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Bootstrap percolation and kinetically constrained models: two-dimensional universality and beyond

Ivailo Hartarsky

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Plan

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Joint with Rob Morris

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For universality see §1, 4, 6, 7, 8, , 12.

Motivation

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Proposition

Everything is useful. □

Two neighbour bootstrap percolation

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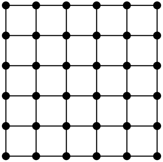
- Geometry: $\mathbb{Z}^2$.



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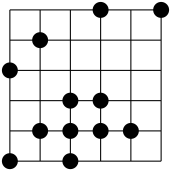

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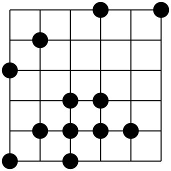

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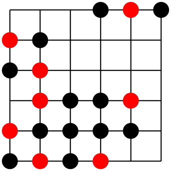

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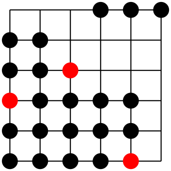

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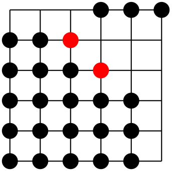

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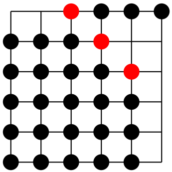

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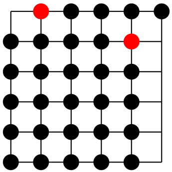

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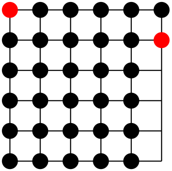

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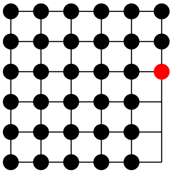

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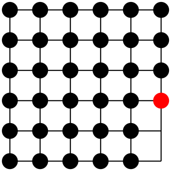

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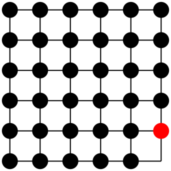

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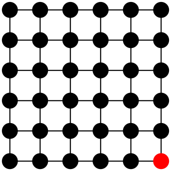

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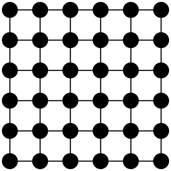

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- Low temperature regime: all bounds will hold a.a.s. as $q \to 0$.

Previous results

- [Van Enter'87] For all $q > 0$ we have $\tau^{\mathrm{BP}} < \infty$ a.s.

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- [Aizenman-Lebowitz'88] For some $c, C > 0$

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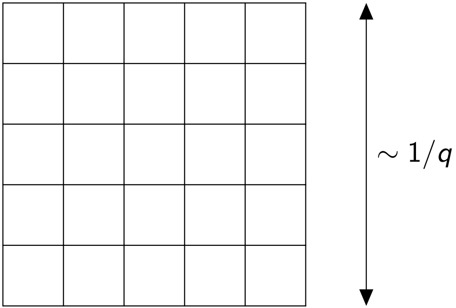

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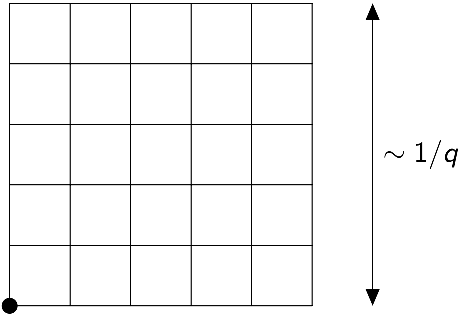

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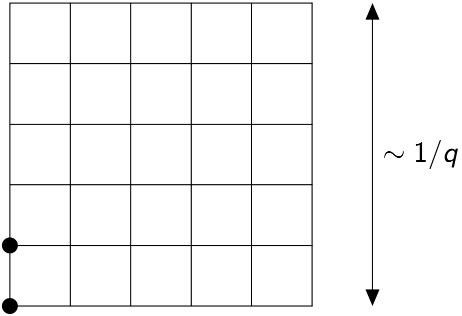

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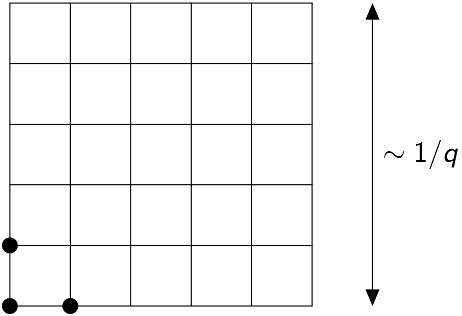

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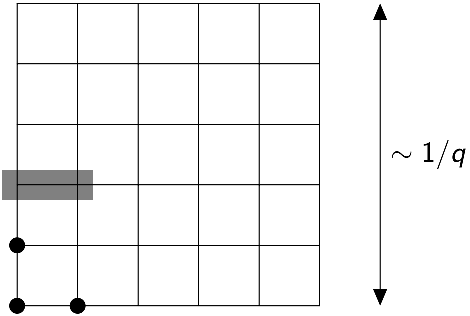

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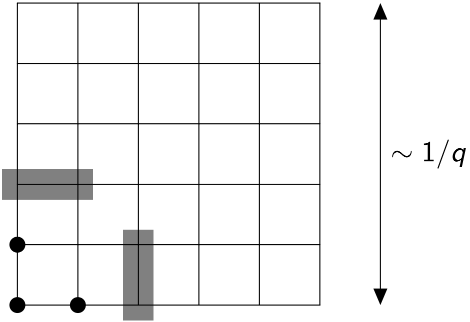

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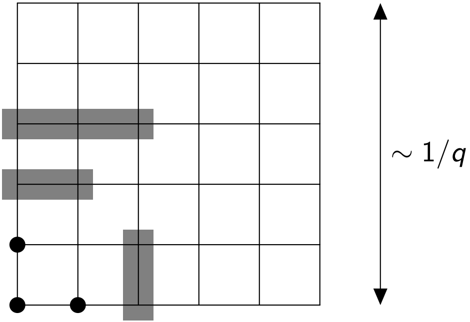

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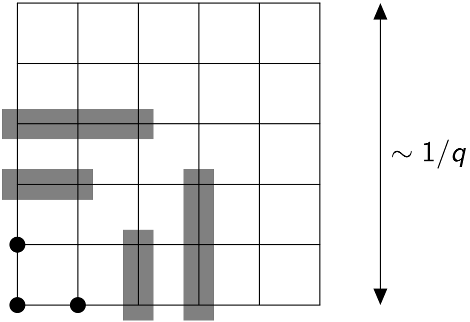

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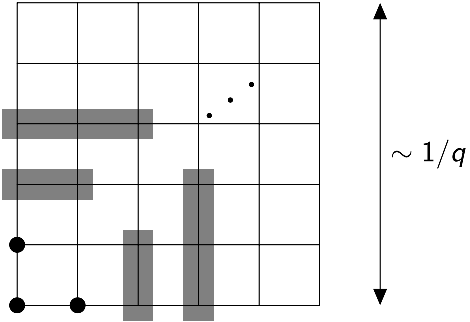

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Lower bound: Rectangles process—if $1/q < \tau^{\mathrm{BP}} < \infty$, there exists an *internally filled* rectangle of size $1/q$.

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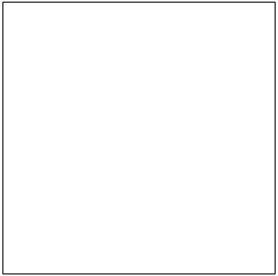

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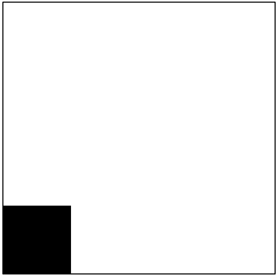

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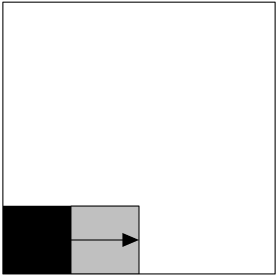

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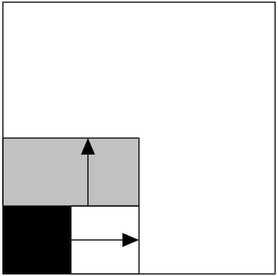

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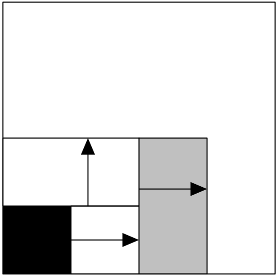

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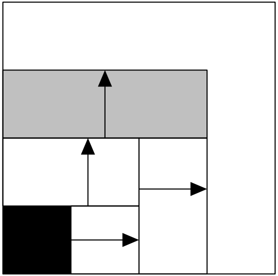

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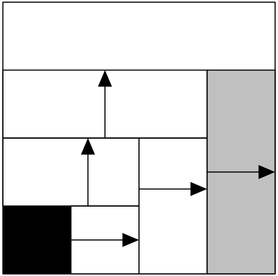

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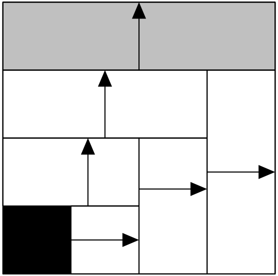

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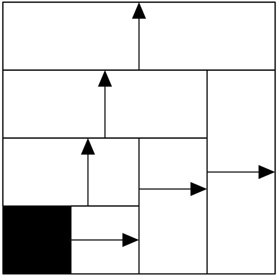

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- [Lenormand–Zarcone'84] 0.33 for $q = 0.086$, size 2.10^3.
- [Nakanishi–Takano'86] 0.27 for $q = 0.03$, size 1.10^3.
- [Adler–Stauffer–Aharony'89] 0.245 ± 0.015 for $q = 0.03$, size 2.10^4.
- [Teomy–Shokef'14] 0.274 for 8 cpu years, q=0.016, size 3.10^7.

But $\pi^2/18 \approx 0.548$.

Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] Sharp threshold
- [Gravner–Holroyd'08] For every $\varepsilon > 0$ and some $c > 0$

$$\exp\left(\frac{\pi^2 - \varepsilon}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 - c\sqrt{q}}{18q}\right).$$

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- [Bringmann–Mahlburg'12] 'Morally,' for some $c, C > 0$

$$\exp\left(\frac{\pi^2 - C(\log(1/q))^{5/2}\sqrt{q}}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 - c\sqrt{q}}{18q}\right).$$

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- [Bringmann–Mahlburg'12] Slightly better lower bound

Theorem (H–Morris'19)

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For some $C > 0$ *and all* $\varepsilon > 0$

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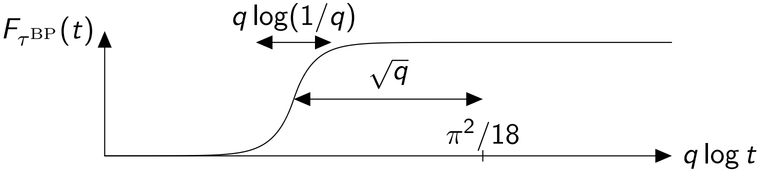

Fredrickson–Andersen 2-spin facilitated model

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- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.

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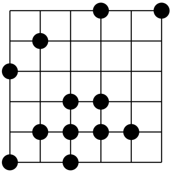

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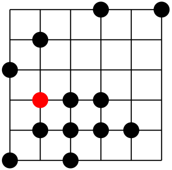

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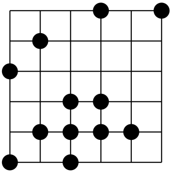

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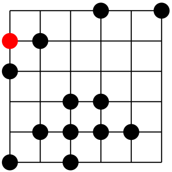

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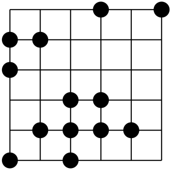

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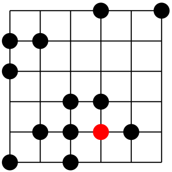

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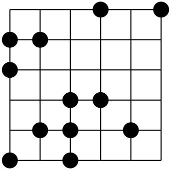

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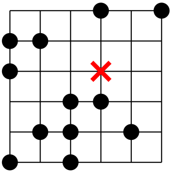

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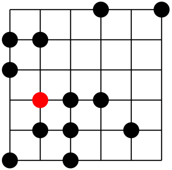

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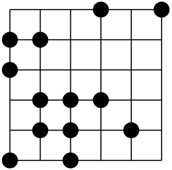

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- $\tau = \inf\{t > 0 : 0 \text{ is } \bullet\} \in [0, \infty]$.
- Initial state $Ber(q)^{\otimes \mathbb{Z}^2}$: equilibrium.
- Low temperature: $q \to 0$.

Previous results

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Idea: the bisection technique. See §2.

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$$\exp\left(\frac{\pi^2 - \varepsilon}{18q}\right) \leqslant \tau \leqslant \exp\left(\frac{C}{q^5}\right).$$

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Idea: FA-2f cannot be faster than 2-neighbour bootstrap percolation.

Previous results

- [Cancrini–Martinelli–Roberto–Toninelli'08] Trivial transition
- [CMRT08] Very rough scaling ($\log\log\tau$ up to constant)
- [Martinelli–Toninelli'19] For some $C > 0$

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Idea: take an infected frame of slightly supercritical size and move it in a FA-1f fashion.

Previous results

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Non-rigorous predictions

- [Nakanishi–Takano'86] For some $C(q) \to \infty$ as $q \to 0$

$$\tau \approx \exp\left(\frac{C(q)}{q}\right).$$

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- [Nakanishi–Takano'86] Log corrections
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Non-rigorous predictions

- [Nakanishi–Takano'86] Log corrections
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- [Toninelli–Biroli–Fisher'05] For some $C \in \mathbb{R}$

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Theorem (H–Martinelli–Toninelli'20+)

For some $C > 0$

$$\exp\left(\frac{\pi^2 - C\sqrt{q}}{9q}\right) \leqslant \tau \leqslant \exp\left(\frac{\pi^2 + C\sqrt{q}(\log(1/q))^3}{9q}\right).$$

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In particular,

$$\tau = \left(\tau^{\mathrm{BP}}\right)^{2+o(1)}.$$

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If $\tau_0 > 0$, then at τ_0 there is an internally filled rectangle of size $\sim \log(1/q)/q$ at the origin.

Theorem (HM19)

The probability of this under the stationary measure is at most

$$\exp\left(\frac{-\pi^2 + C\sqrt{q}}{9q}\right).$$

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- it can move globally without recreating itself.

https://www.youtube.com/watch?v=7pR7TNzJ_pA

Amoeba motion

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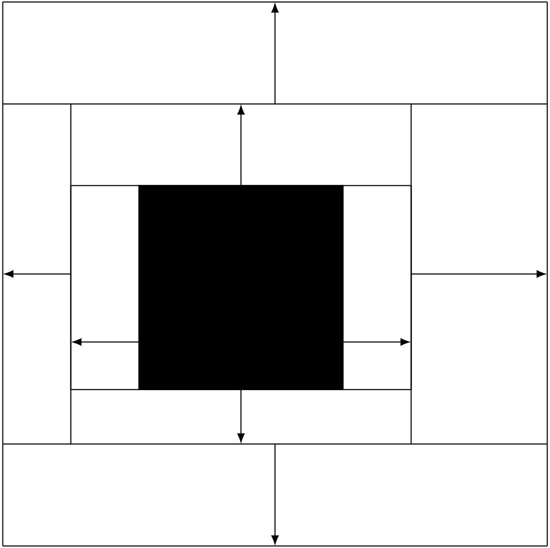

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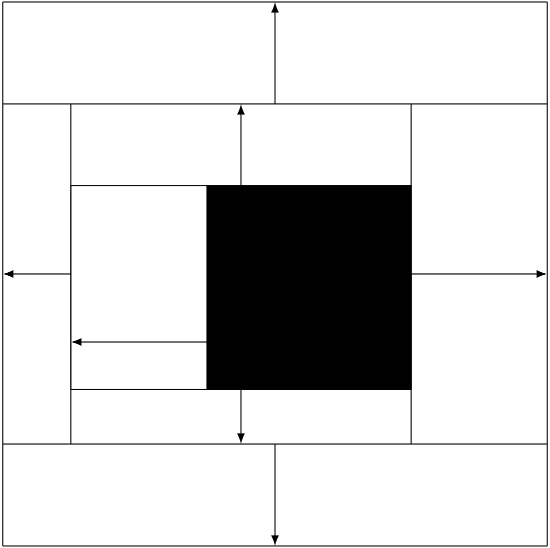

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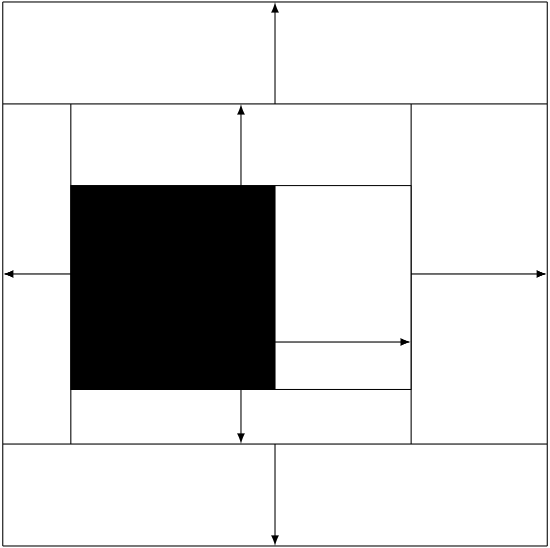

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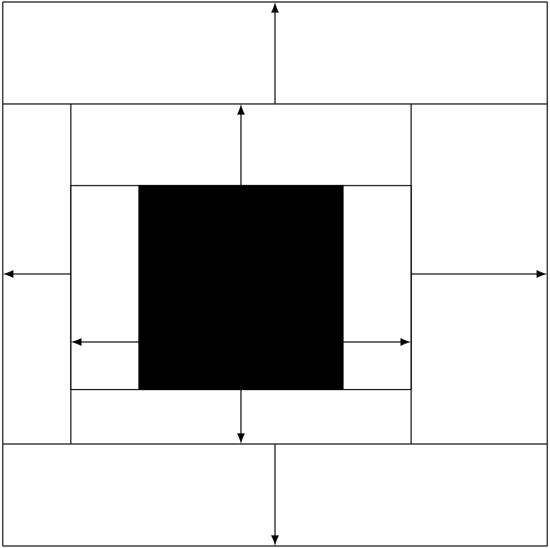

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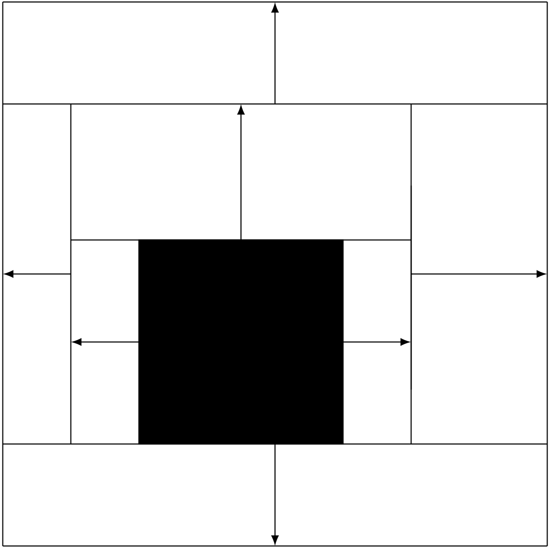

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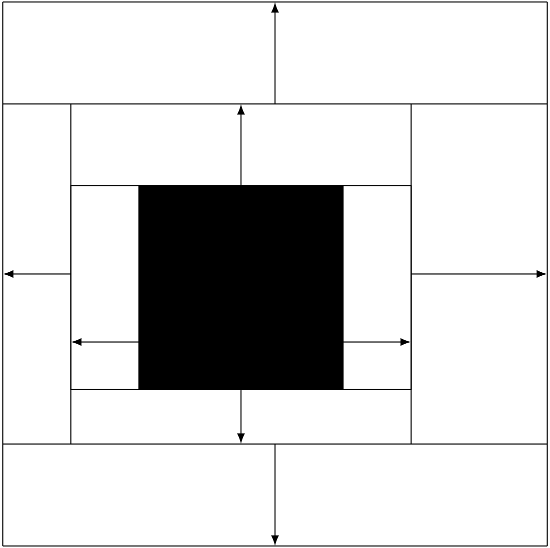

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Theorem (H–Martinelli–Toninelli'22)

The relaxation time on a box of volume V such that there is on average one amoeba is $1/|V|$ (up to logarithms).

?