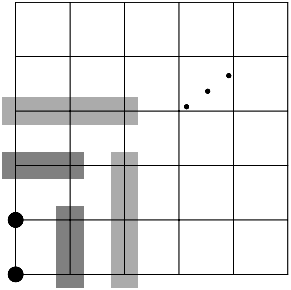

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- [Holroyd'03] For every $\varepsilon > 0$

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- [H–Morris'19] For some $c, C > 0$, for a related model

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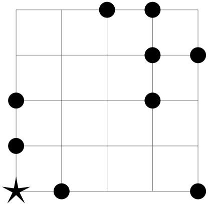

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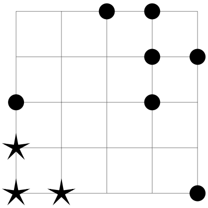

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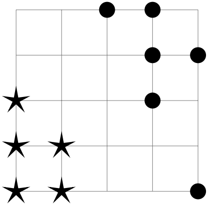

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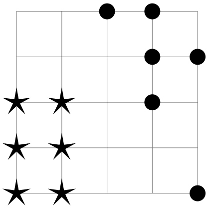

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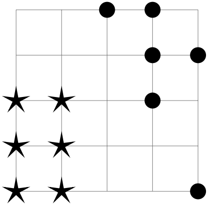

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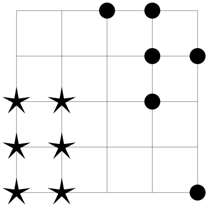

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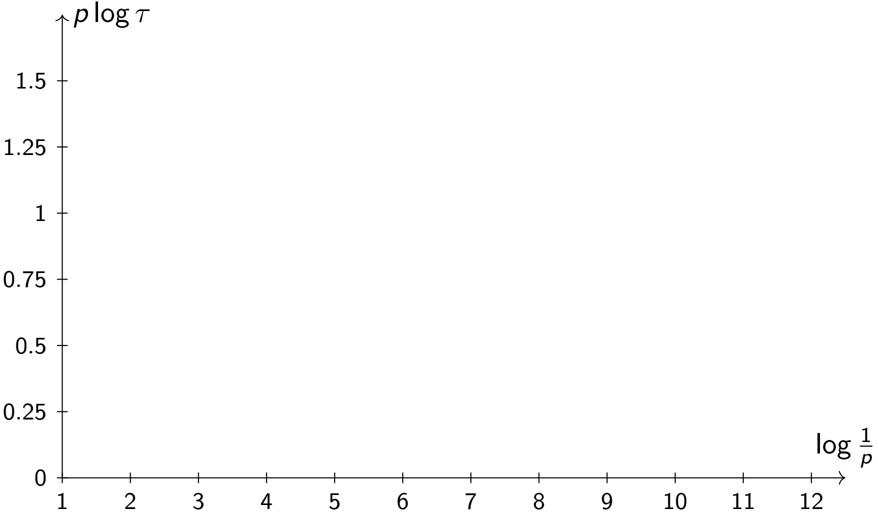

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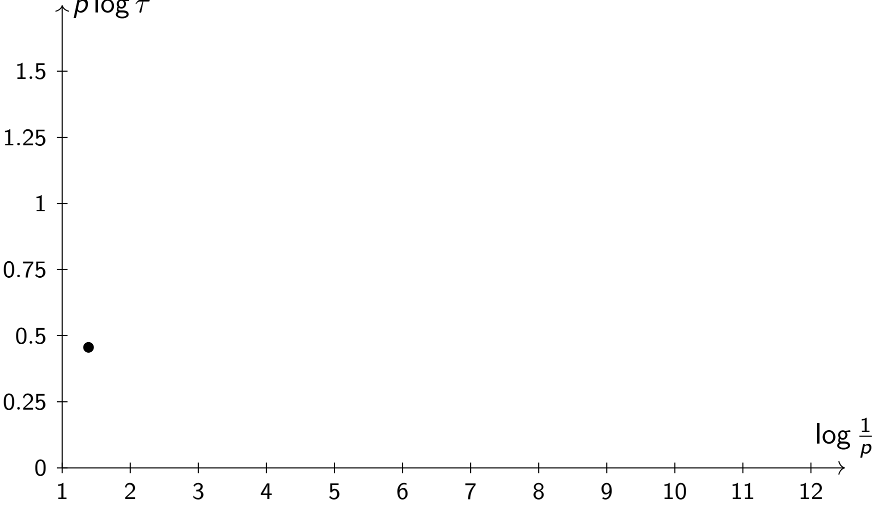

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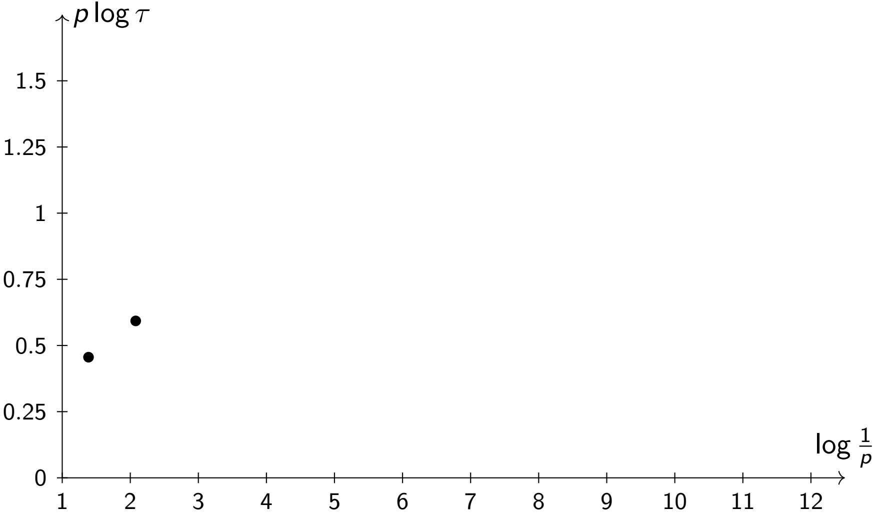

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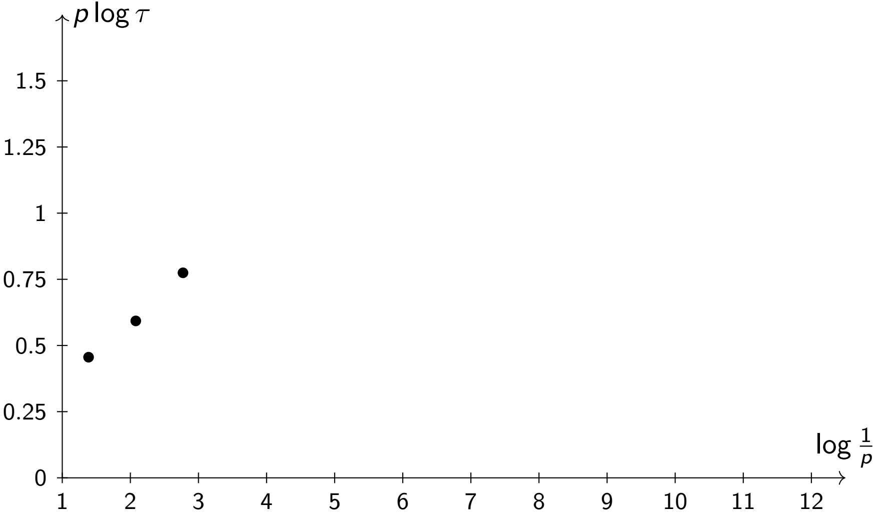

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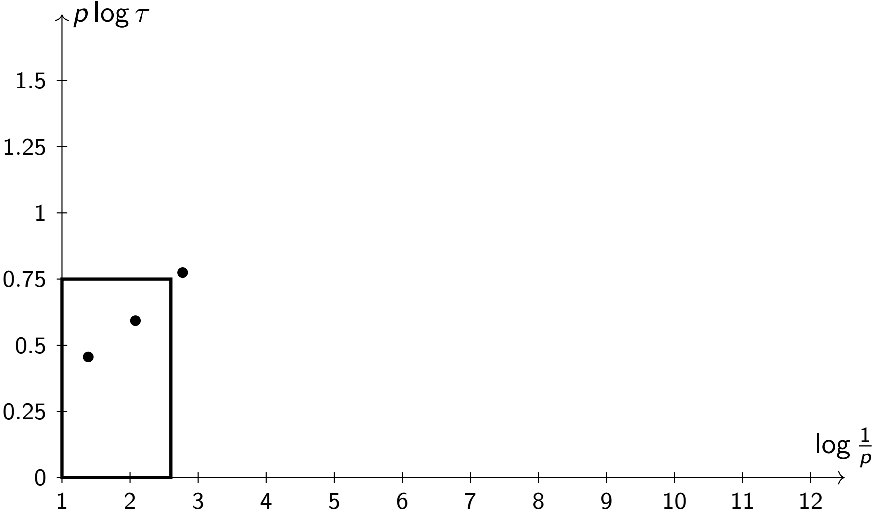

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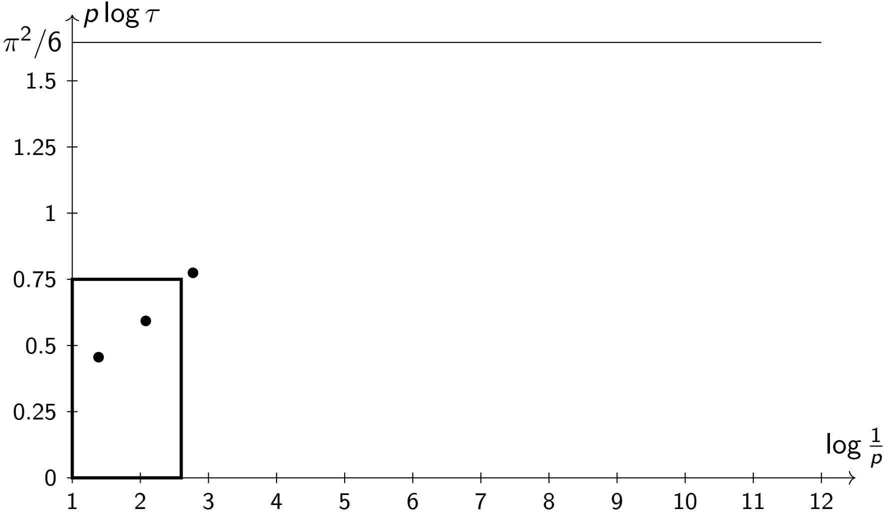

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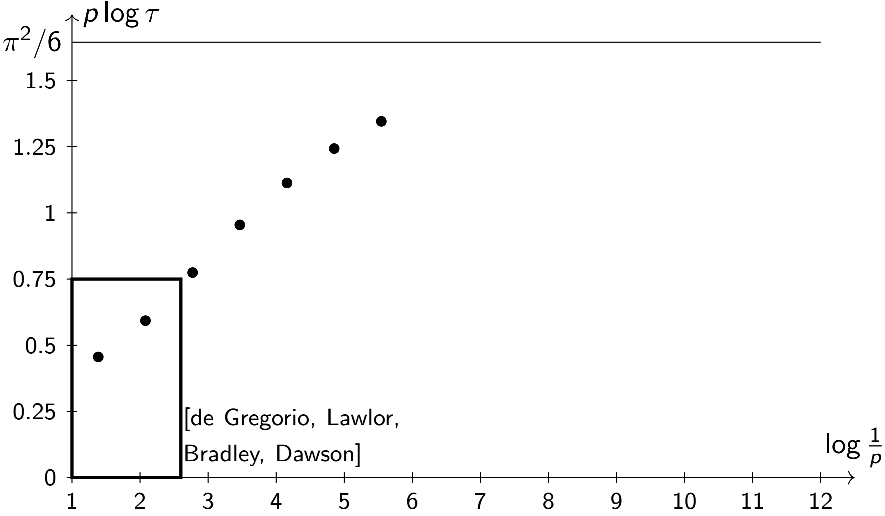

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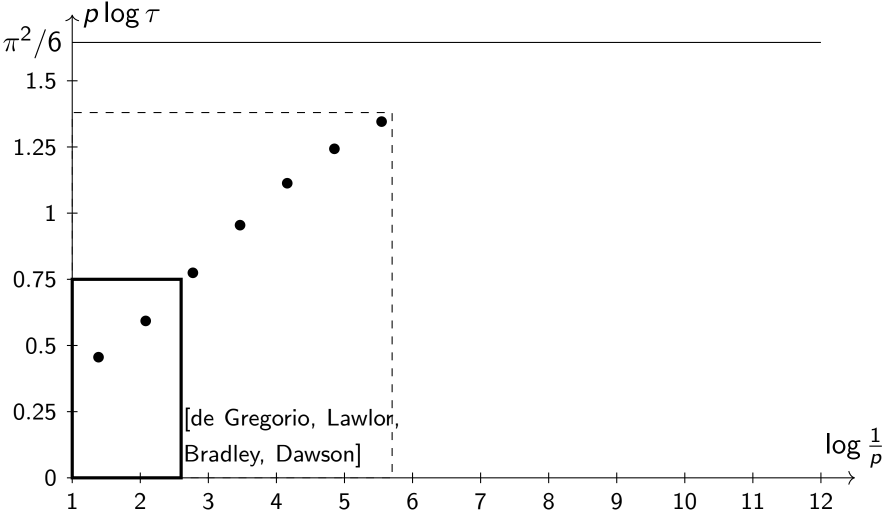

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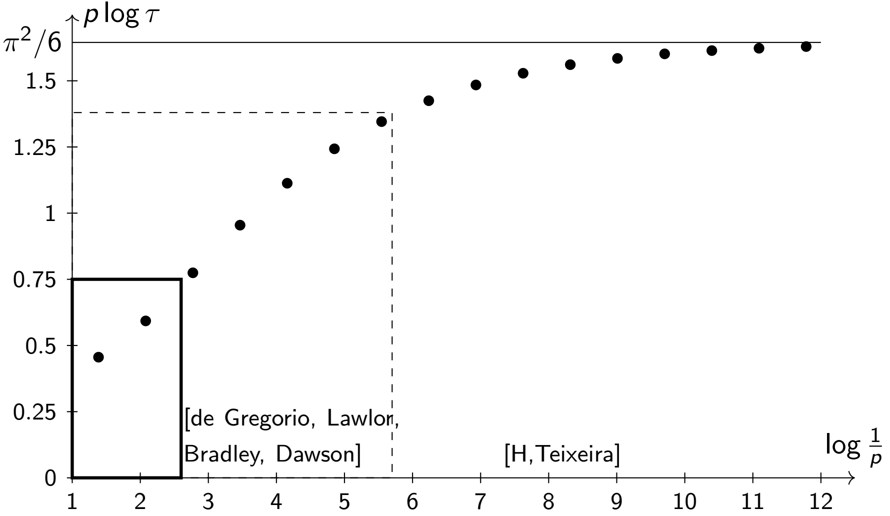

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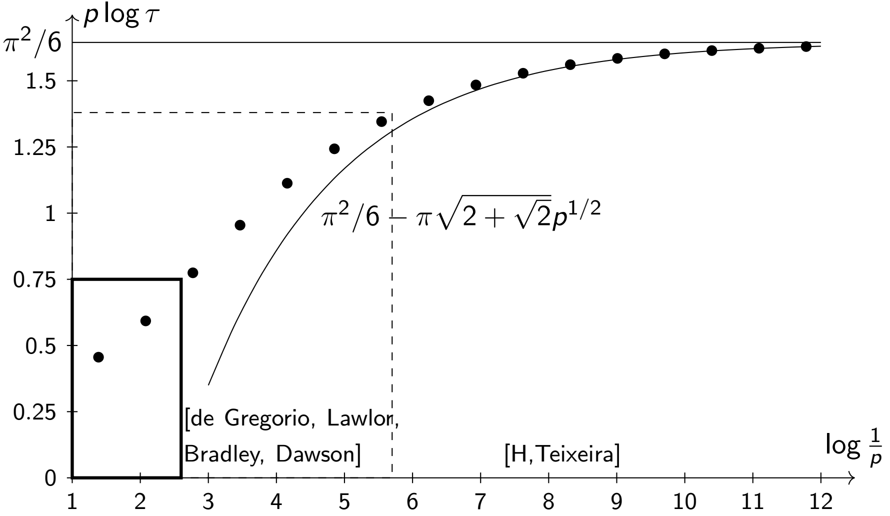

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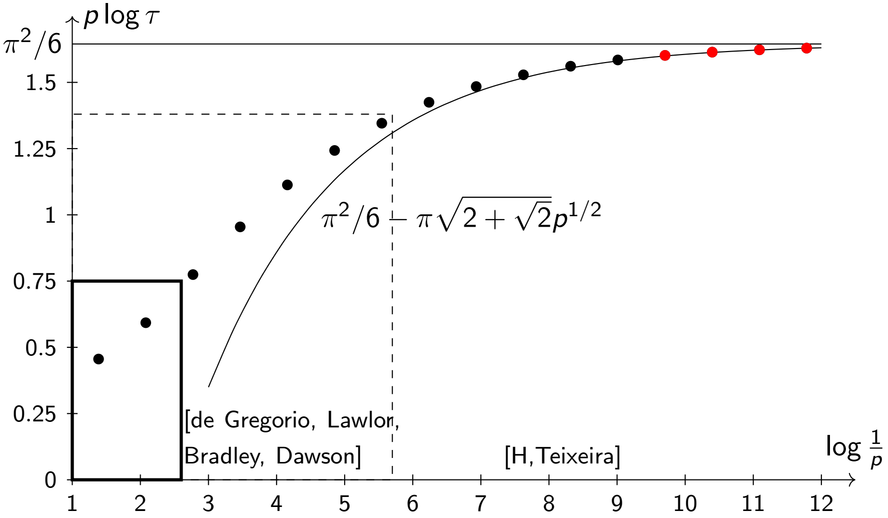

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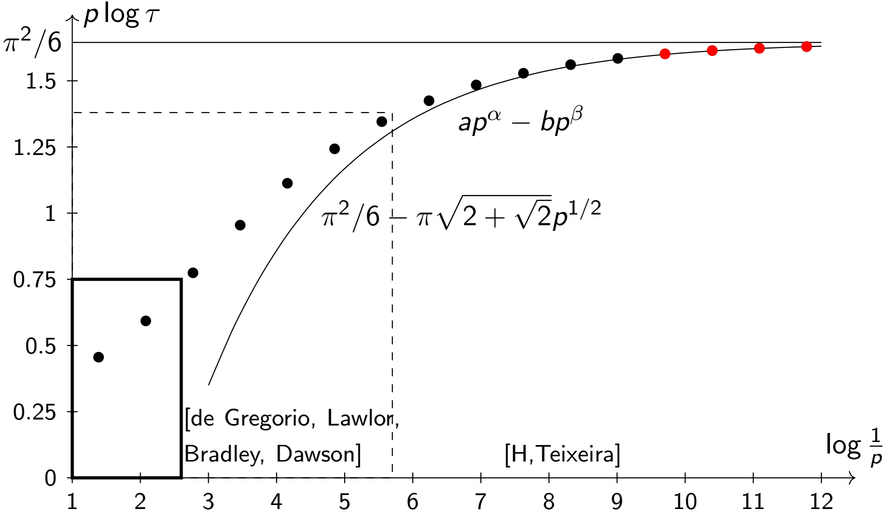

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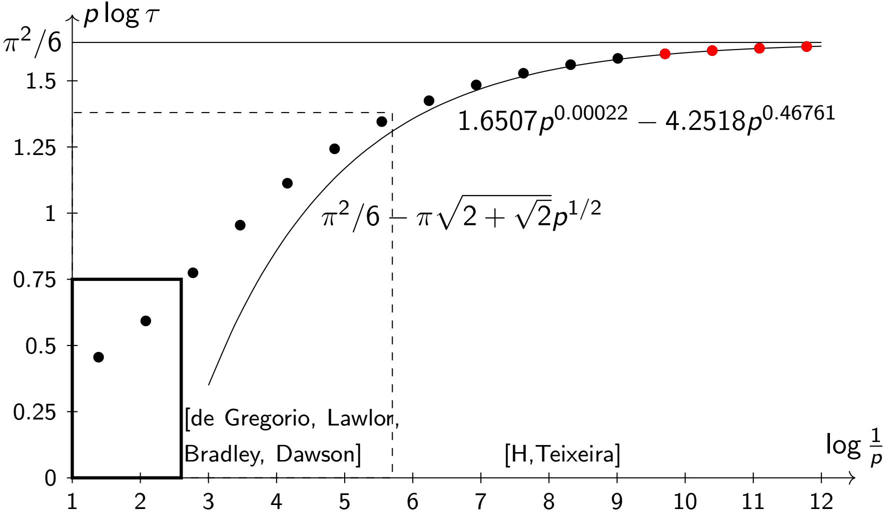

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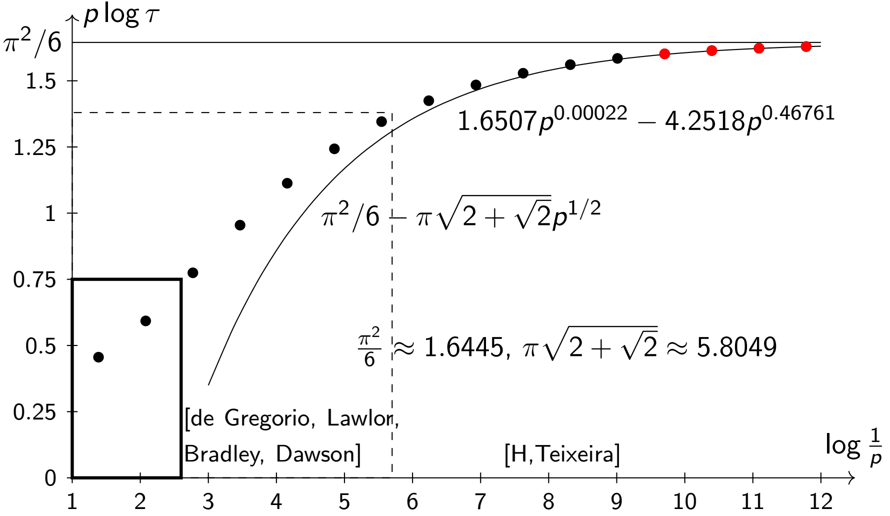

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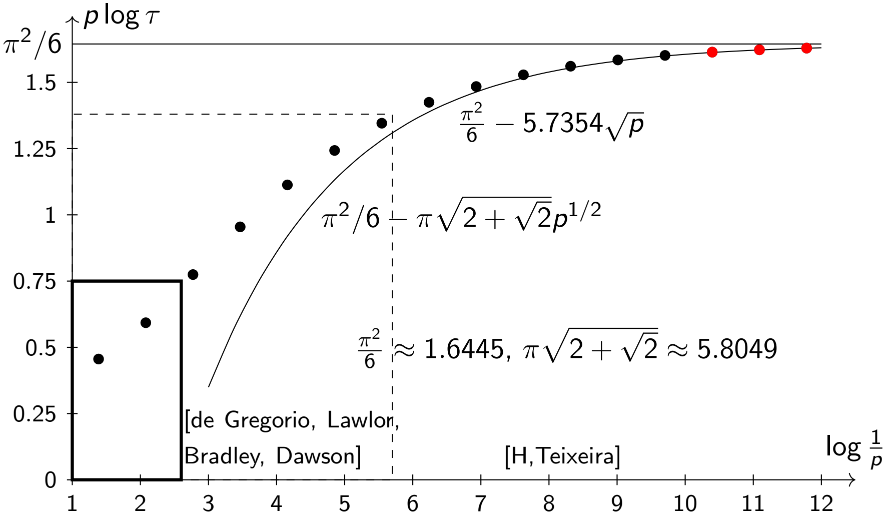

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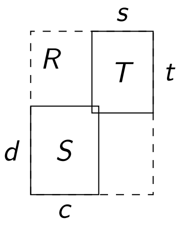

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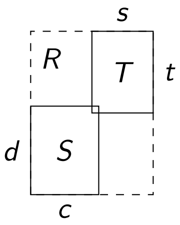

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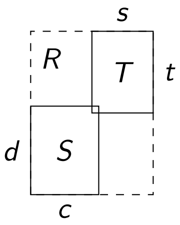

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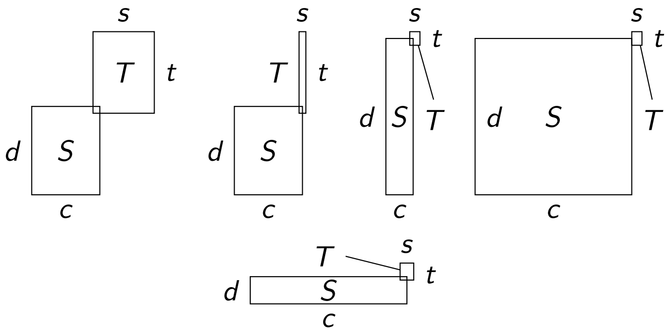
Definition (Internally filled rectangle)

R is *IF*, if • in it are enough to make it completely •.

Lemma

R is IF, iff $\exists S, T \subsetneq R$ disjointly IF such that $S \vee T = R$.



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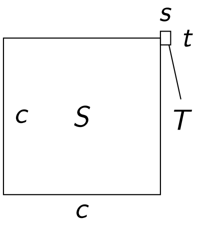

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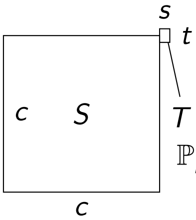

$\mathbb{P}_p(S \to (S \vee T))$

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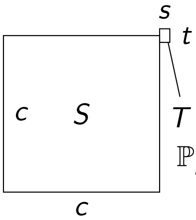

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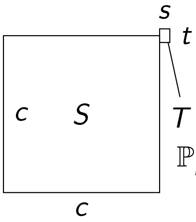

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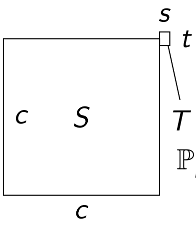

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Lemma (A priori bound)

$$\mathbb{P}_p(T \text{ is } IF) \leqslant (2sp)^{s+t-1}.$$

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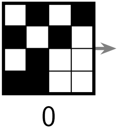

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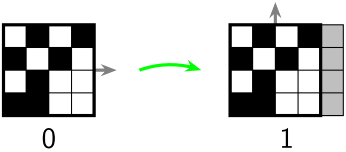

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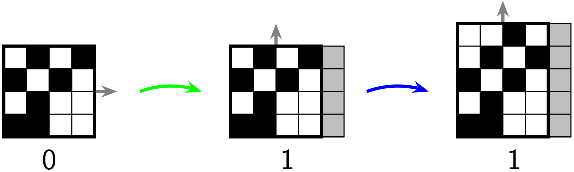

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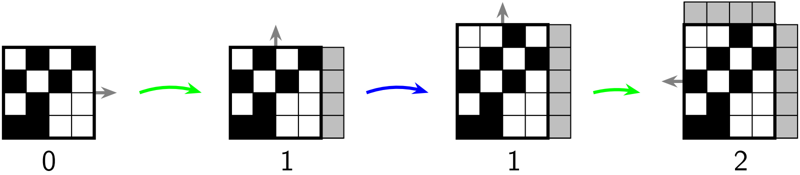

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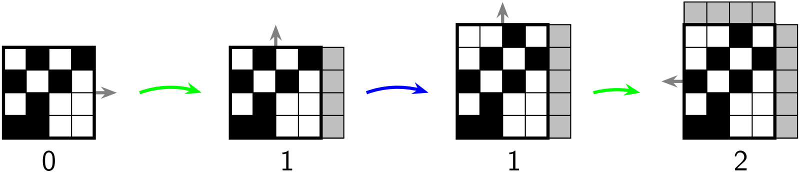

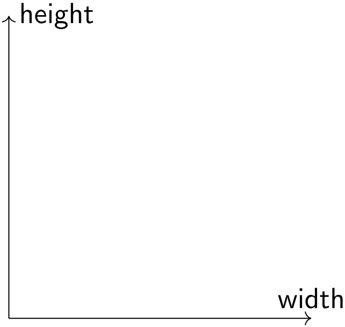

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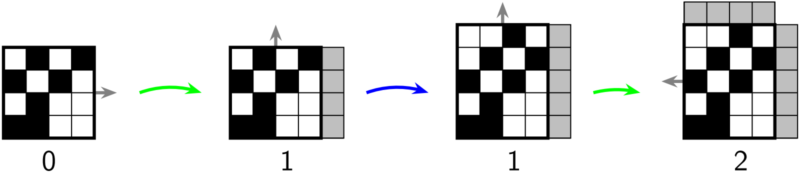

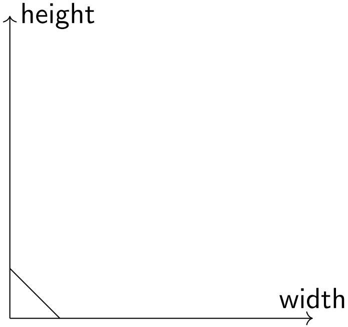

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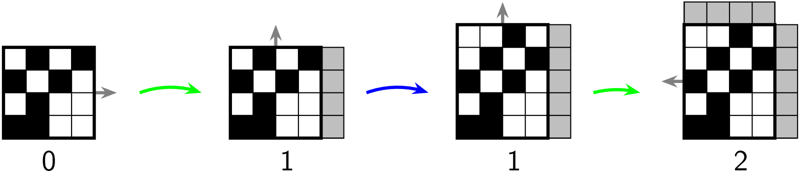

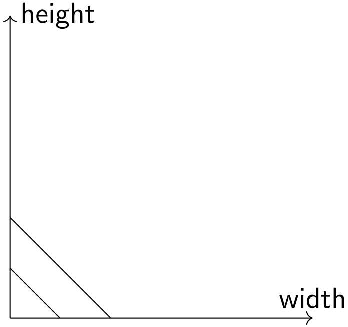

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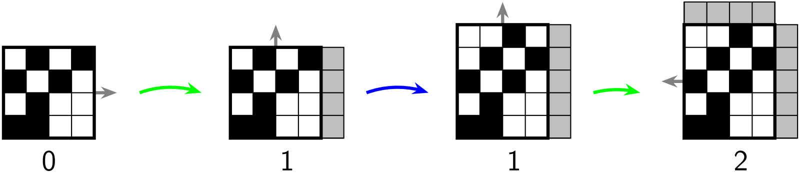

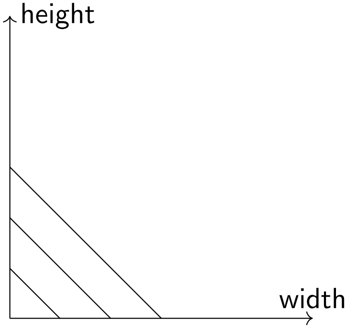

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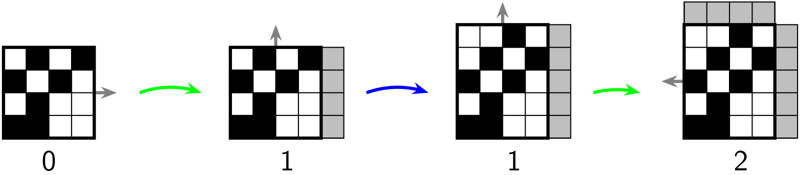

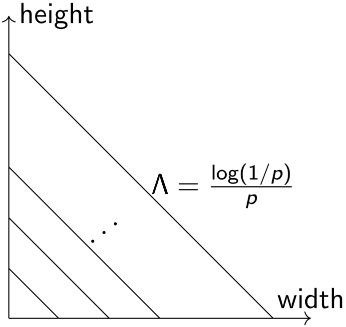

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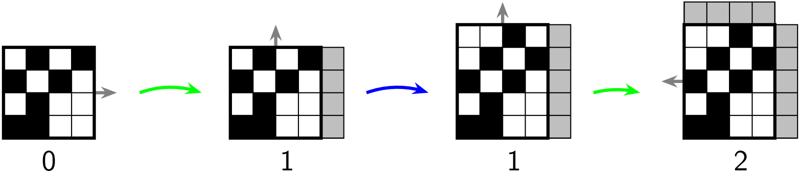
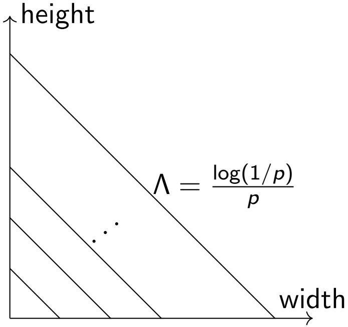
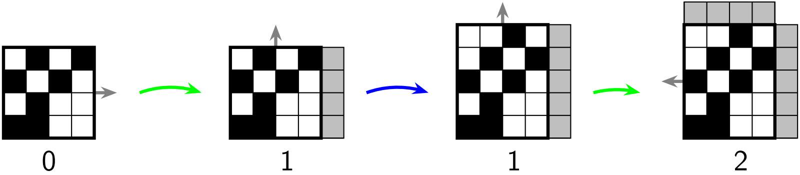
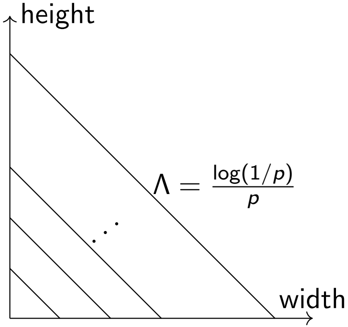




time Λ^2

Algorithm: why is locality useful?





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memory Λ

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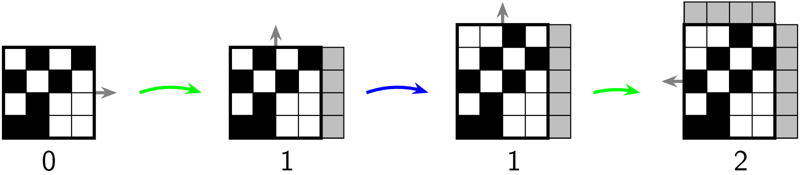

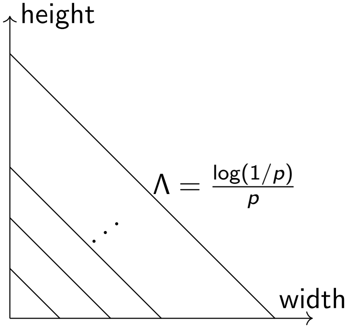

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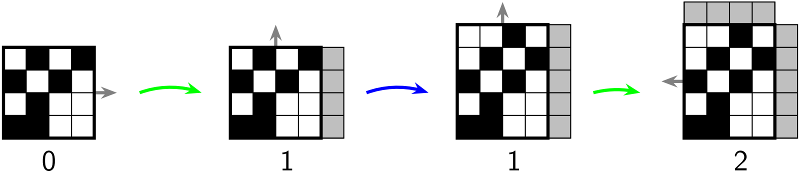

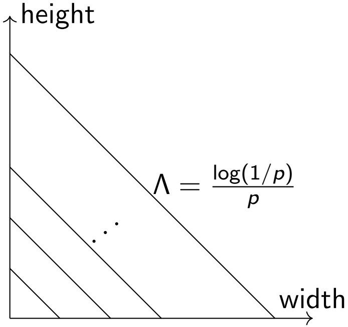

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store logs, $e^a + e^b = e^{a \vee b}(1 + e^{-|a-b|})$

Upper bound: framed rectangle Markov chain

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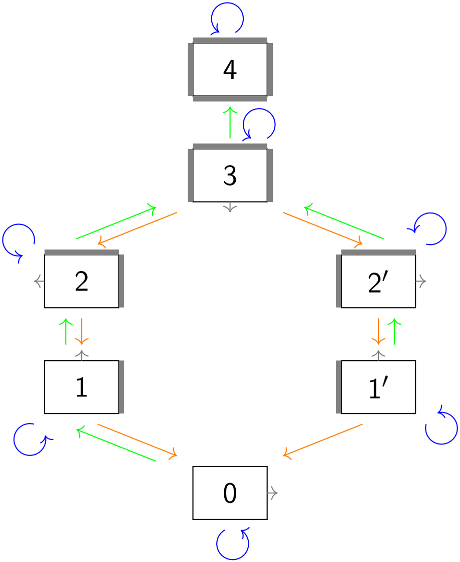

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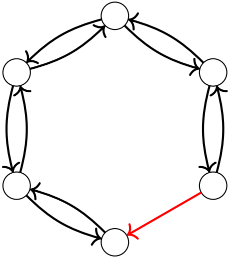

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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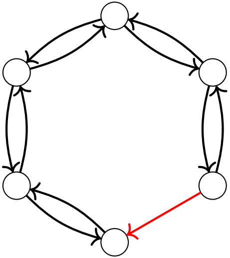

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & {\color{red}0} \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathrm{Spec} = \left\{ -\sqrt{2+\sqrt{2}}, -1, -\sqrt{2-\sqrt{2}}, \sqrt{2-\sqrt{2}}, 1, \sqrt{2+\sqrt{2}} \right\}$$

Lower bound: not much more

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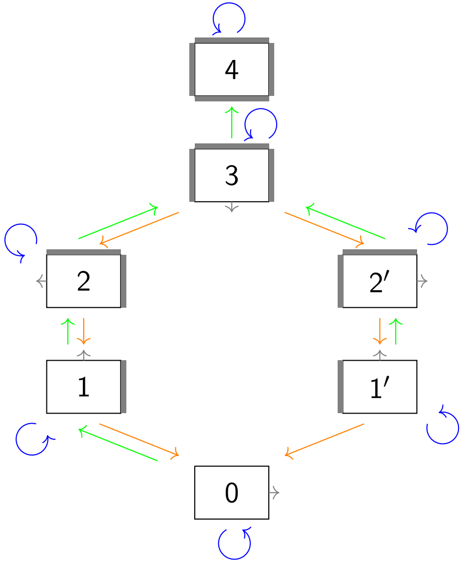
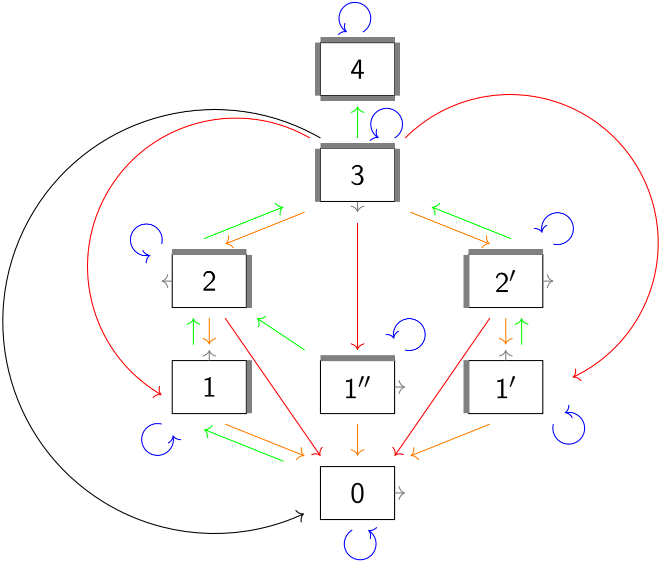


Lower bound: more, but not much more



Lower bound: more, but not much more

$$
\begin{pmatrix}
 & \varepsilon & & & & \\
1/\varepsilon & & \varepsilon & & & \\
 & 1/\varepsilon & & \varepsilon & & \\
 & & 1/\varepsilon & & 1/\varepsilon & \\
 & & & \varepsilon & & 1/\varepsilon \\
1/\varepsilon & & & & \varepsilon &
\end{pmatrix}
$$

Lower bound: more, but not much more

$$\begin{pmatrix}
 & \varepsilon & & & & \\
1/\varepsilon & & \varepsilon & & & \\
\color{red}{1/\varepsilon} & 1/\varepsilon & & \varepsilon & & \\
\color{red}{1/\varepsilon} & \color{red}{2/\varepsilon} & 1/\varepsilon & & 1/\varepsilon & \color{red}{1/\varepsilon} \\
\color{red}{1/\varepsilon} & & & \varepsilon & & 1/\varepsilon \\
1/\varepsilon & & & & \varepsilon &
\end{pmatrix}$$

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$$\begin{pmatrix} & \varepsilon & & & & \\ 1/\varepsilon & & \varepsilon & & & \\ 1/\varepsilon & 1/\varepsilon & & \varepsilon & & \\ 1/\varepsilon & 2/\varepsilon & 1/\varepsilon & & 1/\varepsilon & 1/\varepsilon \\ 1/\varepsilon & & & \varepsilon & & 1/\varepsilon \\ 1/\varepsilon & & & & \varepsilon & \end{pmatrix}$$

$$\mathrm{Spec}' = \mathrm{Spec} + O(\varepsilon)$$

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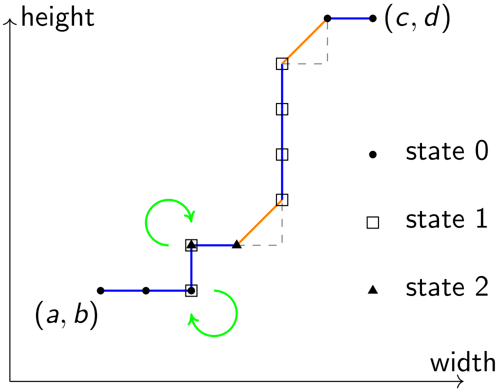

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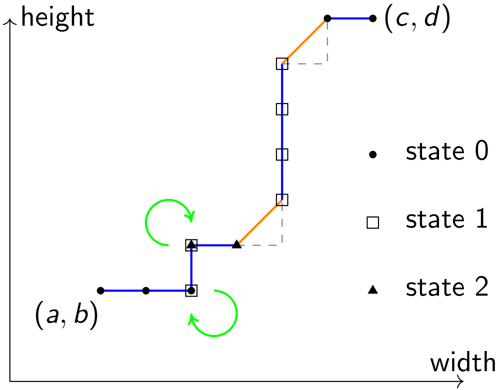

$$f(x) = -\log(1 - e^{-x})$$

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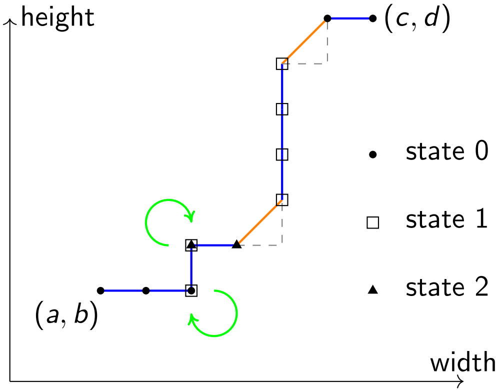

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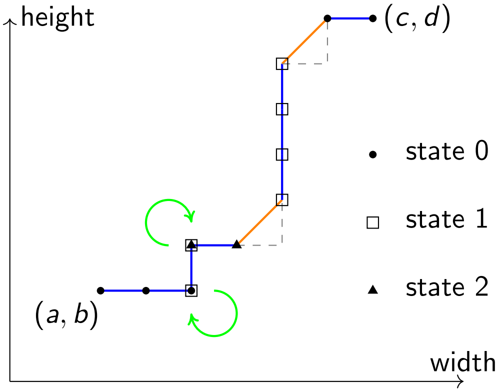

- • state 0
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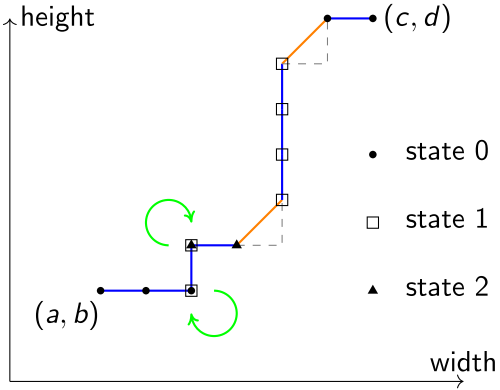

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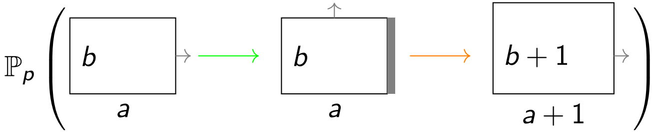

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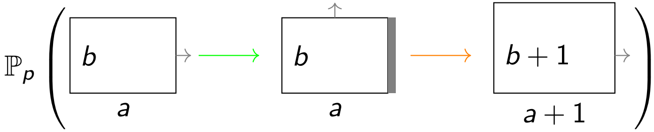

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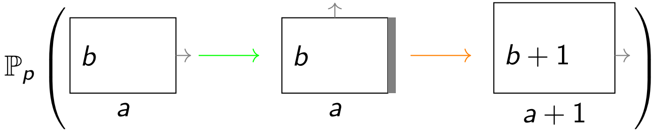

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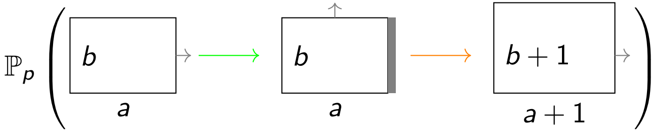

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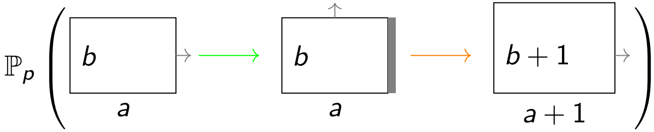

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$$\int_0^\infty \frac{\mathrm{d}x}{\sqrt{e^x-1}} = \pi$$

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?